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DECAY ESTIMATES FOR A VISCOUS HAMILTON-JACOBI EQUATION WITH HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS

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Abstract. Global classical solutions to the viscous Hamilton-Jacobi equation $u_t - \Delta u = a\ |\nabla u|^p$ in $(0,\infty)\times\Omega$ with homogeneous Dirichlet boundary conditions are shown to converge to zero in $W^{1,\infty}(\Omega)$ at the same speed as the linear heat semigroup when $p>1$. For $p=1$, an exponential decay to zero is also obtained in one space dimension but the rate depends on a and differs from that of the linear heat equation. Finally, if $p\in(0,1)$ and $a<0$, finite time extinction occurs for non-negative solutions.

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1. Introduction and main results

We investigate the large time behaviour of solutions to the following initial-boundary value problem

$$\begin{cases} u_t - \Delta u = a|\nabla u|^p & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \text{on} \quad (0,\infty)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $a\in\mathbb{R}$, $a\neq 0$, $p>0$ and Ω is a bounded open subset of $\mathbb{R}^N$ with C^3-smooth boundary $\partial\Omega$. We first recall that several papers have already been devoted to the well-posedness of the Cauchy-Dirichlet problem (1.1) [1, 12, 17, 20]. In particular, when the initial datum is a bounded Radon measure, $p\in[1,(N+2)/(N+1))$ and $a>0$, Alaa proved the existence and uniqueness of weak solutions to (1.1) [1]. When $a>0$ and $p>2$, the non-existence of global solutions is also studied in [1, 37], the latter work providing further information on the way the solution blows up. Using a different approach, Benachour and Dabuleanu have obtained in [12] several results on the existence, uniqueness and regularity of global solutions for non-smooth initial data (typically, u_0 is a bounded Radon measure or belongs to $L^q(\Omega)$ for some $q\geq 1$). These results depend on the sign of a, the value of the exponent $p>0$ and the integrability and sign of the initial datum u_0. Singular initial data had been considered previously by Crandall, Lions and Souganidis in [20] when $a<0$ and $p>1$: using some properties of order-preserving semigroups, a universal bound for non-negative solutions to (1.1) is established in [20] which proves useful to show the existence and uniqueness of solutions to (1.1) when the initial datum u_0 satisfies: $u_0=\infty$ on a bounded open subset $D\subset\overline{D}\subset\Omega$ and $u_0=0$ in $\Omega\setminus\overline{D}$.

The main purpose of this paper is to supplement the above mentioned results by analysing the long time behaviour of global solutions to (1.1). While several results are available for the Cauchy problem [13, 14, 24, 38] and for the Cauchy-Neumann problem [11, 21], this question has only been considered recently in [38] for the Cauchy-Dirichlet problem (1.1): it is shown there that, for $p>2$,

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global solutions converge to zero in $L^\infty(\Omega)$ as time goes to infinity and this property remains true for global solutions which are bounded in $C^1(\overline{\Omega})$ when $p \in (1,2]$ (see also [3] for the one-dimensional case). Besides giving alternative proofs of these results, we shall identify the rate at which this convergence to zero takes place. More precisely, we show that, for $p > 1$, global classical solutions to (1.1) decay to zero in $W^{1,\infty}(\Omega)$ at the same (exponential) rate as the solutions to the linear heat equation with homogeneous Dirichlet boundary conditions. It is only when $p \in (0,1]$ that the gradient term $a|\nabla u|^p$ influences the large time dynamics (see Theorems 1.1 and 1.4 below).

Before stating our results, we introduce some notations: for $T \in (0,\infty]$ we set $Q_T = (0,T) \times \Omega$ and $\Gamma_T = (0,T) \times \partial\Omega$. We denote by $C_0(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$ vanishing on the boundary $\partial\Omega$ and by $C_0^+(\overline{\Omega})$ the positive cone of $C_0(\overline{\Omega})$. Next, $C^{1,2}(Q_T)$ is the space of functions $u \in C(Q_T)$ which are differentiable with respect to $t \in (0,T)$ and twice differentiable with respect to $x \in \Omega$ with derivatives u_t, $(u_{x_i})_{1\le i\le N}$ and $\left(u_{x_ix_j}\right)_{1\le i,j\le N}$ belonging to $C(Q_T)$. For $q \in [1,\infty]$, $\|\ \|_q$ and $\|\ \|_{\partial\Omega,q}$ denote the norms in $L^q(\Omega)$ and $L^q(\partial\Omega)$, respectively, and $W^{1,q}(\Omega)$ the Sobolev space of functions in $L^q(\Omega)$ for which the distributional derivatives $(u_{x_i})_{1\le i\le N}$ also belong to $L^q(\Omega)$. Finally, ν denotes the outward normal unit vector field to Ω and we use the notation $u_\nu(t,x) = \nabla u(t,x)\cdot\nu(x)$ for $(t,x) \in (0,\infty)\times\partial\Omega$ for the normal trace of the gradient of u (when it is well-defined).

Throughout this paper, we only consider classical solutions to (1.1) in the following sense:

Definition 1.1. *Given $u_0 \in C_0(\overline{\Omega})$, $p \in (0,\infty)$, $a \in \mathbb{R}\setminus\{0\}$ and $T \in (0,\infty]$, a classical solution u to (1.1) in Q_T is a function $u \in C([0,T)\times\overline{\Omega}) \cap C^{1,2}(Q_T)$ with $u(0) = u_0$ and satisfying (1.1) pointwisely in Q_T. Such a solution also satisfies*

$$u(t) = e^{t\Delta}u_0 + a\int_0^t e^{(t-s)\Delta}|\nabla u(s)|^p\,ds,\ \ t \in [0,T), \tag{1.2}$$

where $(e^{t\Delta})_{t\ge0}$ denotes the semigroup associated to the linear heat equation with homogeneous Dirichlet boundary conditions.

The results concerning the existence of global classical solutions to (1.1) may then be summarized as follows:

Proposition 1.1. *There exists a unique classical solution u to (1.1) in Q_∞ in the following cases:*

(i) *$p \in (0,2]$, $a \in \mathbb{R}\setminus\{0\}$ and $u_0 \in C_0(\overline{\Omega})$,*

(ii) *$p > 2$, $a < 0$ and $u_0 \in C_0^+(\overline{\Omega})$,*

(iii) *$p > 2$, $a > 0$ and $u_0 \in C_0^+(\overline{\Omega}) \cap C^1(\overline{\Omega})$ with $\|u_0\|_{C^1(\overline{\Omega})} \le \varepsilon_0$ where ε_0 is the constant defined in* [37, Proposition 3.1].

We refer to [12] for the assertion (i), to [20, Theorem 2.1] for (ii) and to [37] for (iii). The size restriction in (iii) is needed to have a global solution since finite gradient blow-up occurs for sufficiently large initial data [37]. We also mention that the previous well-posedness statement is far from being optimal with respect to the regularity of the initial data. Indeed, on the one hand, existence and uniqueness of weak solutions to (1.1) (which are classical solutions for positive times) are established in [12] under much weaker regularity assumptions on u_0, depending on the sign of a and the value of p. On the other hand, it has been shown in [8] that, for any $a \in \mathbb{R} \setminus \{0\}$, $p > 0$ and $u_0 \in C_0(\overline{\Omega})$, the Cauchy-Dirichlet problem (1.1) has a unique continuous viscosity solution, this result being valid for nonhomogeneous continuous Dirichlet boundary conditions as well. Such a solution satisfies the boundary conditions in the viscosity sense and need not satisfy them in the classical sense for $p > 2$ [8, p. 62], the latter phenomenon being in principle related to the possible blow-up of $\|\nabla u(t)\|_\infty$. The restriction on the size of the initial data in Proposition 1.1 thus excludes this behaviour.

We have the following results:

Theorem 1.1. *Assume that $a < 0$, $p \in (0,1)$ and $u_0 \in C_0^+(\overline{\Omega})$. Denoting by u the corresponding classical solution to (1.1), there exists $T^* > 0$ such that*

$$u(t,x) = 0 \quad \text{for each} \quad (t,x) \in (T^*, \infty) \times \Omega. \tag{1.3}$$

This property is called extinction in finite time of the solution to (1.1).

The proof relies on the results of [13, 14, 24] on the long time behaviour of the solution to the Cauchy problem in the whole space $\mathbb{R}^N$. Indeed, when $a < 0$, after an extension by 0 on $\mathbb{R}^N$ of the initial datum, the solution to the Cauchy problem becomes a super-solution to the Cauchy-Dirichlet problem. Thus, from the extinction in finite time of the solutions to the Cauchy problem we deduce the extinction in finite time of the solutions to the Cauchy-Dirichlet problem.

