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HAL Id: irsn-00311696

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Submitted on 20 Aug 2008

UNIFYING PRACTICAL UNCERTAINTY REPRESENTATIONS: I. GENERALIZED P-BOXES

SÉBASTIEN DESTERCKE, DIDIER DUBOIS, AND ERIC CHOJNACKI

ABSTRACT. There exist several simple representations of uncertainty that are easier to handle than more general ones. Among them are random sets, possibility distributions, probability intervals, and more recently Ferson's p-boxes and Neumaier's clouds. Both for theoretical and practical considerations, it is very useful to know whether one representation is equivalent to or can be approximated by other ones. In this paper, we define a generalized form of usual p-boxes. These generalized p-boxes have interesting connections with other previously known representations. In particular, we show that they are equivalent to pairs of possibility distributions, and that they are special kinds of random sets. They are also the missing link between p-boxes and clouds, which are the topic of the second part of this study.

1. INTRODUCTION

Different formal frameworks have been proposed to reason under uncertainty. The best known and oldest one is the probabilistic framework, where uncertainty is modeled by classical probability distributions. Although this framework is of major importance in the treatment of uncertainty due to variability, many arguments converge to the fact that a single probability distribution cannot adequately account for incomplete or imprecise information. Alternative theories and frameworks have been proposed to this end. The three main such frameworks, are, in decreasing order of generality, Imprecise probability theory[43], Random disjunctive sets [12, 40, 32] and Possibility theory [46, 16]. Within each of these frameworks, different representations and methods have been proposed to make inferences and decisions.

This study focuses on uncertainty representations, regarding the relations existing between them, their respective expressiveness and practicality. In the past years, several representation tools have been proposed: capacities [5], credal sets [29], random sets [32], possibility distributions [46], probability intervals [9], p-boxes [21] and, more recently, clouds [34, 35]. Such a diversity of representations motivates the study of their respective expressive power.

The more general representations, such as credal sets and capacities, are expressive enough to embed other ones as particular instances, facilitating their comparison. However, they are generally difficult to handle, computationally demanding and not fitted to all uncertainty theories. As for simpler representations, they are useful in practical uncertainty analysis problems [45, 38, 22]. They come in handy when trading expressiveness (possibly losing some information) against computational efficiency; they are instrumental in elicitation tasks, since requesting less information[1]; they are also instrumental in summarizing complex results of some uncertainty propagation methods [20, 2].

The object of this study is twofold: first, it provides a short review of existing uncertainty representations and of their relationships; second, it studies the ill-explored relationships between more recent simple representations and older ones, introducing a generalized form of p-box. Credal sets and random sets are used as the common umbrella clarifying the relations between simplified models. Finding such formal links facilitates a unified handling and treatment of uncertainty, and suggests how tools used for one theory can eventually be useful in the setting of other theories. We thus regard such a study as an important and necessary preamble to other studies devoted to computational and interpretability issues. Such issues, which still remain a matter of lively debate, are not the main topic of the present work, but we nevertheless provide some comments regarding them. In particular, we feel that is important to recall that a given representation can be interpreted and processed differently according to different theories, which were often independently motivated by specific problems.

This work is made of two companion papers, one devoted to p-boxes, introducing a generalization thereof that subsumes possibility distributions. The second part considers Neumaier's clouds, an even more general representation tool.

This paper first reviews older representation tools, already considered by many authors. A good complement to this first part, although adopting a subjectivist point of view, is provided by Walley [44]. Then, in Section 3, we propose and study a generalized form of p-box extending, among other things, some results by Baudrit and Dubois [1]. As we shall see, this new representation, which consists of two comonotonic distributions, is the missing link between usual p-boxes, clouds and possibility distributions, allowing to relate these three representations. Moreover, generalized p-boxes have interesting properties and are promising uncertainty representations by themselves. In particular,

Section 3.3 shows that generalized p-boxes can be interpreted as a special case of random sets; Section 3.4 studies the relation between probability intervals and generalized p-boxes and discusses transformation methods to extract probability intervals from p-boxes, and vice-versa.

In the present paper, we restrict ourselves to uncertainty representations defined on finite spaces (encompassing the discretized real line) unless stated otherwise. Representations defined on the continuous real line are considered in the second part of this paper. To make the paper easier to read, longer proofs have been moved to an appendix.

2. Non-additive uncertainty theories and some representation tools

To represent uncertainty, Bayesian subjectivists advocate the use of single probability distributions in all circumstances. However, when the available information lacks precision or is incomplete, claiming that a unique probability distribution can faithfully represent uncertainty is debatable[1]. It generally forces to engage in too strong a commitment, considering what is actually known.

Roughly speaking, alternative theories recalled here (imprecise probabilities, random sets, and possibility theory) have the potential to lay bare the existing imprecision or incompleteness in the information. They evaluate uncertainty on a particular event by means of a pair of (conjugate) lower and upper measures rather than by a single one. The difference between upper and lower measures then reflects the lack of precision in our knowledge.

In this section, we first recall basic mathematical notions used in the sequel, concerning capacities and the Möbius transform. Each theory mentioned above is then briefly introduced, with focus on practical representation tools available as of to-date, like possibility distributions, p-boxes and probability intervals, their expressive power and complexity.

2.1. **Basic mathematical notions.** Consider a finite space X containing n elements. Measures of uncertainty are often represented by set-functions called *capacities*, that were first introduced in Choquet's work [5].

Definition 1. A capacity on X is a function μ, defined on the set of subsets of X, such that:

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[1]For instance, the following statement about a coin: "We are not sure that this coin is fair, so the probability for this coin to land on Heads (or Tails) lies between $1/4$ and $3/4$" cannot be faithfully modeled by a single probability.

- $\mu(\emptyset) = 0, \mu(X) = 1$,
- $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

A capacity such that

$$\forall A, B \subseteq X, A \cap B = \emptyset, \mu(A \cup B) \geq \mu(A) + \mu(B)$$

is said to be *super-additive*. The dual notion, called *sub-additivity*, is obtained by reversing the inequality. A capacity that is both *sub-additive* and *super-additive* is called *additive*.

Given a capacity μ, its *conjugate* capacity μ^c is defined as $\mu^c(E) = \mu(X) - \mu(E^c) = 1 - \mu(E^c)$, for any subset E of X, E^c being its complement. In the following, $\mathbb{P}_X$ denotes the set of all additive capacities on space X. We will also denote P such capacities, since they are equivalent to probability measures on X. An additive capacity P is self-conjugate, and $P = P^c$. An additive capacity can also be expressed by restricting it to its distribution p defined on elements of X such that for all $x \in X$, $p(x) = P(\{x\})$. Then $\forall x \in X, p(x) \geq 0$, $\sum_{x \in X} p(x) = 1$ and $P(A) = \sum_{x \in A} p(x)$.

When representing uncertainty, the capacity of a subset evaluates the degree of confidence in the corresponding event. Super-additive and sub-additive capacities are fitted to the representation of uncertainty. The former, being sub-additive, verify $\forall E \subset X, \mu(E) + \mu(E^c) \leq 1$ and can be called *cautious* capacities (since, as a consequence, $\mu(E) \leq \mu^c(E), \forall E$); they are tailored for modeling the idea of certainty. The latter being sub-additive, verify $\forall E \subset X, \mu(E) + \mu(E^c) \geq 1$, can be called *bold* capacities; they account for the weaker notion of plausibility.

The core of a cautious capacity μ is the (convex) set of additive capacities that dominate μ, that is, $\mathcal{P}_\mu = \{P \in \mathbb{P}_X | \forall A \subseteq X, P(A) \geq \mu(A)\}$. This set may be empty even if the capacity is cautious. We need stronger properties to ensure a non-empty core. Necessary and sufficient conditions for non-emptiness are provided by Walley [43, Ch.2]. However, checking that these conditions hold can be difficult in general. An alternative to checking the non-emptiness of the core is to use specific characteristics of capacities that ensure it, such as n-monotonicity.

Definition 2. A super-additive capacity μ defined on X is $n-monotone$, where $n > 0$ and $n \in \mathbb{N}$, if and only if for any set $\mathcal{A} = \{A_i | 0 < i \leq n \ A_i \subset X\}$ of events A_i, it holds that

$$\mu(\bigcup_{A_i \in \mathcal{A}} A_i) \geq \sum_{I \subseteq \mathcal{A}} (-1)^{|I|+1} \mu(\bigcap_{A_i \in I} A_i)$$

An n-monotone capacity is also called a Choquet capacity of order n. Dual capacities are called n-alternating capacities. If a capacity

is n-monotone, then it is also $(n-1)$-monotone, but not necessarily $(n+1)$-monotone. An ∞-monotone capacity is a capacity that is n-monotone for every $n > 0$. On a finite space, a capacity is ∞-monotone if it is n-monotone with $n = |X|$. The two particular cases of 2-monotone (also called convex) capacities and ∞-monotone capacities have deserved special attention in the literature [4, 43, 31]. Indeed, 2-monotone capacities have a non-empty core. ∞-monotone capacities have interesting mathematical properties that greatly increase computational efficiency. As we will see, many of the representations studied in this paper possess such properties. Extensions of the notion of capacity and of n-monotonicity have been studied by de Cooman *et al.* [11].

