Oriented coloring: complexity and approximation

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Abstract. This paper is devoted to an oriented coloring problem motivated by a task assignment model. A recent result established the NP-completeness of deciding whether a digraph is k-oriented colorable; we extend this result to the classes of bipartite digraphs and circuit-free digraphs. Finally, we investigate the approximation of this problem: both positive and negative results are devised.

1 Introduction

1.1 The problem

In this paper, $G = (V(G), E(G))$ denotes a simple graph and $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ a digraph (i.e. a directed graph). A *mixed graph* $M = (V(M), A(M), E(M))$ contains both arcs $(A(M))$ and edges $(E(M))$. Graphs and digraphs can be seen as mixed graphs. We do not allow loops or parallel arcs or edges, but M may have an edge and an arc with the same end-vertices. If S is a subset of $V(M)$, we denote by $M[S]$ the sub-mixed graph of M induced by S. If $v \in V(M)$, $\Gamma^+(v) = \{w|(v,w) \in A(M)\}$ and $\Gamma^-(v) = \{w|(w,v) \in A(M)\}$. Given $U \subset V(M)$, we denote $\Gamma^+(U) = \bigcup_{v \in U} \Gamma^+(v)$; $\Gamma^-(U) = \bigcup_{v \in U} \Gamma^-(v)$.

Let G, G' be graphs, and $\overrightarrow{G}, \overrightarrow{G'}$ be digraphs. An *homomorphism* of G to G' [resp. of $\overrightarrow{G}$ to $\overrightarrow{G'}$] is a mapping $f : V(G) \to V(G')$ [resp. $f : V(\overrightarrow{G}) \to V(\overrightarrow{G'})$] which preserves the edges [resp. the arcs]: i.e. $\{x, y\} \in E(G)$ [resp. $(xy) \in A(G)$] implies $\{f(x), f(y)\} \in E(G')$ [resp. $(f(x), f(y)) \in A(G')$]. Homomorphisms of undirected and directed graphs have been studied as a generalization of graph coloring in the literature [8, 9]. A *k-coloring* of a graph G is equivalent to an homomorphism of G to the complete graph K_k. Therefore, the chromatic number $\chi(G)$ of a graph G is equal to the smallest integer k such that there exists an homomorphism of G to K_k and Min Coloring is to find such an homomorphism.

Generalizing previous definition, an *oriented k-coloring* of $\overrightarrow{G}$ is an homomorphism of $\overrightarrow{G}$ to an oriented graph $\overrightarrow{G'}$ on k vertices. The *oriented chromatic number* of a digraph $\overrightarrow{G}$, denoted by $\chi_o(\overrightarrow{G})$, is the smallest integer k such that there is an oriented k-coloring of $\overrightarrow{G}$. This problem will be called Min Oriented

Coloring. Given an homomorphism c of $\overrightarrow{G}$ to $\overrightarrow{G'}$, the *color digraph* of $\overrightarrow{G}$ (for homomorphism c) will refer to digraph $\overrightarrow{G'}$. For $i \in \{1, 2, ..., |V(\overrightarrow{G'}|\}$, subsets $c^{-1}(i)$ of $V(\overrightarrow{G})$ are independent set of $V(\overrightarrow{G'})$. We call those sets monochromatic classes (for c) of digraph $\overrightarrow{G}$. If there is no possible confusion, we omit the reference to homomorphism c.

An oriented coloring of $\overrightarrow{G}$ can also be define as follows. Given two independent sets S and S' in a graph G, we say that they don't respect the *unidirection-property* if two arcs (ii') and $(j'j)$ exist such that $\{i, j\} \subset S$ and $\{i', j'\} \subset S'$ (we may have $i = j$ or $i' = j'$); in the opposite case, the unidirection-property holds (and we note $S \to S'$). Then, an oriented k-coloring is a partition of the vertex set into k independent sets such that, all pairs of independent sets in this family respect the unidirection-property.

The notion of oriented chromatic number has been first introduced by Nesetril and Sopena ([16, 14]) and has been also studied in [15, 17, 11, 13]. Most of these works focus on upper and lower bounds of the oriented chromatic number. Recently, Klostermeyer and MacGillivray [12] studied its complexity, but to our knowledge, its approximation behavior has not been studied until now. In [12], it is stated that, deciding if the oriented chromatic number of a given digraph is at most k is **NP**-complete for every $k \geq 4$. In section 2, we extend this result to the case of bipartite digraphs or circuit-free digraphs.

