Conjugate and cut loci in averaged orbital transfer

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Abstract

The objective of this Note is to describe the conjugate and cut loci associated with the averaged energy minimization problem in coplanar orbit transfer. *To cite this article: B. Bonnard, J.-B. Caillau, C. R. Acad. Sci. Paris, Ser. I xxx (200x).*

Résumé

Lieu conjugué et lieu de coupure en transfert orbital moyenné. Cette Note décrit le lieu conjugué ainsi que le lieu de coupure du problème de la minimisation de l'énergie en transfert orbital après moyennation. *Pour citer cet article : B. Bonnard, J.-B. Caillau, C. R. Acad. Sci. Paris, Ser. I xxx (200x).*

Version française abrégée

Nous avons établi dans la Note [2] un résultat d'optimalité globale pour une classe particulière de métriques Riemanniennes en dimension trois à l'aide d'une estimation du rayon d'injectivité de la restriction à $\mathbf{S}^2$ d'une telle métrique. Ce résultat s'applique à la métrique (1) obtenue après moyennation pour le problème du transfert orbital à énergie minimale [3]. L'objectif de la présente Note est de décrire le lieu conjugué et le lieu de coupure de la restriction à la sphère de cette métrique.

Après avoir observé que la métrique considérée est analytique sur une deux-sphère de révolution au sens de [8,16], nous calculons la courbure de Gauss qui s'avère être positive au voisinage de l'équateur, et négative autour des pôles. Une remarque importante est l'existence de la symétrie discrète $\varphi \mapsto \pi - \varphi$ pour la métrique, où (θ,φ) sont les coordonnées sphériques usuelles sur $\mathbf{S}^2$. Un résultat général [1] affirme qu'une métrique de la forme de celle étudiée est conforme à la restriction de la métrique plate sur la sphère. Plus précisément, nous montrons qu'on a en transfert orbital conformité avec la restriction de la métrique plate à un ellipsoïde oblat unitaire de demi-petit axe $1/\sqrt{5}$, définissant ainsi une homotopie sur $\mathbf{S}^2$ du cas plat vers celui étudié.

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Le rayon d'injectivité pour le transfert est alors donné, comme sur l'ellipsoïde pour la métrique plate, par le premier point conjugué sur l'équateur et vaut $\pi/\sqrt{5}$. Le lieu conjugué est évaluable soit en utilisant l'intégrabilité au sens de Liouville du flot extrémal, soit numériquement à l'aide de l'algorithme [4]. Pour en déduire le lieu de coupure dont on sait [15] qu'il est, dans le cas analytique sur la sphère, un arbre fini dont les extrémités sont des points conjugués qui constituent des singularités du lieu conjugué, nous rappelons la propriété classique de [14] qu'un domaine bordé par deux géodésiques minimisantes qui s'intersectent contient nécessairement un point conjugué, et utilisons le fait que de telles intersections sont ici localisées sur le parallèle opposé au point considéré à cause de la symétrie antipodale de la métrique. L'estimation des points conjugués dont nous disposons permet d'en déduire que le lieu de coupure en transfert orbital est une branche simple contenue dans le parallèle antipodal du point initial.

1. Introduction

In our previous Note [2] we have presented a global optimality result on Riemannian metrics in dimension three which can be applied to the averaged minimization coplanar transfer between Keplerian orbits [3]. In this case, the metric is given by

$$g = \frac{1}{9n^{1/3}}dn^2 + \frac{2n^{5/3}}{5(1-e^2)}de^2 + \frac{2n^{5/3}}{5-4e^2}e^2 d\theta^2 \qquad (1)$$

where (n,e,θ) are orthogonal coordinates corresponding to orbit elements [12] in the elliptic domain: n is the mean movement ($n = a^{-3/2}$ with a the semi-major axis), e is the eccentricity, and θ the argument of the pericenter. These coordinates are singular for circular orbits ($e = 0$) but the metric is well defined, whereas $e = 1$ is a true singularity corresponding to parabolic orbits at the boundary of the elliptic domain. If we set

