Tanaka Theorem for Inelastic Maxwell Models

François Bolley, José Antonio Carrillo

TANAKA THEOREM FOR INELASTIC MAXWELL MODELS

FRANÇOIS BOLLEY AND JOSÉ A. CARRILLO

ABSTRACT. We show that the Euclidean Wasserstein distance is contractive for inelastic homogeneous Boltzmann kinetic equations in the Maxwellian approximation and its associated Kac-like caricature. This property is as a generalization of the Tanaka theorem to inelastic interactions. Even in the elastic classical Boltzmann equation, we give a simpler proof of the Tanaka theorem than the ones in [25, 27]. Consequences are drawn on the asymptotic behavior of solutions in terms only of the Euclidean Wasserstein distance.

1. Introduction

This work is devoted to contraction and asymptotic properties of the homogeneous Boltzmann-type equations for inelastic interactions in the Maxwellian approximation introduced in [5] and further analyzed in [13, 6, 7, 9, 1, 10, 2, 8]. We are basically concerned with the Boltzmann equation

$$\frac{\partial f}{\partial t} = B\sqrt{\theta(f(t))}\, Q(f,f) \tag{1.1}$$

considered in [5] and its variants. Here, $f(t,v)$ is the density for the velocity $v \in \mathbb{R}^3$ distribution of the molecules at time t, and $Q(f,f)$ is the inelastic Boltzmann collision operator defined by

$$(\varphi, Q(f,f)) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Big[\varphi(v') - \varphi(v)\Big]d\sigma\, dv\, dw \tag{1.2}$$

for any test function φ, where

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|\sigma$$

is the postcollisional velocity, $\sigma \in S^2$, $v, w \in \mathbb{R}^3$ and $0 < e \le 1$ is the constant restitution coefficient. Equation (1.1) preserves mass and momentum, but makes the kinetic energy (or temperature)

$$\theta(f(t)) = \frac{1}{3}\int_{\mathbb{R}^3}\Big|v - \int_{\mathbb{R}^3} v\, f(t,v)\, dv\Big|^2 f(t,v)\, dv$$

decrease towards 0. In particular, solutions to (1.1) tend to the Dirac mass at the mean velocity of the particles [5]. We refer to [5, 7, 28] for the discussion about the relation of this model to the inelastic hard-sphere Boltzmann equation and different ways of writing the operator. Let us just point out that the factor $B\sqrt{\theta(f(t))}$ in front of the operator in (1.1) is chosen for having the same temperature decay law as its hard-sphere counterpart [5] known as the Haff's law.

The convergence towards the monokinetic distribution has been made more precise in [7, 9, 2] by means of homogeneous cooling states. They are self-similar solutions of the homogeneous Boltzmann equation (1.1) describing the long-time asymptotics and presenting power-like tail behavior whose relevance was previously discussed in the physics literature [16, 17].

To avoid the collapse of the solution to the Dirac mass, the authors in [13] suggested the introduction of a stochastic thermostat which, at the kinetic level, is modelled by a linear diffusion term in velocity. In this framework, the density f in the velocity space obeys

$$\frac{\partial f}{\partial t} = B\sqrt{\theta(f(t))}\,Q(f,f) + A\,\theta^p(f(t))\,\triangle_v f \qquad \text{with} \qquad 0 \leq p < \frac{3}{2}\cdot \tag{1.3}$$

Existence and uniqueness for given mean velocity of a steady state to (1.3) have been shown in [15, 6, 1]. The convergence of solutions towards this steady state in all Sobolev norms has also been investigated and quantified by means of Fourier-based distances between probability measures [1].

Fourier techniques are a good toolbox and have been extremely fruitful for studying Maxwellian models in kinetic theory since Bobylev observed [3, 4] that such equations have closed forms in Fourier variables. Fourier distances are not only suitable technical tools to study the long-time asymptotics of models (1.1) and (1.3), but also they represent the first Liapunov functionals known for inelastic Boltzmann-type equations [1, 2]. In the case of the classical elastic Boltzmann equation for Maxwellian molecules, there is another known Liapunov functional, namely, the Tanaka functional [25], apart from the H-functional for which no counterpart is known in inelastic models.

The Tanaka functional is the Euclidean (or quadratic) Wasserstein distance between measures in the modern jargon of optimal mass transport theory. It is defined on the set $\mathcal{P}_2(\mathbb{R}^3)$ of Borel probability measures on $\mathbb{R}^3$ with finite second moment or kinetic energy as

$$W_2(f,g) = \inf_{\pi}\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-w|^2\,d\pi(v,w)\right\}^{1/2} = \inf_{(V,W)}\left\{\mathbb{E}\left[|V-W|^2\right]\right\}^{1/2}$$

where π runs over the set of joint probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ with marginals f and g and (V,W) are all possible couples of random variables with f and g as respective laws. This functional was proven by Tanaka [25] to be non-increasing for the flow of the homogeneous Boltzmann equation in the Maxwellian case. In fact, the Tanaka functional and Fourier-based distances are related to each other [18, 12, 26], and were used to study the trend to equilibrium for Maxwellian gases. On the other hand, related simplified granular models [14] have been shown to be strict contractions for the Wasserstein distance W_2.

With this situation, a natural question arose as an open problem in [2, Remark 3.3] and [28, Section 2.8]: is the Euclidean Wasserstein distance a contraction for the flow of inelastic Maxwell models? The main results of this work answer this question affirmatively. Moreover, we shall not need to introduce Bobylev's Fourier representation of the inelastic Maxwell models working only in the physical space.

We shall show in the next section the key idea behind the proof of all results concerning contractions in W_2 distance for inelastic Maxwell models, namely, the gain part $Q^+(f,f)$

of the collision operator verifies

$$W_2(Q^+(f,f),Q^+(g,g)) \leq \sqrt{\frac{3+e^2}{4}}\, W_2(f,g)$$

for any f,g in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and any restitution coefficient $0 < e \leq 1$. Based on this property, we shall derive contraction and asymptotic properties both for (1.1) and (1.3) in Subsections 3.1 and 3.2. On one hand, we shall prove that the flow for the diffusive equation (1.3) is a strict contraction for W_2, while for the scaled equation associated to (1.1) we shall show that solutions converge in W_2 to a corresponding homogeneous cooling state, without rate but only assuming that initial data have bounded second moment. This improves the Ernst-Brito conjecture [16, 17, 7, 9, 2] since it shows that the basin of attraction of the homogenous cooling state is larger -we avoid the typical assumption of bounded moments of order $2+\delta$- if we do not ask for a rate.

Moreover, a generalization for non constant cross sections including Tanaka's theorem as a particular case will be proven in Section 4. Finally, we shall also show this generic property for the inelastic Kac model introduced in [23] as a dissipative version of Kac's caricature of Maxwellian gases [19, 20].

2. Contraction in W_2 of the gain operator

We start by summarizing the main properties of the Euclidean Wasserstein distance W_2 that we shall make use of in the rest, refering to [11, 27] for the proofs.

Proposition 1. *The space $(\mathcal{P}_2(\mathbb{R}^3), W_2)$ is a complete metric space. Moreover, the following properties of the distance W_2 hold:*

i) **Convergence of measures:** *Given $\{f_n\}_{n\geq 1}$ and f in $\mathcal{P}_2(\mathbb{R}^3)$, the following three assertions are equivalent:*
   a) *$W_2(f_n,f)$ tends to 0 as n goes to infinity.*
   b) *f_n tends to f weakly-* as measures as n goes to infinity and*

$$\sup_{n\geq 1} \int_{|v|>R} |v|^2 f_n(v)\, dv \to 0 \quad as \quad R \to +\infty.$$

   c) *f_n tends to f weakly-* as measures and*

$$\int_{\mathbb{R}^3} |v|^2 f_n(v)\, dv \to \int_{\mathbb{R}^3} |v|^2 f(v)\, dv \quad as \quad n \to +\infty.$$

iii) **Relation to Temperature:** *If f belongs to $\mathcal{P}_2(\mathbb{R}^3)$ and δ_a is the Dirac mass at a in $\mathbb{R}^3$, then*

$$W_2^2(f,\delta_a) = \int_{\mathbb{R}^3} |v-a|^2 df(v).$$

iii) **Scaling:** *Given f in $\mathcal{P}_2(\mathbb{R}^3)$ and $\theta > 0$, let us define*

$$\mathcal{S}_\theta[f] = \theta^{3/2} f(\theta^{1/2} v)$$

for absolutely continuous measures with respect to Lebesgue measure or its corresponding definition by duality for general measures; then for any f *and* g *in* $\mathcal{P}_2(\mathbb{R}^3)$*, we have*

$$W_2(\mathcal{S}_\theta[f], \mathcal{S}_\theta[g]) = \theta^{-1/2}\, W_2(f, g).$$

iv) **Convexity:** *Given* f_1*,* f_2*,* g_1 *and* g_2 *in* $\mathcal{P}_2(\mathbb{R}^3)$ *and* α *in* $[0,1]$*, then*

$$W_2^2(\alpha f_1 + (1-\alpha) f_2, \alpha g_1 + (1-\alpha) g_2) \le \alpha W_2^2(f_1, g_1) + (1-\alpha) W_2^2(f_2, g_2).$$

As a simple consequence, given f, g *and* h *in* $\mathcal{P}_2(\mathbb{R}^3)$*, then*

$$W_2(h * f, h * g) \le W_2(f, g)$$

where $*$ *stands for the convolution in* $\mathbb{R}^3$*.*

Here the convolution of the two measures h and f is defined by duality by

$$(\varphi, h * f) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi(x+y)\, dh(x)\, df(y)$$

for any test function φ on $\mathbb{R}^3$. If f is a Borel probability measure on $\mathbb{R}^3$ we shall let

$$<f>= \int_{\mathbb{R}^3} v\, df(v) = \int_{\mathbb{R}^3} v\, f(v)\, dv$$

denote its mean velocity. We shall use the same notation for densities and measures expecting that the reader will not get confused.