In section 3, we are interested in polynomial time algorithms providing guarantees on the number of colors. Two kinds of approximation ratios are usually used to characterize the performance guarantees of an approximation algorithm $\mathcal{A}$. The most classical one is, for a given instance G, the ratio between the minimum number $\chi_0(\overrightarrow{G})$ of colors required and the number of colors used by the algorithm, denoted by $m_{\mathcal{A}}(\overrightarrow{G})$. Algorithm $\mathcal{A}$ is said to guarantee a ratio of $\rho(G)$ if, for every instance, the related ratio is bounded below by $\rho(G)$. $\mathcal{A}(\overrightarrow{G})$ will denote the solution computed by $\mathcal{A}$ for $\overrightarrow{G}$. The analysis of approximation algorithms for Min Coloring started with Johnson [10] who showed that the greedy algorithm colors k-colorable graphs with $O(n/\log_k n)$ colors, leading to a performance guarantee of $O(n/\log n)$. So far, the best known approximation algorithm achieves a $O(n(\log\log(n)^2/(\log(n))^3$-approximation [5]. Another framework, called differential ratio or also z-approximation, is also widely used [6, 2, 18], particularly for coloring problems [7, 3, 1]; Min Coloring is known to be constant approximable under this ratio although it is hard to approximate in the usual sense. Given an instance G, the differential ratio of an algorithm $\mathcal{A}$ is defined by $[w(G)-m_{\mathcal{A}}(G)]/[w(G)-\beta(G)]$, where $m_{\mathcal{A}}(G), \beta(G)$ and $w(G)$ respectively denote the value of the computed solution, the optimal value of instance G and its worse value. $w(G)$ is obtained by maximizing (minimizing) the same objective under the same constraints for a minimization (maximization) problem. In the frame of Min Oriented Coloring, the worst value of an instance $\overrightarrow{G}$ is the number n of vertices and the ratio for $\overrightarrow{G}$ is $[n - m_{\mathcal{A}}(\overrightarrow{G})]/[n - \chi_o(\overrightarrow{G})]$. For this problem, we can see the differential framework as maximizing the number of unused colors among n potential colors.

1.2 Motivation

Oriented coloring is a natural extension of Min Coloring arising in scheduling models. Indeed, Min coloring models some simple tasks assignment problems. Let us consider a set $V = \{T_1, T_2, ..., T_n\}$ of different tasks to be handled on n identical processors when no preemption is possible. Every processor can perform only one task at a time and every task is supposed to have a unit processing time on any processor. Let $E \subset \{\{t, t'\}/t \in V, t' \in V, t \neq t'\}$ be a set of incompatibilities: two incompatible tasks cannot be performed during the same time by (different) processors. On the other hand, a set of p tasks without incompatibility can be performed at a time by using p processors. Let us consider the incompatibility graph $G = (V, E)$; it is well known that the minimum time required to handle all tasks in V is the chromatic number of G, denoted by $\chi(G)$. Color classes correspond to tasks that are performed simultaneously.

Let us now consider a similar model where incompatibilities are oriented and defined by $\overrightarrow{E} \subset \{(t, t')/t \in V, t' \in V, t \neq t'\}$; an incompatibility $(t, t') \in \overrightarrow{E}$ means that t' cannot be neither performed (on any processor) at the same time as t, nor during the next time unit after t: if t and t' are performed consecutively, then t must be performed after t'. One has to find a feasible scheduling minimizing the total amount of time, that is a proper coloring (in the usual sense) together with the order in which colors have to be performed. If color i is performed just after color j, then only arcs from i to j are allowed. Let us also note that such problem can be defined with a mixed incompatibility graph. Suppose now that such a scheduling is organized in two steps. First, batches of compatible tasks are performed (middle-term decisions) and one wants to minimize their number. During the second step, (short-term step) a subset of p batches with priority is selected and one wants to perform every p selected batches in p time units (without break). The batches defined during the first step correspond to independent sets in the incompatibility graph; a family of p such independent sets corresponds to batches that can be handled in p time units if they can be numbered $S_1, \ldots S_p$ in such a way there is no arc from S_i to S_{i+1}. It is easily shown that such a numbering exists for every family of p sets if and only if every two independent sets satisfy the unidirection-property. So this scheduling problem can be seen as an oriented coloring problem.