Remark 1.1. *Extinction in finite time cannot take place for solutions to (1.1) when $a > 0$, $p \in (0,1)$ and $u_0 \in C_0^+(\overline{\Omega})$, $u_0 \not\equiv 0$. Indeed, in this case, u is greater than the solution to the linear heat equation with initial datum u_0, and thus never vanishes in Ω. Moreover, when $a > 0$ and $p \in (0,1)$, there are non-zero stationary solutions to (1.1) (see Remark 3.1).*

In the next result, we establish the convergence to zero of global solutions to (1.1) for $p \in (1,2]$ and show that it takes place at the same exponential rate as that of the linear heat equation.

Theorem 1.2. *Assume that $p \in (1,2]$, $a \in \mathbb{R} \setminus \{0\}$ and $u_0 \in C_0(\overline{\Omega})$. Denoting by u the corresponding classical solution to (1.1), there is a constant $K > 0$ depending only on the initial datum u_0, the domain Ω and the parameters a and p such that, for $t > 0$,*

$$\|u(t)\|_\infty \le K e^{-t\lambda_1}, \tag{1.4}$$

$$\|\nabla u(t)\|_\infty \le K(1 + t^{-1/2}) e^{-t\lambda_1}, \tag{1.5}$$

where $\lambda_1 > 0$ denotes the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions in Ω.

Recall that, by [36, Lemma 3, p. 25] and [39, p. 35], the solution $e^{t\Delta} u_0$ to the linear heat equation with homogeneous Dirichlet boundary conditions satisfies:

$$\|e^{t\Delta} u_0\|_\infty \le C_0 e^{-t\lambda_1} \|u_0\|_\infty, \tag{1.6}$$

$$\|\nabla e^{t\Delta} u_0\|_\infty \le C_0 (1 + t^{-1/2}) e^{-t\lambda_1} \|u_0\|_\infty, \tag{1.7}$$

for $t > 0$ and $u_0 \in L^\infty(\Omega)$, the constant C depending only on the domain Ω. Basically, the previous result asserts that, for $p \in (1, 2]$, the additional nonlinear term, which depends on the gradient, has no contribution to the large time behaviour of the solution whatever the sign of u_0 is. Moreover, under the additional assumptions $a > 0$ and $u_0 \geq 0$, the temporal decay rate (1.4) is optimal. Indeed, in that case, the comparison principle ensures that $u(t) \geq e^{t\Delta}u_0$ and $t \longmapsto \|e^{t\Delta}u_0\|_\infty$ behaves as $C\ e^{-t\lambda_1}$ for large times. More surprisingly, the temporal decay rate (1.4) is also optimal for a large class of non-negative initial data when $a < 0$, so that the gradient term does not speed up the convergence towards zero in that case, see Proposition 4.2 below. As we shall see, a similar remark is valid when $p > 2$ for global solutions to (1.1) which are bounded in $C^1(\overline{\Omega})$. To prove Theorem 1.2, we will combine the previous decay estimates for the heat semigroup and a fixed point procedure in an appropriate weighted Banach space of Kato-Fujita type.

Remark 1.2. *Let us point out here that the proof of Theorem 1.2 for $p = 2$ is obvious thanks to the Cole-Hopf transformation which reduces (1.1) to the linear heat equation (see (4.11) below).*

We next show that, if $p > 2$, non-negative global solutions to (1.1) are such that $t \longmapsto e^{t\lambda_1}\|u(t)\|_\infty$ is bounded from below and above by positive constants for large times. More precisely, we have the following result:

Theorem 1.3. *Let u_0 be a non-negative function in $C_0^+(\overline{\Omega}) \cap C^1(\overline{\Omega})$ and consider $p > 2$. Assume further that either $a < 0$ or $a > 0$ and $\|u_0\|_{C^1(\overline{\Omega})} \leq \varepsilon_0$ where ε_0 is the constant defined in* [37, Proposition 3.1]. *Denoting by u the unique classical solution to (1.1), there exist non-negative functions $w_0 \in C_0^+(\overline{\Omega}) \cap C^1(\overline{\Omega})$ and $W_0 \in C_0^+(\overline{\Omega}) \cap C^1(\overline{\Omega})$ ($w_0 \not\equiv 0$ and $W_0 \not\equiv 0$ if $u_0 \not\equiv 0$) such that*

$$e^{t\Delta}w_0 \leq u(t) \leq e^{t\Delta}W_0 \quad \text{for} \quad t \quad \text{large enough.} \tag{1.8}$$

Moreover, $w_0 = u_0$ if $a > 0$, $W_0 = u_0$ if $a < 0$ and $\|\nabla u(t)\|_\infty$ satisfies (1.5) .

It turns out that, under the assumptions of either Theorem 1.2 (i) or Theorem 1.3, the estimates (1.4) and (1.5) allow us to be identify more precisely the large time behaviour of the solution u to (1.1). Indeed, in these cases, it follows from (1.5) by classical arguments that

$$\lim_{t\to\infty} \left\| e^{\lambda_1 t}\ u(t) - \alpha_\infty\ e_1 \right\|_\infty = 0\,,$$

where e_1 denotes the first eigenfunction of the Laplace operator with homogeneous Dirichlet boundary conditions (associated to the eigenvalue λ_1 and chosen to be non-negative with $\|e_1\|_2 = 1$) and

$$\alpha_\infty = \int_\Omega u_0(x)\ e_1(x)\ dx + a \int_0^\infty e^{\lambda_1 t} \int_\Omega |\nabla u(t,x)|^p\ e_1(x)\ dxdt\,.$$

Observe that α_∞ is finite by (1.5) since $p > 1$ (but we might have $\alpha_\infty = 0$).

We finally turn to the case $p = 1$ which appears to be a limit case. Indeed, the proof of the decay rates (1.4) and (1.5) obtained in Theorem 1.2 for $p \in (1, 2]$ does not extend to $p = 1$. In fact, as we shall see below in the one-dimensional case, the L^∞-norm of $u(t)$ still decays exponentially but the decay rate depends on a. The case of several dimensions ($N \geq 2$) seems to be an open problem. Nevertheless, it has been shown recently in [27] that there are $\beta \in (0, \lambda_1)$ and $U \in C_0^+(\overline{\Omega}) \cap C^2(\overline{\Omega})$

such that $-\Delta U = a\,|\nabla U| + \beta\, U$ in Ω, so that $(t,x) \longmapsto e^{-\beta t} U(x)$ is a solution to (1.1). We also recall that the case $p=1$ is rather peculiar for the Cauchy problem in $\mathbb{R}^N$ [10, 15, 16, 25, 31].

Theorem 1.4. *Let $\Omega = (-1,1)$, $u_0 \in C_0^+(\overline{\Omega})$, $a \in \mathbb{R}$, $a \neq 0$ and denote by u the unique classical solution to (1.1) with $p=1$. Then there is a positive constant $\gamma(a)$ depending only on a and Ω such that u satisfies*

$$\|u(t)\|_\infty \le \gamma(a)\ \|u_0\|_\infty\ e^{-((a^2/4)+\alpha_1)t}, \quad t \ge 1, \tag{1.9}$$

where α_1 is the first eigenvalue of the unbounded linear operator L of $L^2(0,1)$ with domain

$$D(L) = \left\{ \varphi \in W^{2,2}(0,1) \quad \text{such that} \quad a\varphi(0) + 2\varphi_x(0) = \varphi(1) = 0 \right\}$$

and defined by

$$L(\varphi) = -\varphi_{xx} \quad \text{for} \quad \varphi \in D(L).$$

The restriction to the one-dimensional setting comes from the observation that, if u_0 is an even function in $(-1,1)$ which is non-increasing in $(0,1)$, then u also solves

$$\begin{cases} u_t - u_{xx} = -a u_x & \text{in} \quad (0,\infty) \times (0,1), \\ u_x(t,0) = u(t,1) = 0 & \text{in} \quad (0,\infty), \\ u(0,x) = u_0(x) & \text{in} \quad (0,1), \end{cases}$$

which is a linear convection-diffusion equation. After a suitable change of unknown function, the study of the large time behaviour of u reduces to the spectral decomposition in $L^2(0,1)$ of L whose eigenfunctions form an orthonormal basis in $L^2(0,1)$. The case of general initial data will then follow by a comparison argument.