Let us write the collision operator Q given in (1.3) as

$$Q(f,f) = Q^+(f,f) - f \tag{2.1}$$

where $Q^+(f,f)$ is defined by

$$(\varphi, Q^+(f,f)) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f(v) f(w)\, \varphi(v')\, d\sigma\, dv\, dw \tag{2.2}$$

for any test function φ, where we recall that

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|\sigma.$$

In this section we derive a contraction property in W_2 distance of the gain operator Q^+. For that purpose, let us note that the previous definition of the gain operator can be regarded as follows: given a probability measure f on $\mathbb{R}^3$, the probability measure $Q^+(f,f)$ is defined by

$$(\varphi, Q^+(f,f)) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v)\, f(w)\, (\varphi, \Pi_{v,w})\, dv\, dw$$

where $\Pi_{v,w}$ is the uniform probability distribution on the sphere $S_{v,w}$ with center

$$c_{v,w} = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w)$$

and radius

$$r_{v,w} = \frac{1+e}{4}|v-w|.$$

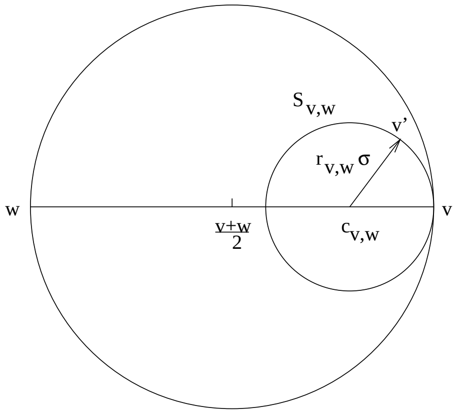

FIGURE 1. Geometry of inelastic collisions

In probabilistic terms, the gain operator is defined as an expectation:

$$Q^+(f,f) = \mathbb{E}\left[\Pi_{V,W}\right]$$

where V and W are independent random variables with law f.

Then the convexity of W_2^2 in Proposition 1 implies

$$\begin{aligned} W_2^2(Q^+(f,f), Q^+(g,g)) &= W_2^2(\mathbb{E}\left[\Pi_{V,W}\right], \mathbb{E}\left[\Pi_{X,Y}\right]) \\ &\leq \mathbb{E}\left[W_2^2(\Pi_{V,W}, \Pi_{X,Y})\right] \end{aligned} \tag{2.3}$$

where X and Y are independent random variables with law g. This observation leads us to consider the W_2 distance between uniform distributions on spheres. To this aim, we have the following general lemma:

Lemma 2. *The squared Wasserstein distance W_2^2 between the uniform distributions on the sphere with center O and radius r and the sphere with center O' and radius r' in $\mathbb{R}^3$ is bounded by $|O'-O|^2 + (r'-r)^2$.*

Proof.- We define a map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ transporting the sphere of center O and radius $r > 0$ onto the sphere with center O' and radius $r' \geq r$ in the following way:

- If $r = r'$, then we just let T be the translation map with vector $O' - O$, i.e., $T(v) = v + O' - O$.
- If $O = O'$, then we just let T be the dilation with factor $\frac{r'}{r}$ centered at O, i.e., $T(v) = \frac{r'}{r}v$.
- If $r \neq r'$, then we consider the only point $\Omega \in \mathbb{R}^3$ verifying that

$$\frac{1}{r}(O - \Omega) = \frac{1}{r'}(O' - \Omega),$$

that is,

$$\Omega = O + \frac{r}{r'-r}(O' - O).$$

Then we let T be the dilation with factor $\frac{r'}{r}$ centered at Ω, that is, we let $T(v) = \Omega + \frac{r'}{r}(v - \Omega)$. Such a construction of the point Ω and the map T is sketched in Figure 2 in the case of non interior spheres.

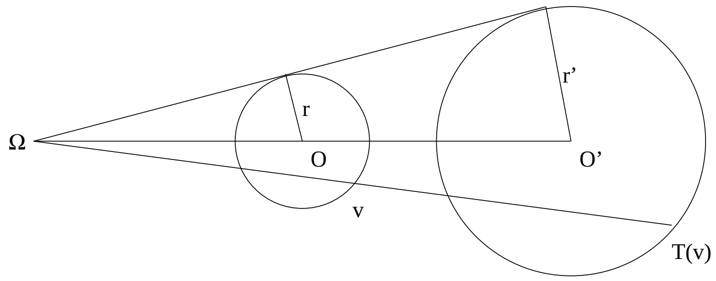

FIGURE 2. Sketch of the computation of the Euclidean cost of transporting spheres to spheres. Transport lines are just rays from the point Ω.

Let $\mathcal{U}_{O,r}$ and $\mathcal{U}_{O',r'}$ denote the uniform distributions on the corresponding spheres. Then the transport plan π given by

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \eta(v,w)\, d\pi(v,w) = \int_{\mathbb{R}^3} \eta(v, T(v))\, d\mathcal{U}_{O,r}(v)$$

for all test functions $\eta(v,w)$ has $\mathcal{U}_{O,r}$ and $\mathcal{U}_{O',r'}$ as marginals by construction of T. Using this transference plan in the definition of the Euclidean Wasserstein distance, we finally conclude

$$W_2^2(\mathcal{U}_{O,r}, \mathcal{U}_{O',r'}) \le \int_{\mathbb{R}^3} |v - T(v)|^2\, d\mathcal{U}_{O,r}(v) = \left(\frac{r'-r}{r}\right)^2 \int_{\mathbb{R}^3} |v - \Omega|^2\, d\mathcal{U}_{O,r}(v)$$

that can be computed explicitly, giving

$$W_2^2(\mathcal{U}_{O,r}, \mathcal{U}_{O',r'}) \le |O' - O|^2 + (r' - r)^2$$

and finishing the proof. $\square$

This lemma, using the notation $a = v - x$ and $b = w - y$, for fixed values v, w, x, y in $\mathbb{R}^3$, implies that

$$\begin{aligned} W_2^2(\Pi_{v,w}, \Pi_{x,y}) &\le |c_{v,w} - c_{x,y}|^2 + |r_{v,w} - r_{x,y}|^2 \\ &\le \left|\frac{3-e}{4}\, a + \frac{1+e}{4}\, b\right|^2 + \left(\frac{1+e}{4}\right)^2 |a-b|^2 \\ &= \frac{5 - 2\,e + e^2}{8}\, |a|^2 + \frac{(1+e)^2}{8}\, |b|^2 + \frac{1-e^2}{4}\, a \cdot b; \end{aligned}$$

here $a \cdot b$ denotes the scalar product between a and b in $\mathbb{R}^3$ and the bound in

$$\begin{aligned} |r_{v,w} - r_{x,y}|^2 &= \left(\frac{1+e}{4}\right)^2 \big||v-w| - |x-y|\big|^2 \\ &\le \left(\frac{1+e}{4}\right)^2 \big|(v-w) - (x-y)\big|^2 = \left(\frac{1+e}{4}\right)^2 |a-b|^2 \end{aligned}$$

follows from the Cauchy-Schwarz inequality

$$(v-w)\cdot(x-y) \le |v-w|\,|x-y|. \tag{2.4}$$

Therefore, by (2.3),

$$W_2^2(Q^+(f,f),Q^+(g,g)) \le \frac{5-2\,e+e^2}{8}\,\mathbb{E}\left[|V-X|^2\right] + \frac{(1+e)^2}{8}\,\mathbb{E}\left[|W-Y|^2\right] + \frac{1-e^2}{4}\,\mathbb{E}\left[(V-X)\cdot(W-Y)\right].$$

Let moreover (V,X) and (W,Y) be two independent optimal couples in the sense that

$$W_2^2(f,g) = \mathbb{E}\left[|V-X|^2\right] = \mathbb{E}\left[|W-Y|^2\right].$$

Then

$$\mathbb{E}\left[(V-X)\cdot(W-Y)\right] = \mathbb{E}\left[(V-X)\right]\,\cdot\,\mathbb{E}\left[(W-Y)\right] = \left|\, <f> - <g>\, \right|^2$$

by independence. Collecting all terms leads to the following key estimate and contraction property:

Proposition 3. *If f and g belong to $\mathcal{P}_2(\mathbb{R}^3)$, then*

$$W_2^2(Q^+(f,f),Q^+(g,g)) \le \frac{3+e^2}{4}\,W_2^2(f,g) + \frac{1-e^2}{4}\left|\, <f> - <g>\, \right|^2$$

for any restitution coefficient $0 < e \le 1$. As a consequence, given f and g in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity, then

$$W_2(Q^+(f,f),Q^+(g,g)) \le \sqrt{\frac{3+e^2}{4}}\,W_2(f,g).$$

The case of equality is addressed in the following statement:

Proposition 4. *Let f and g belong to $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and temperature, where g is absolutely continuous with respect to Lebesgue measure with positive density. If*

$$W_2(Q^+(f,f),Q^+(g,g)) = \sqrt{\frac{3+e^2}{4}}\,W_2(f,g).$$

for some restitution coefficient $0 < e \le 1$, then $f = g$.