2 The complexity of oriented chromatic number

The k-chromatic number problem OCN_k is formally defined as follows: *an instance is an oriented graph $\overrightarrow{G}$ and the question is: does $\overrightarrow{G}$ have an oriented k-coloring ?*

Theorem 1 *([12]) Let k be a fixed positive integer. If $k \leq 3$, then OCN_k can be decided in polynomial time. If $k \geq 4$, then OCN_k is* **NP***-complete, even if the input is restricted to connected digraphs.*

In what follows, we study the complexity of Min Oriented Coloring for particular classes of digraphs.

Proposition 1 *Let G be a tree (with at least one edge), and let $\overrightarrow{G}$ be an orientation of G. Then, $\overrightarrow{G}$ admits an oriented k-coloring, with $k = 2$ or $k = 3$. Consequently OCN_k is polynomial when the input is restricted to an oriented tree.*

Proof is done by induction on the order n. Bipartite or circuit-free digraph are natural generalization of orientation of tree. In what follows, we show that OCN_k is **NP**-complete even if the input graph is supposed to be bipartite or circuit-free.

A *tournament* is a complete antisymmetric digraph. If $\overrightarrow{G}$ is a tournament of order n, then $\chi_o(\overrightarrow{G}) = n$. We denote by $B(\overrightarrow{G})$ the bipartite representation of $\overrightarrow{G}$ defined by: $V(B(\overrightarrow{G})) = \{x_i, y_i / i \in V(\overrightarrow{G})\}$, $A(B(\overrightarrow{G})) = \{(x_i, y_j), (y_i, x_j) / (i, j) \in A(\overrightarrow{G})\}$. Then, the following lemma can be easily shown:

Lemma 1. *$\chi_o(B(\overrightarrow{G})) = n$. Moreover, $\overrightarrow{G}$ is the color-digraph of $B(\overrightarrow{G})$ and the only optimal oriented coloring of $B(\overrightarrow{G})$ is given by: $c(x_i) = c(y_i) = i, \forall i \in V(\overrightarrow{G})$.*

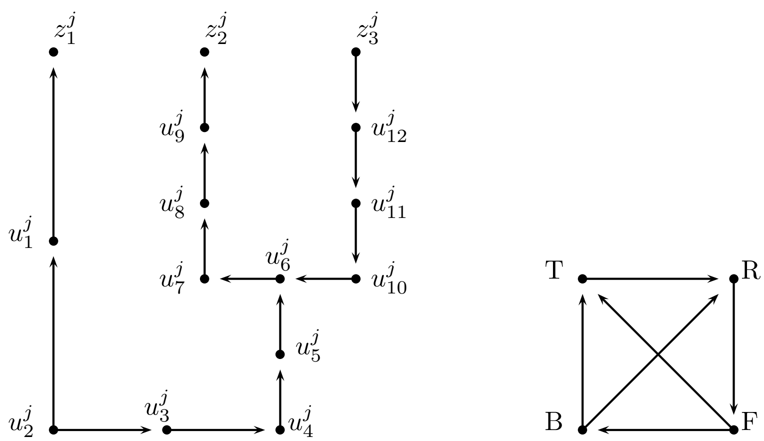

Fig 1. Digraph $\overrightarrow{L_j}$ and its color digraph T_4^1

Let c be an homomorphism of $\overrightarrow{L_j}$ to the tournament T_4^1 (cf. Fig.1) such that $c(z_1^j), c(z_2^j), c(z_3^j) \in \{T, F\}$, then we have:

Lemma 2. *c exists if and only if $(c(z_1^j), c(z_2^j), c(z_3^j)) \neq (F, F, F)$.*

The main argument of the proof is: if $c(z_3^j) = c(z_2^j) = F$, then $c(u_6^j) = F$.

Theorem 2
(i) OCN_4 is **NP***-complete even if the input is restricted to bounded degree bipartite digraphs.*
(ii) OCN_4 is **NP***-complete even if the input is restricted to bounded degree circuit-free digraphs.*

Proof (Sketch): (i) OCN_4 trivially belongs to **NP**. We then reduce 3-Sat to OCN_4. Let us consider an instance (X, C) of 3-Sat: $X = \{x_1, x_2, .., x_n\}$ is a set of boolean variables and $C = \{C_1, ..., C_m\}$ contains m clauses of 3 literals. The main idea is the following: every clause C_j is associated to the gadget $\overrightarrow{L_j}$ guaranteeing that at least one among z_1^j, z_2^j, z_3^j is associated to color "True" and every variable x_i is associated to the gadget $\overrightarrow{H_i}$ defined below guaranteeing that vertices x_i and $\bar{x}_i$ are assigned to color "True" or "False" and have different colors.