$$r = \frac{2}{5}n^{5/6}, \quad \varphi = \arcsin e,$$

the metric is seen to be isometric to

$$g = dr^2 + \frac{r^2}{c^2}(G(\varphi)d\theta^2 + d\varphi^2)$$

with $c = \sqrt{2/5}$ and

$$G(\varphi) = \frac{5\sin^2\varphi}{1+4\cos^2\varphi}\cdot$$

Using homogeneity, we proved in [2] that the global optimality properties can be deduced from the two-dimensional Riemannian metric $g_1 = G(\varphi)d\theta^2 + d\varphi^2$ on $\mathbf{S}^2$ identified with $\{r = c\}$, where (r,θ,φ) are spherical coordinates of $\mathbf{R}^3$. The objective of this Note is to describe the conjugate and cut loci of this metric. It is connected to advanced results in Riemannian geometry [8,16], and we can take advantage of geometric and numerical computations.

2. Preliminaries

We note (M,g) a complete real analytic connected Riemannian manifold, and $z = (x,p)$ are coordinates on the cotangent bundle T^*M. Extremals are integral curves of the Hamiltonian vector field associated with g. If $t \mapsto z(t,x_0,p_0)$ denotes the extremal starting from (x_0,p_0), and if Π is the standard projection of T^*M onto M, the exponential mapping at time t for a fixed x_0 is defined according to

$$\exp_{x_0,t} \;:\; p_0 \mapsto \Pi(z(t,x_0,p_0)).$$

The time t_c is then *conjugated* along the extremal with initial adjoint vector p_0 if $\exp_{x_0,t_c}$ is not an immersion at p_0. The associated critical value $x(t_c)$ is said to be conjugate to x_0. The set of first conjugate points is the *conjugate locus*, $C(x_0)$. The *separating line*, $L(x_0)$, is the set of points where two minimizing extremal curves starting from x_0 intersect. Finally, the *cut point* (if any) along an extremal curve is the last point for which the extremal is minimizing, and the *cut locus*, $\mathrm{Cut}(x_0)$, is the set of such points. The *injectivity radius*, $i(M)$, is the infimum over manifold points of distances between a point and its cut locus.

The properties hereafter are standard on complete Riemannian manifolds [7].

Proposition 2.1 *A cut point belongs either to the separating line or to the conjugate locus. If x_1 is a point which realizes the distance to $\mathrm{Cut}(x_0)$, then either x_1 is conjugate to x_0, or there are two minimizing geodesics joining x_0 to x_1 that form the two halves of the same closed geodesic.*

We now state some properties of the metric $g_1 = G(\varphi)d\theta^2 + d\varphi^2$, $G(\varphi) = 5\sin^2\varphi/(1+4\cos^2\varphi)$. First, we observe that the metric corresponds to an analytic metric on a two-sphere of revolution as introduced in [8,16]. Indeed, it defines a metric on $\mathbf{S}^2$ with two poles at $\varphi = 0$ and π resulting from the singularities of spherical coordinates. The extremals through this pair of poles are meridians $\{\theta = \mathrm{cst}\}$. The curves $\{\varphi = \mathrm{cst}\}$ are parallels, and the extremal $\{\varphi = \pi/2\}$ corresponds to the equator. In orbital transfer, the poles are $e = 0$ (circular orbits) and the equator is $\{e = 1\}$ (parabolic orbits). Moreover, the following holds.

Lemma 2.1 *The Gauss curvature of g_1 is*

$$K = \frac{5(1-8\cos^2\varphi)}{(1+4\cos^2\varphi)^2}$$

which takes positive values near the equator, and negative values near the poles. The transformation $\varphi \mapsto \pi - \varphi$ is an isometry of the metric.