Proof.- It is necessary that the equality holds at each step of the arguments in Proposition 3. In particular, (2.4) holds as an equality, that is,

$$\frac{V-W}{|V-W|} = \frac{X-Y}{|X-Y|}$$

almost surely in the above notation. Then, since g is absolutely continuous with respect to Lebesgue measure with positive density, one can proceed as in [25, Lemma 9.1] to show that $f = g$. We sketch the proof for the sake of the reader. Since g is absolutely continuous with respect to Lebesgue measure, there exists [27] a Borel map $u : \mathbb{R}^3 \to \mathbb{R}^3$ such that

f be the image measure of g by u, and in probabilistic terms $V = u(X)$ and $W = u(Y)$ almost surely. Hence

$$\frac{u(x) - u(y)}{|u(x) - u(y)|} = \frac{x - y}{|x - y|} \tag{2.5}$$

almost everywhere for Lebesgue measure since X and Y are independent and since their law g has positive density. We leave the reader to check [27, Exercise 7.25] that this implies the existence of constants ω_1 and ω_2 such that $u(x) = \omega_1 + \omega_2 x$. First of all $\omega_2^2 = 1$ since f and g have same temperature. Then identity (2.5) forces $\omega_2 = 1$, implying $\omega_1 = 0$ since $<f>=<g>$, and finally $f = g$. □

3. Contractive Estimates for the Inelastic Maxwell Model

In this section, we shall derive contractive estimates in the Euclidean Wasserstein distance for solutions to the inelastic Maxwell models both in the non-diffusive and the diffusive cases.

3.1. **The non-diffusive case.** We are first concerned with solutions $f(t)$ to the Boltzmann equation (1.1) with $0 < e < 1$. After time scaling defined by

$$\tau = \frac{B}{E} \int_0^t \sqrt{\theta(f(w))}\, dw$$

with $E = \dfrac{8}{1 - e^2}$, as in [2], we get a function denoted again $f(\tau)$ for simplicity, solution to

$$\frac{\partial f}{\partial \tau} = E\, Q(f, f). \tag{3.1}$$

Theorem 5. *If f_1 and f_2 are two solutions to* (3.1) *with respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$, then*

$$W_2^2(f_1(\tau), f_2(\tau)) \le \mathrm{e}^{-2\tau}\, W_2^2(f_1^0, f_2^0) + (1 - \mathrm{e}^{-2\tau}) \left| <f_1^0> - <f_2^0> \right|^2 \tag{3.2}$$

for all $\tau \geq 0$.

Proof.- Decomposition (2.1) of the collision operator Q as

$$Q(f, f) = Q^+(f, f) - f$$

allows us to represent the solutions to (3.1) by Duhamel's formula as

$$f_i(\tau) = \mathrm{e}^{-E\tau} f_i^0 + E \int_0^\tau \mathrm{e}^{-E(\tau - s)}\, Q^+(f_i(s), f_i(s))\, ds, \qquad i = 1, 2.$$

Then the convexity of the squared Wasserstein distance in Proposition 1 and Proposition 3 imply

$$\begin{aligned} & W_2^2(f_1(\tau), f_2(\tau)) \\ & \le \mathrm{e}^{-E\tau}\, W_2^2(f_1^0, f_2^0) + E \int_0^\tau \mathrm{e}^{-E(\tau - s)}\, W_2^2\big(Q^+(f_1(s), f_1(s)), Q^+(f_2(s), f_2(s))\big)\, ds \\ & \le \mathrm{e}^{-E\tau}\, W_2^2(f_1^0, f_2^0) + E \int_0^\tau \mathrm{e}^{-E(\tau - s)} \left(\frac{3+e^2}{4}\, W_2^2(f_1(s), f_2(s)) + X \right) ds; \end{aligned}$$

here

$$X = \frac{1-e^2}{4} \, \big| <f_1(s)> - <f_2(s)> \big|^2$$

does not depend on time since the mean velocity is preserved by equation (3.1). In other words, the function $y(\tau) = \mathrm{e}^{E\tau}\, W_2^2(f_1(\tau), f_2(\tau))$ satisfies the inequality

$$y(\tau) \le y(0) + E \int_0^\tau \left(\frac{3+e^2}{4}\, y(s) + X\, \mathrm{e}^{Es} \right) ds$$

and then

$$y(\tau) \le y(0)\, \mathrm{e}^{\gamma E \tau} + \frac{X}{1-\gamma} (\mathrm{e}^{E\tau} - \mathrm{e}^{\gamma E \tau})$$

by Gronwall's lemma with $\gamma = (3+e^2)/4$. This concludes the argument since $(1-\gamma)\, E = 2$. $\square$

Remark 6.

(1) Without further assumptions on the initial data f_1^0 and f_2^0, this result is optimal in the following sense. If f_2^0 is chosen as the Dirac mass at the mean velocity of f_1^0, then inequality (3.2) is actually an equality for all τ; indeed

$$\begin{aligned} W_2^2(f_1(\tau), f_2(\tau)) &= \int_{\mathbb{R}^3} |v - <f_1(\tau)>|^2\, f_1(\tau, v)\, dv = 3\, \theta(f_1(\tau)) \\ &= 3\, \mathrm{e}^{-2\tau}\, \theta(f_1^0) = \mathrm{e}^{-2\,\tau}\, W_2^2(f_1^0, f_2^0) \end{aligned}$$

since $\dfrac{d\theta}{d\tau} = -\,2\,\theta$ by equation (3.1).

(2) In terms of the original time variable t in (1.1), if f_1^0 and f_2^0 are two initial data with the same initial temperature θ_0, then the temperatures of the corresponding solutions f^1 and f^2 to (1.1) follow the law

$$\frac{d\theta}{dt} = -\,\frac{1-e^2}{4}\, B\theta^{\frac{3}{2}} \tag{3.3}$$

and hence are both equal to

$$\theta(t) = \left(\theta_0^{-1/2} + \frac{1-e^2}{8} Bt \right)^{-2}.$$

Then estimate (3.2) reads as

$$W_2^2(f_1(t), f_2(t)) \le \frac{\theta(t)}{\theta_0}\, W_2^2(f_1^0, f_2^0) + \left(1 - \frac{\theta(t)}{\theta_0}\right) \left| <f_1^0> - <f_2^0> \right|^2$$

for all $t \ge 0$.

The convergence of the solutions to (1.1) towards the Dirac measure at their mean velocity has been made precise in [9, 2] by the introduction of self-similar variables and homogeneous cooling states. There the authors prove that the rescaled solutions g defined by

$$g(\tau, v) = \theta^{3/2}(f(\tau))\, f(\tau, \theta^{1/2}(f(\tau))\, v) \tag{3.4}$$

satisfy the strict contraction property

$$d_{2+\varepsilon}(g_1(\tau), g_2(\tau)) \le \mathrm{e}^{-C(\varepsilon)\tau} d_{2+\varepsilon}(g_1^0, g_2^0), \qquad C(\varepsilon) > 0$$

for initial data g_1^0 and g_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and pressure tensor, where $\varepsilon > 0$ and $d_{2+\varepsilon}$ is a Fourier-based distance between probability measures. Moreover, for $\varepsilon = 0$ one has $C(\varepsilon) = 0$ giving a non-strict contraction in d_2 distance. In fact, by the scaling property in Proposition 1, (3.2) reads as

$$W_2(g_1(\tau), g_2(\tau)) \le W_2(g_1^0, g_2^0) \tag{3.5}$$

in the scaled variables. This is consistent with the fact that the distances d_2 and W_2 are "of the same order" [18, 26, 2] up to moment bounds.

A measure $g(\tau, v)$ defined by (3.4) from a solution $f(v, \tau)$ to (3.1) with initial zero mean velocity has zero mean velocity and unit kinetic energy for all τ, and is solution to

$$\frac{\partial g}{\partial \tau} + \nabla \cdot (g\, v) = E\, Q(g, g). \tag{3.6}$$

Moreover it is proven in [9, 2] that (3.6) has a unique stationary solution g_∞ with zero mean velocity and unit kinetic energy; all measure solutions $g(\tau, v)$ to (3.6) with zero mean velocity, unit kinetic energy and bounded moment of order $2 + \varepsilon$ converge to this stationary state g_∞ as τ goes to infinity in the d_2 sense, that is, in the W_2 sense since d_2 and W_2 metrize the same topology on probability measures [26] up to moment conditions. Moreover the convergence has exponential rate in the d_2 sense, and in the W_2 sense if the initial datum has finite fourth order moment. In turn this ensures existence and uniqueness of homogeneous cooling states to (1.1) for given mean velocity and kinetic energy, and algebraic convergence of the solutions $f(t)$ towards them in the original variables.