More precisely, the reduction devises the following digraph $\overrightarrow{G}$: $V(\overrightarrow{G}) = \bigcup_{1 \le j \le m} U_j \cup \bigcup_{1 \le i \le n} V_i$, with $U_j = V(L_j) = \{u_l^j / 1 \le l \le 12\}$ and $V_i = \{x_i, \overline{x_i}, e_{x_i}, a_i^l, x_i^F, y_i^F, x_i^T, y_i^T, x_i^R, y_i^R, x_i^B, y_i^B / 1 \le l \le 16\}$. The arc set of $\overrightarrow{G}$ is $A(\overrightarrow{G}) = \bigcup_{1 \le j \le m} A(\overrightarrow{L_j}) \cup \bigcup_{1 \le i \le n} A(\overrightarrow{H_i})$, where $\overrightarrow{H_i} = (V_i, A(\overrightarrow{H_i}))$, $1 \le i \le n$, is defined by Fig. 2:

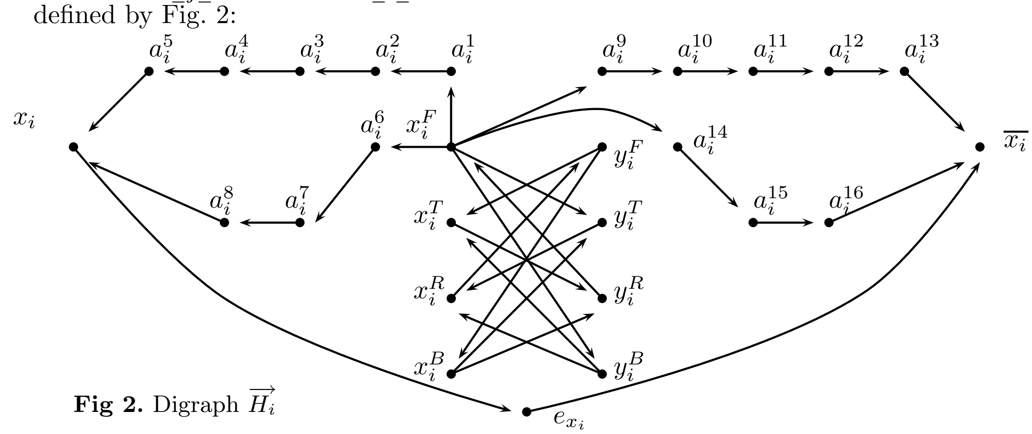

Fig 2. Digraph $\overrightarrow{H_i}$

For each clause $C_j = z_1^j \vee z_2^j \vee z_3^j$, with $z_k^j \in \{x_i, \overline{x_i}, i \in \{1, 2, ..., n\}\}, k = 1, 2, 3$, ($j \in \{1, 2, ..., m\}$), we take a copy of $\overrightarrow{L_j}$, identifying vertices z_1^j, z_2^j, z_3^j to the related vertices of $\bigcup_{1 \le i \le n} V_i$.

The construction of $\overrightarrow{G}$ can be performed in polynomial time. Digraph $\overrightarrow{G}$ is bipartite and its degree is bounded by $Max(p + 3; 7)$, where p denotes the maximum number of occurrences of a literal in clauses.

If c is an oriented coloring of $\overrightarrow{G}$, as x_i and $\overline{x_i}$ are linked by a 2-path, $c(x_i) \neq c(\overline{x_i})$. The sub-digraph $\overrightarrow{H_i}[\{x_i^F, X_i^T, x_i^R, x_i^B, y_i^F, y_i^T, y_i^R, y_i^B\}]$ is isomorphic to $B(T_4^1)$ for any fixed i in $\{1, 2, ..., n\}$. Then, $\chi_o(\overrightarrow{G}) \ge 4$ and if c is an oriented 4-coloring of $\overrightarrow{G}$, its color digraph is tournament $\overrightarrow{T_4^1}$. Moreover, using lemma 1, x_i^F is necessarily colored by F and the existence of a 4-path and a 6-path from x_i^F to x_i and $\overline{x_i}$ imply that $\{c(x_i), c(\overline{x_i})\} = \{T, F\}$.
Given a truth assignment $t : \{x_i, \overline{x_i}, i \in \{1, 2, 3, .., n\}\} \longrightarrow \{True, False\}$, we associate mapping $c : V(\overrightarrow{G}) \to \{T, F\}$ defined as $c(x_i) = T$ if $t(x_i) = True$, $t(x_i) = F$ otherwise. If t satisfies all clauses $\{C_j\}_{1 \le j \le m}$, then applying lemma 2, there exists an homomorphism of $\overrightarrow{G}$ to T_4^1.
Conversely, if such an homomorphism c exists, then we define the truth assign-

ment t by $t(x_i) = True$ if $c(x_i) = T$, $t(x_i) = False$ otherwise. By previous lemma, t satisfies all clauses C_j $(1 \leq j \leq n)$.