Remark 1.3. *It follows from the previous analysis that the gradient term $|\nabla u|^p$ alters the large time dynamics only for $p=1$ which contrasts markedly with the Cauchy problem in $\mathbb{R}^N$ where the effects of the gradient term become preponderant for $p < (N+2)/(N+1)$, see* [13, 14, 24, 31] *and the references therein.*

2. Proof of Theorem 1.1

Let $\tilde{u}_0$ be the extension by 0 of u_0 outside the domain Ω, that is, $\tilde{u}_0(x) = u_0(x)$ if $x \in \Omega$ and $\tilde{u}_0 = 0$ if $x \in \mathbb{R}^N \setminus \Omega$, and denote by $\tilde{u} \in C^{1,2}((0,\infty) \times \mathbb{R}^N) \cap C([0,\infty) \times \mathbb{R}^N)$ the unique solution to the Cauchy problem [25]

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = a|\nabla \tilde{u}|^p & \text{in} \quad (0,\infty) \times \mathbb{R}^N, \\ \tilde{u}(0,.) = \tilde{u}_0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{2.1}$$

Since $a < 0$, $p \in (0,1)$ and $\tilde{u}_0$ is a non-negative continuous and bounded function with compact support in $\mathbb{R}^N$, we infer from [13, 24] that $\tilde{u}$ enjoys the property of extinction in finite time, that is, there exists $T^* > 0$ such that $\tilde{u}(t,x) = 0$ for $(t,x) \in (T^*,\infty) \times \mathbb{R}^N$. On the other hand, $\tilde{u}$ is non-negative by the comparison principle and thus satisfies:

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = a|\nabla \tilde{u}|^p & \text{in } \ Q_\infty, \\ \tilde{u}(t,x) \geq 0 & \text{on } \ \Gamma_\infty, \\ \tilde{u}(0,.) = u_0 & \text{in } \ \Omega. \end{cases} \tag{2.2}$$

Therefore, $\tilde{u}$ is a super-solution to (1.1) and the comparison principle [22, Theorem 16, p. 52] ensures that $0 \leq u \leq \tilde{u}$ in Q_∞. Consequently, there is $T^* > 0$ such that u satisfies (1.3). □

3. Convergence to zero for $p \in (1,2]$

As a preliminary step to the proof of Theorem 1.2 we first establish the convergence to zero in $L^\infty(\Omega)$ of any global classical solution to (1.1) when $p \in (1,2]$.

Proposition 3.1. *Assume that $p \in (1,2]$, $a \in \mathbb{R} \setminus \{0\}$ and $u_0 \in C_0(\overline{\Omega})$. Denoting by u the corresponding classical solution to (1.1) we have*

$$\lim_{t\to\infty} \|u(t)\|_\infty = 0\,.$$

A possible proof of Proposition 3.1 relies on the LaSalle Invariance Principle since it can be shown that the L^∞-norm is a strict Liapunov functional for the dynamical system associated to (1.1) in $C_0(\overline{\Omega})$. However, the following shorter proof relying on the method of relaxed semi-limits in the spirit of [5, Section 8, Exemple 5] has been suggested to us by G. Barles [6].

Proof:

We first introduce some notations: for $\varepsilon \in [0,1)$, $\xi_0 \in \mathbb{R}$, $\xi = (\xi_i)_{\{1\leq i\leq N\}} \in \mathbb{R}^N$ and any symmetric $N \times N$ matrix $S \in \mathcal{M}_N(\mathbb{R})$, we put

$$G^\varepsilon(t,x,r,\xi_0,\xi,S) = \begin{cases} \varepsilon\,\xi_0 - tr(S) - a\,|\xi|^p & \text{if } \ (t,x) \in \mathcal{O}\,, \\ r & \text{if } \ (t,x) \in (0,\infty) \times \partial\Omega\,, \\ r - u_0(x) & \text{if } \ (t,x) \in \{0\} \times \overline{\Omega}\,, \end{cases}$$

and

$$H(x,r,\xi,S) = \begin{cases} -tr(S) - a\,|\xi|^p & \text{if } \ x \in \Omega\,, \\ r & \text{if } \ x \in \partial\Omega\,, \end{cases}$$

where $tr(S)$ denotes the trace of the matrix S and $\mathcal{O} = (0,\infty) \times \Omega$.

Next, we assume that $a < 0$ and put $u^\varepsilon(t,x) = u(t/\varepsilon, x)$ for $\varepsilon \in (0,1)$ and $(t,x) \in \overline{\mathcal{O}}$. It readily follows from (1.1) and Proposition 1.1 that $u^\varepsilon \in C(\overline{\mathcal{O}})$ and solves

$$G^\varepsilon(t,x,u^\varepsilon,u^\varepsilon_t,\nabla u^\varepsilon, D^2u^\varepsilon) = 0 \quad \text{in} \quad \overline{\mathcal{O}} \tag{3.1}$$

in the viscosity sense (D^2u^ε denoting the Hessian matrix of u^ε). In addition, the following bound is a straightforward consequence of the maximum principle

$$\|u^\varepsilon(t)\|_\infty \leq \|u_0\|_\infty\,, \quad (t,\varepsilon) \in [0,\infty) \times (0,1)\,. \tag{3.2}$$

Introducing the semi-limits $\underline{u}$ and $\overline{u}$ defined by

$$\underline{u}(t,x) = \liminf_{(\varepsilon,s,y)\to(0,t,x)} u^\varepsilon(s,y)\,, \quad \overline{u}(t,x) = \limsup_{(\varepsilon,s,y)\to(0,t,x)} u^\varepsilon(s,y)\,,$$

for $(t,x) \in \overline{\mathcal{O}}$, we clearly have

$$\underline{u}(t,x) \le \overline{u}(t,x)\,, \quad (t,x) \in \overline{\mathcal{O}}\,, \tag{3.3}$$

and infer from (3.1), (3.2) and [4, Théorème 4.1] that $\overline{u}$ is an upper semicontinuous viscosity subsolution to $G^0 = 0$ in $\overline{\mathcal{O}}$ while $\underline{u}$ is a lower semicontinuous viscosity supersolution to $G^0 = 0$ in $\overline{\mathcal{O}}$. Since u^ε is obtained from u by a time dilatation, we realize that $\underline{u}$ and $\overline{u}$ actually do not depend on the time variable, that is,

$$\underline{u}(t,x) = \underline{u}(1,x) = \underline{v}(x) \quad \text{and} \quad \overline{u}(t,x) = \overline{u}(1,x) = \overline{v}(x) \quad \text{for} \quad (t,x) \in (0,\infty) \times \overline{\Omega}\,. \tag{3.4}$$

We therefore deduce from the properties of $\underline{u}$ and $\overline{u}$ that $\overline{v}$ is an upper semicontinuous viscosity subsolution to $H = 0$ in $\overline{\Omega}$ while $\underline{v}$ is a lower semicontinuous viscosity supersolution to $H = 0$ in $\overline{\Omega}$. In addition, since the boundary of Ω is smooth, we may proceed as in the proof of [7, Corollary 2.1] or apply [26, Theorem 2.1] to deduce that

$$\overline{v}(x) \le 0 \le \underline{v}(x) \quad \text{for} \quad x \in \partial\Omega\,. \tag{3.5}$$

Indeed, H clearly fulfils [7, (F5)] and [26, Theorem 2.1 (iv)]. We are now in a position to apply the strong comparison principle stated in Proposition 3.2 below to conclude that

$$\overline{v}(x) \le \underline{v}(x) \quad \text{for} \quad x \in \overline{\Omega}\,. \tag{3.6}$$

Combining (3.3), (3.4) and (3.6), we conclude that $\overline{v} = \underline{v} = v$ in $\overline{\Omega}$ and, thanks to [4, Lemma 4.1], we obtain that $v \in C(\overline{\Omega})$ is a viscosity solution to $H = 0$ and (u^ε) converges towards v in $C((0,\infty) \times \overline{\Omega})$. In particular, $(u^\varepsilon(1,.))$ converges towards v in $C(\overline{\Omega})$ from which we deduce that

$$\lim_{t\to\infty} \|u(t) - v\|_\infty = 0\,.$$

On the other hand, $x \mapsto 0$ is also a continuous viscosity solution to $H = 0$ and it readily follows from (3.5) and Proposition 3.2 that $v = 0$, which completes the proof of Proposition 3.1 when $a < 0$.

Finally, if $a > 0$, it is straightforward to check that $-u$ solves (1.1) with $-a$ instead of a and $-u_0$ instead of u_0. We then apply the previous analysis to $-u$ to complete the proof of Proposition 3.1. □

We now turn to the cornerstone of the previous proof, namely the strong comparison principle [19, Theorem 3.3]. We cannot however apply directly [19, Theorem 3.3] because H lacks some coercivity with respect to r and thus does not fulfil [19, (3.13)]. Still, the fact that [19, Theorem 3.3] holds true without this assumption in some cases is already well-known and a strategy to bypass this assumption is sketched in [19, Section 5.C] (see also [5, Section 4.4.1]). We will thus only give the required modification of the proof of [19, Theorem 3.3]. Actually, Proposition 3.2 is a particular case of [7, Theorem 1] which applies to a wider class of functions H with no dependence on u but is more complicated to prove.

