On points at infinity of real spectra of polynomial rings

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Abstract

Let R be a real closed field and $A = R[x_1, \dots, x_n]$. Let Sper A denote the real spectrum of A. There are two kinds of points in Sper A: finite points (those for which all of $|x_1|,\dots,|x_n|$ are bounded above by some constant in R) and points at infinity. In this paper we study the structure of the set of points at infinity of Sper A and their associated valuations. Let T be a subset of $\{1, \dots, n\}$. For $j \in \{1, \dots, n\}$, let $y_j = x_j$ if $j \notin T$ and $y_j = \frac{1}{x_j}$ if $j \in T$. Let $B_T = R[y_1, \dots, y_n]$. We construct a finite partition Sper $A = \coprod\limits_T U_T$ and a homeomorphism of each of the sets U_T with a subspace of the space of finite points of Sper B_T. For each point δ at infinity in U_T, we describe the associated valuation ν_{δ^*} of its image $\delta^* \in$ Sper B_T in terms of the valuation ν_δ associated to δ. Among other things we show that the valuation ν_{δ^*} is composed with ν_δ (in other words, the valuation ring R_δ is a localization of R_{δ^*} at a suitable prime ideal).

1 Introduction

Let R be a real closed field and $z_0, \dots, z_n$ independent variables. A basic fact of life in mathematics is the way the n-dimensional projective space Proj $R[z_0, \dots, z_n]$ and other rational projective schemes such as $\left(\mathbb{P}^1_R\right)^n$ are glued together from affine charts of the form Spec $R[x_1, \dots, x_n]$. Given two such coordinate charts Spec $R[x_1, \dots, x_n]$ and Spec $R[y_1, \dots, y_n]$, it is often easy to write down formulae describing the coordinate transformation from the x to the y coordinates. The subject of this paper is a part of the analogous story for real spectra (see Definition 1.1 below), which is more interesting, because the real spectrum Sper $R[x_1, \dots, x_n]$ already contains much information "at infinity".

To explain this in more detail, we first recall the definition of real spectrum and other related objects, studied in this paper.

Notation and conventions. All the rings in this paper will be commutative with 1. For a prime ideal $\mathfrak{p}$ in a ring B, $\kappa(\mathfrak{p})$ will denote the residue field of the local ring $B_\mathfrak{p}$: $\kappa(\mathfrak{p}) = \frac{B_\mathfrak{p}}{\mathfrak{p}B_\mathfrak{p}}$.

Let B be a ring. A point α in the real spectrum of B is, by definition, the data of a prime ideal $\mathfrak{p}$ of B, and a total ordering $\leq$ of the quotient ring $B/\mathfrak{p}$, or, equivalently, of the field of fractions of $B/\mathfrak{p}$. Another way of defining the point α is as a homomorphism from B to a real closed field, where two homomorphisms are identified if they have the same kernel $\mathfrak{p}$ and induce the same total ordering on $B/\mathfrak{p}$.

The ideal $\mathfrak{p}$ is called the support of α and denoted by $\mathfrak{p}_\alpha$, the quotient ring $B/\mathfrak{p}_\alpha$ by $B[\alpha]$, its field of fractions by $B(\alpha)$ and the real closure of $B(\alpha)$ by $k(\alpha)$. The total ordering of $B(\alpha)$ is denoted by $\leq_\alpha$. Sometimes we write $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$.

Definition 1.1 *The real spectrum of B, denoted by Sper B, is the collection of all pairs $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of B and $\leq_\alpha$ is a total ordering of $B/\mathfrak{p}_\alpha$.*

Given a point $\delta \in \mathrm{Sper}(A)$ and an element $f \in A$, the notation $|f|_\delta$ will mean f if $f \geq_\delta 0$, $-f$ if $f \leq_\delta 0$. When no confusion is possible, we will write simply $|f|$, with δ understood.

Two kinds of points occur in Sper B: finite points and points at infinity.

Definition 1.2 *Let B be an R-algebra and α a point of Sper B. We say that α is* **finite** *if for each $y \in B[\alpha]$ there exists $N \in R$ such that $|y|_\alpha <_\alpha N$. Otherwise, we say that α is* **a point at infinity***.*

Notation: The subset of Sper B consisting of all the finite points will be denoted by $\mathrm{Sper}^* B$.

It is known, as we explain in detail in §2, that Sper B is closely related to the space $\bigcup\limits_{\mathfrak{p} \in \mathrm{Spec}\ B} S_\mathfrak{p}$, where $S_\mathfrak{p}$ denotes the Zariski–Riemann surface of the residue field $\kappa(\mathfrak{p})$. Namely, one can associate to every point $\delta \in$ Sper B a valuation ν_δ of $\kappa(\mathfrak{p}_\delta)$ (where $\mathfrak{p}_\delta$ is the support of δ) with totally ordered residue field k_δ. Conversely, given a prime ideal $\mathfrak{p} \subset B$ and a valuation ν of $\kappa(\mathfrak{p})$ with totally ordered residue field, one can define a point $\delta \in$ Sper $R[x_1, \dots, x_n]$ with $\mathfrak{p}_\delta = \mathfrak{p}$ and $\nu_\delta = \nu$ by specifying the signs of finitely many elements of $\kappa(\mathfrak{p})$ with respect to the total ordering $\leq_\delta$ (see Remark 2.2 below).

The real spectrum Sper B is endowed with the **spectral (or Harrison) topology**. By definition, this topology has basic open sets of the form

$$U(f_1, \dots, f_k) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0\}$$

with $f_1, ..., f_k \in B$. Here and below, we commit the following standard abuse of notation: for an element $f \in B$, $f(\alpha)$ stands for the natural image of f in $B[\alpha]$ and the inequality $f(\alpha) > 0$ really means $f(\alpha) >_\alpha 0$.

Denote by $\mathrm{Maxr}(A)$ the set of points $\alpha \in \mathrm{Sper}(A)$ such that $\mathfrak{p}_\alpha$ is a maximal ideal of A. We view $\mathrm{Maxr}(A)$ as a topological subspace of $\mathrm{Sper}(A)$ with the spectral (respectively, constructible) topology. We may naturally identify R^n with $\mathrm{Maxr}(A)$: a point $(a_1, \dots, a_n) \in R^n$ corresponds to the point $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha) \in \mathrm{Sper}(A)$, where $\mathfrak{p}_\alpha$ is the maximal ideal

$$\mathfrak{p}_\alpha = (x_1 - a_1, \dots, x_n - a_n)$$

and $\leq_\alpha$ is the unique order on R. The spectral topology on $\mathrm{Sper}(A)$ induces the euclidean topology on R^n.