Given a capacity μ on X, one can obtain multiple equivalent representations by applying various (bijective) transformations [23] to it. One such transformation, useful in this paper, is the Möbius inverse:

Definition 3. Given a capacity μ on X, its Möbius transform is a mapping $m : 2^{|X|} \to \mathbb{R}$ from the power set of X to the real line, which associates to any subset E of X the value

$$m(E) = \sum_{B \subset E} (-1)^{|E-B|} \mu(B)$$

Since $\mu(X) = 1$, $\sum_{E \in X} m(E) = 1$ as well, and $m(\emptyset) = 0$. Moreover, it can be shown [40] that the values $m(E)$ are non-negative for all subsets E of X (hence $\forall E \in X, 1 \geq m(E) \geq 0$) if and only if the capacity μ is ∞-monotone. Then m is called a mass assignment. Otherwise, there are some (non-singleton) events E for which $m(E)$ is negative. Such a set-function m is actually the unique solution to the set of 2^n equations

$$\forall A \subseteq X, \mu(A) = \sum_{E \subseteq A} m(E),$$

given any capacity μ. The Möbius transform of an additive capacity P coincides with its distribution p, assigning positive masses to singletons only. In the sequel, we focus on pairs of conjugate cautious and bold capacities. Clearly only one of the two is needed to characterize an uncertainty representation (by convention, the cautious one).

2.2. **Imprecise probability theory.** The theory of imprecise probabilities has been systematized and popularized by Walley's book [43]. In this theory, uncertainty is modeled by lower bounds (called coherent lower previsions) on the expected value that can be reached by bounded real-valued functions on X (called gambles). Mathematically speaking, such lower bounds have an expressive power equivalent to closed convex sets $\mathcal{P}$ of (finitely additive) probability measures P on X. In the

rest of the paper, such convex sets will be named *credal sets* (as is often done [29]). It is important to stress that, even if they share similarities (notably the modeling of uncertainty by sets of probabilities), Walley's behavioral interpretation of imprecise probabilities is different from the one of classical robust statistics[2] [25].

Imprecise probability theory is very general, and, from a purely mathematical and static point of view, it encompasses all representations considered here. Thus, in all approaches presented here, the corresponding credal set can be generated, making the comparison of representations easier. To clarify this comparison, we adopt the following terminology:

Definition 4. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ denote two uncertainty representation frameworks, a and b particular representatives of such frameworks, and $\mathcal{P}_a, \mathcal{P}_b$ the credal sets induced by these representatives a and b. Then:

- Framework $\mathbb{F}_1$ is said to *generalize* framework $\mathbb{F}_2$ if and only if for all $b \in \mathbb{F}_2$, $\exists a \in \mathbb{F}_1$ such that $\mathcal{P}_a = \mathcal{P}_b$ (we also say that $\mathbb{F}_2$ is a special case of $\mathbb{F}_1$).
- Frameworks $\mathbb{F}_1$ and $\mathbb{F}_2$ are said to be *equivalent* if and only if for all $b \in \mathbb{F}_2$, $\exists a \in \mathbb{F}_1$ such that $\mathcal{P}_a = \mathcal{P}_b$ and conversely.
- Framework $\mathbb{F}_2$ is said to be *representable* in terms of framework $\mathbb{F}_1$ if and only if for all $b \in \mathbb{F}_2$, there exists a subset $\{a_1, \ldots, a_k | a_i \in \mathbb{F}_1\}$ such that $\mathcal{P}_b = \mathcal{P}_{a_1} \cap \ldots \cap \mathcal{P}_{a_k}$
- A representative $a \in \mathbb{F}_1$ is said to outer-approximate (inner-approximate) a representative $b \in \mathbb{F}_2$ if and only if $\mathcal{P}_b \subseteq \mathcal{P}_a$ ($\mathcal{P}_a \subseteq \mathcal{P}_b$)

2.2.1. *Lower/upper probabilities.* In this paper, lower probabilities (lower previsions assigned to events) are sufficient to our purpose of representing uncertainty. We define a *lower probability* $\underline{P}$ on X as a super-additive capacity. Its conjugate $\overline{P}(A) = 1 - \underline{P}(A^c)$ is called an upper probability. The (possibly empty) credal set $\mathcal{P}_{\underline{P}}$ induced by a given lower probability is its core:

$$\mathcal{P}_{\underline{P}} = \{P \in \mathbb{P}_X | \forall A \subset X, \ P(A) \geq \underline{P}(A)\}.$$

Conversely, from any given non-empty credal set $\mathcal{P}$, one can consider a lower envelope P_* on events, defined for any event $A \subseteq X$ by $P_*(A) = \min_{P \in \mathcal{P}} P(A)$. A lower envelope is a super-additive capacity, and consequently a lower probability. The upper envelope

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[2]Roughly speaking, in Walley's approach, the primitive notions are lower and upper previsions or sets of so-called desirable gambles describing epistemic uncertainty, and the fact that there always exists a "true" precise probability distribution is not assumed.

$P^*(A) = \max_{P \in \mathcal{P}} P(A)$ is the conjugate of P_*. In general, a credal set $\mathcal{P}$ is included in the core of its lower envelope: $\mathcal{P} \subseteq \mathcal{P}_{P_*}$, since $\mathcal{P}_{P_*}$ can be seen as a projection of $\mathcal{P}$ on events.

Coherent lower probabilities $\underline{P}$ are lower probabilities that coincide with the lower envelopes of their core, i.e. for all events A of X, $\underline{P}(A) = \min_{P \in \mathcal{P}_{\underline{P}}} P(A)$. Since all representations considered in this paper correspond to particular instances of coherent lower probabilities, we will restrict ourselves to such lower probabilities. In other words, credal sets $\mathcal{P}_{\underline{P}}$ in this paper are entirely characterized by their lower probabilities on events and are such that for every event A, there is a probability distribution P in $\mathcal{P}_{\underline{P}}$ such that $P(A) = \underline{P}(A)$.

A credal set $\mathcal{P}_{\underline{P}}$ can also be described by a set of constraints on probability assignments to elements of X:

$$\underline{P}(A) \leq \sum_{x \in A} p(x) \leq \overline{P}(A).$$

Note that $2^{|X|} - 2$ values ($|X|$ being the cardinality of X), are needed in addition to constraints $\underline{P}(X) = 1, \underline{P}(\emptyset) = 0$ to completely specify $\mathcal{P}_{\underline{P}}$.

2.2.2. *Simplified representations.* Representing general credal sets induced or not by coherent lower probabilities is usually costly and dealing with them presents many computational challenges (See, for example, Walley [44] or the special issue [3]). In practice, using simpler representations of imprecise probabilities often alleviates the elicitation and computational burden. P-boxes and interval probabilities are two such simplified representations.

P-boxes

Let us first recall some background on cumulative distributions. Let P be a probability measure on the real line $\mathbb{R}$. Its *cumulative distribution* is a non-decreasing mapping from $\mathbb{R}$ to $[0, 1]$ denoted F^P, such that for any $r \in \mathbb{R}$, $F^P(r) = P((-\infty, r])$. Let F_1 and F_2 be two cumulative distributions. Then, F_1 is said to stochastically dominate F_2 if only if F_1 is point-wise lower than F_2: $F_1 \leq F_2$.

A p-box [21] is then defined as a pair of (discrete) cumulative distributions $[\underline{F}, \overline{F}]$ such that $\underline{F}$ stochastically dominates $\overline{F}$. A p-box induces a credal set $\mathcal{P}_{[\underline{F}, \overline{F}]}$ such that:

$$(1) \qquad \mathcal{P}_{[\underline{F}, \overline{F}]} = \{P \in \mathbb{P}_{\mathbb{R}} | \forall r \in \mathbb{R}, \ \underline{F}(r) \leq P((-\infty, r]) \leq \overline{F}(r)\}$$

We can already notice that since sets $(-\infty, r]$ are nested, $\mathcal{P}_{[\underline{F}, \overline{F}]}$ is described by constraints that are lower and upper bounds on such nested sets (as noticed by Kozine and Utkin [26], who discuss the problem of

building p-boxes from partial information). This interesting character-
istic will be crucial in the generalized form of p-box we introduce in
section 3. Conversely we can extract a p-box from a credal set $\mathcal{P}$ by con-
sidering its lower and upper envelopes restricted to events of the form
$(-\infty, r]$, namely, letting $\underline{F}(r) = P_*((-\infty, r]), \overline{F}(r) = P^*((-\infty, r])$.
$\mathcal{P}_{[\underline{F}, \overline{F}]}$ is then the tightest outer-approximation of $\mathcal{P}$ induced by a p-
box.