Consequently, there exists a truth assignment $t : \{x_i, \overline{x_i}, i \in \{1, 2, 3, .., n\}\} \longrightarrow \{True, False\}$ satisfying all clauses $\{C_j\}_{1 \leq j \leq m}$, if and only if $\overrightarrow{G}$ admits an oriented 4-coloring. As $\overrightarrow{G}$ is bipartite, statement (i) of the theorem is proved.

Proof of statement (ii.) is similar by replacing $\overrightarrow{H_i}$ by $\overrightarrow{H_i'}$:

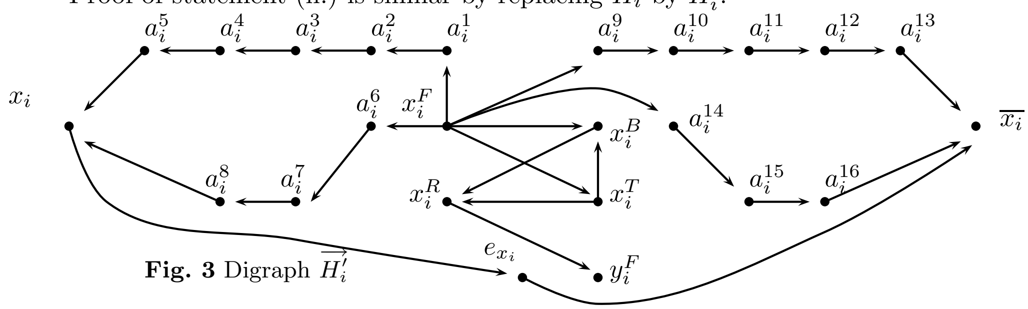

Fig. 3 Digraph $\overrightarrow{H_i'}$

The resulting digraph $\overrightarrow{G}$ is circuit-free and its color digraph is T_4^1.

2.1 Case of complete multipartite digraphs

The bipartite representation of a tournament gives existence of bipartite graphs of order $2n$ with $\chi_o(B) = n$. We focus here on the analysis of general multipartite digraphs. Let $\overrightarrow{G} = (V_1 \cup V_2 \cup ... \cup V_l, A(\overrightarrow{G}))$ be a complete multipartite digraph. Given a digraph $\overrightarrow{G}$, we define the mixed graph $M(\overrightarrow{G})$ associated to $\overrightarrow{G}$ by: $V(M(\overrightarrow{G})) = V(\overrightarrow{G})$, $A(M(\overrightarrow{G})) = A(\overrightarrow{G})$ and $E(M(\overrightarrow{G})) = \{\{x, y\} | \exists z \in V(\overrightarrow{G}), (x, z), (z, y) \in A(\overrightarrow{G})$ or $(y, z), (z, x) \in A(\overrightarrow{G})\}$.

Proposition 2
(i) $\chi_o(\overrightarrow{G}) = \sum_{i=1}^{l} \chi(M(\overrightarrow{G})[V_i])$.
(ii) for $i \in \{1, 2, \ldots, l\}$, *if* $x, y \in M(\overrightarrow{G})[V_i]$,

$$\{y, x\} \notin E(M(\overrightarrow{G})) \Rightarrow \forall z, \{x, z\} \in E(M(\overrightarrow{G})), \{y, z\} \in E(M(\overrightarrow{G}))$$

(iii) Min Oriented Coloring is polynomial for complete multipartite digraphs.