Let $A = R[x_1, \dots, x_n]$. Take a point $\delta \in$ Sper A. In §3 we associate to δ three disjoint subsets $I_\delta, F_\delta, G_\delta \subset \{1, \dots, n\}$, as follows. By definition, the set $I_\delta \coprod F_\delta \coprod G_\delta$ is the set of all $j \in \{1, \dots, n\}$ such that

$$\nu_\delta(x_j) = 0. \tag{1}$$

The set G_δ consists of all j such that $|x_j|_\delta$ is bounded below by all the elements of R, I_δ—of all j such that (1) holds and $|x_j|_\delta$ is smaller than any strictly positive constant in R and F_δ of all j such that $|x_j|_\delta$ is bounded both above and below by strictly positive constants from R. We show that $I_\delta = \emptyset$ whenever $G_\delta = \emptyset$.

Let T be a set such that $G_\delta \subset T \subset G_\delta \cup F_\delta$. For $j \in \{1,\dots,n\}$, let $y_j = x_j$ if $j \notin T$ and $y_j = \frac{1}{x_j}$ if $j \in T$. Let $B_T = R[y_1,\dots,y_n]$. We associate to δ a point δ^* in $\mathrm{Sper}^* B_T$ such that $A(\delta) = B_T(\delta^*)$. We show that R_δ is a localization of R_{δ^*} at a prime ideal.

Let I, F, G be three disjoint subsets of $\{1,\dots,n\}$, such that if $G = \emptyset$ then $I = \emptyset$. Let $U_{I,F,G}$ denote the set of all points of Sper A such that $I = I_\delta$, $F = F_\delta$ and $G = G_\delta$. The main theorem, Theorem 3.1, describes a homeomorphism between $U_{I,F,G}$ and a certain explicitly described subspace $U^*_{I,F,G} \subset \mathrm{Sper}^*(B_T)$, where T is a set satisfying $G \subset T \subset F \cup G$. At the end of section §3, we describe a partition

$$\mathrm{Sper}(A) = \coprod_{I,F,G} U_{I,F,G}, \tag{2}$$

where I, F, G runs over all the triples of disjoint subsets of $\{1,\dots,n\}$ such that $I = \emptyset$ whenever $G = \emptyset$, and each $U_{I,F,G}$ is homeomorphic to a subspace $U^*_{I,F,G} \subset \mathrm{Sper}^*(B_T)$, as above.

This paper originally grew out of the authors' joint work with J.J. Madden [7] on the Pierce–Birkhoff conjecture. Certain definitions and constructions only worked for finite points of Sper A, so a need naturally arose to cover Sper$A$ by subspaces, each of which is homeomorphic to a subspace of $\mathrm{Sper}^* B$ for some other polynomial ring B. Eventually, we found another way of getting around this difficulty and were able to deal in a uniform way with all the points of Sper A, whether finite or infinite. However, we hope that the decomposition (2) may some day come in useful to someone who is faced with finiteness problems similar to ours. Also, since in [7] we are interested in proving connectedness of certain subsets of Sper A, we gave a variation of the decomposition (2) into sets which are *not* disjoint; we derive it as an easy consequence of (2).

2 The valuation associated to a point in the real spectrum

Let B be a ring and α a point in Sper B. In this section we define the valuation ν_α of $B(\alpha)$, associated to α. We also give a geometric interpretation of points in the real spectrum as semi-curvettes.

Terminology: If B is an integral domain, the phrase "valuation of B" will mean "a valuation of the field of fractions of B, non-negative on B". Also, we will sometimes commit the following abuse of notation. Given a ring B, a prime ideal $\mathfrak{p} \subset B$, a valuation ν of $\frac{B}{\mathfrak{p}}$ and an element $x \in B$, we will write $\nu(x)$ instead of $\nu(x \mod \mathfrak{p})$, with the usual convention that $\nu(0) = \infty$, which is taken to be greater than any element of the value group.

For a point α in Sper B, we define the valuation ring R_α by

$$R_\alpha = \{x \in B(\alpha) \mid \exists z \in B[\alpha], |x|_\alpha \leq_\alpha z\}.$$

That R_α is, in fact, a valuation ring, follows because for any $x \in B(\alpha)$, either $x \in R_\alpha$ or $\frac{1}{x} \in R_\alpha$. The maximal ideal of R_α is $M_\alpha = \left\{x \ \in B(\alpha) \middle| \ |x|_\alpha < \frac{1}{|z|_\alpha}, \ \forall z \in B[\alpha] \setminus \{0\}\right\}$; its residue field k_α comes equipped with a total ordering, induced by $\leq_\alpha$. For a ring B let $U(B)$ denote the multiplicative group of units of B. Recall that $\Gamma_\alpha \cong \dfrac{B(\alpha)\setminus\{0\}}{U(R_\alpha)}$ and that the valuation ν_α can be identified with the natural homomorphism

$$B(\alpha)\setminus\{0\} \to \frac{B(\alpha)\setminus\{0\}}{U(R_\alpha)}.$$

By definition, we have a natural ring homomorphism

$$B \to R_\alpha \tag{3}$$

whose kernel is $\mathfrak{p}_\alpha$. The valuation ν_α has the following properties:

(1) $\nu_\alpha(B[\alpha]) \geq 0$

(2) If B is an R-algebra then for any positive elements $y, z \in B(\alpha)$,

$$\nu_\alpha(y) < \nu_\alpha(z) \implies y > Nz, \ \forall N \in R \tag{4}$$

(an example at the end of the paper shows that the converse implication in (4) is not true in general).

Remark 2.1 *Let B be an R-algebra and take a point $\alpha \in Sper^* B$ (see Definition 1.2). Then*

$$R_\alpha = \{x \in B(\alpha) \ \mid \exists N \in R, |x| \leq_\alpha N\}. \tag{5}$$

Thus for points $\alpha \in Sper^* B$ the valuation ν_α of $B(\alpha)$ depends on the ordering $\leq_\alpha$ but not on the ring $B[\alpha]$ (this means that given another R-algebra $\tilde{B}$, a point $\tilde{\alpha} \in Sper^* \tilde{B}$ and an order-preserving isomorphism $\phi : B(\alpha) \cong \tilde{B}(\tilde{\alpha})$, we have $\phi(R_\alpha) = R_{\tilde{\alpha}}$).

Remark 2.2 *([1], [6], [2] 10.1.10, p. 217) Conversely, the point α can be reconstructed from the ring R_α by specifying a certain number of sign conditions (finitely many conditions when B is noetherian), as we now explain. Take a prime ideal $\mathfrak{p} \subset B$ and a valuation ν of $\kappa(\mathfrak{p}) := \frac{B_\mathfrak{p}}{\mathfrak{p}B_\mathfrak{p}}$, with value group Γ. Let*

$$r = \dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$$

(if B is not noetherian, it may happen that $r = \infty$). Let $x_1, \dots, x_r$ be elements of $\kappa(\mathfrak{p})$ such that $\nu(x_1), \dots, \nu(x_r)$ induce a basis of the $\mathbb{F}_2$-vector space $\Gamma/2\Gamma$. Then for every $x \in \kappa(\mathfrak{p})$, there exist $f \in \kappa(\mathfrak{p})$ and a unit u of R_ν such that $x = u x_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ with $\epsilon_i \in \{0, 1\}$ (to see this, note that for a suitable choice of f and ϵ_j the value of the quotient u of x by the product $x_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ is 0, hence u is invertible in R_ν). Now, specifying a point $\alpha \in Sper\ B$ supported at $\mathfrak{p}$ amounts to specifying a valuation ν of $\frac{B}{\mathfrak{p}}$, a total ordering of the residue field k_ν of R_ν, and the sign data $sgn\ x_1, \dots, sgn\ x_r$. For $x \notin \mathfrak{p}$, the sign of x is given by the product $sgn(x_1)^{\epsilon_1} \cdots sgn(x_r)^{\epsilon_r} sgn(u)$, where $sgn(u)$ is determined by the ordering of k_ν.