Cumulative distributions are often used to elicit probabilistic knowl-
edge from experts [6]. P-boxes can thus directly benefit from such
methods and tools, with the advantages of allowing some imprecision in
the representation (e.g., allowing experts to give imprecise percentiles).
P-boxes are also sufficient to represent final results produced by im-
precise probability models when only a threshold violation has to be
checked. Working with p-boxes also allows, via so-called probabilistic
arithmetic [45], for very efficient numerical methods to achieve some
particular types of (conservative) inferences.

Probability intervals

Probability intervals, extensively studied by De Campos *et al.* [9], are
defined as lower and upper bounds of probability distributions. They
are defined by a set of numerical intervals $L = \{[l(x), u(x)] | x \in X\}$ such
that $l(x) \leq p(x) \leq u(x), \forall x \in X$, where $p(x) = P(\{x\})$. A probability
interval induces the following credal set:

$$\mathcal{P}_L = \{P \in \mathbb{P}_X | \forall x \in X, \ l(x) \leq p(x) \leq u(x)\}$$

A probability interval L is called reachable if the credal set $\mathcal{P}_L$ is
not empty and if for each element $x \in X$, we can find at least one
probability measure $P \in \mathcal{P}_L$ such that $p(x) = l(x)$ and one for which
$p(x) = u(x)$. In other words, each bound can be reached by a prob-
ability measure in $\mathcal{P}_L$. Non-emptiness and reachability respectively
correspond to the conditions [9]:

$$\sum_{x \in X} l(x) \leq 1 \leq \sum_{x \in X} u(x) \qquad \text{non-emptiness}$$

$$u(x) + \sum_{y \in X \setminus \{x\}} l(y) \leq 1 \text{ and } l(x) + \sum_{y \in X \setminus \{x\}} u(y) \geq 1 \qquad \text{reachability}$$

If a probability interval L is non-reachable, it can be transformed into
a probability interval L', by letting $l'(x) = \inf_{P \in \mathcal{P}_L}(p(x))$ and $u'(x) = \sup_{P \in \mathcal{P}_L}(p(x))$. More generally, coherent lower and upper probabilities
$\underline{P}(A), \overline{P}(A)$ induced by $\mathcal{P}_L$ on all events $A \subset X$ are easily calculated

by the following expressions
(2)
$$\underline{P}(A) = \max(\sum_{x \in A} l(x), 1 - \sum_{x \in A^c} u(x)), \overline{P}(A) = \min(\sum_{x \in A} u(x), 1 - \sum_{x \in A^c} l(x)).$$

De Campos *et al.* [9] have shown that these lower and upper probabilities are Choquet capacities of order 2.

Probability intervals, which are modeled by $2|X|$ values, are very convenient tools to model uncertainty on multinomial data, where they can express lower and upper confidence bounds. They can thus be derived from a sample of small size [30]. On the real line, discrete probability intervals correspond to imprecisely known histograms. Probability intervals can be extracted, as useful information, from any credal set $\mathcal{P}$ on a finite set X, by constructing $L_{\mathcal{P}} = \{[\underline{P}(\{x\}), \overline{P}(\{x\})], x \in X\}$. $L_{\mathcal{P}}$ then represents the tightest probability interval outer-approximating $\mathcal{P}$. Numerical and computational advantages that probability intervals offer are discussed by De Campos *et al.* [9].

2.3. **Random disjunctive sets.** A more specialized setting for representing partial knowledge is that of random sets. Formally a random set is a family of subsets of X each bearing a probability weight. Typically, each set represents an incomplete observation, and the probability bearing on this set should potentially be shared among its elements, but is not by lack of sufficient information.

2.3.1. *Belief and Plausibility functions.* Formally, a random set is defined as a mapping $\Gamma : \Omega \to \wp(X)$ from a probability space $(\Omega, \mathcal{A}, P)$ to the power set $\wp(X)$ of another space X (here finite). It is also called a multi-valued mapping Γ. Insofar as sets $\Gamma(\omega)$ represent incomplete knowledge about a single-valued random variable, each such set contains mutually exclusive elements and is called a *disjunctive* set[3]. Then this mapping induces the following coherent lower and upper probabilities on X for all events A [12] (representing all probability functions on X that could be found if the available information were complete):

$$\underline{P}(A) = P(\{\omega \in \Omega | \Gamma(\omega) \subseteq A\}); \quad \overline{P}(A) = P(\{\omega \in \Omega | \Gamma(\omega) \cap A \neq \emptyset\}),$$

where $\{\omega \in \Omega | \Gamma(\omega) \cap A \neq \emptyset\} \in \mathcal{A}$ is assumed. When X is finite, a random set can be represented as a mass assignment m over the power set $\wp(X)$ of X, letting $m(E) = P(\{\omega, \Gamma(\omega) = E\}), \forall E \in X$. Then, $\sum_{E \subseteq X} m(E) = 1$ and $m(\emptyset) = 0$. A set E that receives strict positive mass is called a focal set, and the mass $m(E)$ can be interpreted as the

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[3]as opposed to sets as collections of objects, i.e. sets whose elements are jointly present, such as a region in a digital image.

probability that the most precise description of what is known about a particular situation is of the form "$x \in E$". From this mass assignment, Shafer [40] define two set functions, called *belief and plausibility functions*, respectively:

$$Bel(A) = \sum_{E, E \subseteq A} m(E); \qquad Pl(A) = 1 - Bel(A^c) = \sum_{E, E \cap A \neq \emptyset} m(E).$$

The mass assignment being positive, a belief function is an ∞-monotone capacity. The mass assignment m is indeed the Möbius transform of the capacity Bel. Conversely, any ∞-monotone capacity is induced by one and only one random set. We can thus speak of *the* random set underlying Bel. In the sequel, we will use this notation for lower probabilities stemming from random sets (Dempster and Shafer definitions being equivalent on finite spaces). Smets [42] has studied the case of continuous random intervals defined on the real line $\mathbb{R}$, where the mass function is replaced by a mass density bearing on pairs of interval endpoints.

Belief functions can be considered as special cases of coherent lower probabilities, since they are ∞-monotone capacities. A random set thus induces the credal set $\mathcal{P}_{Bel} = \{P \in \mathbb{P}_X | \forall A \subseteq X, Bel(A) \leq P(A)\}$.

Note that Shafer [40] does not refer to an underlying probability space, nor does he uses the fact that a belief function is a lower probability: in his view, extensively taken over by Smets [41], $Bel(A)$ is supposed to quantify an agent's belief per se with no reference to a probability. However, the primary mathematical tool common to Dempster's upper and lower probabilities and to the Shafer-Smets view is the notion of (generally finite) random disjunctive set.

2.3.2. *Practical aspects.* In general, $2^{|X|} - 2$ values are still needed to completely specify a random set, thus not clearly reducing the complexity of the model representation with respect to capacities. However, simple belief functions defined by only a few positive focal elements do not exhibit such complexity. For instance, a simple support belief function is a natural model of an unreliable testimony, namely an expert stating that the value of a parameter x belong to set $A \subseteq X$. Let α be the reliability of the expert testimony, i.e. the probability that the information is irrelevant. The corresponding mass assignment is $m(A) = \alpha, m(X) = 1 - \alpha$. Imprecise results from some statistical experiments are easily expressed by means of random sets, $m(A)$ being the probability of an observation of the form $x \in A$.

As practical models of uncertainty, random sets present many advantages. First, as they can be seen as probability distributions over

subsets of X, they can be easily simulated by classical methods such as Monte-Carlo sampling, which is not the case for other Choquet capacities. On the real line, a random set is often restricted to a finite collection of closed intervals with associated weights, and one can then easily extend results from interval analysis [33] to random intervals [18, 24].

2.4. **Possibility theory.** The primary mathematical tool of possibility theory is the possibility distribution, which is a set-valued piece of information where some elements are more plausible than others. To a possibility distribution are associated specific measures of certainty and plausibility.

2.4.1. *Possibility and necessity measures.* A possibility distribution is a mapping $\pi : X \to [0, 1]$ from X to the unit interval such that $\pi(x) = 1$ for at least one element x in X. Formally, a possibility distribution is equivalent to the membership function of a normalized fuzzy set [46][4]. Twenty years earlier, Shackle [39] had introduced an equivalent notion called distribution of potential surprise (corresponding to $1 - \pi(x)$) for the representation of non-probabilistic uncertainty.