Proof:(i) Any given optimal oriented coloring of $\overrightarrow{G}$ induces a (usual) coloring of (undirected) graphs $\{M(\overrightarrow{G})[V_i]\}_{1 \leq i \leq l}$. As no oriented color class contains vertices from both V_i and V_j for $1 \leq i \neq j \leq l$, we have: $\chi_o(\overrightarrow{G}) \geq \sum_{i=1}^{l} \chi(M(\overrightarrow{G})[V_i])$. For $i \in \{1, 2, \ldots, l\}$, let c_i be a k_i-coloring of $M(\overrightarrow{G})[V_i]$. Any couple of monochromatic classes in $\{c_i\}_{1 \leq i \leq l}$ satisfies the unidirection property in $\overrightarrow{G}$. Indeed, let $1 \leq i < j \leq l$ and let $\{x, y\} \in M(\overrightarrow{G})[V_i]$ and $\{z, t\} \in M(\overrightarrow{G})[V_j]$. Without lost of generality, we suppose $(x, z) \in A(\overrightarrow{G})$. As $\{x, y\} \notin E(M(\overrightarrow{G})[V_i])$, then $(x, t) \in A(\overrightarrow{G})$. As $\{z, t\} \notin E(M(\overrightarrow{G})[V_j])$, then $(y, t) \in A(\overrightarrow{G})$. Then

$\{x, y\} \to \{z, t\}$, and the unidirection property is verified. Mapping $c : V(\overrightarrow{G}) \to \{1, 2, ..., k_1 + k_2 + .. + k_l\}$ defined by $c(x) = c_i(x) + \sum_{j=1,...,i} k_j$ if $x \in V_i$, is an oriented $(k_1 + k_2 + \cdots + k_l)$-coloring of $\overrightarrow{G}$.
(ii) Let x, y, z be vertices of $M(\overrightarrow{G})[V_i]$ $(1 \leq i \leq l)$, such that $\{y, x\} \notin E(M(\overrightarrow{G}))$ and $\{z, x\} \in E(M(\overrightarrow{G}))$. Without lost of generality, we suppose that the 2-path from x to z (x, α, z) exists. As $\{y, x\} \notin E(M(\overrightarrow{G}))$, $\{y, \alpha\} \in E(\overrightarrow{G})$, then $\{z, y\} \in E(M(\overrightarrow{G}))$.
(iii) Note finally that graphs $\{M(\overrightarrow{G})[V_i]\}_{1 \leq i \leq l}$ are cographs (P_4-free) and consequently their chromatic number can be computed in polynomial time ([4]).

3 Approximation

As OCN_k is $\mathcal{NP}$-complete, for $k \geq 4$ and for various classes of digraphs, we are interested in approximate this problem. The objective of the first subsection is to obtain negative results by the use of a reduction from a well known problem: Maximum Independent Set. In the second subsection, we obtain positive result by the analysis of a greedy algorithm.

3.1 Reduction from Maximum Independent Set

Let $G = (V, E)$ be an instance of the Maximum Independent Set problem with $V = \{1, 2, 3, ..., n\}$. Let us define a digraph $\overrightarrow{G'}$ as follows:
$V(\overrightarrow{G'}) = X \cup Y \cup Z$ with: $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$ and $Z = \{z_1, z_2, ..., z_n\}$. $A(\overrightarrow{G'}) = A_{XY} \cup A_{XZ} \cup A_{YZ}$ with: $A_{XZ} = \{(x_i, z_j), i \leq j\} \cup \{(z_i, x_j), i < j\}$, $A_{YZ} = \{(y_i, z_j), i \leq j\} \cup \{(z_i, y_j), i < j\}$ and $A_{XY} = \{(x_i, y_j), i < j\} \cup \{(x_j, y_i), i < j \textit{ and } (j, i) \in E\} \cup \{(y_i, x_j), i < j, (i, j) \notin E\}$.

Let us note that digraphs $\overrightarrow{G'}[X \cup Z]$ and $\overrightarrow{G'}[Y \cup Z]$ are isomorphic, that $\overrightarrow{G'}[X \cup Z]$ and $\overrightarrow{G'}[Y \cup Z]$ are complete bipartite digraphs and that $\overrightarrow{G'}[X \cup Y] \cup \{(x_i, y_i)/1 \leq i \leq n\}$ is a complete bipartite digraph. Moreover, $\chi_o(\overrightarrow{G'}[X \cup Z]) = \chi_o(\overrightarrow{G'}[Y \cup Z]) = 2n$: indeed, as there is always an oriented 2-path from x_i to x_j $(i < j)$, and from z_i to z_j, color classes contain only one vertex.