We conclude this section by proving this convergence result using only the W_2 distance, and without assuming that the initial data has more than two finite moments. This in turn shows that the Euclidean Wasserstein distance W_2 between solutions of (3.6) converges to zero as t goes to infinity, improving over (3.5) that does not *a priori* yield any information on the long-time behavior of the solutions g. As a drawback, this argument does not provide any rate of convergence as does the Fourier-based argument in [2].

Theorem 7. *Let g_1^0 and g_2^0 be two Borel probability measures on $\mathbb{R}^3$ with zero mean velocity and unit kinetic energy, and let $g_1(\tau)$ and $g_2(\tau)$ be the solutions to* (3.6) *with respective initial data g_1^0 and g_2^0. Then the map $\tau \mapsto W_2(g_1(\tau), g_2(\tau))$ is non-increasing and tends to* 0 *as τ goes to infinity.*

Proof.- It is based on the argument in [27] to Tanaka's theorem. The first statement is a simple consequence of (3.5). Then we turn to the second part of the theorem which by triangular inequality for the W_2 distance is enough to prove when g_2^0, and hence $g_2(\tau)$, is the unique stationary state g_∞ to (3.6) with zero mean velocity and unit kinetic energy.

Step 1.- Let us first assume that the fourth moment of the initial datum is bounded, i.e.,

$$\int_{\mathbb{R}^3} |v|^4 \, g_1^0(v) \, dv < \infty.$$

Then Proposition 15 in the appendix ensures that

$$\sup_{\tau \geq 0} \int_{\mathbb{R}^3} |v|^4 \, g_1(\tau, v) \, dv < \infty,$$

so that

$$\sup_{\tau \geq 0} \int_{|v|>R} |v|^2 \, g_1(\tau, v) \, dv$$

tends to 0 as R goes to infinity. Prohorov's compactness theorem and Proposition 1 imply the existence of a sequence $\tau_k \to \infty$ as $k \to \infty$ and a probability measure μ^0 on $\mathbb{R}^3$ with zero mean velocity and unit kinetic energy such that $W_2(g_1(\tau_k), \mu^0) \to 0$ as $k \to \infty$. We want to prove that $\mu^0 = g_\infty$.

Without loss of generality, we can assume that the diverging time sequence satisfies $\tau_k + 1 \leq \tau_{k+1}$ for all k. Now, since g_∞ is a stationary solution, it follows from the first part of the theorem that

$$W_2(g_1(\tau_{k+1}), g_\infty) \leq W_2(g_1(\tau_k + 1), g_\infty) \leq W_2(g_1(\tau_k), g_\infty). \tag{3.7}$$

On one hand, both $W_2(g_1(\tau_k), g_\infty)$ and $W_2(g_1(\tau_{k+1}), g_\infty)$ tend to $W_2(\mu^0, g_\infty)$ as k goes to infinity by triangular inequality. Then, if $\mu(\tau)$ denotes the solution to (3.6) with initial datum μ^0, the first point again ensures that

$$W_2(g_1(\tau_k + 1), \mu(1)) \leq W_2(g_1(\tau_k), \mu^0)$$

which tends to 0. Hence $W_2(g_1(\tau_k + 1), g_\infty)$ tends to $W_2(\mu(1), g_\infty)$ by triangular inequality, and finally

$$W_2(\mu(1), g_\infty) = W_2(\mu^0, g_\infty)$$

by passing to the limit in k in (3.7). By the non-increasing character of W_2 along the flow, we deduce that

$$W_2(\mu(1), g_\infty) = W_2(\mu(\tau), g_\infty) = W_2(\mu^0, g_\infty)$$

for all $\tau \in [0, 1]$.

Consequently $\mu(\tau)$ and g_∞ are two solutions to (3.6) with zero mean velocity and unit temperature, whose W_2 distance is constant on the time interval $[0,1]$. This is possible only if equality holds at each step in the proof of Theorem 5 in the original space variables; in particular

$$W_2(Q^+(\mu(\tau),\mu(\tau)),Q^+(g_\infty,g_\infty)) = \sqrt{\frac{3+e^2}{4}} W_2(\mu(\tau),g_\infty)$$

for all τ, and especially for $\tau = 0$. But μ^0 and g_∞ have same mean velocity and temperature, and, according to [7, Theorem 5.3], g_∞ is absolutely continuous with respect to Lebesgue measure, with positive density. Hence Proposition 4 ensures that $\mu^0 = g_\infty$.

In particular $W_2(g_1(\tau_k), g_\infty) \to 0$ as $k \to \infty$, and then $W_2(g_1(\tau), g_\infty) \to 0$ as $\tau \to \infty$ since it is a non increasing function.

Step 2.- Let us now remove the hypothesis on the boundedness of the initial fourth order moment. Let $(g^{0n})_n$ be a sequence in $\mathcal{P}_2(\mathbb{R}^3)$ with zero mean velocity, unit kinetic energy, finite fourth order moment and converging to g_1^0 in the weak sense of probability measures; in particular it converges to g_1^0 in the W_2 distance sense by Proposition 1. Such a g^{0n} can be obtained by successive truncation of g_1^0 to a ball of radius n in $\mathbb{R}^3$, translation to keep the mean property, and dilation centered at 0 to keep the kinetic energy equal to 1.

Then, if $g^n(\tau)$ is the solution to (3.6) with initial datum g^{0n}, the triangular inequality for W_2 and (3.5) ensure that

$$\begin{aligned} W_2(g_1(\tau),g_\infty) &\le W_2(g_1(\tau),g^n(\tau)) + W_2(g^n(\tau),g_\infty) \\ &\le W_2(g_1^0,g^{0n}) + W_2(g^n(\tau),g_\infty). \end{aligned}$$

Given $\varepsilon > 0$, the first term in the right hand side is bounded by ε for some n large enough, and for this now fixed n, the second term is bounded by ε for all τ larger than some constant by the first step. This ensures that $W_2(g_1(\tau), g_\infty)$ tends to 0 as τ goes to infinity. □

3.2. **The diffusive case.** We now turn to the diffusive version (1.3) of (1.1). Again by the change of time

$$\tau = \frac{B}{E}\int_0^t \sqrt{\theta(f(w))}\,dw$$

with $E = \dfrac{8}{1-e^2}$ we are brought to studying the equation

$$\frac{\partial f}{\partial \tau} = E\,Q(f,f) + \Theta^2(f(\tau))\,\Delta_v f \tag{3.8}$$

where

$$\Theta^2(f(\tau)) = \frac{E\,A}{B}\left[\theta(f(\tau))\right]^{p-1/2}.$$

As in the nonviscous case of (3.1) we shall prove

Theorem 8. *If f_1 and f_2 are two solutions to* (3.8) *for the respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with same kinetic energy, then*

$$W_2^2(f_1(\tau), f_2(\tau)) \le \mathrm{e}^{-2\tau}\, W_2^2(f_1^0, f_2^0) + (1-\mathrm{e}^{-2\tau})\, \big| < f_1^0 > - < f_2^0 > \big|^2 \tag{3.9}$$

for all $\tau \ge 0$.

Proof.- We again start by giving a Duhamel's representation of the solutions. To this aim we write (3.8) as

$$\frac{\partial f}{\partial \tau} = E\, F - E\, f + \Theta^2(f(\tau))\, \Delta f$$

where $F = Q^+(f,f)$, that is,

$$\frac{\partial \hat f}{\partial \tau} + \big(E + |k|^2\, \Theta^2(f(\tau))\big)\, \hat f = E\, \hat F.$$

Here, we are using the convention

$$\hat\mu(k) = \int_{\mathbb{R}^3} \mathrm{e}^{-i\, k\cdot x}\, d\mu(x)$$

for the Fourier transform of the measure μ on $\mathbb{R}^3$. Hence the solutions satisfy

$$\hat f(\tau, k) = \mathrm{e}^{-E\,\tau}\, \hat f^0(k)\, \mathrm{e}^{-\Sigma(f,\tau)\,|k|^2} + E \int_0^\tau \mathrm{e}^{-E(\tau-s)}\, \hat F(s,k)\, \mathrm{e}^{-(\Sigma(f,\tau)-\Sigma(f,s))|k|^2}\, ds$$

where $\Sigma(f,\tau) = \displaystyle\int_0^\tau \Theta^2(f(s))\, ds$, and thus

$$\begin{aligned} f(\tau, v) &= \mathrm{e}^{-E\,\tau}\, (f^0 * \Gamma_{2\Sigma(f,\tau)})(v) + E \int_0^\tau \mathrm{e}^{-E(\tau-s)}\, (F(s) * \Gamma_{2(\Sigma(f,\tau)-\Sigma(f,s))})(v)\, ds \\ &:= \mathrm{e}^{-E\,\tau}\, \tilde f(\tau, v) + E \int_0^\tau \mathrm{e}^{-E(\tau-s)}\, \tilde F(\tau, s, v)\, ds. \end{aligned}$$

Here

$$\Gamma_\alpha(v) = \frac{1}{(2\pi\alpha)^{3/2}}\, \mathrm{e}^{-|v|^2/2\alpha}$$

is the centered Maxwellian with temperature $\alpha/3 > 0$. Moreover f_1 and f_2 have same temperature at all times, so that $\Sigma(f_1,\tau) = \Sigma(f_2,\tau)$. Then the convexity of the squared Wasserstein distance and its non-increasing character by convolution with a given measure, see Proposition 1, imply that

$$\begin{aligned} W_2^2(f_1(\tau), f_2(\tau)) &\le \mathrm{e}^{-E\tau} W_2^2(\tilde f_1(\tau), \tilde f_2(\tau)) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)} W_2^2(\tilde F_1(\tau,s), \tilde F_2(\tau,s))ds \\ &\le \mathrm{e}^{-E\tau}\, W_2^2(f_1^0, f_2^0) + E \int_0^\tau \mathrm{e}^{-E(\tau-s)}\, W_2^2(F_1(s), F_2(s))\, ds. \end{aligned}$$

In other words the squared distance $W_2^2(f_1(\tau), f_2(\tau))$ satisfies the same bound as in the nonviscous case of Theorem 5, and we can conclude analogously. □

Remark 9.