Points of Sper B admit the following geometric interpretation (we refer the reader to [3], [4], [8], p. 89 and [9] for the construction and properties of generalized power series rings and fields).

Definition 2.1 *Let k be a field and Γ an ordered abelian group. The generalized formal power series ring $k\left[\left[t^\Gamma\right]\right]$ is the ring formed by elements of the form $\sum\limits_\gamma a_\gamma t^\gamma$, $a_\gamma \in k$ such that the set $\{\gamma \mid a_\gamma \neq 0\}$ is well ordered.*

The ring $k\left[\left[t^\Gamma\right]\right]$ is equipped with the natural t-adic valuation v with values in Γ, defined by $v(f) = \inf\{\gamma \mid a_\gamma \neq 0\}$ for $f = \sum\limits_\gamma a_\gamma t^\gamma \in k\left[\left[t^\Gamma\right]\right]$. Specifying a total ordering on k and $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$ sign conditions defines a total ordering on $k\left[\left[t^\Gamma\right]\right]$. In this ordering $|t|$ is smaller than any positive element of k. For example, if $t^\gamma > 0$ for all $\gamma \in \Gamma$ then $f > 0$ if and only if $a_{v(f)} > 0$.

For an ordered field k, let $\bar{k}$ denote the real closure of k. The following result is a variation on a theorem of Kaplansky ([4], [5]) for valued fields equipped with a total ordering.

Theorem 2.1 ([9], p. 62, Satz 21) *Let K be a real valued field, with residue field k and value group Γ. There exists an injection $K \hookrightarrow \bar{k}\left(\left(t^\Gamma\right)\right)$ of real valued fields.*

Let $\alpha \in \text{Sper } B$ and let Γ_α be the value group of ν_α. In view of (3) and the Remark above, specifying a point $\alpha \in \text{Sper } B$ is equivalent to specifying a total order of k_α, a morphism

$$B[\alpha] \to \bar{k}_\alpha\left[\left[t^{\Gamma_\alpha}\right]\right]$$

and $\dim_{\mathbb{F}_2}(\Gamma_\alpha/2\Gamma_\alpha)$ sign conditions as above.

We may pass to usual spectra to obtain morphisms

$$\text{Spec }\left(\bar{k}_\alpha\left[\left[t^{\Gamma_\alpha}\right]\right]\right) \to \text{Spec } B[\alpha] \to \text{Spec } B.$$

In particular, if $\Gamma_\alpha = \mathbb{Z}$, we obtain a **formal curve** in Spec B (an analytic curve if the series are convergent). This motivates the following definition:

Definition 2.2 *Let k be an ordered field. A k-**curvette** on Sper(B) is a morphism of the form*

$$\alpha : B \to k\left[\left[t^{\Gamma}\right]\right],$$

*where Γ is an ordered group. A k-**semi-curvette** is a k-curvette α together with a choice of the sign data sgn x_1,..., sgn x_r, where $x_1, ..., x_r$ are elements of B whose t-adic values induce an $\mathbb{F}_2$-basis of $\Gamma/2\Gamma$.*

We have thus explained how to associate to a point α of Sper B a $\bar{k}_\alpha$-semi-curvette. Conversely, given an ordered field k, a k-semi-curvette α determines a prime ideal $\mathfrak{p}_\alpha$ (the ideal of all the elements of B which vanish identically on α) and a total ordering on $B/\mathfrak{p}_\alpha$ induced by the ordering of the ring $k\left[\left[t^{\Gamma}\right]\right]$ of formal power series. These two operations are inverse to each other. This establishes a one-to-one correspondence between semi-curvettes and points of Sper B.

Below, we will sometimes describe points in the real spectrum by specifying the corresponding semi-curvettes.

Example: Consider the curvette $R[x, y] \to R[[t]]$ defined by $x \mapsto t^2$, $y \mapsto t^3$, and the semi-curvette given by declaring, in addition, that t is positive. This semi-curvette is nothing but the upper branch of the cusp.

Later in the paper, we will need, for a certain number $p \in \{0, 1, \dots, n\}$ and two points δ, δ^* living in different spaces, to compare $(n-p)$-tuples of elements such as $(\nu_\delta(x_{p+1}), \dots, \nu_\delta(x_n)) \in \Gamma_\delta^{n-p}$ and $(\nu_{\delta^*}(y_{p+1}), \dots, \nu_{\delta^*}(y_n)) \in \Gamma_{\delta^*}^{n-p}$ and to be able to say that they are in some sense "equivalent". To do this, we need to embed Γ_δ in some "universal" ordered group.

Notation and convention: Let us denote by Γ the ordered group $\mathbb{R}^n_{lex}$. This means that elements of Γ are compared as words in a dictionary: we say that $(a_1, \dots, a_n) < (a'_1, \dots, a'_n)$ if and only if there exists $j \in \{1, \dots, n\}$ such that $a_q = a'_q$ for all $q < j$ and $a_j < a'_j$.

The reason for introducing Γ is that by Abhyankar's inequality we have rank $\nu_\delta \leq \dim A = n$ for all $\delta \in \text{Sper } A$, so the value group Γ_δ of ν_δ can be embedded into Γ as an ordered subgroup (of course, this embedding is far from being unique). Let Γ_+ be the semigroup of non-negative elements of Γ.

Fix a strictly positive integer ℓ. In order to deal rigourously with ℓ-tuples of elements of Γ_δ despite the non-uniqueness of the embedding $\Gamma_\delta \subset \Gamma$, we introduce the category $\mathcal{OGM}(\ell)$, as follows. An object in $\mathcal{OGM}(\ell)$ is an ordered abelian group G together with ℓ fixed generators $a_1, \dots, a_\ell$ (such an object will be denoted by $(G, a_1, \dots, a_\ell)$). A morphism from $(G, a_1, \dots, a_\ell)$ to $(G', a'_1, \dots, a'_\ell)$ is a homomorphism $G \to G'$ of ordered groups which maps a_j to a'_j for each j.

Given $(G, a_1, \ldots, a_\ell)$, $(G', a'_1, \ldots, a'_\ell) \in Ob(\mathcal{OGM}(\ell))$, the notation

$$(a_1, \ldots, a_\ell) \underset{\circ}{\sim} (a'_1, \ldots, a'_\ell) \tag{6}$$

will mean that $(G, a_1, \ldots, a_\ell)$ and $(G', a'_1, \ldots, a'_\ell)$ are isomorphic in $\mathcal{OGM}(\ell)$.

