One-parameter family of Clairaut-Liouville metrics with application to optimal control*

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Abstract

Riemannian metrics with singularities are considered on the 2-sphere of revolution. The analysis of such singularities is motivated by examples stemming from mechanics and related to projections of higher dimensional (regular) sub-Riemannian distributions. An unfolding of the metrics in the form of an homotopy from the canonical metric on $\mathbf{S}^2$ is defined which allows to analyze the singular case as a limit of standard Riemannian ones. A bifurcation of the conjugate locus for points on the singularity is finally exhibited.

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MSC classification. 49K15, 53C20, 70Q05

Introduction

We call Clairaut-Liouville metrics on the 2-sphere of revolution, that is Riemannian metrics of the form

$$g = G(\varphi)d\theta^2 + d\varphi^2$$

where θ and φ are the standard angular coordinates on $\mathbf{S}^2$, G being such that $G(\pi - \varphi) = G(\varphi)$. We are interested in such metrics with singularities, meaning that G becomes infinite, for instance at the equator, $\varphi = \pi/2$. Such singularities can be obtained by considering projections of higher dimensional (constant rank) sub-Riemannian distributions, as illustrated by examples arising from space or quantum mechanics discussed in §1. Using an unfolding which consists in an homotopy from the restriction of the flat case on the sphere (that is, algebraically,

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from the unit—see §2), the analysis can be performed by means of techniques borrowed from standard Riemannian geometry. Because of singularities, the classical Riemannian invariants do not necessarily give a full account of the system. The curvature may for instance be negative wherever it is defined, but conjugate points exist however. The complete information may in some sense be retrieved thanks to the unfolding, though a bifurcation in the conjugate locus may occur as illustrated by §3 computations.

1 Motivating examples

As we shall see, singularities of the metric are naturally classified according to the order of the pole of an underlying meromorphic function. We begin by recalling a first example originating in space mechanics which turns to provide the simplest kind of singularity we will consider, namely a first order one.

The two-input coplanar orbit transfer system [7] is modelled by a periodic sub-Riemannian distribution on a three dimensional space manifold, X,

$$\frac{dx}{dl} = u_1 F_1(l,x) + u_2 F_2(l,x),$$

geometric coordinates for the state x being for instance $x = (n, e, \theta)$, where n is the *mean motion*, e the *eccentricity*, and θ the *argument of pericenter*. Such coordinates account for the geometry of the osculating ellipse at each value of the *longitude*, l, which measures the angular length of the transfer. The two vector fields F_1 and F_2 are periodically parameterized by the longitude and define functions from $\mathbf{S}^1 \times X$ to TX ($F_i(l,x) \in T_x X$, $i = 1,2$). When considering energy minimization, that is L^2-norm minimization of the control, only normal extremals have to be considered (the structure of the Lie algebra defined by F_1 and F_2 prevents the system from having abnormal extremals), and the corresponding (maximized) Hamiltonian

$$H(l,x,p) = \frac{1}{2}(H_1^2 + H_2^2)(l,x,p)$$

is written using the Hamiltonian lifts $H_i = \langle p, F_i(l,x)\rangle$ of the two fields. The long time behaviour of the system is approximated by the new system associated with the averaged Hamiltonian,

$$\overline{H}(x,p) = \frac{1}{2\pi}\int_0^{2\pi} H(l,x,p)dl,$$

and trajectories of both systems are ε-close for the $\mathscr{C}^0$-topology on intervals of length $1/\varepsilon$. As averaging with respect to the periodic variable generates brackets of the initial vector fields, the averaged Hamiltonian turns to be a full rank quadratic form in the adjoint variable, p, thus associated with a Riemannian metric in three dimensions. We recall its expression [4]:

$$\overline{g} = \frac{1}{9n^{1/3}}dn^2 + \frac{2n^{5/3}}{5(1-e^2)}de^2 + \frac{2n^{5/3}}{5-4e^2}e^2 d\theta^2.$$