(1) As pointed out to us by C. Villani the result can also be obtained by a splitting argument between the collision term and the diffusion term.

(2) As proven in [1], the temperature $\theta(f(t))$ of the solution f in the original time variable t converges towards

$$\theta_\infty = \Big(\frac{8\,A}{B(1-e^2)}\Big)^{\frac{2}{3-2p}}$$

as t goes to infinity, and satisfies $\theta(f(t)) \geq \min(\theta(f(0)), \theta_\infty)$. In particular

$$\tau = \frac{B}{E}\int_0^t \sqrt{\theta(f(s))}\, ds \geq \frac{C_1}{E}\, t$$

if $C_1 = B\, \min(\theta(f(0)), \theta_\infty)^{1/2}$. Writing (3.9) in the original variable t for initial data with equal mean velocity and temperature, we recover the contraction property

$$W_2(f_1(t), f_2(t)) \leq W_2(f_1^0, f_2^0)\, \mathrm{e}^{-(1-\gamma)\mathrm{C}_1 \mathrm{t}},$$

that coincides with (3.1) in [1] for the Fourier-based d_2 distance exactly with the same rate. For $p = 1$ one can exactly compute τ and also recover (3.2) in [1] but for the distance W_2.

(3) The existence of unique diffusive equilibria for each given value of the initial mean velocity can be obtained from this contraction property of the W_2 distance analogously to the arguments done in [1] with the Fourier-based distance d_2.

4. General cross section

In this section, we consider the more general case of a variable collision cross section when the gain term Q^+ is defined by

$$(\varphi, Q^+(f,f)) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)\, f(w)\, \varphi(v')\, b\Big(\frac{v-w}{|v-w|}\cdot\sigma\Big)\, d\sigma\, dv\, dw$$

where again the post-collisional velocity v' is given by

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|\,\sigma$$

and the cross section b satisfies the normalized cut-off assumption

$$\int_{S^2} b(k\cdot\sigma)\, d\sigma = \int_0^{2\pi}\int_0^{\pi} b(\cos\theta)\, \sin\theta\, d\theta\, d\phi = 1 \tag{4.1}$$

for any k in S^2. Then we shall prove the following extension of Proposition 3 for non constant cross sections b:

Theorem 10. *If f and g in $\mathcal{P}_2(\mathbb{R}^3)$ have equal mean velocity, then*

$$W_2^2(Q^+(f,f), Q^+(g,g)) \leq \Big(\frac{3+e^2}{4} + \frac{1-e^2}{2}\pi\int_0^{\pi} b(\cos\theta)\cos\theta\, \sin\theta\, d\theta\Big)\, W_2^2(f,g).$$

Before going onto the proof, we draw the main consequence. Let $f_1 = f_1(\tau, v)$ and $f_2 = f_2(\tau, v)$ be two solutions to the Boltzmann equation

$$\frac{\partial f}{\partial \tau} = Q(f, f) = Q^+(f, f) - f$$

with respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$, where Q^+ is defined as above. Then, as in Section 3.1, Duhamel's representation formula

$$f(\tau) = \mathrm{e}^{-\tau} f^0 + \int_0^\tau \mathrm{e}^{-(\tau - s)} Q^+(f(s), f(s))\, ds$$

of the solutions and the convexity of W_2^2 ensure the contraction property

$$W_2(f_1(\tau), f_2(\tau)) \leq \mathrm{e}^{-(1-\gamma_b)\tau/2}\, W_2(f_1^0, f_2^0) \tag{4.2}$$

for all τ, where

$$\gamma_b = \frac{3 + e^2}{4} + \frac{1 - e^2}{2}\, \pi \int_0^\pi b(\cos\theta)\, \cos\theta\, \sin\theta\, d\theta$$

is bounded by 1 by (4.1).

In the elastic case when $e = 1$, $\gamma_b = 1$, one recovers Tanaka's non-strict contraction result [25] for the solutions to the homogeneous elastic Boltzmann equation for Maxwellian molecules, at least under the cut-off assumption, but with a somehow simpler argument than those given in [25] and [27].

Proof.- By definition

$$\begin{aligned}(\varphi, Q^+(f, f)) &= 2\pi \int_0^\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Big\{ \int_0^{2\pi} \varphi(v') \frac{d\phi}{2\pi} \Big\} f(v)\, f(w)\, dv\, dw\, b(\cos\theta)\, \sin\theta\, d\theta \\ &= 2\pi \int_0^\pi \mathbb{E}\left[(\varphi, \mathcal{U}_{V,W,\theta})\right] b(\cos\theta)\, \sin\theta\, d\theta\end{aligned}$$

where V and W are independent random variables distributed according to f and, given v, w in $\mathbb{R}^3$, $\mathcal{U}_{v,w,\theta}$ is the uniform probability measure on the circle $C_{v,w,\theta}$ with center

$$c_{v,w,\theta} = \frac{1}{2}(v + w) + \Big(\frac{1 - e}{4} + \frac{1 + e}{4} \cos\theta\Big)(v - w),$$

radius

$$r_{v,w,\theta} = \frac{1 + e}{4} |v - w| \sin\theta$$

and axis

$$k = \frac{v - w}{|v - w|}\cdot$$

Let also g be a Borel probability measure on $\mathbb{R}^3$ and X, Y be independent random variables with law g. Then, by the normalization assumption (4.1), the convexity of the squared Wasserstein distance with respect to both arguments ensures that

$$W_2^2(Q^+(f, f), Q^+(g, g)) \leq 2\pi \int_0^\pi \mathbb{E}\left[W_2^2(\mathcal{U}_{V,W,\theta}, \mathcal{U}_{X,Y,\theta})\right] b(\cos\theta)\, \sin\theta\, d\theta. \tag{4.3}$$

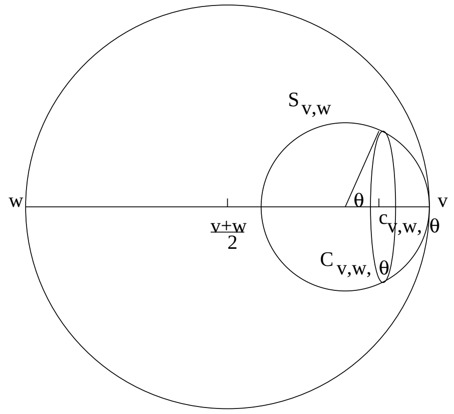

FIGURE 3.

We now let v, w, x, y and θ be fixed in $\mathbb{R}^3$ and $[0, \pi]$ respectively, and give an upper bound to $W_2^2(\mathcal{U}_{v,w,\theta}, \mathcal{U}_{x,y,\theta})$. This consists in estimating the transport cost of a circle in $\mathbb{R}^3$ onto another one, for which we have the following general bound:

Lemma 11. [27] *The squared Wasserstein distance between the uniform distributions on the circles with centers c and c', radii r and r' and axes k and k' is bounded by*

$$|c - c'|^2 + r^2 + r'^2 - rr'(1 + |k \cdot k'|).$$

Hence, using the notations $a = v - x$, $b = w - y$, $\tilde{a} = v - w$ and $\tilde{b} = x - y$ in our case we get

$$\begin{aligned} W_2^2(\mathcal{U}_{v,w,\theta}, \mathcal{U}_{x,y,\theta}) &\leq \Big| \Big(\frac{3-e}{4} + \frac{1+e}{4} \cos\theta \Big) a + \frac{1+e}{4} (1 - \cos\theta)\, b \Big|^2 \\ &\quad + \Big(\frac{1+e}{4} \Big)^2 \sin^2\theta \Big[|\tilde{a}|^2 + |\tilde{b}|^2 - |\tilde{a}||\tilde{b}| \Big(1 + \Big(\frac{\tilde{a}}{|\tilde{a}|} \cdot \frac{\tilde{b}}{|\tilde{b}|} \Big) \Big] \\ &\leq \Big[\Big(\frac{3-e}{4} + \frac{1+e}{4} \cos\theta \Big)^2 + \Big(\frac{1+e}{4} \Big)^2 \sin^2\theta \Big] |a|^2 \\ &\quad + 2 \Big(\frac{1+e}{4} \Big)^2 \Big[\Big(\frac{3-e}{1+e} + \cos\theta \Big) (1 - \cos\theta) - 2 \sin^2\theta \Big] a \cdot b \\ &\quad + \Big(\frac{1+e}{4} \Big)^2 \big[(1 - \cos\theta)^2 + \sin^2\theta \big] |b|^2 \end{aligned} \tag{4.4}$$

where we have used the bound

$$|\tilde{a}|^2 + |\tilde{b}|^2 - |\tilde{a}||\tilde{b}| - \tilde{a} \cdot \tilde{b} \leq |\tilde{a}|^2 + |\tilde{b}|^2 - 2\, \tilde{a} \cdot \tilde{b} = |\tilde{a} - \tilde{b}|^2 = |a - b|^2.$$

Assume now that (V, X) and (W, Y) are two independent couples of random variables, optimal in the sense that

$$W_2^2(f, g) = \mathbb{E}\big[|V - X|^2\big] = \mathbb{E}\big[|W - Y|^2\big].$$

Note that

$$\mathbb{E}\big[(V - X) \cdot (W - Y)\big] = \mathbb{E}\big[(V - X)\big] \cdot \mathbb{E}\big[(W - Y)\big] = 0$$

since (V, X) and (W, Y) are independent and since f and g have same mean velocity. Then from (4.4):

$$\mathbb{E}\left[W_2^2(\mathcal{U}_{V,W,\theta}, \mathcal{U}_{X,Y,\theta})\right] \le \gamma(\theta)\, W_2^2(f, g)$$

where

$$\begin{aligned}\gamma(\theta) &= \Big(\frac{3-e}{4} + \frac{1+e}{4}\cos\theta\Big)^2 + \Big(\frac{1+e}{4}\Big)^2\big[(1-\cos\theta)^2 + 2\sin^2\theta\big] \\ &= \frac{3+e^2}{4} + \frac{1-e^2}{4}\cos\theta.\end{aligned}$$

One concludes the argument after averaging over θ as in (4.3) and taking (4.1) into account. □

5. Inelastic Kac Model

In this last section we consider a simple one-dimensional model introduced in [23] which can be seen as a dissipative version of the Kac caricature of a Maxwellian gas [19, 20]. Let us remark that the definition and properties of the Euclidean Wasserstein distance W_2 discussed above generalizes equally well to any dimension. Tanaka himself [24] showed that the Euclidean Wasserstein distance is a non strict contraction for the elastic classical Kac model. In the inelastic Kac model, the evolution of the density function f is governed by the equation

$$\frac{\partial f}{\partial t} = Q(f, f) \tag{5.1}$$

in which the collision term $Q(f, f)$ is defined by

$$(\varphi, Q(f, f)) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f(v)f(w)\Big[\varphi(v') - \varphi(v)\Big]\,\frac{d\theta}{2\pi}\,dv\,dw$$

for any test function φ, where

$$v' = v\,\cos\theta\,|\cos\theta|^p - w\,\sin\theta\,|\sin\theta|^p$$

is the postcollisional velocity and $p > 0$ measures the inelasticity. Equation (5.1) preserves mass but makes the momentum and kinetic energy decrease to 0 at an exponential rate, $\theta(f(t)) = \mathrm{e}^{-2\beta t}\theta(f^0) + (\mathrm{e}^{-2\beta t} - \mathrm{e}^{-2t}) <f^0>$ with $\beta > 0$ given below. In particular, solutions to (5.1) tend to the Dirac mass at 0.

As in the inelastic Maxwell model discussed above, we start by deriving a contraction property for the gain operator Q^+ defined by

$$(\varphi, Q^+(f, f)) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f(v)f(w)\,\varphi(v')\,\frac{d\theta}{2\pi}\,dv\,dw.$$

Proposition 12. *If f and g belong to $\mathcal{P}_2(\mathbb{R})$, then*

$$W_2^2(Q^+(f, f), Q^+(g, g)) \le \left[\int_0^{2\pi}\big(|\cos\theta|^{2(p+1)} + |\sin\theta|^{2(p+1)}\big)\,\frac{d\theta}{2\pi}\right]\, W_2^2(f, g).$$

In terms of solutions $f(t)$ and $g(t)$ to the modified Kac equation (5.1) with finite initial energy only, the above proposition yields, as in previous sections, the bound

$$W_2(f(t), g(t)) \leq \mathrm{e}^{-\beta t}\, W_2(f^0, g^0)$$

where

$$2\,\beta = 1 - \int_0^{2\pi} \left(|\cos\theta|^{2(p+1)} + |\sin\theta|^{2(p+1)} \right) \frac{d\theta}{2\pi} > 0.$$

This bound is optimal without further assumptions on the initial data f^0 and g^0 since equality holds in the case when $< f^0 >= 0$ and $g^0 = \delta_0$ analogously to previous cases.

Proof.- Given a vector (v, w) in $\mathbb{R}^2$, let $\mathcal{C}_{v,w}$ denote the curve

$$\{(v'(\theta), w'(\theta)), \theta \in [0, 2\,\pi]\}$$

where

$$\begin{array}{rcl} v'(\theta) & = & v\,\cos\theta\,|\cos\theta|^p - w\,\sin\theta\,|\sin\theta|^p \\ w'(\theta) & = & v\,\sin\theta\,|\sin\theta|^p + w\,\cos\theta\,|\cos\theta|^p. \end{array} \tag{5.2}$$

Let also $\mathcal{U}_{v,w}$ be the uniform probability distribution on $\mathcal{C}_{v,w}$.

Given V and W two independent random variables distributed according to f, we note that $Q^+(f, f)$ is the first marginal on $\mathbb{R}$ of $\mathbb{E}\big[\mathcal{U}_{V,W}\big]$, but also its second marginal by symmetry. Then, we have the following result, which is the analogous of Lemmas 2 and 11 for this model:

Lemma 13. *Given two vectors (v, w) and (x, y) in $\mathbb{R}^2$, the squared Wasserstein distance between the distributions $\mathcal{U}_{v,w}$ and $\mathcal{U}_{x,y}$ is bounded by*

$$(1 - 2\beta)\left(|v - x|^2 + |w - y|^2\right).$$

Proof.- One can transport the curve $\mathcal{C}_{v,w}$ onto $\mathcal{C}_{x,y}$ by the linear map

$$(a, b) \mapsto T(a, b) = \frac{r'}{r}(a\,\cos\omega - b\,\sin\omega, a\,\sin\omega + b\,\cos\omega)$$

where $r = \sqrt{v^2 + w^2}$, $r' = \sqrt{x^2 + y^2}$ and ω is the angle between the vectors (v, w) and (x, y) in case they do not vanish. We leave the reader discuss the case when either (x, y) or (v, w) are zero. Then, analogously to the proof of Lemma 2, one can define a transport plan associated to the transport map T to get

$$W_2^2(\mathcal{U}_{v,w}, \mathcal{U}_{x,y}) \leq \int_{\mathbb{R}^2} |T(a, b) - (a, b)|^2\, d\mathcal{U}_{v,w}(a, b).$$

Furthermore, for all (a, b) in $\mathbb{R}^2$,

$$\begin{aligned} |T(a,b) - (a,b)|^2 &= \left|\frac{r'}{r}\big(a\,\cos\omega - b\,\sin\omega\big) - a\right|^2 + \left|\frac{r'}{r}\big(a\,\sin\omega + b\,\cos\omega\big) - b\right|^2 \\ &= \left(\left(\frac{r'}{r}\right)^2 - 2\,\frac{r'}{r}\,\cos\omega + 1\right)\big(a^2 + b^2\big) \\ &= \frac{|v - x|^2 + |w - y|^2}{v^2 + w^2}\,\big(a^2 + b^2\big). \end{aligned}$$

Hence, we deduce

$$\begin{aligned} W_2^2(\mathcal{U}_{v,w},\mathcal{U}_{x,y}) &\le \frac{|v-x|^2+|w-y|^2}{v^2+w^2}\int_{\mathbb{R}^2}(a^2+b^2)\,d\mathcal{U}_{v,w}(a,b) \\ &= \frac{|v-x|^2+|w-y|^2}{v^2+w^2}\int_0^{2\pi}(v'(\theta)^2+w'(\theta)^2)\,\frac{d\theta}{2\pi}\cdot \end{aligned}$$