Take an element

$$a = (a_1, \ldots, a_\ell) \in \Gamma_+^\ell.$$

Let $G \subset \Gamma$ be the ordered group generated by $a_1, \ldots, a_\ell$. Then $(G, a_1, \ldots, a_\ell) \in Ob(\mathcal{OGM}(\ell))$. For each $\delta \in \mathrm{Sper}(A)$, let Γ_δ denote the value group of the associated valuation ν_δ and Γ_δ^* the subgroup of Γ_δ generated by $\nu_\delta(x_1)$, $\ldots$, $\nu_\delta(x_n)$. In this way, we associate to δ an object $(\Gamma_\delta^*, \nu_\delta(x_1), \ldots, \nu_\delta(x_n)) \in Ob(\mathcal{OGM}(n))$.

Notation. Let Γ be an ordered group. Consider an ℓ-tuple $a = (a_1, \ldots, a_\ell) \in \Gamma^\ell$. We denote by $Rel(a)$ the set

$$Rel(a) = \left\{ (m_1, \ldots, m_\ell, m_{\ell+1}, \ldots, m_{2\ell}) \in \mathbb{Z}^{2\ell} \;\middle|\; \sum_{j=1}^{\ell} m_j a_j > 0 \text{ and } \sum_{j=\ell+1}^{2\ell} m_j a_{j-\ell} = 0 \right\}.$$

Remark 2.3 *Let Γ and a be as above and let G be the subgroup of Γ generated by $a_1, \ldots, a_\ell$, so that $(G, a_1, \ldots, a_\ell) \in Ob(\mathcal{OGM}(\ell))$. The set $Rel(a)$ completely determines the isomorphism class of $(G, a_1, \ldots, a_\ell)$ in $\mathcal{OGM}(\ell)$ and vice-versa; the set $Rel(a)$ and the isomorphism class of $(G, a_1, \ldots, a_\ell)$ are equivalent sets of data.*

3 Points at infinity of Sper(A)

In this section, we study the structure of the set of points at infinity in $\mathrm{Sper}(A)$. Take a point $\delta \in \mathrm{Sper}(A)$. Renumbering the coordinates if necessary, we may assume there exists p, $0 \leq p \leq n$, such that

$$\nu_\delta(x_j) = 0 \text{ for } 1 \leq i \leq p \text{ and } \nu_\delta(x_j) > 0 \text{ for } j > p. \tag{7}$$

For a subset T of $\{1, \ldots, p\}$, let $B_T = R[y_1, \ldots, y_n]$ where

$$\begin{aligned} y_j &= x_j && \text{if } j \in \{1, \ldots, n\} \setminus T && (8)\\ &= 1/x_j && \text{if } j \in T. && (9) \end{aligned}$$

For certain subsets $T \subset \{1, \ldots, p\}$ we will associate to δ a point δ^* in $\mathrm{Sper}^*(B_T)$ such that $A(\delta) = B_T(\delta^*)$. We will define a new valuation ν_{δ^*} of $A(\delta)$, such that R_δ is a localization of R_{δ^*} at a suitable prime ideal. At the end of this section, we will use these results to cover $\mathrm{Sper}(A)$ by sets, each of which is homeomorphic to a certain subspace of $\mathrm{Sper}^*(B_T)$ for some T.

First, take any subset $T \subset \{1, \ldots, n\}$ whatsoever. Let B_T be defined as in (8)–(9).

Notation : The notation A_f stands for the localization of A by f, the ring $A[1/f]$.

Remark : We have a natural homeomorphism

$$\mathrm{Sper}(A) \setminus \{f = 0\} \xrightarrow{\sim} \mathrm{Sper}(A_f) \tag{10}$$

Consider the natural isomorphism $A_{\prod x_j, j\in T} \cong (B_T)_{\prod y_j, j\in T}$. It induces a homeomorphism

$$\begin{array}{cccc} \psi: & \mathrm{Sper}(A) \setminus \left\{\prod_{j\in T} x_j = 0\right\} & \to & \mathrm{Sper}(B_T) \setminus \left\{\prod_{j\in T} y_j = 0\right\} \\ & \| & & \| \\ & \mathrm{Sper}(A_{\prod_{j\in T} x_j}) & \to & \mathrm{Sper}((B_T)_{\prod_{j\in T} y_j}) \end{array} \tag{11}$$

which we describe explicitly for future reference. Take a point $\delta \in \mathrm{Sper}(A_{\prod_{j\in T} x_j})$. We will now describe the point $\delta^* = \psi(\delta)$ in $\mathrm{Sper}(B_T)$, as follows. The ideal $\mathfrak{p}_{\delta^*}$ is the prime ideal of B_T such that

$$\mathfrak{p}_\delta A_{\prod x_j, j\in T} \cong \mathfrak{p}_{\delta^*}(B_T)_{\prod y_j, j\in T} \tag{12}$$

Then (12) implies the existence of a canonical isomorphism

$$\phi : \kappa(\mathfrak{p}_\delta) \cong \kappa(\mathfrak{p}_{\delta^*}). \tag{13}$$

The total order $\leq_{\delta^*}$ is the order induced by δ on $\kappa(\mathfrak{p}_{\delta^*})$ via the isomorphism (13). This describes ψ; the inverse map ψ^{-1} is described in a completely analogous way.

Here and below, $R_{>0}$ will denote the set of strictly positive elements of R.

Take a $\delta \in \mathrm{Sper}(A)$ and let p be as in (7). We associate to δ a partition

$$\{1, \dots, p\} = I_\delta \coprod F_\delta \coprod G_\delta,$$

as follows:

$$j \in I_\delta \iff |x_j|_\delta <_\delta \epsilon, \ \forall \epsilon \in R_{>0} \tag{14}$$
$$j \in F_\delta \iff \exists c_1, c_2 \in R_{>0} \quad \text{such that } c_1 <_\delta |x_j|_\delta <_\delta c_2 \tag{15}$$
$$j \in G_\delta \iff |x_j|_\delta >_\delta N, \ \forall N \in R. \tag{16}$$

Remark 3.1 *We have $\delta \in Sper^*(A)$ if and only if $G_\delta = \emptyset$. Below, we show that in this case necessarily $I_\delta = \emptyset$.*

Take a set T such that

$$G_\delta \subset T \subset G_\delta \cup F_\delta. \tag{17}$$

Let B_T be the ring defined by (8) and (9). It follows from (15), (16) and (17) that $x_j \notin \mathfrak{p}_\delta$ for $j \in T$. Let $\delta^* = \psi(\delta)$. It is immediate from the definition that δ^* is finite in $\mathrm{Sper}(B_T)$.