Proposition 3.2. *Let U be an upper semicontinuous viscosity subsolution to $H = 0$ in $\overline{\Omega}$ and V be a lower semicontinuous viscosity supersolution to $H = 0$ in $\overline{\Omega}$ such that $U \le V$ on $\partial\Omega$. If $a < 0$, then $U \le V$ in $\overline{\Omega}$.*

Proof:
According to [19, Section 5.C] and [5, Section 4.4.1], it suffices to construct a sequence $(U_\delta)_{\delta\in(0,1)}$ such that (i) $U_\delta \le V$ on $\partial\Omega$, (ii) U_δ is an upper semicontinuous viscosity subsolution to $H+\eta(\delta)=0$ for some $\eta(\delta)>0$, (iii) $\eta(\delta)\to 0$ and $\|U_\delta - U\|_\infty \to 0$ as $\delta\to 0$.
We now construct such a sequence: since Ω is bounded, there exists $\lambda>0$ such that $e^{(p-1)(x_1-\lambda)} \le 1/(2|a|)$. We then put

$$M = \max_{x\in\overline{\Omega}} e^{x_1-\lambda}, \quad m = \min_{x\in\overline{\Omega}} e^{x_1-\lambda}$$

and

$$U_\delta(x) = (1-\delta)\, U(x) + \delta\, \left(e^{x_1-\lambda} - M + \min_{y\in\overline{\Omega}} V(y) \right), \quad x\in\overline{\Omega}.$$

Owing to the convexity of H with respect to ξ and S (recall that $a<0$), U_δ satisfies the requirements (i), (ii) and (iii) listed above with $\eta(\delta) = (m\ \delta)/2$. At this point, one then proceeds as in the proof of [19, Section 5.C] to conclude that $U_\delta \le V$ in $\overline{\Omega}$ for each $\delta\in(0,1)$ and then pass to the limit as $\delta\to 0$ to complete the proof. □

Remark 3.1. *It follows from Proposition 3.1 that zero is the only bounded stationary solution to (1.1) when $p\in(1,2]$. This property no longer holds true for $p\in(0,1)$. Indeed, in the particular case where Ω is the unit ball $B_1(0)$ of $\mathbb{R}^N$ and $a=1$, there are at least two stationary solutions to (1.1) as the function*

$$w(x) = \frac{(1-p)^{(2-p)/(1-p)}}{(2-p)(N-(N-1)p)^{1/(1-p)}} \left(1-|x|^{(2-p)/(1-p)}\right), \quad x\in B_1(0),$$

is a non-zero stationary solution to (1.1) in $\Omega = B_1(0)$ (for $p>1$, similar solutions exist but are singular [2, 33]*). Furthermore, if $N=1$, there is a continuum of non-negative stationary solutions and the convergence towards these stationary solutions is investigated in* [30]*. For general domains Ω in several space dimensions, the large time behaviour seems to be an open problem.*

4. Proof of Theorem 1.2 - Decay estimates

We denote by X the Banach space of functions $u\in C([0,\infty), C_0(\overline{\Omega}))\cap C((0,\infty), C^1(\overline{\Omega}))$ such that

$$\|u\|_X = \max\left\{ \sup_{t\in(0,\infty)} e^{t\lambda_1}\|u(t)\|_\infty,\ \sup_{t\in(0,\infty)} \frac{t^{1/2}}{1+t^{1/2}} e^{t\lambda_1}\|\nabla u(t)\|_\infty \right\} < \infty,$$

where λ_1 is the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions in Ω. Similar weighted spaces related to the heat semigroup have been previously used by, e.g., Kato [28], Kato and Fujita [29], Brezis and Cazenave [18] and Ben-Artzi, Souplet and Weissler [9].
For $u\in X$, $u_0\in C_0(\overline{\Omega})$ and $t>0$, we introduce the maps

$$\mathcal{G}u(t) = \int_0^t e^{(t-s)\Delta} |\nabla u(s)|^p\, ds \tag{4.1}$$

and

$$\mathcal{F}u(t) = e^{t\Delta}u_0 + a\mathcal{G}u(t). \tag{4.2}$$

Clearly, a solution to (1.1) is a fixed point of $\mathcal{F}$ and we will use a fixed point procedure to show that some solutions to (1.1) belong to some suitable bounded subsets of X. More precisely, we have the following result:

Proposition 4.1. *If $p \in (1,2)$, there is a positive constant $K_1 > 0$ depending only on p, a, λ_1 and Ω such that, if $u_0 \in C_0(\overline{\Omega})$ satisfies $\|u_0\|_\infty \le K/(2C_0)$ for some $K \in (0, K_1]$, the corresponding classical solution u to (1.1) belongs to $X_K = \{u \in X; \|u\|_X \le K\}$ (recall that C_0 is the constant occurring in (1.6) and (1.7)).*

Proof:
Recalling that the Laplace operator with homogeneous Dirichlet boundary conditions in Ω generates an analytic semigroup in $C_0(\overline{\Omega})$ and $C_0(\overline{\Omega}) \cap C^1(\overline{\Omega})$ (see, e.g., [34, Definition 2.0.2, Corollary 3.1.21 & Theorem 3.1.25]), both $\mathcal{G}u$ and $\mathcal{F}u$ belong to $C([0,\infty), C_0(\overline{\Omega})) \cap C((0,\infty), C^1(\overline{\Omega}))$ for $u \in X$. We next check that $\mathcal{G}u$ and $\mathcal{F}u$ map X into itself. Let $u \in X$. On the one hand, taking into account (1.6) we have

$$
\begin{aligned}
\|\mathcal{G}u(t)\|_\infty &\le \int_0^t \|e^{(t-s)\Delta}|\nabla u(s)|^p\|_\infty \, ds \le C_0 \int_0^t e^{-(t-s)\lambda_1} \|\nabla u(s)\|_\infty^p \, ds \\
&\le C_0 \|u\|_X^p \int_0^t e^{-(t-s)\lambda_1} (1+s^{-1/2})^p e^{-sp\lambda_1} \, ds \\
&\le C_0 \|u\|_X^p e^{-t\lambda_1} \int_0^t (1+s^{-1/2})^p e^{-s(p-1)\lambda_1} \, ds,
\end{aligned}
$$

which implies

$$
e^{t\lambda_1} \|\mathcal{G}u(t)\|_\infty \le C_0 \; I_1(p) \|u\|_X^p \quad \text{with} \quad I_1(p) = \int_0^\infty (1+s^{-1/2})^p e^{-s(p-1)\lambda_1} \, ds < \infty, \tag{4.3}
$$

the integral $I_1(p)$ being finite since $p \in (1,2)$. On the other hand, we infer from (1.7) that

$$
\frac{t^{1/2}}{1+t^{1/2}} e^{t\lambda_1} \|\nabla \mathcal{G}u(t)\|_\infty \le C_0 \|u\|_X^p \frac{t^{1/2}}{1+t^{1/2}} \int_0^t (1+(t-s)^{-1/2})(1+s^{-1/2})^p e^{-s(p-1)\lambda_1} \, ds.
$$

Since $p \in (1,2)$, we have

$$\begin{aligned}
&\frac{t^{1/2}}{1+t^{1/2}} \int_0^t (1+(t-s)^{-1/2})(1+s^{-1/2})^p e^{-s(p-1)\lambda_1}\, ds \\
\leq\ & \frac{2t^{1/2}}{1+t^{1/2}} \int_0^t \left(1+(t-s)^{-1/2}+s^{-p/2}+(t-s)^{-1/2}\ s^{-p/2}\right) e^{-s(p-1)\lambda_1}\, ds \\
\leq\ & 2\int_0^t \left(1+(t-s)^{-1/2}+s^{-p/2}+\frac{t^{1/2}}{1+t^{1/2}}(t-s)^{-1/2}\ s^{-p/2}\right) e^{-s(p-1)\lambda_1}\, ds \\
\leq\ & 2\int_0^\infty \left(1+s^{-p/2}\right) e^{-s(p-1)\lambda_1}\, ds + 2\sup_{r\geq 0}\left\{r^{1/2}\ e^{-r(p-1)\lambda_1}\right\} \int_0^1 (1-s)^{-1/2} s^{-1/2}\, ds \\
+\ & 2\frac{t^{(2-p)/4}}{1+t^{1/2}}\ \sup_{r\geq 0}\left\{r^{(2-p)/4}\ e^{-r(p-1)\lambda_1}\right\} \int_0^1 (1-s)^{-1/2}\ s^{-(p+2)/4}\, ds,
\end{aligned}$$

and the right-hand side of the above inequality is bounded since $p \in (1,2)$ and $t^{(2-p)/4} \leq 1+t^{1/2}$ for $t \geq 0$. Consequently,

$$I_2(p) = \sup_{t\geq 0}\left\{\frac{t^{1/2}}{1+t^{1/2}} \int_0^t (1+(t-s)^{-1/2})(1+s^{-1/2})^p e^{-s(p-1)\lambda_1}\, ds\right\} < \infty \tag{4.4}$$

and

$$\frac{t^{1/2}}{1+t^{1/2}} e^{t\lambda_1} \|\nabla \mathcal{G}u(t)\|_\infty \leq C_0 I_2(p) \|u\|_X^p. \tag{4.5}$$