A normal form of the metric is obtained [3] by setting

$$n = (5r/2)^{6/5}, \quad e = \sin\varphi,$$

so that

$$\bar{g} = dr^2 + (r^2/c^2)g,$$

where $c = \sqrt{2/5}$ and $g = G(\varphi)d\theta^2 + d\varphi^2$ is a Clairaut-Liouville metric on $\mathbf{S}^2$ with

$$G(\varphi) = \frac{\sin^2\varphi}{1 - (4/5)\sin^2\varphi}\cdot$$

As explained in the previous reference [3], the analysis is essentially reduced to studying the Clairaut-Liouville metric, which is in fact related to the following unfolding. Consider the simplest rational fraction with an order one pole in $X = 1$,

$$R = \frac{1}{1-X},$$

and set

$$R_\lambda(X) = R(\lambda X), \quad g_\lambda = (XR_\lambda \circ \sin^2)(\varphi)d\theta^2 + d\varphi^2$$

for λ in $[0, 1]$. The algebraic homotopy from $R_0 = 1$ to $R = R_1 = 1/(1 - X)$ defines an unfolding of the Clairaut-Liouville metric with singularity at the equator ($\varphi = \pi/2$)

$$g_1 = \tan^2\varphi d\theta^2 + d\varphi^2, \tag{1}$$

with two remarkable values: $g_0 = \sin^2\varphi d\theta^2 + d\varphi^2$ is the restriction of the flat metric on $\mathbf{S}^2$, while the averaged metric of the two-input coplanar transfer is obtained for $\lambda = 4/5$.

It is worth noticing that the metric (1) also appears in quantum mechanics, see [9] (*Grušin model* on $\mathbf{S}^2$).

Since $1 - \sin^2(\varphi) \sim (\pi/2 - \varphi)^2$ when φ tends to $\pi/2$, a local model for the singularity at the equator is

$$ds^2 = dx^2 + \frac{dy^2}{x^2}\cdot$$

This metric is actually obtained by projecting the Heisenberg sub-Riemmanian distribution (see also [2] where the concept of *almost-Riemannian* structure is introduced). This distribution is indeed defined, up to a renormalization, by the following two vector fields on $\mathbf{R}^3$ (see [8]):

$$F_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad F_2 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial z},$$

and the corresponding sub-Riemannian Hamiltonian is

$$H = \frac{1}{2}\left[(p_x^2 + p_y^2) + 2p_z(xp_y - yp_x) + (x^2 + y^2)p_z^2\right],$$

which strongly suggests to use cylindrical coordinates. In these variables,

$$H = \frac{1}{2}\left[p_r^2 + (p_\theta/r + rp_z)^2\right].$$

As θ and z are cyclic, the system is integrable in dimension three, and projects onto a Hamiltonian in the (r, z)-space with the desired singularity,

$$H = \frac{1}{2}(p_r^2 + r^2 p_z^2),$$

when restricting to $p_\theta = 0$.

If we now come back to the original bi-entry system modelling a coplanar orbit transfer and fix the direction of the control, we obtain a single-input periodic sub-Riemannian,

$$\frac{dx}{dl} = uF(l, x).$$

As before, averaging can be used and in the so-called *tangential case* where the control has to be directed by the Cartesian velocity, we obtain again a full rank averaged Hamiltonian. A remarkable feature of the system is that the same geometric coordinates remain orthogonal for the new metric which writes [5]

$$\overline{g}_t = \frac{1}{9n^{1/3}}dn^2 + n^{5/3}\left[\frac{1+\sqrt{1-e^2}}{4(1-e^2)^{3/2}}de^2 + \frac{1+\sqrt{1-e^2}}{4(1-e^2)}e^2 d\theta^2\right].$$