Proposition 3.1 *The valuation ν_{δ^*} of $B_T(\delta^*)$ associated to δ^* has the following properties:*

(1) $\nu_{\delta^*}(y_j) = 0$ for $j \in F_\delta$;

(2) $\nu_{\delta^*}(y_j) > 0$ for $j \in I_\delta \cup G_\delta$;

(3) there exists $q \in G_\delta$ and a strictly positive integer N such that, for all $j \in I_\delta$,

$$N\nu_{\delta^*}(y_q) > \nu_{\delta *}(y_j). \tag{18}$$

In particular, if $I_\delta \neq \emptyset$ then $G_\delta \neq \emptyset$.

(4) The valuation ring R_δ is the localization of R_{δ^*} at a prime ideal; this gives rise to a surjective order-preserving group homomorphism $\tilde{\phi} : \Gamma_{\delta^*} \to \Gamma_\delta$ whose kernel is an isolated subgroup.

(5) For all $j \in \{1, \dots, n\}$, $\tilde{\phi}(\nu_{\delta^*}(y_j)) = \nu_\delta(x_j)$.

(6) For $j \in \{1, \dots, p\}$, $\nu_{\delta^*}(y_j) \in \ker(\tilde{\phi})$. In particular, given any $j \in \{1, \dots, p\}$, $t \in \{p+1, \dots, n\}$ and $N' \in \mathbb{N}$, we have $N'\nu_{\delta^*}(y_j) < \nu_{\delta^*}(y_t)$.

(7) Assume that $\nu_\delta(x_{p+1}), \dots, \nu_\delta(x_n)$ are $\mathbb{Q}$-linearly independent. Then

$$(\nu_{\delta^*}(y_{p+1}), \dots, \nu_{\delta^*}(y_n)) \underset{\circ}{\sim} (\nu_\delta(x_{p+1}), \dots, \nu_\delta(x_n))$$

in $\mathcal{OGM}(n-p)$.

Proof : (1) Take $j \in F_\delta$. We have $1/|y_j|_{\delta^*} <_{\delta^*} c$ for some $c \in R$ by definition of F_δ (15). Hence $\frac{1}{y_j} \in R_{\delta^*}$ and the result follows.

(2) Take $j \in I_\delta \cup G_\delta$. By definition of I_δ (14), G_δ (16) and (17), $|y_j|_{\delta^*} <_{\delta^*} \epsilon$ for every $\epsilon \in R_{>0}$, so $1/|y_j|_{\delta^*} >_{\delta^*} N$ for every $N \in R$. By the boundedness of δ^*, for each $f \in B_T$, we have $|f(y_1, \dots, y_n)|_{\delta^*} <_{\delta^*} N'$ for some $N' \in R$. Hence $1/|y_j|_{\delta^*} >_{\delta^*} f(y_1, \dots, y_n)$ for each $f \in B_T$, so $1/y_j \notin R_{\delta^*}$. This proves that $\nu_{\delta^*}(y_j) > 0$.

(3) Take a $j \in I_\delta$. Since $\nu_\delta(x_j) = 0$, we have $1/x_j \in R_\delta$. This means that there exists $z \in A[\delta]$ such that $1/|x_j|_\delta <_\delta |z|_\delta$. Now, z is a polynomial in the x_k, $k = 1, \dots, n$, and taking x_q, $q \in G_\delta$, such that $|x_q|_\delta \geq_\delta |x_k|_\delta$ for all $k \in G_\delta$ (and hence $|x_q|_\delta \geq_\delta |x_k|_\delta$ for all $k \in \{1, \dots, n\}$), there exists $N > 0$ such that $1/|x_j|_\delta <_\delta |x_q|_\delta^N$ for all $j \in I_\delta$, so that $|y_j|_{\delta^*} >_{\delta^*} |y_q|_{\delta^*}^N$. Then $N\nu_{\delta^*}(y_q) \geq \nu_{\delta^*}(y_j)$ by equation (4). Replacing N by $N+1$, we can make the inequality (18) strict.

(4) It is well known that every homomorphism between two valuation rings having the same field of fractions is a localization at a prime ideal. Thus it is sufficient to show that $R_{\delta^*} \subset R_\delta$. Take $f \in R_{\delta^*}$. By definition, this means that $|f|_{\delta^*}$ is bounded above by a polynomial in the y_j with respect to $\leq_{\delta^*}$, and hence also by a monomial ω in the y_j. Then $\phi^{-1}(\omega)$ is bounded above with respect to $\leq_\delta$ by a monomial in the x_j, in which x_j with $j \in T$ appear with non-positive exponents. Since each $\frac{1}{|x_j|_\delta}$, $j \in T$, is bounded above by a constant in R, replacing factors of the form $x_j^{-\gamma_j}$, $j \in T$, $\gamma_j \in \mathbb{N}$ by a suitable constant in R, we obtain that $\phi^{-1}(f)$ is bounded above with respect to $\leq_{\delta^*}$ by a monomial in y with non-negative exponents. This proves that $\phi^{-1}(f) \in R_\delta$ as desired.

The last statement of (4) follows immediately by the general theory of composition of valuations ([11], Chapter VI, §10, p. 43). Alternatively, recall that $\Gamma_\delta \cong \dfrac{A(\delta) \setminus \{0\}}{U(R_\delta)}$ and that the valuation ν_δ can be identified with the natural homomorphism

$$A(\delta) \setminus \{0\} \to \frac{A(\delta) \setminus \{0\}}{U(R_\delta)}.$$

Similarly, ν_{δ^*} can be thought of as

$$B_T(\delta^*) \setminus \{0\} \to \frac{B_T(\delta^*) \setminus \{0\}}{U(R_{\delta^*})} \cong \Gamma_{\delta^*}.$$

From the isomorphism ϕ and the inclusion $R_{\delta^*} \hookrightarrow R_\delta$, we obtain a natural surjective homomorphism of ordered groups

$$\tilde{\phi} : \frac{B_T(\delta^*) \setminus \{0\}}{U(R_{\delta^*})} \to \frac{A(\delta) \setminus \{0\}}{U(R_\delta)}. \tag{19}$$

(5) If $j \notin T$, the fact that $\phi(x_j) = y_j$ implies that

$$\tilde{\phi}(y_j \mod U(R_{\delta^*})) = x_j \mod U(R_\delta).$$

If $j \in T$, we have $\phi(x_j) = 1/y_j$, hence

$$\tilde{\phi}(\nu_{\delta^*}(y_j)) = \nu_\delta(1/x_j) = 0 = \nu_\delta(x_j).$$

(6) is an immediate consequence of (5) and the fact that $\nu_\delta(x_1) = \dots = \nu_\delta(x_p) = 0$.