Combining (4.3) and (4.5) we conclude that

$$\|\mathcal{G}(u)\|_X \leq C_2 \|u\|_X^p,$$

where C_2 is a constant depending only on p, λ_1 and Ω. Applying once more the estimates (1.6) and (1.7) and using the above inequality we obtain

$$\|\mathcal{F}(u)\|_X \leq C_0\ \|u_0\|_\infty + |a| C_2 \|u\|_X^p. \tag{4.6}$$

We have thus established that $\mathcal{F}u \in X$ for $u \in X$.
Next, for $K > 0$, we consider two functions u_1 and u_2 in X_K, where X_K is defined in Proposition 4.1. By (1.6) we have

$$\begin{aligned}
\|\mathcal{F}u_1(t) - \mathcal{F}u_2(t)\|_\infty \ &\leq\ |a| \int_0^t \|e^{(t-s)\Delta}(|\nabla u_1|^p(s) - |\nabla u_2|^p(s))\|_\infty\, ds \\
&\leq\ |a| p C_0 \int_0^t e^{-(t-s)\lambda_1} (\|\nabla u_1(s)\|_\infty^{p-1} + \|\nabla u_2(s)\|_\infty^{p-1}) \|\nabla (u_1-u_2)(s)\|_\infty\, ds \\
&\leq\ |a| p C_0 I_1(p) (\|u_1\|_X^{p-1} + \|u_2\|_X^{p-1}) \|u_1 - u_2\|_X e^{-t\lambda_1},
\end{aligned}$$

whence

$$e^{t\lambda_1} \|\mathcal{F}u_1(t) - \mathcal{F}u_2(t)\|_\infty \leq C_3 K^{p-1} \|u_1 - u_2\|_X, \tag{4.7}$$

where C_3 is a constant depending only on p, λ_1 and Ω. Likewise we deduce from (1.7) and (4.4) that

$$\frac{t^{1/2}}{1+t^{1/2}} e^{t\lambda_1} \|\nabla\mathcal{F}u_1(t) - \nabla\mathcal{F}u_2(t)\|_\infty \le C_3 K^{p-1} \|u_1 - u_2\|_X \tag{4.8}$$

for a possibly larger constant C_3. Combining (4.7) and (4.8) the functional $\mathcal{F}$ satisfies

$$\|\mathcal{F}u_1 - \mathcal{F}u_2\|_X \le C_3 K^{p-1} \|u_1 - u_2\|_X \tag{4.9}$$

for $u_1 \in X_K$ and $u_2 \in X_K$. Introducing $K_1 > 0$ given by

$$K_1^{p-1} = \min\left\{ \frac{1}{2|a|C_2}, \frac{1}{2C_3} \right\}, \tag{4.10}$$

we infer from (4.6) and (4.9) that, if $K \in (0, K_1]$ and $\|u_0\|_\infty \le K/(2C_0)$, then $\mathcal{F}(u) \in X_K$ for $u \in X_K$ and $\|\mathcal{F}u_1 - \mathcal{F}u_2\|_X \le \|u_1 - u_2\|_X /2$ for $u_1 \in X_K$ and $u_2 \in X_K$. Consequently, under these assumptions on K and u_0, $\mathcal{F}$ is a strict contraction from X_K into X_K. By the Banach fixed point theorem, $\mathcal{F}$ has a unique fixed point $u \in X_K$. Then u satisfies (1.2) and [12, Theorems 3.2 & 3.3] warrant that u is the unique classical solution to (1.1), which completes the proof of Proposition 4.1. □

Proof of Theorem 1.2 (decay estimates):
Assume first that $p \in (1,2)$. We consider $u_0 \in C_0(\overline{\Omega})$ and denote by u the corresponding classical solution to (1.1). We have already established in the previous section that $\|u(t)\|_\infty \longrightarrow 0$ as $t \to \infty$. Therefore, there exists $t_0 > 0$ such that $\|u(t_0)\|_\infty \le K_1/(2C_0)$ and we infer from Proposition 4.1 with $K = K_1$ that $u(. + t_0)$ belongs to X_{K_1}, i.e.,

$$\|u(t)\|_\infty \le K_1 e^{-(t-t_0)\lambda_1} \quad \text{and} \quad \|\nabla u(t)\|_\infty \le K_1 (1 + (t-t_0)^{-1/2} e^{-(t-t_0)\lambda_1}$$

for any $t > t_0$. On the other hand, it follows from the analysis in [12] that $\|u(t)\|_\infty \le \|u_0\|_\infty$ and $\|\nabla u(t)\|_\infty \le C(t_0, u_0) t^{-1/2}$ for $t \in (0, t_0]$. Combining these two facts yields (1.4) and (1.5) for u.

It remains to study the case $p = 2$. Introducing $U = e^{au} - 1$, it follows from (1.1) that $U_t = \Delta U$ in $(0,\infty) \times \Omega$ with $U = 0$ on $(0,\infty) \times \partial\Omega$ and $U(0) = U_0 = e^{au_0} - 1$ in Ω. Consequently, $U(t) = e^{t\Delta} U_0$ and

$$u(t) = \frac{1}{a} \log\left(1 + e^{t\Delta} U_0\right) \quad \text{and} \quad \nabla u(t) = \frac{1}{a} \frac{\nabla e^{t\Delta} U_0}{1 + e^{t\Delta} U_0}, \quad t \ge 0. \tag{4.11}$$

Since $\|e^{t\Delta} U_0\|_\infty \longrightarrow 0$ as $t \to \infty$, the temporal decay estimates (1.4) and (1.5) follow from (1.6) and (1.7) and the previous formulae for $u(t)$ and $\nabla u(t)$. □

As already mentioned, when $a > 0$, $p \in (1,2]$ and $u_0 \in C_0^+(\overline{\Omega})$, the temporal decay rate (1.4) is optimal since $u(t) \ge e^{t\Delta} u_0$ by the comparison principle and $t \longmapsto \|e^{t\Delta} u_0\|_\infty$ behaves as $C\, e^{-t\lambda_1}$ for large times. It turns out that the temporal decay rate (1.4) is still optimal when $a < 0$, $p \in (1,2)$ and $u_0 \in C_0^+(\overline{\Omega})$.

Proposition 4.2. *Consider $a < 0$, $p \in (1,2)$ and $u_0 \in C_0^+(\overline{\Omega})$ such that $u_0 \ge \alpha e_1$ for some $\alpha > 0$, where e_1 denotes the eigenfunction of the Laplace operator with homogeneous Dirichlet boundary conditions in Ω associated to the eigenvalue λ_1 and normalized such that $e_1 \ge 0$ and $\|e_1\|_\infty = 1$.*

Then, there exist two positive constants $C_4(u_0) > 0$ and $C_5(u_0) > 0$ depending on u_0 such that the classical solution u to (1.1) satisfies:

$$C_4(u_0)e^{-t\lambda_1} \leq \|u(t)\|_\infty \leq C_5(u_0)e^{-t\lambda_1} \quad \textit{for} \quad t > 0. \tag{4.12}$$

Proof:
The second inequality in (4.12) readily follows from the comparison principle and the properties of the linear heat equation since u is a subsolution to the linear heat equation. As for the first inequality, we proceed as follows: consider $n \geq 1$ large enough such that

$$n \geq K_1^{-1} \quad \text{and} \quad n^{p-1} > 4|a|C_0^2 I_1(p), \tag{4.13}$$

the parameters C_0, $I_1(p)$ and K_1 being defined in (1.6), (4.3) and Proposition 4.1, respectively. By (4.13), we have

$$\frac{1}{2C_0 n} > \frac{2|a|C_0 I_1(p)}{n^p}$$

and we fix

$$\beta_n \in \left(\frac{2|a|C_0 I_1(p)}{n^p}, \frac{1}{2C_0 n} \right). \tag{4.14}$$