Several set-functions can be defined from a possibility distribution π [15]:

$$(3) \qquad \text{Possibility measures:} \quad \Pi(A) \quad = \sup_{x \in A} \pi(x).$$

$$(4) \qquad \text{Necessity measures:} \quad N(A) \quad = 1 - \Pi(A^c).$$

$$(5) \qquad \text{Sufficiency measures:} \quad \Delta(A) \quad = \inf_{x \in A} \pi(x).$$

The possibility degree of an event A evaluates the extent to which this event is plausible, i.e., consistent with the available information. Necessity degrees express the certainty of events, by duality. In this context, distribution π is potential (in the spirit of Shackle's), i.e. $\pi(x) = 1$ does not guarantee the existence of x. Their characteristic property are: $N(A \cap B) = \min(N(A), N(B))$ and $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$ for any pair of events A, B of X. On the contrary $\Delta(A)$ measures the extent to which all states of the world where A occurs are plausible. Sufficiency[5] distributions, generally denoted by δ, express *actual* possibility. They are understood as degree of empirical support and obey an opposite convention: $\delta(x) = 1$ guarantees (= is sufficient for) the existence of x.

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[4]The membership function of a fuzzy set ν is a mapping $\nu : X \to [0, 1]$

[5]also called guaranteed possibility distributions [15].

2.4.2. *Relationships with previous theories.* A necessity measure (resp a possibility measure) is formally a particular case of belief function (resp. a plausibility function) induced by a random set with nested focal sets (already in [40]). Given a possibility distribution π and a degree $\alpha \in [0, 1]$, strong and regular α-cuts are subsets respectively defined as $A_{\overline{\alpha}} = \{x \in X | \pi(x) > \alpha\}$ and $A_\alpha = \{x \in X | \pi(x) \geq \alpha\}$. These α-cuts are nested, since if $\alpha > \beta$, then $A_\alpha \subseteq A_\beta$. On finite spaces, the set $\{\pi(x), x \in X\}$ is of the form $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_M = 1$. There are only M distinct α-cuts. A possibility distribution π then induces a random set having, for $i = 1, \ldots, M$, the following focal sets E_i with masses $m(E_i)$:

(6)
$$\begin{cases} E_i = \{x \in X | \pi(x) \geq \alpha_i\} = A_{\alpha_i} \\ \quad m(E_i) = \alpha_i - \alpha_{i-1} \end{cases}$$

In this nested situation, the same amount of information is contained in the mass function m and the possibility distribution $\pi(x) = Pl(\{x\})$, called the *contour* function of m. For instance a simple support belief function such that $m(A) = \alpha, m(X) = 1 - \alpha$ forms a nested structure, and yields the possibility distribution $\pi(x) = 1$ if $x \in A$ and $1 - \alpha$ otherwise. In the general case, m cannot be reconstructed only from its contour function. Outer and inner approximations of general random sets in terms of possibility distributions have been studied by Dubois and Prade in [17].

Since the necessity measure is formally a particular case of belief function, it is also an ∞-monotone capacity, hence a particular coherent lower probability. If the necessity measure is viewed as a coherent lower probability, its possibility distribution induces the credal set $\mathcal{P}_\pi = \{P \in \mathbb{P}_X | \forall A \subseteq X, \ P(A) \geq N(A)\}$. We recall here a result, proved by Dubois *et al.* [19, 14] and by Couso *et al.* [8] in a more general setting, which links probabilities P that are in $\mathcal{P}_\pi$ with constraints on α-cuts, that will be useful in the sequel:

Proposition 1. *Given a possibility distribution π and the induced convex set $\mathcal{P}_\pi$, we have for all α in $(0, 1]$, $P \in \mathcal{P}_\pi$ if and only if*

$$1 - \alpha \leq P(\{x \in X | \pi(x) > \alpha\})$$

This result means that the probabilities P in the credal set $\mathcal{P}_\pi$ can also be described in terms of constraints on strong α-cuts of π (i.e. $1 - \alpha \leq P(A_{\overline{\alpha}})$).

2.4.3. *Practical aspects.* At most $|X| - 1$ values are needed to fully assess a possibility distribution, which makes it the simplest uncertainty

representation explicitly coping with imprecise or incomplete knowledge. This simplicity makes this representation very easy to handle. This also implies less expressive power, in the sense that, for any event A , either $\Pi(A) = 1$ or $N(A) = 0$ (i.e. intervals $[N(A), \Pi(A)]$ are of the form $[0, \alpha]$ or $[\beta, 1]$). This means that, in several situations, possibility distributions will be insufficient to reflect the available information.

Nevertheless, the expressive power of possibility distributions fits various practical situations. Moreover, a recent psychological study [37] shows that sometimes people handle uncertainty according to possibility theory rules. Possibility distributions on the real line can be interpreted as a set of nested intervals with different confidence degrees [14] (the larger the set, the highest the confidence degree), which is a good model of, for example, an expert opinion concerning the value of a badly known parameter. Similarly, it is natural to view nested confidence intervals coming from statistics as a possibility distribution. Another practical case where uncertainty can be modeled by possibility distributions is the case of vague linguistic assessments concerning probabilities [10].

2.5. **P-boxes and probability intervals in the uncertainty landscape.** P-boxes, reachable probability intervals, random sets and possibility distributions can all be modeled by credal sets and define coherent lower probabilities. Kriegler and Held [27] show that random sets generalize p-boxes (in the sense of Definition 4), but that the converse do not hold (i.e. credal sets induced by different random sets can have the same upper and lower bounds on events of the type $(\infty, r]$, and hence induce the same p-boxes).

There is no specific relationship between the frameworks of possibility distributions, p-boxes and probability intervals, in the sense that none generalize the other. Some results comparing possibility distributions and p-boxes are given by Baudrit and Dubois [1]. Similarly, there is no generalization relationship between probability intervals and random sets. Indeed upper and lower probabilities induced by reachable probability intervals are order 2 capacities only, while belief functions are ∞-monotone. In general, one can only approximate one representation by the other.

Transforming a belief function Bel into the tightest probability interval L outer-approximating it (i.e. $\mathcal{P}_{Bel} \subset \mathcal{P}_L$, following Definition 4) is simple, and consists of taking for all $x \in X$:

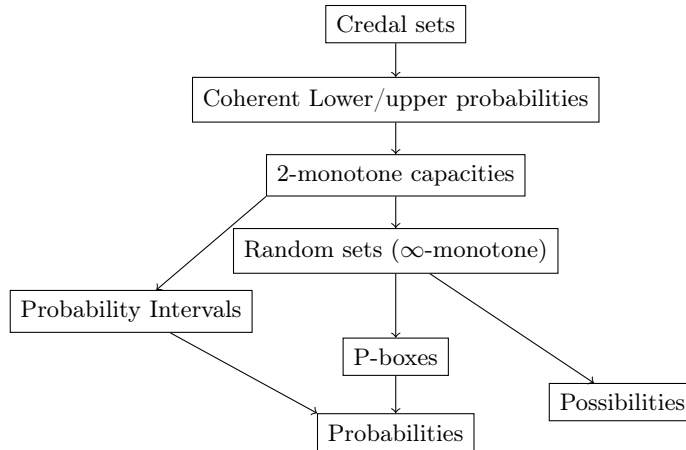$$l(x) = Bel(\{x\}) \text{ and } u(x) = Pl(\{x\})$$

FIGURE 1. Representation relationships: summary
$A \longrightarrow B$: A generalizes B

and since belief and plausibility functions are the lower envelope of the induced credal set $\mathcal{P}_{Bel}$, we are sure that the so-built probability interval L is reachable.

The converse problem, i.e. to transform a set L of probability intervals into an inner-approximating random set was studied by Lemmer and Kyburg [28]. On the contrary, Denoeux [13] extensively studies the problem of transforming a probability interval L into a random set outer-approximating L (i.e., $\mathcal{P}_L \subset \mathcal{P}_{Bel}$). The transformation of a given probability interval L into an outer-approximating possibility distribution is studied by Masson and Denoeux [30], who propose efficient methods to achieve such a transformation.

The main relations existing between imprecise probabilities, lower/upper probabilities, random sets, probability intervals, p-boxes and possibility distributions, are pictured on Figure 1. From top to bottom, it goes from the more general, expressive and complex theories to the less general, less expressive but simpler representations. Arrows are directed from a given representation to the representations it generalizes.