The standard metric on $\mathbf{S}^2$ induced by the flat metric takes the form $g_0 = \sin^2\varphi d\theta^2 + d\varphi^2$ in spherical coordinates. The metric g_1 is then known [1] to be isometric to $f(z)g_0$ where $f(z)$ is a smooth positive function of the vertical coordinate z in the closed interval $[-1,1]$. A more specific construction is available.

Proposition 2.2 *The metric g_1 is conformal to the restriction of the flat metric to an oblate ellipsoid with semi-minor axis equal to $1/\sqrt{5}$.*

To prove this, we write g_1 as

$$g_1 = \frac{1}{E_\mu}(\sin^2\varphi d\theta^2 + E_\mu(\varphi)d\varphi^2)$$

with

$$E_\mu(\varphi) = \mu^2 + (1-\mu^2)\cos^2\varphi \quad \text{and} \quad \mu = \frac{1}{\sqrt{5}}\cdot$$

The metric $\sin^2\varphi d\theta^2 + E_\mu(\varphi)d\varphi^2$ is the restriction of the flat metric to the ellipsoid of revolution with unit semi-major axis and semi-minor axis $\mu \le 1$. This construction defines a geometric homotopy between the usual metric on $\mathbf{S}^2$ where $\mu_0 = 1$, and the case $\mu_1 = 1/\sqrt{5}$, for instance taking the path

$$\mu_\lambda^2 = (1-\lambda)\mu_0^2 + \lambda\mu_1^2, \quad \lambda \in [0,1].$$

This reduction allows at each step to compare the conjugate and cut loci, the case of the ellipsoid being well-known.

3. The conjugate and cut loci

Before stating the results, we make the following remarks. For $\mu < 1$ fixed, we can compare both metrics. The curvature in the flat case is

$$K = \frac{\mu^2}{[\mu^2 + (1-\mu^2)\cos^2\varphi]^2},$$

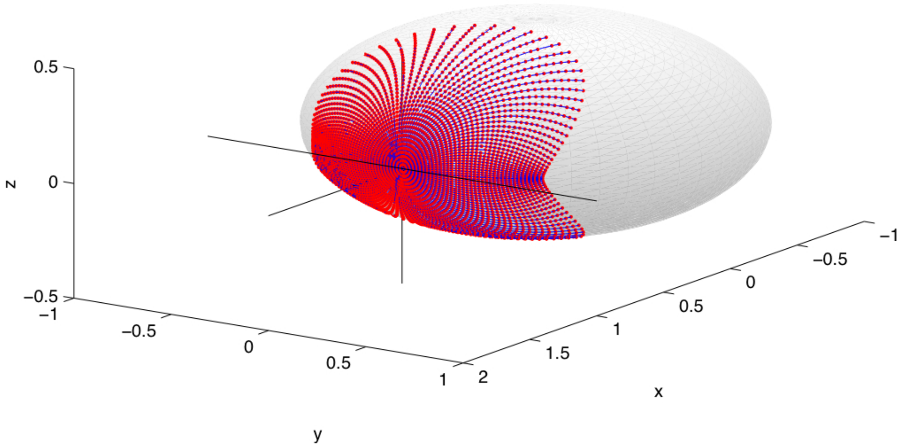

Figure 1. Geodesics and spheres of the transfer metric, $g_1 = G(\varphi)d\theta^2 + d\varphi^2$, up to the radius of injectivity, $\pi/\sqrt{5}$.

and the maximum $1/\mu^2$ is reached along the equator, $\{\varphi = \pi/2\}$. There, the first conjugate point for $\mu = 1/\sqrt{5}$ has thus length $\pi/\sqrt{5}$, as in the transfer case, and this gives the injectivity radius: $i(\mathbf{S}^2) = \pi/\sqrt{5}$ in both cases (see Fig. 1).

To compute the conjugate loci, we can use Liouville integrability of the extremal flow. This computation is standard in the flat case, and the same can be done in orbital transfer where the extremal curves can be integrated using elementary functions [3]. Another approach consists in using numerical simulations thanks to the method presented in [4] which allows to compute conjugate points.