We next take the initial datum in (1.1) to be $u_{0,n} = \beta_n e_1$ and denote by u_n the corresponding solution to (1.1). Owing to (1.2) and (1.6) we have

$$\begin{aligned} u_n(t) &\geq \beta_n e^{t\Delta} e_1 - |a| \int_0^t \|e^{(t-s)\Delta} |\nabla u_n(s)|^p\|_\infty \, ds \\ &\geq \beta_n e^{-t\lambda_1} e_1 - |a| C_0 \int_0^t e^{-\lambda_1(t-s)} \|\nabla u_n(s)\|_\infty^p \, ds. \end{aligned}$$

Since $\|u_{0,n}\|_\infty = \beta_n \leq 1/(2C_0 n)$ and $1/n \leq K_1$, we deduce from Proposition 4.1 that $u \in X_{1/n}$. Consequently,

$$u_n(t) \geq \beta_n e^{-t\lambda_1} e_1 - |a| C_0 n^{-p} \int_0^t e^{-\lambda_1(t-s)} (1+s^{-1/2})^p e^{-\lambda_1 p s} \, ds \geq e^{-t\lambda_1} \left(\beta_n e_1 - \frac{|a|C_0 I_1(p)}{n^p} \right),$$

whence, since $\|e_1\|_\infty = 1$ and β_n fulfils (4.14),

$$\|u_n(t)\|_\infty \geq e^{-t\lambda_1} \left(\beta_n - \frac{|a|C_0 I_1(p)}{n^p} \right) \geq e^{-t\lambda_1} \frac{\beta_n}{2} = e^{-t\lambda_1} \frac{\|u_{0,n}\|_\infty}{2}. \tag{4.15}$$

Consider now u_0 as in Proposition 4.2 and denote by u the corresponding classical solution to (1.1). Since $\beta_n \to 0$ as $n \to \infty$, there is n_0 fulfilling (4.13) such that $u_0 \geq \alpha e_1 \geq \beta_{n_0} e_1$. By the comparison principle, we have $u(t) \geq u_{n_0}(t)$ and Proposition 4.2 is then a straightforward consequence of (4.15). □

5. Proof of Theorem 1.3

We first consider the case $a < 0$. On the one hand, u is a subsolution of the linear heat equation with the same initial datum and the comparison principle entails that

$$0 \leq u \leq v \quad \text{in} \quad Q_\infty \quad \text{with} \quad v(t) = e^{t\Delta} u_0, \quad t \geq 0. \tag{5.1}$$

On the other hand, by the maximum principle (see [22, 32] and also [37, Remark 3.3]) we have

$$\|\nabla u(t)\|_\infty = \|\nabla u(t)\|_{\partial\Omega,\infty} = \|u_\nu(t)\|_{\partial\Omega,\infty}, \tag{5.2}$$

and

$$\|u_0\|_{C^1(\overline{\Omega})} \ge \|\nabla v(t)\|_\infty = \|\nabla v(t)\|_{\partial\Omega,\infty} = \|v_\nu(t)\|_{\partial\Omega,\infty}, \tag{5.3}$$

the last equality in (5.2) and (5.3) being a consequence of the homogeneous Dirichlet boundary conditions. Owing to the homogeneous Dirichlet boundary conditions, we deduce from (5.1) that $v_\nu(t) \le u_\nu(t) \le 0$ for $t \ge 0$, which, together with (5.2) and (5.3), yields

$$\|\nabla u(t)\|_\infty = \|u_\nu(t)\|_{\partial\Omega,\infty} \le \|v_\nu(t)\|_{\partial\Omega,\infty} = \|\nabla v(t)\|_\infty \le \|u_0\|_{C^1(\overline{\Omega})} \quad \text{for} \quad t > 0. \tag{5.4}$$

Consequently, since $p > 2$,

$$u_t - \Delta u = -|a||\nabla u|^{p-2}|\nabla u|^2 \ge -|a|\|u_0\|_{C^1(\overline{\Omega})}^{p-2}|\nabla u|^2 \quad \text{in} \quad Q_\infty.$$

Denote by $C_6 = |a|\|u_0\|_{C^1(\overline{\Omega})}^{p-2}$ and let w be the solution to the following initial boundary value problem:

$$\begin{cases} w_t - \Delta w = -C_6|\nabla w|^2 \ \text{ in } \quad Q_\infty, \\ w(t,x) = 0 \ \text{ on } \ \Gamma_\infty, \\ w(0,.) = u_0 \ \text{ in } \ \Omega. \end{cases} \tag{5.5}$$

Then $z = 1 - e^{-C_6 w}$ satisfies the linear heat equation in Q_∞ with homogeneous Dirichlet boundary conditions and initial datum $z(0) = 1 - e^{-C_6 u_0}$, so that $w(t) = -C_6^{-1} \log\left(1 - e^{t\Delta} z(0)\right)$ for $t \ge 0$, while the comparison principle ensures that

$$-\frac{1}{C_6} \log\left(1 - e^{t\Delta} z(0)\right) \le u(t) \le e^{t\Delta} u_0, \quad \text{for} \quad t > 0.$$

Since $-\log(1-r) \ge r$ for $r \in (0,1)$ and $e^{t\Delta} z(0) \in (0,1)$ for $t \ge 0$, we conclude that

$$\frac{1}{C_6} e^{t\Delta} z(0) \le u(t) \le e^{t\Delta} u_0, \quad \text{for} \quad t > 0,$$

and (1.8) holds true with $w_0 = (1 - e^{-C_6 u_0})/C_6$ which is a positive function in $C_0^1(\overline{\Omega})$ and $W_0 = u_0$. Furthermore, the large time behaviour of $\nabla u(t)$ is a consequence of (1.7) and (5.4):

$$\|\nabla u(t)\|_\infty \le \|\nabla e^{t\Delta} u_0\|_\infty \le C_0(1 + t^{-1/2}) e^{-\lambda_1 t} \|u_0\|_\infty,$$

and the proof of Theorem 1.3 is complete for $a < 0$.
We now turn to the case $a > 0$. By [37, Proposition 3.1], the condition $\|u_0\|_{C^1(\overline{\Omega})} \le \varepsilon$ not only warrants that the corresponding classical solution u to (1.1) is global but also that it is bounded in $C^1(\overline{\Omega})$. Consequently, there is a positive constant $C_7 > 0$ such that

$$\|\nabla u(t)\|_\infty \le C_7 \ \text{ for } \ t \ge 0.$$

Thanks to this property, we may proceed as in the previous case and deduce from the comparison principle that

$$e^{t\Delta} u_0 \le u(t) \le w(t) = \frac{1}{C_8} \log(1 + e^{t\Delta} z(0)) \ \text{ for } \ t > 0, \tag{5.6}$$

where $C_8 = aC_7^{p-2}$, $z(0) = e^{C_8 u_0} - 1$ and w is the solution to the following initial boundary value problem:

$$\begin{cases} w_t - \Delta w = C_8 |\nabla w|^2 \quad \text{in} \quad Q_\infty, \\ w(t,x) = 0 \quad \text{on} \quad \Gamma_\infty, \\ w(0,.) = u_0 \quad \text{in} \quad \Omega. \end{cases} \tag{5.7}$$

Since $\log(1+r) \le r$ for $r \ge 0$ and $e^{t\Delta} z(0) \ge 0$ for $t \ge 0$, we infer from (5.6) that (1.8) is satisfied with $w_0 = u_0$ and $W_0 = (e^{C_8 u_0} - 1)/C_8$. In addition, owing to (5.6) and [37, Remark 3.3] we have that:

$$\begin{aligned} & \|\nabla u(t)\|_\infty = \|\nabla u(t)\|_{\partial\Omega,\infty} = \|u_\nu(t)\|_{\partial\Omega,\infty} \\ \le \ & \|w_\nu(t)\|_{\partial\Omega,\infty} = \|\nabla w(t)\|_{\partial\Omega,\infty} = \|\nabla w(t)\|_\infty = \frac{1}{C_8} \left\| \frac{\nabla e^{t\Delta} z(0)}{1 + e^{t\Delta} z(0)} \right\|_\infty , \end{aligned}$$