3. GENERALIZED P-BOXES

As recalled in Section 2.2, p-boxes are useful representations of uncertainty in many practical applications[7, 21, 27]. So far, they only make sense on the (discretized) real line equipped with the natural ordering of numbers. P-boxes are instrumental to extract interpretable information from imprecise probability representations. They provide faithful estimations of the probability that a variable $\tilde{x}$ violates a threshold θ,

i.e., upper and lower estimates of the probability of events of the form $\tilde{x} \geq \theta$. However, they are much less adequate to compute the probability that some output remains close to a reference value ρ [1], which corresponds to computing upper and lower estimates of the probability of events of the form $|\tilde{x} - \rho| \geq \theta$. The rest of the paper is devoted to the study of a generalization p-boxes, to arbitrary (finite) spaces, where the underlying ordering relation is arbitrary, and that can address this type of query. Generalized p-boxes will also be instrumental to characterize a recent representation proposed by Neumaier [34], studied in the second part of this paper.

Generalized p-boxes are defined in Section 3.1. We then proceed to show the link between generalized p-boxes, possibility distributions and random sets. We first show that the former generalize possibility distributions and are representable (in the sense of Definition 4) by pairs thereof. Connections between generalized p-boxes and probability intervals are also explored.

3.1. **Definition of generalized p-boxes.** The starting point of our generalization is to notice that any two cumulative distribution functions modelling a p-box are comonotonic. Two mappings f and f' from a space X to the real line are said to be comonotonic if and only if, for any pair of elements $x, y \in X$, we have $f(x) < f(y) \Rightarrow f'(x) \leq f'(y)$. In other words, given an indexing of $X = \{x_1, \ldots, x_n\}$, there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $f(x_{\sigma(1)}) \geq f(x_{\sigma(2)}) \geq \cdots \geq f(x_{\sigma(n)})$ and $f'(x_{\sigma(1)}) \geq f'(x_{\sigma(2)}) \geq \cdots \geq f'(x_{\sigma(n)})$. We define a generalized p-box as follows:

Definition 5. A generalized p-box $[\underline{F}, \overline{F}]$ defined on X is a pair of comonotonic mappings $\underline{F}, \overline{F}, \underline{F} : X \to [0, 1]$ and $\overline{F} : X \to [0, 1]$ from X to $[0, 1]$ such that $\underline{F}$ is pointwise less than $\overline{F}$ (i.e. $\underline{F} \leq \overline{F}$) and there is at least one element x in X for which $\overline{F}(x) = \underline{F}(x) = 1$.

Since each distribution $\underline{F}, \overline{F}$ is fully specified by $|X| - 1$ values, it follows that $2|X| - 2$ values completely determine a generalized p-box. Note that, given a generalized p-box $[\underline{F}, \overline{F}]$, we can always define a *complete* pre-ordering $\leq_{[\underline{F}, \overline{F}]}$ on X such that $x \leq_{[\underline{F}, \overline{F}]} y$ if $\underline{F}(x) \leq \underline{F}(y)$ and $\overline{F}(x) \leq \overline{F}(y)$, due to the comonotonicity condition. If X is a subset of the real line and if $\leq_{[\underline{F}, \overline{F}]}$ is the natural ordering of numbers, then we retrieve classical p-boxes.

To simplify notations in the sequel, we will consider that, given a generalized p-box $[\underline{F}, \overline{F}]$, elements x of X are indexed such that $i < j$ implies that $x_i \leq_{[\underline{F}, \overline{F}]} x_j$. We will denote $(x)_{[\underline{F}, \overline{F}]}$ the set of the form $\{x_i | x_i \leq_{[\underline{F}, \overline{F}]} x\}$. The credal set induced by a generalized p-box $[\underline{F}, \overline{F}]$

can then be defined as

$$\mathcal{P}_{[\underline{F},\overline{F}]} = \{P \in \mathbb{P}_X | i = 1, \ldots, n, \ \underline{F}(x_i) \le P((x_i]_{[\underline{F},\overline{F}]}) \le \overline{F}(x_i)\}.$$

It induces coherent upper and lower probabilities such that $\underline{F}(x_i) = \underline{P}((x_i]_{[\underline{F},\overline{F}]})$ and $\overline{F}(x_i) = \overline{P}((x_i]_{[\underline{F},\overline{F}]})$. When $X = \mathbb{R}$ and $\le_{[\underline{F},\overline{F}]}$ is the natural ordering on numbers, then $\forall r \in \mathbb{R}$, $(r]_{[\underline{F},\overline{F}]} = (-\infty, r]$, and the above equation coincides with Equation (1).

In the following, sets $(x_i]_{[\underline{F},\overline{F}]}$ are denoted A_i, for all $i = 1, \ldots, n$. These sets are nested, since $\emptyset \subset A_1 \subseteq \ldots \subseteq A_n = X^6$. For all $i = 1, \ldots, n$, let $\underline{F}(x_i) = \alpha_i$ and $\overline{F}(x_i) = \beta_i$. With these conventions, the credal set $\mathcal{P}_{[\underline{F},\overline{F}]}$ can now be described by the following n constraints bearing on probabilities of nested sets A_i:

(7) $i = 1, \ldots, n \qquad \alpha_i \le P(A_i) \le \beta_i$

with $0 = \alpha_0 \le \alpha_1 \le \ldots \le \alpha_n = 1$, $0 = \beta_0 < \beta_1 \le \beta_2 \le \ldots \le \beta_n = 1$ and $\alpha_i \le \beta_i$.

As a consequence, a generalized p-box can be generated in two different ways:

- By starting from two comonotone functions $\underline{F} \le \overline{F}$ defined on X, the pre-order being induced by the values of these functions,
- or by assigning upper and lower bounds on probabilities of a prescribed collection of nested sets A_i.

Note that the second approach is likely to be more useful in practical assessments and elicitation of generalized p-boxes.

Example 1. *All along this section, we will use this example to illustrate results on generalized p-boxes. Let $X = \{x_1, \ldots, x_6\}$. These elements could be, for instance, the facets of a biased die. For various reasons, we only have incomplete information about the probability of some subsets $A_1 = \{x_1, x_2\}$, $A_2 = \{x_1, x_2, x_3\}$, $A_3 = \{x_1, \ldots, x_5\}$, or $X(= A_4)$. An expert supplies the following confidence bounds on the frequencies of these sets:*

$$P(A_1) \in [0, 0.3] \qquad P(A_2) \in [0.2, 0.7] \qquad P(A_3) \in [0.5, 0.9]$$

The uncertainty can be modeled by the generalized p-box pictured on Figure 2:

	x_1	x_2	x_3	x_4	x_5	x_6
$\overline{F}$	0.3	0.3	0.7	0.9	0.9	1
$\underline{F}$	0	0	0.2	0.5	0.5	1.

---

[6]Since $\le_{[\underline{F},\overline{F}]}$ is a complete pre-order on X, we can have $x_i =_{[\underline{F},\overline{F}]} x_{i+1}$ and $A_i = A_{i+1}$, which explains the non-strict inclusions. They would be strict if $<_{[\underline{F},\overline{F}]}$ were a linear order.
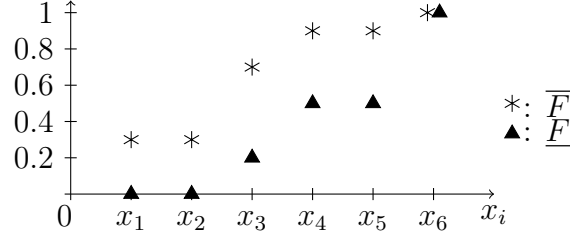
FIGURE 2. Generalized p-box $[\underline{F}, \overline{F}]$ of Example 1

3.2. Connecting generalized p-boxes with possibility distributions.

It is natural to search for a connection between generallized p-boxes and possibility theory, since possibility distributions can be interpreted as a collection of nested sets with associated lower bounds, while generalized p-boxes correspond to lower and upper bounds also given on a collection of nested sets. Given a generalized p-box $[\underline{F}, \overline{F}]$, the following proposition holds:

Proposition 2. *Any generalized p-box $[\underline{F}, \overline{F}]$ on X is representable by a pair of possibility distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$, such that $\mathcal{P}_{[\underline{F}, \overline{F}]} = \mathcal{P}_{\pi_{\overline{F}}} \cap \mathcal{P}_{\pi_{\underline{F}}}$, where:*

$$\pi_{\overline{F}}(x_i) = \beta_i \text{ and } \pi_{\underline{F}}(x_i) = 1 - \max\{\alpha_j | \alpha_j < \alpha_i; j = 0, \dots, i\}$$

for $i = 1, \dots, n$, with $\alpha_0 = 0$.

Proof of Proposition 2. Consider the set of constraints given by Equation (7) and defining the convex set $\mathcal{P}_{[\underline{F}, \overline{F}]}$. These constraints can be separated into two distinct sets: $(P(A_i) \leq \beta_i)_{i=1,n}$ and $(P(A_i^c) \leq 1 - \alpha_i)_{i=1,n}$. Now, rewrite constraints of Proposition 1, in the form $\forall \alpha \in (0, 1]: P \in \mathcal{P}_\pi$ if and only if $P(\{x \in X | \pi(x) \leq \alpha\}) \leq \alpha$.

