Formulary of geodesics of the projected averaged Kepler Hamiltonian

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Abstract

This short note gives the quadratures of the geodesics of the averaged Hamiltonian of the controlled Kepler equation (energy criterion) projected on $\mathbf{S}^2$. The endpoints of the corresponding cut locus are also deduced, as well as the injectivity radius of the associated Riemannian metric on the 2-sphere.

Keywords. Averaged Kepler equation, geodesics on $\mathbf{S}^2$, cut locus.

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The Hamiltonian is

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{G(\varphi)} + p_\varphi^2\right)$$

with

$$G(\varphi) = \frac{\sin^2\varphi}{1-(1-\mu^2)\sin^2\varphi}.$$

Herebefore, μ is a parameter which is equal to $1/\sqrt{5}$ in the case of the Kepler equation [1, 2, 3, 4]. The system is clearly integrable, and the geodesics on the level set $\{H = 1/2\}$ of the corresponding Riemannian metric are as parameterized by p_θ (note that θ is cyclic) as follows:

$$\varphi = \operatorname{asin}\sqrt{\frac{(1+b)-(1-b)\cos(a(t-t_1))}{2}},$$

and

$$\theta = \operatorname{sign}(p_\theta)\left[\operatorname{atan}\left(\frac{\tan\left(a(t-t_1)/2\right)}{\sqrt{b}}\right)\right]_0^t - (1-\mu^2)p_\theta t.$$

These two quadratures are valid for t in $[t_1, t_1+T/4]$ with nonegative $p_{\varphi 0}$, and extended on $[t_1, t_1+T]$ by using the obvious symmetries of the Hamiltonian (a $-2t_1$ time translation gives the case when $p_{\varphi 0}$ is negative). The coordinate

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φ and the time derivative $\dot{\theta}$ are T-periodic and hence clearly extended to the whole real line. The period T, the time t_1, and the parameters a and b have the following expression (θ_0 is assumed to be zero by symmetry):

$$\begin{aligned} a &= 2\sqrt{1+(1-\mu^2)p_\theta^2}, \\ b &= p_\theta^2/(1+(1-\mu^2)p_\theta^2), \\ t_1 &= (1/a)\left[-\pi/2 - \operatorname{asin}\left(\frac{2\sin^2\varphi_0 - (1+b)}{1-b}\right)\right], \\ T &= 4\pi/a. \end{aligned}$$

The cut locus [2] of the Riemannian metric is then easily deduced. For symmetry reasons, it is included in the antipodal parallel $\{\pi-\varphi_0\}$, and the coordinates of its left endpoint are

$$\begin{aligned} \theta_l &= \pi(1-(1-\mu^2)\sin\varphi_0), \\ \varphi_l &= \pi-\varphi_0. \end{aligned}$$

The time, that is the distance to the cut locus is

$$t_l = \pi\sqrt{1-(1-\mu^2)\sin^2\varphi_0},$$

which gives the injectivity radius of the Riemannian metric $G(\varphi)d\theta^2 + d\varphi^2$ on $\mathbf{S}^2$, that is the infimum of distances from a point to its cut locus (clearly reached on the equator, for $\varphi_0 = \pi/2$),

$$i(\mathbf{S}^2) = \mu\pi.$$

As consequence of the computations, any geodesic is—up to a rotation in θ—a *pseudo-equator*, that is a geodesic generated by $p_\varphi = 0$ on $\{H = 1/2\}$. In Kepler case, every pseudo-equator starting from a rational initial eccentricity is closed. There exist five simple closed geodesic modulo rotations with respect to θ, and the shortest closed geodesics are the meridians whose length is 2π.

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