from which the estimate (1.5) follows. □

6. Proof of Theorem 1.4

We first prove Theorem 1.4 for non-negative initial data $u_0 \in C_0([-1,1])$ which are profiled, that is, u_0 is a non-decreasing function on $(-1,0)$ and a non-increasing function on $(0,1)$. From [25, Corollary 4.4] we know that this property is preserved throughout time evolution, so that $u(t)$ is a non-decreasing function on $(-1,0)$ and a non-increasing function on $(0,1)$ for any $t > 0$. Since $u(t) \in C^1([-1,1])$ for $t > 0$, an alternative formulation of this property is $|u_x(t,x)| = -\text{sign}(x) u_x(t,x)$ for $(t,x) \in (0,\infty) \times (-1,1)$. Therefore, u also solves

$$\begin{cases} u_t - u_{xx} = -a u_x & \text{in} \quad (0,\infty) \times (0,1), \\ u_x(t,0) = u(t,1) = 0 & \text{in} \quad (0,\infty), \\ u(0,x) = u_0(x) & \text{in} \quad (0,1), \end{cases} \tag{6.1}$$

and a similar equation on $(-1,0)$. Conversely, as a consequence of the uniqueness of the solution to (1.1), solving (6.1) on $(0,1)$ and $(-1,0)$ gives back the solution to (1.1). We shall therefore study the solution to (6.1). Using the transformation

$$v(t,x) = e^{a^2 t/4} e^{-ax/2} u(t,x) , \quad (t,x) \in (0,\infty) \times (0,1) , \tag{6.2}$$

then v satisfies the following problem

$$\begin{cases} v_t - v_{xx} = 0 & \text{in} \quad (0,\infty) \times (0,1), \\ 2v_x(t,0) + av(t,0) = v(t,1) = 0 & \text{in} \quad (0,\infty), \\ v(0,x) = v_0(x) = e^{-ax/2} u_0(x) & \text{in} \quad (0,1), \end{cases} \tag{6.3}$$

which also reads $v_t = Lv$ with $v(0) = v_0$, the unbounded linear operator L being defined in Theorem 1.4. The initial boundary value problem (6.3) being linear, the large time behaviour of its solutions is determined by the spectrum of L. First, classical results ensure that the spectrum is an increasing sequence $(\alpha_n)_{n\ge1}$ of eigenvalues converging to ∞ and the corresponding normalized eigenfunctions $(\varphi_n)_{n\ge1}$ form an orthonormal basis of $L^2(0,1)$ (see, e.g., [35, Théorème 6.2-1 and Remarque 6.2-2]). The next step is to identify the eigenvalues and eigenfunctions of L.

Proposition 6.1. *For $a \neq 0$, the equation $\tan(z) = 2z/a$ has a countably infinite number of positive solutions and we denote by $\mathcal{Z}_a$ the set of these solutions.*

(i) *If $a \in (-\infty, 2) \setminus \{0\}$, we have $\{\sqrt{\alpha_n}; n \geq 1\} = \mathcal{Z}_a$ with $\sqrt{\alpha_n} \in (((2n-1)\pi)/2, n\pi)$ if $a < 0$ and $\sqrt{\alpha_n} \in ((n-1)\pi, ((2n-1)\pi)/2)$ if $a \in (0, 2)$ for $n \geq 1$.*

(ii) *If $a \in [2, \infty)$, we have $\{\sqrt{\alpha_n}; n \geq 2\} = \mathcal{Z}_a$ and $\sqrt{\alpha_n} \in ((n-1)\pi, ((2n-1)\pi)/2)$ for $n \geq 2$. In addition, if $a = 2$ then $\alpha_1 = 0$ and the corresponding eigenfunction is given by $\varphi_1(x) = \sqrt{3}(1-x)$.*
If $a > 2$ then $(-\alpha_1)$ is the unique positive real number satisfying

$$\frac{a + 2\sqrt{-\alpha_1}}{a - 2\sqrt{-\alpha_1}} = e^{2\sqrt{-\alpha_1}} \quad \text{with} \quad \alpha_1 \in \left(-\frac{a^2}{4}, -\frac{a(a-2)}{4}\right),$$

and the corresponding eigenfunction is given by $\varphi_1(x) = A_1(a)\sinh\left(\sqrt{-\alpha_1}(1-x)\right)$, the parameter $A_1(a)$ being a positive constant such that $\|\varphi_1\|_2 = 1$ and $\varphi_1 > 0$ in $(0,1)$.

Moreover, for $n \geq 1$ such that $\sqrt{\alpha_n} \in \mathcal{Z}_a$, the corresponding eigenfunction φ_n is given by $\varphi_n(x) = A_n(a)\sin(\sqrt{\alpha_n}(1-x))$ where $A_n(a)$ is chosen such that $\|\varphi_n\|_2 = 1$. If $n = 1$, we also choose $A_1(a)$ such that $\varphi_1 > 0$ in $(0,1)$.

Remark 6.1. *By Proposition 6.1, all the eigenvalues of L are positive if $a < 2$, $a \neq 0$. A direct proof of this fact can be performed as follows: let α be an eigenvalue of L with corresponding eigenfunction φ, so that $L\varphi = \alpha\varphi$. Multiplying this identity by φ and integrating over $(0,1)$ we have:*

$$\int_0^1 |\varphi_x(x)|^2 dx - \frac{a}{2}\varphi(0)^2 = \alpha \int_0^1 \varphi(x)^2 \, dx.$$

Since $\varphi(1) = 0$, an elementary computation shows that

$$\varphi(0)^2 = \left(\int_0^1 \varphi_x(x) dx\right)^2 \leq \int_0^1 |\varphi_x(x)|^2 dx,$$

so that the left-hand side of the above identity is positive if $a < 2$.

Proof of Proposition 6.1:
Let α be an eigenvalue of L with corresponding eigenfunction φ. Then

$$-\varphi_{xx} = \alpha\varphi \quad \text{in} \quad (0,1) \quad \text{and} \quad 2\varphi_x(0) + a\varphi(0) = \varphi(1) = 0\,. \tag{6.4}$$

In order to solve (6.4) we distinguish among the cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.
1) If $\alpha < 0$, solving the first equation in (6.4) gives

$$\varphi(x) = Ae^{x\sqrt{-\alpha}} + Be^{-x\sqrt{-\alpha}}, \quad x \in (0,1),$$

for some yet unspecified real numbers A and B. To comply with the boundary conditions in (6.4), we deduce that α has to verify the following equation:

$$\frac{a + 2\sqrt{-\alpha}}{a - 2\sqrt{-\alpha}} = e^{2\sqrt{-\alpha}}. \tag{6.5}$$

Now, it is easy to check that the equation $e^z = (a+z)/(a-z)$ has a unique positive solution ϱ_a if and only if $a > 2$, and $\varrho_a \in \left(\sqrt{a(a-2)}, a\right)$. Consequently, if $a > 2$, we have $\alpha_1 = -\varrho_a^2/4 \in (-a^2/4, -a(a-2)/4)$ and $\varphi_1(x) = A_1(a)\sinh\left(\sqrt{-\alpha_1}(1-x)\right)$ for $x \in (0,1)$ with

$$A_1(a) = 2\,\left(\frac{\sinh\left(2\sqrt{-\alpha_1}\right)}{\sqrt{-\alpha_1}} - 2\right)^{-1/2}, \tag{6.6}$$

so that $\|\varphi_1\|_2 = 1$ and φ_1 is positive in $(0,1)$.

2) For (6.4) to have a non-zero solution with $\alpha = 0$, it is necessary that $a = 2$. Hence, if $a = 2$, we have $\alpha_1 = 0$ with $\varphi_1(x) = \sqrt{3}(1-x)$, $x \in (0,1)$.

3) If $\alpha > 0$, solving the first equation in (6.4) leads to

$$\varphi(x) = A\sin(\sqrt{\alpha}x) + B\cos(\sqrt{\alpha}x), \quad x \in (0,1),$$

for some yet unspecified real numbers A and B. Requiring that φ fulfils the boundary conditions in (6.4) implies that $\alpha \in \mathcal{Z}_a$. Then either $a \geq 2$ and, since α_1 has already been determined, we have $\{\sqrt{\alpha_n}; n \geq 2\} = \mathcal{Z}_a$. Or $a < 2$ ($a \neq 0$) and $\{\sqrt{\alpha_n}; n \geq 1\} = \mathcal{Z}_a$. In both cases, $\varphi_n(x) = A_n(a)\sin(\sqrt{\alpha_n}(1-x))$ for $x \in (0,1)$ with

$$A_n(a) = 2\,\left(2 - \frac{\sin\left(2\sqrt{\alpha_n}\right)}{\sqrt{\alpha_n}}\right)^{-1/2}, \tag{6.7}$$

chosen such that $\|\varphi_n\|_2 = 1$ and φ_1 is positive in $(0,1)$. □

As a direct consequence of formulae (6.6) and (6.7), we next derive some properties of $(A_n(a))_{n\geq 1}$ according to the values of a.