But

$$v'(\theta)^2+w'(\theta)^2 = \big(|\cos\theta|^{2(p+1)}+|\sin\theta|^{2(p+1)}\big)(v^2+w^2)$$

by (5.2), so that

$$W_2^2(\mathcal{U}_{v,w},\mathcal{U}_{x,y}) \le \left[\int_0^{2\pi}\big(|\cos\theta|^{2(p+1)}+|\sin\theta|^{2(p+1)}\big)\,\frac{d\theta}{2\pi}\right]\,\big(|v-x|^2+|w-y|^2\big)$$

which is the bound given by the lemma. □

We now continue the *proof of Proposition 12.* First of all, let (V,X) and (W,Y) be two independent couples of random variables, with V and X distributed according to f, W and Y according to g, optimal in the sense that

$$W_2^2(f,g) = \mathbb{E}\big[|V-W|^2\big] = \mathbb{E}\big[|X-Y|^2\big].$$

Then, by convexity of the squared Wasserstein distance again, it follows from Lemma 13 that

$$\begin{aligned} W_2^2\big(\mathbb{E}[\mathcal{U}_{V,W}],\mathbb{E}[\mathcal{U}_{X,Y}]\big) &\le \mathbb{E}\big[W_2^2(\mathcal{U}_{V,W},\mathcal{U}_{X,Y})\big] \\ &\le (1-2\beta)\big(\mathbb{E}\big[|V-W|^2\big]+\mathbb{E}\big[|X-Y|^2\big]\big) \\ &= 2\,(1-2\beta)\,W_2^2(f,g). \end{aligned} \tag{5.3}$$

Next, we remark that the measure $\mathcal{U}_{V,W}$ on $\mathbb{R}^2$ has first **and** second marginals equal by symmetry of the curve $\mathcal{C}_{V,W}$ by a $\pi/2$ rotation. This implies that the first and second marginals of $\mathbb{E}\big[\mathcal{U}_{V,W}\big]$ on $\mathbb{R}^2$ are equal to $Q^+(f,f)$, and likewise for the measure $\mathbb{E}\big[\mathcal{U}_{X,Y}\big]$ with marginals $Q^+(g,g)$. We shall conclude the argument of Proposition 12 by using the following general result:

Lemma 14. *If the Borel probability measures μ_j^i on $\mathbb{R}$ are the successive one-dimensional marginals of the measure μ^i on $\mathbb{R}^N$, for $i=1,2$ and $j=1,\dots,N$, then*

$$\sum_{j=1}^N W_2^2(\mu_j^1,\mu_j^2) \;\le\; W_2^2(\mu^1,\mu^2).$$

Proof.- Let π be a measure on $\mathbb{R}_v^N\times\mathbb{R}_w^N$ with marginals μ^1 and μ^2, optimal in the sense that

$$W_2^2(\mu^1,\mu^2) = \iint_{\mathbb{R}^N\times\mathbb{R}^N}|v-w|^2\,d\pi(v,w).$$

Then its marginal π_j on $\mathbb{R}_{v_j} \times \mathbb{R}_{w_j}$ has itself marginals μ_j^1 and μ_j^2, so

$$W_2^2(\mu_j^1, \mu_j^2) \leq \iint_{\mathbb{R}\times\mathbb{R}} |v_j - w_j|^2 \, d\pi_j(v_j, w_j).$$

The lemma follows by noting that $\displaystyle\sum_{j=1}^N |v_j - w_j|^2 = |v - w|^2$. □

In our particular case, Lemma 14 ensures that

$$2\, W_2^2(Q^+(f,f), Q^+(g,g)) \leq W_2^2\big(\mathbb{E}[\mathcal{U}_{V,W}], \mathbb{E}[\mathcal{U}_{X,Y}]\big)$$

which concludes the proof of Proposition 12 taking (5.3) into account. □

Appendix: Uniform in time Propagation of fourth order Moments

In this appendix we derive a uniform propagation of fourth order moments $\displaystyle\int_{\mathbb{R}^3} |v|^4 \, g(\tau, v) \, dv$ of solutions g to

$$\frac{\partial g}{\partial \tau} + \nabla \cdot (g\, v) = E\, Q(g,g) \tag{5.4}$$

where the operator $Q(g,g)$ is defined as in (1.2) for $0 < e < 1$ and $E = \dfrac{8}{1-e^2}\cdot$

This result has been used in the proof of Theorem 7.

Proposition 15. *If g^0 is a Borel probability measure on $\mathbb{R}^3$ such that*

$$\int_{\mathbb{R}^3} |v|^4 \, g^0(v) \, dv < \infty,$$

then the solution g to (5.4) *with initial datum g^0 verifies*

$$\sup_{\tau \geq 0} \int_{\mathbb{R}^3} |v|^4 \, g(\tau, v) \, dv < \infty.$$

Proof.- Without loss of generality we can assume that g^0, and hence $g(\tau)$ for all $\tau \geq 0$, has zero mean velocity. We let

$$m_4(\tau) = \int_{\mathbb{R}^3} |v|^4 \, g(\tau, v) \, dv$$

denote the fourth order moment of $g(\tau)$. Then, using the weak formulation of the inelastic Boltzmann equation, we have:

$$\frac{dm_4(\tau)}{d\tau} = \int_{\mathbb{R}^3} \nabla(|v|^4) \cdot v \, g(\tau, v) \, dv + E \int_{\mathbb{R}^3} |v|^4 \, Q(g(\tau), g(\tau))(v) \, dv. \tag{5.5}$$

While the first term in the right hand side is simply $4\, m_4(\tau)$, the second term is computed by

Lemma 16. *There exist some constants* μ_1 *and* μ_2*, depending only on* e*, such that*

$$\int_{\mathbb{R}^3} |v|^4 \, Q(g,g)(v) \, dv = -\,\lambda \int_{\mathbb{R}^3} |v|^4 \, g(v) \, dv + \mu_1 \Big(\int_{\mathbb{R}^3} |v|^2 \, g(v) \, dv \Big)^2 + \mu_2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w)^2 \, g(v) \, g(w) \, dv \, dw$$

for any probability measure g *on* $\mathbb{R}^3$ *with finite moment of order* 4 *and zero mean velocity, where*

$$\lambda = \frac{1}{3}(1 + 4\,\varepsilon - 7\,\varepsilon^2 + 4\,\varepsilon^3 - 2\,\varepsilon^4) \qquad \text{and} \qquad \varepsilon = \frac{1-e}{2}\cdot$$

With this lemma in hand, (5.5) reads

$$\frac{dm_4(\tau)}{d\tau} = \Big(4 - E\,\lambda\Big) m_4(\tau) + m(\tau) \tag{5.6}$$

where $m(\tau)$ is a combination of second order moments, which are bounded in time since the kinetic energy is preserved by equation (5.4). Moreover one can check from the expression of E and λ in terms of $\varepsilon = (1-e)/2$ that

$$4 - E\,\lambda = \frac{2}{3\,\varepsilon(1-\varepsilon)}\,[-1 + 2\,\varepsilon + \varepsilon^2 - 4\,\varepsilon^3 + 2\,\varepsilon^4]$$

which is negative for any $0 < \varepsilon < 1/2$, that is, for any $0 < e < 1$. By Gronwall's lemma this ensures that $m_4(\tau)$ is bounded uniformly in time if initially finite, and concludes the argument to Proposition 15. □

Let us remark that identity (5.6) is also useful to understand that moments are not created by this equation in contrast to the hard-spheres case [21, 22]. In fact, if initially moments are infinite, they will remain so. Thus, this is another reason why homogeneous cooling states have only certain number of moments bounded (see [7]).

We now turn to the *proof of Lemma 16*, whose result is given in [6, Section 4] only in the radial isotropic case, i.e., whenever $g(v)$ depends only on $|v|$. By symmetry we start by writing

$$\int_{\mathbb{R}^3} |v|^4 \, Q(g,g)(v) \, dv = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} g(v) g(w) \frac{1}{2} [|v'|^4 + |w'|^4 - |v|^4 - |w|^4] \, d\sigma \, dv \, dw$$

where

$$\begin{aligned} v' &= \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|\sigma \\ w' &= \frac{1}{2}(v+w) - \frac{1-e}{4}(v-w) - \frac{1+e}{4}|v-w|\sigma. \end{aligned}$$

Then we introduce the notation

$$u = \frac{v+w}{2}, \quad U = \frac{v-w}{2}, \quad \varepsilon = \frac{1-e}{2}, \quad \varepsilon' = 1 - \varepsilon = \frac{1+e}{2}$$

in which

$$v' = u + \varepsilon\, U + \varepsilon' \, |U| \, \sigma, \quad v' = u - \varepsilon\, U - \varepsilon' \, |U| \, \sigma, \quad v = u + U, \quad w = u - U.$$