The first set of constraints $(P(A_i) \leq \beta_i)_{i=1,n}$ defines a credal set $\mathcal{P}_{\pi_{\overline{F}}}$ that is induced by the possibility distribution $\pi_{\overline{F}}$, while the second set of constraints $(P(A_i^c) \leq 1 - \alpha_i)_{i=1,n}$ defines a credal set $\mathcal{P}_{\pi_{\underline{F}}}$ that is induced by the possibility distribution $\pi_{\underline{F}}$, since $A_i^c = \{x_k, \dots x_n\}$, where $k = \max\{j | \alpha_j < \alpha_i\}$. The credal set of the generalized p-box $[\underline{F}, \overline{F}]$, resulting from the two sets of constraints, namely $i = 1, \dots, n$, $\beta_i \leq P(A_i) \leq \alpha_i$, is thus $\mathcal{P}_{\pi_{\overline{F}}} \cap \mathcal{P}_{\pi_{\underline{F}}}$. □

Example 2. *The possibility distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$ for the generalized p-box defined in Example 1 are:*

	x_1	x_2	x_3	x_4	x_5	x_6
$\pi_{\overline{F}}$	0.3	0.3	0.7	0.9	0.9	1
$\pi_{\underline{F}}$	1	1	1	0.8	0.8	0.5

Note that, when $\underline{F}$ is injective, $<_{[\underline{F},\overline{F}]}$ is a linear order, and we have $\pi_{\underline{F}}(x_i) = 1 - \alpha_{i-1}$. So, generalized p-boxes allow to model uncertainty in terms of pairs of comonotone possibility distributions. In this case, contrary to the case of only one possibility distribution, the two bounds enclosing the probability of a particular event A can be tighter, i.e. no longer restricted to the form $[0, \alpha]$ or $[\beta, 1]$, but contained in the intersection of intervals of this form.

An interesting case is the one where, for all $i = 1, \ldots, n-1$, $\underline{F}(x_i) = 0$ and $\underline{F}(x_n) = 1$. Then, $\pi_{\underline{F}} = 1$ and $\mathcal{P}_{\pi_{\overline{F}}} \cap \mathcal{P}_{\pi_{\underline{F}}} = \mathcal{P}_{\pi_{\overline{F}}}$ and we retrieve the single distribution $\pi_{\overline{F}}$. We recover $\pi_{\underline{F}}$ if we take for all $i = 1, \ldots, n$, $\overline{F}(x_i) = 1$. This means that generalized p-boxes also generalize possibility distributions, and are representable by them in the sense of Definition 4.

3.3. **Connecting Generalized p-boxes and random sets.** We already mentioned that p-boxes are special cases of random sets, and the following proposition shows that it is also true for generalized p-boxes.

Proposition 3. *Generalized p-boxes are special cases of random sets, in the sense that for any generalized p-box $[\underline{F}, \overline{F}]$ on X, there exist a belief function Bel such that $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{Bel}$.*

In order to prove Proposition 3, we show that the lower probabilities induced by a generalized p-box and by the belief function given by Algorithm 1 coincide on every event. To do that, we use the partition of X induced by nested sets A_i, and compute lower probabilities of elements of this partition. We then check that the lower probabilities on all events induced by the generalized p-box coincide with the belief function induced by Algorithm 1. The detailed proof can be found in the appendix.

Algorithm 1 below provides an easy way to build the random set encoding a given generalized p-box. It is similar to existing algorithms [27, 38], and extends them to more general spaces. The main idea of the algorithm is to use the fact that a generalized p-box can be seen as a random set whose focal sets are obtained by thresholding the cumulative distributions (as in Figure 2). Since the sets A_i are nested, they induce a partition of X whose elements are of the form $G_i = A_i \setminus A_{i-1}$. The focal sets of the random set equivalent to the generalized p-box are made of unions of consecutive elements of this partition. Basically, the procedure comes down to progressing a threshold $\theta \in [0, 1]$. When $\alpha_{i+1} > \theta \geq \alpha_i$ and $\beta_{j+1} > \theta \geq \beta_j$, then, the corresponding focal set is $A_{i+1} \setminus A_j$, with mass

$$(8) \qquad m(A_{i+1} \setminus A_j) = \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j).$$

Algorithm 1: R-P-box $\to$ random set transformation

Input: Generalized p-box $[\underline{F}, \overline{F}]$ and corresponding nested sets
$\emptyset = A_0, A_1, \ldots, A_n = X$, lower bounds α_i and upper
bounds β_i

Output: Equivalent random set

for $i = 1, \ldots, n$ **do**
$\quad \lfloor$ Build partition $G_i = A_i \setminus A_{i-1}$

Build set
$\{\gamma_l | l = 1, \ldots, 2n - 1\} = \{\alpha_i | i = 1, \ldots, n\} \cup \{\beta_i | i = 1, \ldots, n\}$
With γ_l indexed such that
$\gamma_1 \leq \ldots \leq \gamma_l \leq \ldots \leq \gamma_{2n-1} = \beta_n = \alpha_n = 1$
Set $\alpha_0 = \beta_0 = \gamma_0 = 0$ and focal set $E_0 = \emptyset$

for $k = 1, \ldots, 2n - 1$ **do**
$\quad$ **if** $\gamma_{k-1} = \alpha_i$ **then**
$\quad\quad \lfloor$ $E_k = E_{k-1} \cup G_{i+1}$
$\quad$ **if** $\gamma_{k-1} = \beta_i$ **then**
$\quad\quad \lfloor$ $E_k = E_{k-1} \setminus G_i$
$\quad$ Set $m(E_k) = \gamma_k - \gamma_{k-1}$

We can also give another characterization of the random set (8): let
us note $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_M = 1$ the distinct values taken by $\underline{F}, \overline{F}$
over elements x_i of X (note that M is finite and $M < 2n$). Then, for
$j = 1, \ldots, M$, the random set defined as:

(9)
$$\begin{cases} E_j = \{x_i \in X | (\pi_{\overline{F}}(x_i) \geq \gamma_j) \wedge (1 - \pi_{\underline{F}}(x_i) < \gamma_j)\} \\ m(E_j) = \gamma_j - \gamma_{j-1} \end{cases}$$

is the same as the one built by using Algorithm 1, but this formulation
lays bare the link between Equation (6) and the possibility distributions
$\pi_{\overline{F}}, \pi_{\underline{F}}$.

Example 3. *This example illustrates the application of Algorithm 1,
by applying it to the generalized p-box given in Example 1. We have:*

$$G_1 = \{x_1, x_2\} \qquad G_2 = \{x_3\} \qquad G_3 = \{x_4, x_5\} \qquad G_4 = \{x_6\}$$

and

$$0 \leq 0 \leq 0.2 \leq 0.3 \leq 0.5 \leq 0.7 \leq 0.9 \leq 1$$
$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \alpha_3 \leq \beta_2 \leq \beta_3 \leq \alpha_4$$
$$\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4 \leq \gamma_5 \leq \gamma_6 \leq \gamma_7$$

which finally yields the following random set

$$m(E_1) = m(G_1) = 0 \qquad m(E_2) = m(G_1 \cup G_2) = 0.2$$
$$m(E_3) = m(G_1 \cup G_2 \cup G_3) = 0.1 \qquad m(E_4) = m(G_2 \cup G_3) = 0.2$$
$$m(E_5) = m(G_2 \cup G_3 \cup G_4) = 0.2 \qquad m(E_6) = m(G_3 \cup G_4) = 0.2$$
$$m(E_7) = m(G_4) = 0.1$$

This random set can then be used as an alternative representation of the provided information.

Propositions 3 and 2 together indicate that generalized p-boxes are more expressive than single possibility distributions and less expressive than random sets, but, as recalled before, less expressive (and, in this sense, simpler) models are often easier to handle in practice. The following explicit expression for lower probabilities induced by generalized p-boxes $[\underline{F}, \overline{F}]$ on X shows that we can expect it to be the case (see appendix):

(10) $$\underline{P}(\bigcup_{k=i}^{j} G_k) = \max(0, \alpha_j - \beta_{i-1}).$$

Let us call a subset E of X $[\underline{F}, \overline{F}]$-*connected* if it can expressed as an union of consecutive elements G_k, i.e. $E = \bigcup_{k=i}^{j} G_k$, with $0 < i < j \leq n$. For any event A, let $A_* = \bigcup_{E \subseteq A} E$, with E all maximal $[\underline{F}, \overline{F}]$-*connected* subsets included in A. We know (see appendix) that $\underline{P}(A) = \underline{P}(A_*)$. Then, the explicit expression for $\underline{P}(A)$ is $\underline{P}(A_*) = \sum_{E \subseteq A} \underline{P}(E)$, which remains quite simple to compute, and more efficient than computing the belief degree by checking which focal elements are included in A.