Lemma 6.1. *If $a \neq 0$ and $n \geq 2$, we have*

$$A_n(a) \leq \sqrt{\pi}, \tag{6.8}$$

and

$$\lim_{a\to 0} A_1(a) = \sqrt{2}\,, \quad \lim_{a\to 2} A_1(a) = \infty\,, \quad \lim_{a\to\infty} A_1(a) = 0\,.$$

Moreover, if $a < 0$, $A_1(a) \leq \sqrt{\pi}$.

Proof of Theorem 1.4:

Since the normalised eigenfunctions $(\varphi_n)_{n\geq 1}$ of L form an orthonormal basis in $L^2(0,1)$, the solution to (6.3) is given by

$$v(t,x) = \sum_{n\geq 1} e^{-\alpha_n t} < v_0, \varphi_n >_{L^2(0,1)} \varphi_n(x) \text{ in } (0,\infty)\times(0,1),$$

where $< .,. >_{L^2(0,1)}$ denotes the usual scalar product in $L^2(0,1)$. From (6.2) we deduce that

$$u(t,x) = \sum_{n\geq 1} e^{-((a^2/4)+\alpha_n)t} e^{ax/2} < v_0, \varphi_n >_{L^2(0,1)} \varphi_n(x) \text{ in } (0,\infty)\times(0,1). \tag{6.9}$$

Changing x to $-x$ we obtain a similar identity on the interval $(-1,0)$

$$u(t,x) = \sum_{n\geq 1} e^{-((a^2/4)+\alpha_n)t} e^{-ax/2} < \tilde{v}_0, \varphi_n >_{L^2(0,1)} \varphi_n(-x) \text{ in } (0,\infty)\times(-1,0), \tag{6.10}$$

where $\tilde{v}_0(y) = e^{-ay/2}u_0(-y)$ for $y \in (0,1)$.

Thanks to the properties of the eigenvalues $(\alpha_n)_{n\geq 1}$ and to relations (6.6), (6.7) and (6.8), a simple computation shows that, if $a > 0$,

$$\begin{aligned}
\|u(t)\|_\infty &\leq e^{a/2}\|v_0\|_\infty \sum_{n\geq 1} e^{-((a^2/4)+\alpha_n)t}\|\varphi_n\|_\infty^2 \\
&\leq e^{a/2}\|u_0\|_\infty e^{-((a^2/4)+\alpha_1)t}\left(\|\varphi_1\|_\infty^2 + \sum_{n\geq 2} e^{-(\alpha_n-\alpha_1)t}A_n(a)^2\right) \\
&\leq e^{a/2}\|u_0\|_\infty e^{-((a^2/4)+\alpha_1)t}\left(\|\varphi_1\|_\infty^2 + \pi\sum_{n\geq 2} e^{-(2n-3)(2n-1)\pi^2 t/4}\right) \\
&\leq e^{a/2}\|u_0\|_\infty e^{-((a^2/4)+\alpha_1)t}\left(\|\varphi_1\|_\infty^2 + \pi\sum_{n\geq 1} e^{-n\pi^2 t/2}\right) \\
&\leq e^{a/2}\left(\|\varphi_1\|_\infty^2 + \frac{\pi}{e^{\pi^2 t/2}-1}\right)\|u_0\|_\infty e^{-((a^2/4)+\alpha_1)t},
\end{aligned}$$

for $t > 0$, whence (1.9) for $t \geq 1$.
If $a < 0$ we notice that (6.8) holds for all $n \geq 1$. Therefore, we have

$$\begin{aligned}
\|u(t)\|_\infty &\leq \|v_0\|_\infty \sum_{n\geq 1} e^{-((a^2/4)+\alpha_n)t}A_n(a)^2 \\
&\leq e^{|a|/2}\|u_0\|_\infty e^{-((a^2/4)+\alpha_1)t}\pi\left(1 + \sum_{n\geq 2} e^{-(\alpha_n-\alpha_1)t}\right) \\
&\leq e^{|a|/2}\frac{\pi e^{\pi^2 t/2}}{e^{\pi^2 t/2}-1}\|u_0\|_\infty e^{-((a^2/4)+\alpha_1)t},
\end{aligned}$$

whence (1.9) for $a < 0$. We have thus established (1.9) for a profiled function u_0.

In the general case, if $u_0 \in C_0^+([-1,1])$, we define $\bar{u}_0$ by:

$$\bar{u}_0(x) = \sup\{\ u_0(y);\quad |y| \geq |x|\ \}.$$

Thus, $\bar{u}_0 \in C_0([-1,1])$ is a profiled function such that $u_0 \leq \bar{u}_0$. Denoting by $\bar{u}$ the solution to (1.1) corresponding to the initial datum $\bar{u}_0$, we infer from the comparison principle and the estimate (1.9) for $\bar{u}$ that

$$\|u(t)\|_\infty \leq \|\bar{u}(t)\|_\infty \leq \gamma(a)\ \|\bar{u}_0\|_\infty\ e^{-((a^2/4)+\alpha_1)t},\quad t \geq 1.$$

Since $\|u_0\|_\infty = \|\bar{u}_0\|_\infty$, we deduce that (1.9) is fulfilled in the general case too. □

The following corollary is a direct consequence of formulae (6.9) and (6.10).

Corollary 6.1. *Let $u_0 \in C_0^+(\overline{\Omega})$ be a profiled function. Under the hypotheses of Theorem 1.4, the solution u to (1.1) also satisfies*

$$\left|e^{((a^2/4)+\alpha_1)t}e^{-ax/2}u(t,x) - < v_0, \varphi_1 >_{L^2(0,1)} \varphi_1(x)\right| \leq \gamma(a)\ \|u_0\|_\infty\ e^{-(\alpha_2-\alpha_1)t} \tag{6.11}$$

for $t \geq 1$ and $x \in (0,1)$ and a similar inequality on $(-1,0)$ with $\tilde{v}_0$ instead of v_0.

As a final comment, we emphasize that the large time behaviour of solutions to (1.1) is rather peculiar in the case $p = 1$ and $N = 1$, since it is the only situation where we observe a real difference between the solution to the linear heat equation and the solution to (1.1). The nonlinear term actually plays an important role whatever the sign of a is. Indeed, recalling that the first eigenvalue λ_1 of the Laplace operator with homogeneous Dirichlet boundary conditions is given by $\lambda_1 = \pi^2/4$ in the particular case $\Omega = (-1, 1)$, we denote by $r_1(a) = (a^2/4)+\alpha_1$ the exponent which gives the decay rate in (1.9), α_1 being the first eigenvalue of the operator L defined in Theorem 1.4 and thus depending on a. We then aim at comparing $r_1(a)$ and λ_1.

(i) if $a < 0$ we have $r_1(a) > \lambda_1$ and the absorption term $-|a||u_x|$ drives the solutions to (1.1) to zero at a faster rate than the solutions to the linear heat equation. Moreover, we have $r_1(a) \searrow \lambda_1$ as $a \nearrow 0$ and $r_1(a) \nearrow \infty$ as $a \searrow -\infty$.

(ii) if $a > 0$, we have $r_1(a) \in (0, \lambda_1)$ and the source term $a|u_x|$ slows down the convergence to zero of solutions to (1.1). Furthermore, $r_1(a) \nearrow \lambda_1$ as $a \searrow 0$ and $r_1(a) \searrow 0$ as $a \nearrow \infty$. Indeed, if $a \in (0, 2)$, we have $\sqrt{\alpha_1} \in (0, \pi/2)$ and $tan(\sqrt{\alpha_1}) = 2\sqrt{\alpha_1}/a$ by Proposition 6.1 (i), whence

$$r_1(a) = \left(\frac{\sqrt{\alpha_1}}{\sin(\sqrt{\alpha_1})}\right)^2 \leq \frac{\pi^2}{4} = \lambda_1,$$

while, for $a > 2$, it follows from Proposition 6.1 (ii) and (6.5) that $\alpha_1 \in (-a^2/4, -a(a-2)/4)$ and

$$r_1(a) = -\frac{4\alpha_1 e^{2\sqrt{-\alpha_1}}}{(e^{2\sqrt{-\alpha_1}} - 1)^2} \leq 1 \leq \lambda_1.$$

Finally, by (6.5), $r_1(a)$ also satisfies

$$r_1(a) = \frac{1}{4}(a + 2\sqrt{-\alpha_1})(a - 2\sqrt{-\alpha_1}) = \frac{1}{4}(a + 2\sqrt{-\alpha_1})^2 e^{-2\sqrt{-\alpha_1}},$$

from which we deduce that $r_1(a) \searrow 0$ as $a \nearrow \infty$.

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