Then

$$\begin{aligned} |v'|^2 &= |u|^2 + (\varepsilon^2+\varepsilon'^2)|U|^2 + 2\,\varepsilon\,\varepsilon'\,|U|\,(U\cdot\sigma) + 2\,\varepsilon\,(u\cdot U) + 2\,\varepsilon'\,|U|\,(u\cdot\sigma) \\ |w'|^2 &= |u|^2 + (\varepsilon^2+\varepsilon'^2)|U|^2 + 2\,\varepsilon\,\varepsilon'\,|U|(U\cdot\sigma) - 2\,\varepsilon\,(u\cdot U) - 2\,\varepsilon'\,|U|\,(u\cdot\sigma) \\ |v|^2 &= |u|^2 + |U|^2 + 2\,(u\cdot U) \\ |w|^2 &= |u|^2 + |U|^2 - 2\,(u\cdot U) \end{aligned}$$

and eventually

$$\begin{aligned} &\frac{1}{2}[|v'|^4+|w'|^4-|v|^4-|w|^4] \\ &\qquad = \big[(\varepsilon^2+\varepsilon'^2)^2-1\big]\,|U|^4 + 2(\varepsilon^2+\varepsilon'^2-1)|u|^2|U|^2 + 4(\varepsilon^2-1)(u\cdot U)^2 \\ &\qquad\quad + 4\,\varepsilon^2\,\varepsilon'^2\,|U|^2\,(U\cdot\sigma)^2 + 4\,\varepsilon'^2\,|U|^2\,(u\cdot\sigma)^2 \\ &\qquad\quad + 4\,\varepsilon\,\varepsilon'\,|U|\,\big[|u|^2+(\varepsilon^2+\varepsilon'^2)|U|^2\big]\,(U\cdot\sigma) + 8\,\varepsilon\,\varepsilon'\,|U|\,(u\cdot U)\,(u\cdot\sigma). \end{aligned}$$

Integrating with respect to σ in S^2 and taking the identities

$$\int_{S^2} 1\,\frac{d\sigma}{4\pi} = 1, \qquad \int_{S^2} (k\cdot\sigma)\,\frac{d\sigma}{4\pi} = 0, \qquad \int_{S^2} (k\cdot\sigma)^2\,\frac{d\sigma}{4\pi} = \frac{|k|^2}{3}$$

into account, we obtain

$$\int_{S^2} \frac{1}{2}[|v'|^4+|w'|^4-|v|^4-|w|^4]\,\frac{d\sigma}{4\pi} = \alpha\,|U|^4 + \beta\,|u|^2|U|^2 + \gamma\,(u\cdot U)^2$$

where

$$\alpha = (\varepsilon^2+\varepsilon'^2)^2 - 1 + \frac{4}{3}\varepsilon^2\varepsilon'^2, \quad \beta = 2\big[\varepsilon^2+\varepsilon'^2-1+\frac{2}{3}\varepsilon'^2\big], \quad \gamma = 4(\varepsilon^2-1).$$

Then, by definition of u and U in terms of v and w, the identities

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |U|^4\,g(v)\,g(w)\,dv\,dw = \frac{1}{8}[m_4 + m_2^2 + 2\,\overline{m_2^2}],$$

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |u|^2\,|U|^2\,g(v)\,g(w)\,dv\,dw = \frac{1}{8}[m_4 + m_2^2 - 2\,\overline{m_2^2}]$$

and

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} (u\cdot U)^2\,g(v)\,g(w)\,dv\,dw = \frac{1}{8}[m_4 - m_2^2]$$

hold with

$$m_4 = \int_{\mathbb{R}^3} |v|^4\,g(v)\,dv, \;\; m_2 = \int_{\mathbb{R}^3} |v|^2\,g(v)\,dv, \;\; \overline{m_2^2} = \iint_{\mathbb{R}^3\times\mathbb{R}^3} (v\cdot w)^2\,g(v)\,g(w)\,dv\,dw$$

since g has zero mean velocity. Collecting all terms, we obtain

$$\int_{\mathbb{R}^3} |v|^4\,Q(g,g)(v)\,dv = -\lambda\,m_4 + \mu_1\,m_2^2 + \mu_2\,\overline{m_2^2}$$

where

$$\lambda = -\frac{1}{8}(\alpha+\beta+\gamma) = \frac{1}{3}(1 + 4\,\varepsilon - 7\,\varepsilon^2 + 4\,\varepsilon^3 - 2\,\varepsilon^4),$$

$$\mu_1 = \frac{1}{8}(\alpha + \beta - \gamma) \quad \text{and} \quad \mu_2 = \frac{1}{4}(\alpha - \beta)$$

depend only on ε, that is, only on e. This concludes the proof of Lemma 16. □

Acknowledgements.- The authors are grateful to Laurent Desvillettes, Giuseppe Toscani and Cédric Villani for stimulating discussions and fruitful comments. JAC acknowledges the support from DGI-MEC (Spain) project MTM2005-08024.

References

[1] M. Bisi, J.A. Carrillo, G. Toscani, "Contractive Metrics for a Boltzmann equation for granular gases: Diffusive equilibria", *J. Statist. Phys.* **118** (2005), 301–331.

[2] M. Bisi, J.A. Carrillo, G. Toscani, "Decay rates in probability metrics towards homogeneous cooling states for the inelastic Maxwell model", to appear in *J. Statist. Phys.* (2006).

[3] A.V. Bobylev, "Exact solutions of the Boltzmann equation", *Doklady Akad. Nauk SSSR* **231** (1975), 571–574.

[4] A.V. Bobylev, "The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules", *Sov. Sci. Rev. C. Math. Phys.* **7** (1988), 111–233.

[5] A.V. Bobylev, J.A. Carrillo, I. Gamba, "On some properties of kinetic and hydrodynamic equations for inelastic interactions", *J. Statist. Phys.* **98** (2000), 743–773.

[6] A.V. Bobylev, C. Cercignani, "Moment equations for a Granular Material in a Thermal Bath", *J. Statist. Phys.* **106** (2002), 547–567.

[7] A.V. Bobylev, C. Cercignani, "Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions", *J. Statist. Phys.* **110** (2003), 333-375.

[8] A.V. Bobylev, C. Cercignani, I. Gamba, "Generalized Maxwell models and self-similar asymptotics", personal communication and work in progress.

[9] A.V. Bobylev, C. Cercignani, G. Toscani, "Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials", *J. Statist. Phys.* **111** (2003), 403-417.

[10] A.V. Bobylev, I. Gamba, "Boltzmann equations for mixtures of Maxwell gases: exact solutions and power-like tails", to appear in *J. Statist. Phys.* (2006).

[11] F. Bolley, "Separability and completeness for the Wasserstein distance", to appear in *Séminaire de probabilités.* Lecture Notes in Math. (2006).

[12] E.A. Carlen, E. Gabetta, G. Toscani, "Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous Maxwellian gas", *Commun. Math. Phys.* **305** (1999), 521–546.

[13] J. A. Carrillo, C. Cercignani, I. Gamba, "Steady states of a Boltzmann equation for driven granular media", *Phys. Rev. E.* **62** (2000), 7700–7707.

[14] J. A. Carrillo, R. J. McCann, C. Villani, "Contractions in the 2-Wasserstein length space and thermalization of granular media", *Arch. Rat. Mech. Anal.* **179** (2006), 217–263.

[15] C. Cercignani, R. Illner, C. Stoica, "On Diffusive Equilibria in Generalized Kinetic Theory", *J. Statist. Phys.* **105** (2001), 337–352.

[16] M.H. Ernst, R. Brito, "High energy tails for inelastic Maxwell models", *Europhys. Lett.* **58** (2002), 182-187.

[17] M.H. Ernst, R. Brito, "Scaling solutions of inelastic Boltzmann equation with over-populated high energy tails", *J. Statist. Phys.* **109** (2002), 407-432.

[18] E. Gabetta, G. Toscani, W. Wennberg, "Metrics for Probability Distributions and the Trend to Equilibrium for Solutions of the Boltzmann Equation", *J. Statist. Phys.* **81** (1995), 901–934.

[19] M. Kac, *Probability and Related Topics in the Physical Sciences*, Interscience, London-New York, 1959.

[20] H. P. McKean, Jr., "Speed of approach to equilibrium for Kac's caricature of a maxwellian gas", *Arch. Rat. Mech. Anal.* **21** (1966), 343–367.
[21] S. Mischler, C. Mouhot, "Cooling process for inelastic Boltzmann equations for hard spheres, Part I: The Cauchy problem", to appear in *J. Statist. Phys.* (2006).
[22] S. Mischler, C. Mouhot, M. Rodriguez-Ricard, "Cooling process for inelasic Boltzmann equations for hard spheres, Part II: Self-similar solutions and tail behavior", to appear in *J. Statist. Phys.* (2006).
[23] A. Pulvirenti, G. Toscani, "Asymptotic properties of the inelastic Kac model", *J. Statist. Phys.* **114** (2004), 1453–1480.
[24] H. Tanaka, "An inequality for a functional of probability distributions and its applications to Kac's one-dimensional model of a Maxwellian gas", *Z. Wahrsch. Verw. Gebiete* **27** (1973), 47–52.
[25] H. Tanaka, "Probabilistic treatment of the Boltzmann equation of Maxwellian molecules", *Z. Wahrsch. Verw. Gebiete* **46**, 1 (1978/79), 67–105.
[26] G. Toscani, C. Villani, "Probability Metrics and Uniqueness of the Solution to the Boltzmann Equation for a Maxwell Gas", *J. Statist. Phys.* **94** (1999), 619–637.
[27] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003.
[28] C. Villani, "Mathematics of granular materials", to appear in *J. Statist. Phys.* (2006).

Institut de Mathématiques, LSP (UMR C5583), Université Paul Sabatier, Route de Narbonne, F-31062 Toulouse cedex 9
E-mail address: bolley@cict.fr

ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
E-mail address: carrillo@mat.uab.es