To deduce the cut locus, we need several theoretical results. First, in the analytical case on $\mathbf{S}^2$, the cut locus is a finite tree whose extremities are conjugate points and singularities of the conjugate locus [15,9,10,11,13,17]. Secondly, in some situations, this tree turns to be reduced to a single branch. It is the case on the ellipsoid, but also on a general surface of revolution [6] under the following assumptions [16]. If the metric is isometric for the transformation $\varphi \mapsto \pi - \varphi$, two extremals of same length starting from the same initial point $x_0 = (\theta_0, \varphi_0)$ (where θ_0 can be normalized to zero by symmetry) intersect on the antipodal parallel, $\{\varphi = \pi - \varphi_0\}$. Morevover, if the Gauss curvature is non-constant and monotone non-decreasing along half meridians from the north pole to the equator, the cut locus of a point is a subarc of this antipodal parallel. In orbit transfer however, the Gauss curvature is increasing from $-5/3$ to 5 between $\pi/4$ and $\pi/2$, but slightly decreasing from $-7/5$ to $-5/3$ between 0 and $\pi/4$. To prove that the cut locus shares a similar property, we use estimates of the conjugate points. Indeed, the interior of a domain bounded by two intersecting minimizing curves must contain a conjugate point [14]. The conjugate and cut loci for different initial conditions are pictured at Fig. 2. In conclusion, we are able to characterize the cut locus in orbital transfer [5].

Theorem 3.1 *In orbital transfer, the cut locus of a point $(0, \varphi_0)$ is a subarc of the antipodal parallel $\{\varphi = \pi - \varphi_0\}$ whose extremities are cusp points of the conjugate locus.*

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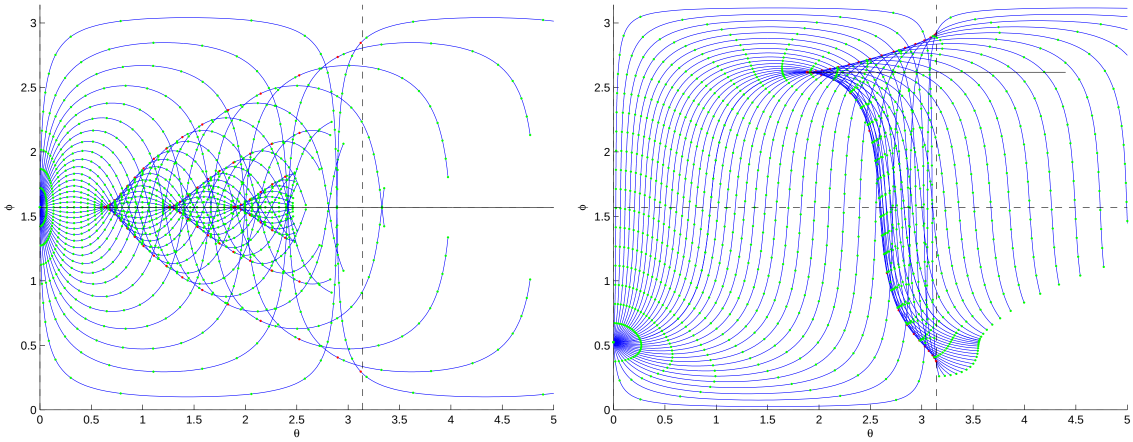

Figure 2. Conjugate and cut loci of metric g_1 (in red dots and black line, respectively), for $\varphi_0 = \pi/2$ (left) and $\varphi_0 = \pi/6$ (right). In both cases, the first cusp of the conjugate loci corresponding to the injectivity radius is clearly observed, while the entire loci have four such cusps forming singularities of an astroid-like set [5]. The cut loci are contained in the antipodal parallels, symmetric to the initial point with respect to the equator $\{\varphi = \pi/2\}$.

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