(7) By Remark 2.3 at the end of the previous section, it suffices to prove that

$$Rel(\nu_{\delta^*}(y_{\ell+1}), \dots, \nu_{\delta^*}(y_n)) = Rel(\nu_\delta(x_{\ell+1}), \dots, \nu_\delta(x_n)). \tag{20}$$

The fact that $\nu_\delta(x_{p+1}),\dots,\nu_\delta(x_n)$ are $\mathbb{Q}$-linearly independent and (5) of the Proposition imply that so are $\nu_{\delta^*}(y_{p+1}),\ \dots,\ \nu_{\delta^*}(y_n)$. Hence, using (5) of the Proposition again, for any $(n-p)$-tuple, $(m_{p+1},\dots,m_n)\in\mathbb{Z}^{n-p}$, we have $\sum_{i=p+1}^n m_j\nu_\delta(x_j)>0$ if and only if $\sum_{j=p+1}^n m_j\nu_{\delta^*}(y_j)>0$. Together with the linear independence of $\nu_\delta(x_{p+1}),\dots,\nu_\delta(x_n)$ and of $\nu_{\delta^*}(y_{p+1}),\dots,\nu_{\delta^*}(y_n)$, this proves the desired equality (20). □

Let G be an ordered group of rank r and ℓ a positive integer. Take ℓ elements $a_1,\dots,a_\ell\in G$. Let $(0)=\Delta_r\subsetneqq\Delta_{r-1}\subsetneqq\dots\subsetneqq\Delta_0=G$ be the isolated subgroups of G. Renumbering the a_j if necessary, we may assume that there exist integers $i_0,i_1,\dots,i_r$ with

$$\ell=i_0\ge i_1\ge i_2\ge\dots\ge i_r=0,$$

such that $a_{i_{q+1}},\dots,a_{i_q}\in\Delta_q-\Delta_{q+1}$ for $q\in\{0,\dots,r-1\}$.

Definition 3.1 *We say that $a_1,\dots,a_\ell$ are scalewise $\mathbb{Q}$-linearly independent if, for each $q\in\{0,\dots,r-1\}$, the images of $a_{i_{q+1}},\dots,a_{i_q}$ in $\dfrac{\Delta_q}{\Delta_{q+1}}$ are $\mathbb{Q}$-linearly independent.*

Remark 3.2 *Let the notation be as above and assume that $a_1,\dots,a_\ell$ are scalewise $\mathbb{Q}$-linearly independent. Let $\lambda:G\to G'$ be a homomorphism of ordered groups. Then $\lambda(a_1),\dots,\lambda(a_\ell)$ are scalewise $\mathbb{Q}$-linearly independent if and only if they are $\mathbb{Q}$-linearly independent if and only if all of them are non-zero. This is precisely the form in which we will use scalewise $\mathbb{Q}$-linear independence in the sequel.*

Fix an integer $p\in\{1,\dots,n\}$ and two decompositions

$$\{1,\dots,p\}=H\coprod T=I\coprod F\coprod G,\tag{21}$$

where $I=\emptyset$ whenever $G=\emptyset$,

$$\begin{aligned} I &\subset H \quad\text{and} &(22)\\ G &\subset T. &(23)\end{aligned}$$

Fix an n-tuple $(a_1,\dots,a_n)\in\Gamma_+^n$ such that $a_1=\dots=a_p=0$. Let

$$U_{I,F,G}=\left\{\delta\in\mathrm{Sper}(A)\ \middle|\ \begin{array}{l}\forall j\in I,\forall c\in R_{>0},\ |x_j|_\delta<_\delta c\\ \forall j\in F\exists c_1,c_2\in R_{>0}, c_1<_\delta|x_j|_\delta<_\delta c_2\\ \forall j\in G,\forall N\in R,\ |x_j|_\delta>_\delta N\\ \nu_\delta(x_1)=\ldots=\nu_\delta(x_p)=0\\ \nu_\delta(x_{p+1})>0,\dots,\nu_\delta(x_n)>0\end{array}\right\},\tag{24}$$

$$U^*_{I,F,G}=\left\{\delta^*\in\mathrm{Sper}^*(B_T)\ \middle|\ \begin{array}{l}\forall j\in F\exists c\in R_{>0},|y_j|_{\delta^*}>_{\delta^*}c\\ \exists q\in G,N\in\mathbb{N}\text{ s.t. }\forall j\in I,\ N\nu_{\delta^*}(y_q)>\nu_{\delta*}(y_j)\\ \forall j\in\{1,\dots,p\},\ \forall t\in\{p+1,\dots,n\},\forall N'\in\mathbb{N},\\ \qquad\qquad N'\nu_{\delta^*}(y_j)<\nu_{\delta^*}(y_t)\\ \nu_{\delta^*}(y_j)>0\ \forall j\in I\cup G\end{array}\right\},\tag{25}$$

$$U_{a,I,F,G}=\left\{\delta\in U_{I,F,G}\ \middle|\ (\nu_\delta(x_1),\dots,\nu_\delta(x_n))\underset{\circ}{\sim}(a_1,\dots,a_n)\right\},\tag{26}$$

$$U^*_{a,I,F,G}=\left\{\delta^*\in U^*_{I,F,G}\ \middle|\ (\nu_{\delta^*}(y_{p+1}),\dots,\nu_{\delta^*}(y_n))\underset{\circ}{\sim}(a_{p+1},\dots,a_n)\right\},\tag{27}$$

$$U_{H,T}=\left\{\delta\in\operatorname{Sper}(A)\ \middle|\ \begin{array}{l}\exists c\in R,\ |x_j|_\delta<_\delta c,\forall j\in H\\ \exists\epsilon\in R_{>0},\ |x_j|_\delta>_\delta\epsilon,\forall j\in T\\ \nu_\delta(x_1)=\ldots=\nu_\delta(x_p)=0\\ \nu_\delta(x_{p+1})>0,\ldots,\nu_\delta(x_n)>0\end{array}\right\}\tag{28}$$

and

$$U^*_{H,T}=\left\{\delta^*\in\operatorname{Sper}^*(B)\ \middle|\ \begin{array}{c}\exists q\in T,N\in\mathbb{N}\text{ s.t. }\forall j\in H,\ N\nu_{\delta^*}(y_q)>\nu_{\delta*}(y_j)\\ \forall j\in\{1,\ldots,p\},\ \forall t\in\{p+1,\ldots,n\},\forall N'\in\mathbb{N},\\ N'\nu_{\delta^*}(y_j)<\nu_{\delta^*}(y_t)\end{array}\right\}.\tag{29}$$

We view $U_{I,F,G}$, $U_{a,I,F,G}$ and $U_{H,T}$ (resp. $U^*_{I,F,G}$, $U^*_{a,I,F,G}$ and $U^*_{H,T}$) as topological subspaces of $\operatorname{Sper}(A)$ (resp. $\operatorname{Sper}^*(B_T)$) with the spectral topology. Clearly, for each I and G satisfying (21) we have

$$U_{I,F,G}=\bigcup_{\substack{a\in\Gamma^n_+\\ a_1=\cdots=a_p=0\\ a_{p+1}>0,\ldots,a_n>0}}U_{a,I,F,G}$$

and

$$U^*_{I,F,G}=\bigcup_{\substack{a\in\Gamma^n_+\\ a_1=\cdots=a_p=0\\ a_{p+1}>0,\ldots,a_n>0}}U^*_{a,I,F,G}$$