Notice that Equation (10) can be restated in terms of the two possibility distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$, rewriting $\underline{P}(E)$ as

$$\underline{P}(E) = \max(0, N_{\pi_{\underline{F}}}(\bigcup_{k=1}^{j} G_k) - \Pi_{\pi_{\overline{F}}}(\bigcup_{k=1}^{i-1} G_k)),$$

where $N_{\pi_i}(A), \Pi_{\pi_i}(A)$ are respectively the necessity and possibility degree of event A (given by Equations (3)) with respect to a distribution π_i. It makes $\underline{P}(A_*)$ even easier to compute.

3.4. **Probability intervals and generalized p-boxes.** As in the case of random sets, there is no direct relationship between probability intervals and generalized p-boxes. The two representations have comparable complexities, but do not involve the same kind of events. Nevertheless, given previous results, we can state how a probability

interval L can be transformed into a generalized p-box $[\underline{F}, \overline{F}]$, and vice-versa.

First consider a probability interval L and some indexing $\{x_1, \ldots, x_n\}$ of elements in X. A generalized p-box $[\underline{F}', \overline{F}']$ outer-approximating the probability interval L can be computed by means of Equations (2) as follows:

$$(11) \qquad \underline{F}'(x_i) = \underline{P}(A_i) = \alpha_i' = \max\left(\sum_{x_i \in A_i} l(x_i), 1 - \sum_{x_i \notin A_i} u(x_i)\right)$$

$$\overline{F}'(x_i) = \overline{P}(A_i) = \beta_i' = \min\left(\sum_{x_i \in A_i} u(x_i), 1 - \sum_{x_i \notin A_i} l(x_i)\right)$$

where $\underline{P}, \overline{P}$ are respectively the lower and upper probabilities of $\mathcal{P}_L$. Recall that $A_i = \{x_1, \ldots, x_i\}$. Note that each permutation of elements of X provide a different generalized p-box and that there is no tightest outer-approximation among them.

Next, consider a generalized p-box $[\underline{F}, \overline{F}]$ with nested sets $A_1 \subseteq \ldots \subseteq A_n$. The probability interval L' on elements x_i corresponding to $[\underline{F}, \overline{F}]$ is given by:

$$(12) \qquad\qquad \underline{P}(\{x_i\}) = l'(x_i) = \max(0, \alpha_i - \beta_{i-1})$$

$$\overline{P}(\{x_i\}) = u'(x_i) = \beta_i - \alpha_{i-1}$$

where $\underline{P}, \overline{P}$ are the lower and upper probabilities of the credal set $\mathcal{P}_{[\underline{F}, \overline{F}]}$ on elements of X, $\alpha_i = \overline{F}(A_i)$, $\beta_i = \underline{F}(A_i)$ and $\beta_0 = \alpha_0 = 0$. This is the tightest probability interval outer-approximating the generalized p-box, and there is only such set.

Of course, transforming a probability interval L into a p-box $[\underline{F}, \overline{F}]$ and vice-versa generally induces a loss of information. But we can show that probability intervals are representable (see definition 4) by generalized p-boxes: let Σ_σ the set of all possible permutations σ of elements of X, each defining a linear order. A generalized p-box according to permutation σ is denoted $[\underline{F}', \overline{F}']_\sigma$ and called a σ-p-box. We then have the following proposition (whose proof is in the appendix):

Proposition 4. *Let L be a probability interval, and let $[\underline{F}', \overline{F}']_\sigma$ be the σ-p-box obtained from L by applying Equations (11). Moreover, let L''_σ denote the probability interval obtained from the σ-p-box $[\underline{F}', \overline{F}']_\sigma$ by applying Equations (12). Then, the various credal sets thus defined satisfy the following property:*

$$(13) \qquad\qquad \mathcal{P}_L = \bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{[\underline{F}', \overline{F}']_\sigma} = \bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{L''_\sigma}$$
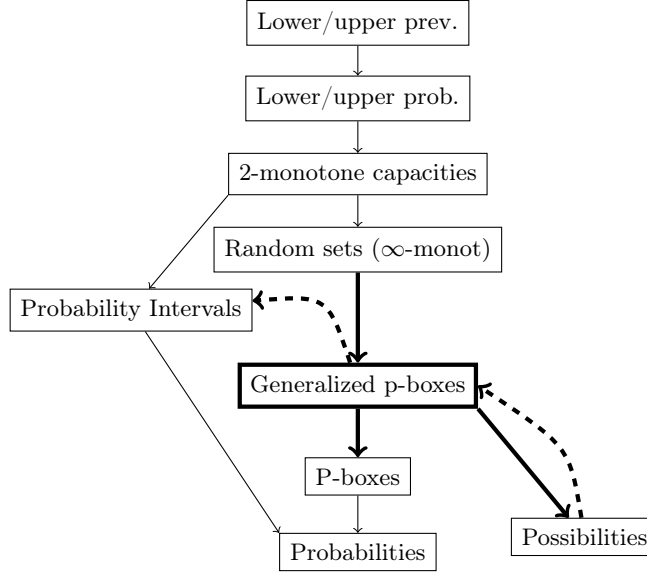
FIGURE 3. Representation relationships: summary with generalized p-boxes. $A \longrightarrow B$: A generalizes B. $A \dashrightarrow B$: B is representable by A

This means that the information modeled by a set L of probability intervals can be entirely recovered by considering sets of σ-p-boxes. Note that not all $|X|!$ such permutations need to be considered, and that in practice, L can be exactly recovered by means of a reduced set $\mathcal{S}$ of $|X|/2$ permutations, provided that $\{x_{\sigma(1)}, \sigma \in \mathcal{S}\} \cup \{x_{\sigma(n)}, \sigma \in \mathcal{S}\} = X$. Since $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{\pi_{\underline{F}}} \cap \mathcal{P}_{\pi_{\overline{F}}}$, then it is immediate from Proposition 4, that, in terms of credal sets, $\mathcal{P}_L = \bigcap_{\sigma \in \Sigma_\sigma} \left(\mathcal{P}_{\pi_{\underline{F}_\sigma}} \cap \mathcal{P}_{\pi_{\overline{F}_\sigma}} \right)$, where $\pi_{\underline{F}_\sigma}, \pi_{\overline{F}_\sigma}$ are respectively the possibility distributions corresponding to $\underline{F}_\sigma$ and $\overline{F}_\sigma$.

4. CONCLUSION

This paper introduces a generalized notion of p-box. Such a generalization allows to define p-boxes on finite (pre)-ordered spaces as well as discretized p-boxes on multi-dimensional spaces equipped with an arbitrary (pre)-order. On the real line, this preorder can be of the form $x \leq_\rho y$ if and only if $|x - \rho| \leq |y - \rho|$, thus accounting for events of the form "close to a prescribed value ρ". Generalized p-boxes are representable by a pair of comonotone possibility distributions. They are special case of random sets, and the corresponding mass assignment has been laid bare. Generalized p-boxes are thus more expressive than single possibility distributions and likely to be more tractable than general

random sets. Moreover, the fact that they can be interpreted as lower and upper confidence bounds over nested sets makes them quite attractive tools for subjective elicitation. Finally, we showed the relation existing between generalized p-boxes and sets of probability intervals. Figure 3 summarizes the results of this paper, by placing generalized p-boxes inside the graph of Figure 1. New relationships and representations obtained in this paper are in bold lines. Computational aspects of calculations with generalized p-boxes need to be explored in greater detail (as is done by De Campos *et al.* [9] for probability intervals) as well as their application to the elicitation of imprecise probabilities. Another issue is to extend presented results to more general spaces, to general lower/upper previsions or to cases not considered here (e.g. continuous p-boxes with discontinuity points), possibly using existing results [42, 11]. Interestingly, the key condition when representing generalized p-boxes by two possibility distributions is their comonotonicity. In the second part of this paper, we pursue the present study by dropping this assumption. We then recover so-called clouds, recently proposed by Neumaier [34].