Lemma 3. *Let $n = |G|$, then, $\chi_o(\overrightarrow{G}) = 3n - \alpha(G)$, where $\alpha(G)$ denotes the independent number of G, and every k-oriented coloring of $\overrightarrow{G'}$ allows us to compute in polynomial time an independent set of G of size $3n - k$.*

Proof: Any color class of $\overrightarrow{G'}$ is either a single vertex or the pair $\{x_i, y_i\}$ for $i \in \{1, 2, ..., n\}$. Consequently $\chi_o(\overrightarrow{G'}) = k$ with $2n \leq k \leq 3n$. Any k-oriented coloring of $\overrightarrow{G'}$ is formed by $(3n - k)$ pairs of vertices and $(2k - 3n)$ single sets. Let S be the set $\{i \in V/\{x_i, y_i\}$ *is a color* $\}$. $\forall (i, j) \in S \times S, i < j$, both definition of A_{XY} and the unidirection property imply that $\{(x_i, y_j), (y_i, x_j)\} \subset A(\overrightarrow{G'})$; consequently $(i, j) \notin E$. Then, S is an independent set of G.

Conversely, let $S \subset V$ be an independent set of G. By definition of $\overrightarrow{G'}$, it is straightforward to verify that we can define an oriented coloring c of $\overrightarrow{G'}$ as follows: color by a same color x_i and y_i, for $i \in S$, and color every other vertex by a new color. Consequently, there is a bijection between the oriented colorings of $\overrightarrow{G'}$ and the independent sets of G, which achieves the lemma.

Theorem 3 *There exists a reduction from Maximum Independent Set to Min Oriented Coloring transforming any differential ratio $\rho(n)$ for the Min Oriented Coloring into a $\rho(3n)$-standard approximation for the Maximum Independent Set.*

Proof: Let $\mathcal{A}$ be an algorithm guaranteeing a differential ratio of $\rho(n)$ for the Min Oriented Coloring. Let G be a graph. We define $\overrightarrow{G'}$ as previously. We denote by $\chi'_o(\overrightarrow{G'})$ be the number of color classes used by algorithm $\mathcal{A}$ for instance $\overrightarrow{G'}$. By lemma 3, we get an independent set of G of size $\alpha'(G) = 3n - \chi'_o(\overrightarrow{G'})$. So we have: $\alpha'(G)/\alpha(G) = (3n - \chi'_o(\overrightarrow{G'}))/(3n - \chi_o(\overrightarrow{G'})) \geq \rho(3n)$, which concludes the proof.

Corollary 1. *If* $\mathbf{P} \neq \mathbf{NP}$*, then Min Oriented Coloring is not approximable within a constant differential approximation ratio. If* $\mathbf{P} \neq \mathbf{ZPP}$*, then Min Oriented Coloring is not approximable within a differential ratio of* $O(n^{\epsilon-1})$*,* $\epsilon > 0$*.*

3.2 A greedy algorithm

In this section, we propose a natural generalization of the usual greedy algorithm consisting in iteratively applying a greedy independent set algorithm [10]. The main difference for the oriented case arises from the fact that an oriented coloring of a sub-digraph cannot systematically be completed into an oriented coloring of the whole digraph (two vertices of the same color in the sub-digraph can be connected by a 2-path in the whole graph). To overcome this difficulty, the algorithm is devised in the framework of mixed graphs.

We first introduce a generalization of oriented coloring to mixed graph. A *mixed k-coloring* of a mixed graph $M = (V, A, E)$ is a mapping $c : V(M) \rightarrow \{1, 2, ..., k\}$ such that, for all $1 \leq i \leq k$, sub-mixed graph $M[c^{-1}(i)]$ of M contains no arc nor edge, and for all $1 \leq i \leq k$, color classes $c^{-1}(i)$ and $c^{-1}(j)$ are in unidirection in M. Given a mixed graph $M = (V, A, E)$ and a vertex $v \in V$, we define $B(v, 2)$ as the set of vertices y such that $c(v) \neq c(y)$ for all mixed coloring c of M: $B(v, 2) = \{y | [\{v, y\} \in E] \vee [(v, y) \in A] \vee [(y, v) \in A] \vee [\exists z \in V, (v, z), (z, y) \in A] \vee [\exists z \in V, (y, z), (z, v) \in A]\}$.