Also, we have

$$U_{H,T}=\coprod_{\substack{\{1,\ldots,p\}=I\coprod F\coprod G\\ I\subset H,G\subset T}}U_{I,F,G},\tag{30}$$

and

$$U^*_{H,T}=\coprod_{\substack{\{1,\ldots,p\}=I\coprod F\coprod G\\ I\subset H,G\subset T}}U^*_{I,F,G}.\tag{31}$$

Theorem 3.1 *The map ψ which sends δ to δ^*, defined above, induces homeomorphisms*

$$U_{I,F,G}\tilde{\to}U^*_{I,F,G}\tag{32}$$

and

$$U_{H,T}\tilde{\to}U^*_{H,T}.\tag{33}$$

If, in addition, $a_{p+1},\ldots,a_n$ are scalewise $\mathbb{Q}$-linearly independent, we also have a homeomorphism

$$U_{a,I,F,G}\tilde{\to}U^*_{a,I,F,G}.\tag{34}$$

Proof: To show (32) and (34), we have to prove that

$$\psi(U_{I,F,G})\subset U^*_{I,F,G}.\tag{35}$$

$$\psi(U_{a,I,F,G})\subset U^*_{a,I,F,G},\tag{36}$$

$$\psi^{-1}(U^*_{I,F,G}) \subset U_{I,F,G} \tag{37}$$

and

$$\psi^{-1}(U^*_{a,I,F,G}) \subset U_{a,I,F,G}. \tag{38}$$

First, take a point $\delta \in U_{I,F,G}$. By definitions, we have $I_\delta = I$, $F_\delta = F$ and $G_\delta = G$. For all $j \in F = F_\delta$ there exists $c \in R_{>0}$ such that $|y_j|_{\delta^*} >_{\delta^*} c$ by definition of F_δ and B_T. The condition

$$\forall j \in \{1, \ldots, p\},\ \forall t \in \{p+1, \ldots, n\}, \forall N' \in \mathbb{N},\ N'\nu_{\delta^*}(y_j) < \nu_{\delta^*}(y_t) \tag{39}$$

is nothing but Proposition 3.1 (6).

By Proposition 3.1 (3), there exist $q \in G$ and $N \in \mathbb{N},\ N > 0$ such that for all $j \in I$ we have

$$N\nu_{\delta^*}(y_q) > \nu_{\delta*}(y_j). \tag{40}$$

By Proposition 3.1 (2), we have $\nu_{\delta^*}(y_j) > 0$ for all $j \in I \cup G$. This completes the proof of the inclusion (35).

Next, assume that $\delta \in U_{a,I,F,G}$ and that $a_{p+1}, \ldots, a_n$ are $\mathbb{Q}$-linearly independent. The isomorphism

$$(\nu_{\delta^*}(y_{p+1}), \ldots, \nu_{\delta^*}(y_n)) \underset{\circ}{\sim} (a_{p+1}, \ldots, a_n)$$

is given by Proposition 3.1 (7). This proves the inclusion (36).

To prove the opposite inclusions, take any $\delta^* \in U^*_{I,F,G}$. The existence of $c, N \in R_{>0}$ such that $|x_j|_\delta <_\delta c$ for all $j \in I$ and

$$|x_j|_\delta >_\delta N, \quad \text{for all } j \in G \tag{41}$$

follow from the facts that δ^* is bounded, $x_j = y_j$ for $j \in I$ and $x_j = 1/y_j$ for $j \in G$. For $j \in F$ we have either $x_j = y_j$ or $x_j = \frac{1}{y_j}$, but in both cases the fact that $\delta^* \in U^*_{I,F,G}$ implies the existence of $c_1, c_2 \in R_{>0}$ such that

$$c_1 <_\delta |x_j|_\delta <_\delta c_2. \tag{42}$$

To prove the inclusion (37), it remains to prove that

$$\nu_\delta(x_1) = \ldots = \nu_\delta(x_p) = 0 \tag{43}$$

and

$$\nu_\delta(x_t) > 0 \text{ for all } t \in \{p+1, \ldots, n\}. \tag{44}$$

Equation (43) is equivalent to saying that

$$1/|x_j|_\delta \in R_\delta \qquad \text{for } 1 \le j \le p. \tag{45}$$

First, if $j \in G$, $|x_j|_\delta = \frac{1}{|y_j|_{\delta^*}}$ is bounded below by a positive constant by (41), hence (45) holds for $j \in G$.

If $j \in I$, the assumed existence of $q \in G$ and a positive $N \in \mathbb{N}$ such that for all $j \in I$ we have $N\nu_{\delta^*}(y_q) > \nu_{\delta*}(y_j)$ implies that $|y_j|_{\delta^*} >_{\delta^*} |y_q|^N_{\delta^*}$ by equation (4), in other words, $|x_j|_\delta >_\delta 1/|x_q|^N_\delta$ or, equivalently, $1/|x_j|_\delta <_\delta |x_q|^N_\delta$. This proves (45) for $j \in I$. For $j \in F$, (45) follows from (42). Thus (45) holds for all $j \in \{1, \ldots, p\}$, which proves (43).

Take an index $t \in \{p+1, \ldots, n\}$. To prove (44), it suffices to show that

$$1/x_t \notin R_\delta, \tag{46}$$

that is, that $1/|x_t|_\delta$ is not bounded above (with respect to $\leq_\delta$) by any polynomial in $x_1, \dots, x_n$. By the triangle inequality, this is equivalent to saying that $1/|x_t|_\delta$ is not bounded above by any monomial in $x_1, \dots, x_n$, or, equivalently, by any element of the form cx_j^N with $j \in \{1, \dots, n\}$, $N \in \mathbb{N}$ and $c \in R$. We prove this last statement by contradiction. Suppose there was an inequality of the form

$$1/|x_t|_\delta <_\delta cx_j^N \tag{47}$$

with $N \in \mathbb{N}$, $c \in R$ and $j \in \{1, \dots, n\}$. Since $\nu_{\delta^*}(y_t) > 0$, we have $|y_t|_{\delta^*} <_{\delta^*} \epsilon$ for all positive $\epsilon \in R$, so $|x_t|_\delta <_\delta \epsilon$ and $1/|x_t|_\delta > 1/\epsilon$ for all positive $\epsilon \in R$. On the other hand, if $j \in I \cup \{p+1, \dots, n\}$, we have $\nu_{\delta^*}(y_j) > 0$, hence $|x_j|_\delta = |y_j|_{\delta^*} <_{\delta^*} \theta$ for all positive $\theta \in R$ and if $j \in F$ then $|x_j|_\delta$ is bounded above by a constant from R by (42). This proves that $j \in G$ in (47).

Now, the hypotheses (39) implies that for any constant $d \in R$ and any $N' \in \mathbb{N}$ we have $d|y_j|_{\delta^*}^{N'} >_{\delta^*} |y_t|_{\delta^*}$, so $d/|x_j|_\delta^{N'} >_\delta |x_t|_\delta$, which contradicts (47). This completes the proof of (46) and (44). The inclusion (37) is proved.