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Appendix

Proof of Proposition 3. From the nested sets $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n = X$ we can build a partition s.t. $G_1 = A_1, G_2 = A_2 \setminus A_1, \ldots, G_n = A_n \setminus A_{n-1}$. Once we have a finite partition, every possible set $B \subseteq X$ can be approximated from above and from below by pairs of sets $B_* \subseteq B^*$ [36]:

$$B^* = \bigcup \{G_i, G_i \cap B \neq \emptyset\}; \quad B_* = \bigcup \{G_i, G_i \subseteq B\}$$

made of a finite union of the partition elements intersecting or contained in this set B. Then $\underline{P}(B) = \underline{P}(B_*), \overline{P}(B) = \overline{P}(B^*)$, so we only have to care about unions of elements G_i in the sequel. Especially, for each event $B \subset G_i$ for some i, it is clear that $\underline{P}(B) = 0 = Bel(B)$ and $\overline{P}(B) = \overline{P}(G_i) = Pl(B)$. So, to prove Proposition 3, we have to show that lower probabilities given by a generalized p-box $[\underline{F}, \overline{F}]$ and by the corresponding random set built through algorithm 1 coincide on

unions of elements G_i. We will first concentrate on unions of conscutive elements G_i, and then to any union of such elements.

Let us first consider union of consecutive elements $\bigcup_{k=i}^{j} G_k$ (when $k = 1$, we retrieve the sets A_j). Finding $\underline{P}(\bigcup_{k=i}^{j} G_k)$ is equivalent to computing the minimum of $\sum_{k=i}^{j} P(G_k)$ under the constraints

$$i = 1, \ldots, n \qquad \alpha_i \leq P(A_i) = \sum_{k=1}^{i} P(G_k) \leq \beta_i$$

which reads

$$\alpha_j \leq P(A_{i-1}) + \sum_{k=i}^{j} P(G_k) \leq \beta_j$$

so $\sum_{k=i}^{j} P(G_k) \geq \max(0, \alpha_j - \beta_{i-1})$. This lower bound is optimal, since it is always reachable: if $\alpha_j > \beta_{i-1}$, take P s.t. $P(A_{i-1}) = \beta_{i-1}$, $P(\bigcup_{k=i}^{j} G_k) = \alpha_j - \beta_{i-1}$, $P(\bigcup_{k=j+1}^{n} G_k) = 1 - \alpha_j$. If $\alpha_j \leq \beta_{i-1}$, take P s.t. $P(A_{i-1}) = \beta_{i-1}$, $P(\bigcup_{k=i}^{j} G_k) = 0$, $P(\bigcup_{k=j+1}^{n} E_k) = 1 - \beta_{i-1}$. And we can see, by looking at Algorithm 1, that $Bel(\bigcup_{k=i}^{j} G_k) = \max(0, \alpha_j - \beta_{i-1})$: focal elements of Bel are subsets of $\bigcup_{k=i}^{j} G_k$ if $\beta_{i-1} < \alpha_j$ only.

Now, let us consider a union A of non-consecutive elements s.t. $A = (\bigcup_{k=i}^{i+l} G_k \cup \bigcup_{k=i+l+m}^{j} G_k)$ with $m > 1$. As in the previous case, we must compute $\min \left(\sum_{k=i}^{i+l} P(G_k) + \sum_{k=i+l+m}^{j} P(G_k) \right)$ to find the lower probability on $\underline{P}(A)$. An obvious lower bound is given by

$$\min \left(\sum_{k=i}^{i+l} P(G_k) + \sum_{k=i+l+m}^{j} P(G_k) \right) \geq \min \left(\sum_{k=i}^{i+l} P(G_k) \right) + \min \left(\sum_{k=i+l+m}^{j} P(G_k) \right)$$

and this lower bound is equal to

$$\max(0, \alpha_{i+l} - \beta_{i-1}) + \max(0, \alpha_j - \beta_{i+l+m-1}) = Bel(A)$$

Consider the two following cases with probabilistic mass assignments showing that bounds are attained:

- $\alpha_{i+l} < \beta_{i-1}$, $\alpha_j < \beta_{i+l+m-1}$ and probability masses:
  $P(A_{i-1}) = \beta_{i-1}$, $P(\bigcup_{k=i}^{i+l} G_k) = \alpha_{i+l} - \beta_{i-1}$, $P(\bigcup_{k=i+l+1}^{i+l+m-1} G_k) = \beta_{i+l+m-1} - \alpha_{i+l}$, $P(\bigcup_{k=i+l+m}^{j} G_k) = \alpha_j - \beta_{i+l+m-1}$ and $P(\bigcup_{k=j+1}^{n} G_k) = 1 - \alpha_j$.
- $\alpha_{i+l} > \beta_{i-1}$, $\alpha_j > \beta_{i+l+m-1}$ and probability masses:
  $P(A_{i-1}) = \beta_{i-1}$, $P(\bigcup_{k=i}^{i+l} G_k) = 0$, $P(\bigcup_{k=i+l+1}^{i+l+m-1} G_k) = \alpha_j - \beta_{i-1}$, $P(\bigcup_{k=i+l+m}^{j} E_k) = 0$ and $P(\bigcup_{k=j+1}^{n} G_k) = 1 - \alpha_j$.

The same line of thought can be followed for the two remaining cases. As in the consecutive case, the lower bound is reachable without violating any of the restrictions associated to the generalized p-box. We

have $\underline{P}(A) = Bel(A)$ and the extension of this result to any number n of "discontinuities" in the sequence of G_k is straightforward. The proof is complete, since for every possible union A of elements G_k, we have $\underline{P}(A) = Bel(A)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of Proposition 4. To prove this proposition, we must first recall a result given by De Campos *et al.* [9]: given two probability intervals L and L' defined on a space X and the induced credal sets $\mathcal{P}_L$ and $\mathcal{P}_{L'}$, the conjunction $\mathcal{P}_{L\cap L'} = \mathcal{P}_L \cap \mathcal{P}_{L'}$ of these two sets can be modeled by the set $(L \cap L')$ of probability intervals that is such that for every element x of X,

$$l_{(L\cap L')}(x) = \max(l_L(x), l_{L'}(x)) \text{ and } u_{(L\cap L')}(x) = \min(u_L(x), u_{L'}(x))$$

and these formulas extend directly to the conjunction of any number of probability intervals on X.

To prove Proposition 4, we will show, by using the above conjunction, that $\mathcal{P}_L = \bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{L''_\sigma}$. Note that we have, for any $\sigma \in \Sigma_\sigma$, $\mathcal{P}_L \subset \mathcal{P}_{[\underline{F'},\overline{F'}]_\sigma} \subset \mathcal{P}_{L''_\sigma}$, thus showing this equality is sufficient to prove the whole proposition.

Let us note that the above inclusion relationships alone ensure us that
$\mathcal{P}_L \subseteq \bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{[\underline{F'},\overline{F'}]_\sigma} \subseteq \bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{L''_\sigma}$. So, all we have to show is that the inclusion relationship is in fact an equality.

Since we know that both $\mathcal{P}_L$ and $\bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{L''_\sigma}$ can be modeled by probability intervals, we will show that the lower bounds l on every element x in these two sets coincide (and the proof for upper bounds is similar).

For all x in X, $l_{L''_\Sigma}(x) = \max_{\sigma \in \Sigma_\sigma}\{l_{L''_\sigma}(x)\}$, with L''_Σ the probability interval corresponding to $\bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{L''_\sigma}$ and L''_σ the probability interval corresponding to a particular permutation σ. We must now show that, for all x in X, $l_{L''_\Sigma}(x) = l_L(x)$.

Given a ranking of elements of X, and by applying successively Equations (11) and (12), we can express the differences between bounds $l''(x_i)$ of the set L'' and $l(x_i)$ of the set L in terms of set of bounds L. This gives, for any $x_i \in X$:

(14)
$$l(x_i) - l''(x_i) = \min(l(x_i), 0 + \sum_{x_i \in A_{i-1}} (u(x_i) - l(x_i)),$$

$$0 + \sum_{x_i \in A_i^c} (u(x_i) - l(x_i)), (l(x_i) + \sum_{\substack{x_j \neq x_i \\ x_j \in X}} u(x_j)) - 1, 1 - \sum_{x_i \in X} l(x_i))$$

We already know that, for any permutation σ and for all x in X, we have $l_{L''_\sigma}(x) \leq l_L(x)$. So we must now show that, for a given x in X, there is one permutation σ such that $l_{L''_\sigma}(x) = l_L(x)$. Let us consider the permutation placing the given element at the front. If x is the first element $x_{\sigma(1)}$, then Equation (14) has value 0 for this element, and we thus have $l_{L''_\sigma}(x) = l_L(x)$. Since if we consider every possible ranking, every element x of X will be first in at least one of these rankings, this completes the proof. $\qquad\square$

Institut de Radioprotection et Sûreté nucléaire, Bât 720, 13115 St-Paul lez Durance, FRANCE
  E-mail address: sdestercke@gmail.com

Université Paul Sabatier, IRIT, 118 Route de Narbonne, 31062 Toulouse
  E-mail address: dubois@irit.fr

Institut de Radioprotection et Sûreté nucléaire, Bât 720, 13115 St-Paul lez Durance, FRANCE
  E-mail address: eric.chojnacki@irsn.fr