It is obvious that an oriented k-coloring of $\overrightarrow{G}$ is also a mixed k-coloring of $M(\overrightarrow{G})$, and conversely. Note also that notions of Γ^+ and Γ^- in $\overrightarrow{G}$ and $M(\overrightarrow{G})$ coincide. It is straightforward to verify that the following proposition holds for mixed coloring of $M(\overrightarrow{G})$ and does not hold for oriented coloring of $\overrightarrow{G}$. Nevertheless, every mixed k-coloring of $M(\overrightarrow{G})$ induces an oriented k-coloring of $\overrightarrow{G}$.

Proposition 3 *Let $\overrightarrow{G}$ be a digraph and $z \in V(\overrightarrow{G})$. Every mixed k-coloring c of $M(\overrightarrow{G})[V(\overrightarrow{G}) \setminus \{z\}]$ can be completed into a mixed $(k+1)$-coloring of $M(\overrightarrow{G})$.*

We then consider `Greed-monochromatic` (`GMC`) algorithm which can be seen as an adaptation of the usual greedy independent set algorithm:

Proposition 4 *: Let $\overrightarrow{G}$ be a directed graph and $M(\overrightarrow{G})$ its associated mixed graph. `GMC` computes an independent set S of $(M(\overrightarrow{G}))$ (and hence of $\overrightarrow{G}$) such that $|S| \geq log_{\chi_o(\overrightarrow{G})}(|\overrightarrow{G}|)$ and $\forall z \in V(\overrightarrow{G})$, $\{z\}$ and S verify the unidirection property implying: $\Gamma^+(S) \cap \Gamma^-(S) = \emptyset$*

The proof is a simple adaptation of the usual analysis of greedy independent set algorithm [10].

`GMC`
Input: A mixed graph $MG = (V, A, E)$.
Output: `GMC`(MG) is an independent set S of MG.
(0) $S \leftarrow \emptyset$, $U \leftarrow V$;
(1) While $U \neq \emptyset$ do:
(2) Let v minimizing $|B(v, 2)|$ in $MG[U]$;
(3) $S \leftarrow S \cup \{v\}$; $U \leftarrow U \setminus B(v, 2)$

Let us now consider algorithm `Greed-Oriented-Coloring` (`GOC`) that iteratively calls `GMC`:

`GOC`
Input A digraph $\overrightarrow{G} = (V, A)$.
Output `GOC`($\overrightarrow{G}$) is a mixed coloring of $\overrightarrow{G}$.
(0) Construct $M(\overrightarrow{G})$; $U \leftarrow V$, $i \leftarrow 1$.
(a) While $|U| > 0$ do:
(b) Select at most $\log(|U|)$ vertices in `GMC`($G[U]$) for color i.
(c) Let V_{min} be the subset of minimum order between Γ^+(`GMC`($G[U]$)) and Γ^-(`GMC`($G[U]$)).
(d) Every vertex of V_{min} receives a different color in $\{i+1, ..., i+|V_{min}|\}$.
(e) $U \leftarrow U \setminus$ (`GMC`($G[U]$) $\cup V_{min}$); $i \leftarrow i + |V_{min}| + 1$.

Let G_i denote the mixed graph $G[U]$ at the i^{th} iteration of inner loop. Let $n_i = |G_i|$ and $\lambda_i = Min\{\log(n_i); |\texttt{GMC}(G_i)|\}$ and let $k = \chi_o(\overrightarrow{G})$. Then we have: $\log_k(n_i) \leq |\texttt{GMC}(G_i)| \leq \log(n_i)$ and $n_{i+1} \geq \frac{n_i - \lambda_i}{2} \geq \frac{n_i - \log(n_i)}{2} \geq \frac{n_i}{3}$ if $n_i \geq 5$. Thus, with $p = \lfloor \log_3(n) \rfloor$ calls of algorithm `GMC`, the number of vertices colored by these p colors is at least:

$$\log_k(n) + \log_k(\frac{n}{3}) + \log_k(\frac{n}{3^2}) + ... \log_k(\frac{n}{3^{p-1}}) = O(\frac{\log^2(n)}{\log(k)})$$

Then, the number of colors used by the algorithm `GOC` is at most $\log_3(n) + n - O(\frac{\log^2(n)}{\log(k)})$. We deduce : $(n - \lambda)/(n - k) \geq O[(\log^2(n))/(n \log k)]$. So we have:

Theorem 4 *Min-Oriented-Coloring admits a differential* $O[(\log^2(n))/(n\log(\chi_o(\overrightarrow{G})))]$*-algorithm. In particular, if* $\chi_o(\overrightarrow{G})$ *is bounded, then a differential ratio of* $O[(\log^2(n))/n]$ *is guaranteed.*

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