Assume that $\delta^* \in U^*_{a,I,G}$. To prove the inclusion (38), it remains to prove the isomorphism

$$(\nu_\delta(x_1), \dots, \nu_\delta(x_n)) \underset{\circ}{\sim} (a_1, \dots, a_n). \tag{48}$$

By Proposition 3.1 (5), (44), the assumed scalewise $\mathbb{Q}$-linear independence of $a_{p+1}, \dots, a_n$ and the Remark following Definition 3.1, $\nu_\delta(x_{p+1}), \dots, \nu_\delta(x_n)$ are also scalewise $\mathbb{Q}$-linearly independent. Now (48) follows from Proposition 3.1 (7). The inclusion (38) is proved.

Finally, the homeomorphism (33) follows from (32), (30) and (31) by letting I, F, G run over all the triples of disjoint subsets, satisfying (21), (22) and (23), such that I is empty whenever G is empty. $\square$

Of course, Theorem 3.1 is true with $\{1, \dots, p\}$ replaced by any other subset of $\{1, \dots, n\}$. In the next Corollary we drop the assumption (21) and let I, F, G run over all the triples of disjoint subsets of $\{1, \dots, n\}$ such that $I = \emptyset$ whenever $G = \emptyset$. Similarly, H, T will run over all the pairs of disjoint subsets of $\{1, \dots, n\}$.

Corollary 3.1 *We have finite coverings*

$$\textit{Sper } A = \coprod_{I,F,G} U_{I,F,G},$$

and

$$\textit{Sper } A = \bigcup_{H,T} U_{H,T}$$

For each I, F, G as above, the set $U_{I,F,G}$ is homeomorphic to the subset $U^*_{I,F,G}$ of the set Sper* B_G of finite points of Sper B_G. For each H, T as above, the set $U_{H,T}$ is homeomorphic to the subset $U^*_{H,T}$ of the set Sper* B_T of finite points of Sper B_T.

Remark 3.3 *The assumption of scalewise $\mathbb{Q}$-linear independence of $a_{p+1}, \dots, a_n$, is needed in Theorem 3.1 only for the inclusion (38). The usual $\mathbb{Q}$-linear independence is needed for the inclusion (36) and for Proposition 3.1. Although at first glance these assumptions seem rather restrictive, we remark that any point $\delta \in$ Sper A can be transformed into one for which these assumptions hold by a sequence of blowings up. We refer the reader to Corollary 6.2 of [7] for details. Corollary 6.2 of [7] shows how to achieve usual $\mathbb{Q}$-linear independence of $a_{p+1}, \dots, a_n$, but it also works for scalewise $\mathbb{Q}$-linear independence after some minor and obvious modifications.*

Example. Let $n = 5$. Let $\delta \in \text{Sper } A$ be the point given by the following semi-curvette. We let $\Gamma = \mathbb{Z}^2_{\text{lex}}$ and $k_\delta = R(z, w)$, where z and w are independent variables. Let the total order on k_δ be given by the following inequalities:

$$\begin{aligned} 0 &<_\delta \quad w <_\delta c <_\delta z \qquad \text{for all } \; c \in R_{>0} && (49) \\ \frac{1}{w^N} &<_\delta \quad z \qquad\qquad\quad \text{for all } N \in \mathbb{N}. && (50) \end{aligned}$$

As usual, we define the total order on $k_\delta\left(\left(t^\Gamma\right)\right)$ by declaring t to be positive. Define the map $\delta : A \to k_\delta\left(\left(t^\Gamma\right)\right)$ by

$$\begin{aligned} \delta(x_1) &= w && (51) \\ \delta(x_2) &= 1 + t^{(0,1)} && (52) \\ \delta(x_3) &= z && (53) \\ \delta(x_4) &= t^{(1,0)} && (54) \\ \delta(x_5) &= zt^{(1,0)}. && (55) \end{aligned}$$

We have $\nu_\delta(x_1) = \nu_\delta(x_2) = \nu_\delta(x_3) = 0$,

$$\nu_\delta(x_4) = \nu_\delta(x_5) = (1,0) > 0, \tag{56}$$

so $p = 3$. Moreover, $I_\delta = \{1\}$, $F_\delta = \{2\}$, $G_\delta = \{3\}$. Let $T = G_\delta$ and let $\delta^* = \psi(\delta) \in \text{Sper}^* B_T$. We have $\Gamma_{\delta^*} = \mathbb{Z}^4_{\text{lex}}$ and $k_{\delta^*} = R$. The semi-curvette δ^* can be defined by the map

$$\begin{aligned} \delta^*(y_1) &= t^{(0,0,0,1)} && (57) \\ \delta^*(y_2) &= 1 + t^{(0,1,0,0)} && (58) \\ \delta^*(y_3) &= t^{(0,0,1,0)} && (59) \\ \delta^*(y_4) &= t^{(1,0,0,0)} && (60) \\ \delta^*(y_5) &= t^{(1,0,1,0)}. && (61) \end{aligned}$$

In this example, $\nu_\delta(x_4)$ and $\nu_\delta(x_5)$ are not $\mathbb{Q}$-linearly independent (56), and the conclusion of Proposition 3.1 does not hold: we do *not* have the equivalence

$$(\nu_{\delta^*}(y_4), \nu_{\delta^*}(y_5)) \underset{\circ}{\sim} (\nu_\delta(x_4), \nu_\delta(x_5)).$$

Let $A' = R[x'_1, x'_2, x'_3, x'_4, x'_5]$. Consider the map $\pi : A \to A'$ defined by

$$\begin{aligned} \pi(x_j) &= x'_j \quad \text{for } j \in \{1,2,3,4\}, && (62) \\ \pi(x_5) &= x'_4 x'_5. && (63) \end{aligned}$$

Let δ' be the unique preimage of δ under the natural map $\pi^* : \text{Sper } A' \to \text{Sper } A$ of the real spectra, induced by π (in the terminology of [7], π is an affine monomial blowing up along the ideal (x_4, x_5) with respect to δ and δ' is the transform of δ by π). Explicitly, we have $\Gamma_{\delta'} = \mathbb{Z}^2_{\text{lex}}$, $k_{\delta'} = R(z, w)$, as above, and δ is given by the semi-curvette

$$\begin{aligned} \delta(x_1) &= w && (64) \\ \delta(x_2) &= 1 + t^{(0,1)} && (65) \\ \delta(x_3) &= z && (66) \\ \delta(x_4) &= t^{(1,0)} && (67) \\ \delta(x_5) &= z. && (68) \end{aligned}$$

This is an example of the fact that every point $\delta \in \text{Sper } A$ can be transformed, after a sequence $\text{Sper } A' \to \text{Sper } A$ of affine monomial blowings up with respect to δ, into a point $\delta' \in \text{Sper } A'$ such that the non-zero elements of the set $\{\nu_{\delta'}(x_1), \dots, \nu_{\delta'}(x_n)\}$ are (scalewise) $\mathbb{Q}$-linearly independent.

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