In the two-input case, the change of variables $e = \sin\varphi$ only consisted in lifting the Poincaré disk—on which (e, θ) are polar coordinates—onto $\mathbf{S}^2$ where (θ, φ) are the standard angles. We set again $n = (5r/2)^{6/5}$ and slightly twist the previous lifting according to

$$e = \sin\varphi\sqrt{1+\cos^2\varphi}$$

to obtain the preceding normal form anew:

$$\overline{g}_t = dr^2 + (r^2/c_t^2)g_t,$$

with $c_t = c^2 = 2/5$ (note the reduction of the constant, measuring the loss of convexity—connected to existence issues—compared to the two-input case), $g_t = G_t(\varphi)d\theta^2 + d\varphi^2$, and

$$G_t(\varphi) = \sin^2\varphi\left(\frac{1-(1/2)\sin^2\varphi}{1-\sin^2\varphi}\right)^2.$$

Hence, $g_t = (XR \circ \sin^2)(\varphi)d\theta^2 + d\varphi^2$, where now

$$\begin{aligned} R &= \left(\frac{1-X/2}{1-X}\right)^2, \\ &= \frac{1}{4}\left(1 + \frac{2}{1-X} + \frac{1}{(1-X)^2}\right), \end{aligned}$$

and the singularity at the equator is associated with a pole of order two.

The local model is

$$ds^2 = dx^2 + \frac{dy^2}{x^4},$$

which is connected to the flat Martinet sub-Riemannian distribution. Consider indeed the two vector fields on $\mathbf{R}^3$ (see [1])

$$F_1 = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial z}, \quad F_2 = \frac{\partial}{\partial y}.$$

The sub-Riemannian Hamiltonian is

$$H = \frac{1}{2}\left[p_y^2 + (p_x + y^2 p_z)^2\right].$$

The two coordinates x and z are cyclic, and the Hamiltonian projects onto $H = (1/2)(p_y^2 + y^4 p_z^2)$ in the (y,z)-space when restricting to $p_x = 0$, providing the higher order singularity.

2 Unfolding of Clairaut-Liouville metrics

Let R be a meromorphic function not zero at the origin. We consider the Clairaut-Liouville metric

$$g = (XR \circ \sin^2)(\varphi) d\theta^2 + d\varphi^2$$

and define the following unfolding.

For λ in $[0,1]$, $R_\lambda(X) = R(\lambda X)$ is an homotopy between the unit polynomial (up to a renormalization not changing the metric, we can assume $R(0) = 1$) and R. Then, setting

$$g_\lambda = (XR_\lambda \circ \sin^2)(\varphi) d\theta^2 + d\varphi^2, \tag{2}$$

one connects the restriction of the flat metric on $\mathbf{S}^2$ to $g = g_1$. For λ small enough, the poles of R_λ lie outside $[0,1]$ and g_λ is a Riemannian metric (without singularity).

The Gauss curvature of a Clairaut-Liouville metric, $ds^2 = G(\varphi)d\theta^2 + d\varphi^2$, is

$$K = -\frac{1}{\sqrt{G}}(\sqrt{G})''.$$

The following is clear.

Lemma 2.1. *Let R be a meromorphic function not zero at the origin. The Gauss curvature of the Clairaut-Liouville metric defined by (2) is $K = Q \circ \sin^2$ where Q is a meromorphic function homogeneously depending on R,*

$$Q = 1 + (4X - 3)(R'/R) + X(1-X)(R'/R)^2 - 2X(1-X)(R''/R).$$

In particular, the curvature of the unfolding of g is given by

$$Q_\lambda = 1 + \lambda(4X-3)(R'/R) + \lambda^2 X(1-X)(R'/R)^2 - 2\lambda^2 X(1-X)(R''/R)$$

and we have the following estimates.

Proposition 2.1. *Let R be a meromorphic function not zero at the origin. Then*

$$Q_\lambda \sim 1 + \lambda(4X-3)(R'/R)(0) + O(\lambda^2) \quad \text{as} \quad \lambda \to 0.$$

Moreover, if $X = 1$ is a pole of finite order of R,

$$\begin{aligned} Q &\sim -\frac{p(p+1)}{1-X} \quad as \quad X \to 1, \\ Q_\lambda(1) &\sim \frac{p}{1-\lambda} \quad as \quad \lambda \to 1, \end{aligned}$$

where p is the order of the pole.

Thus, if the metric has a singularity at the equator defined by a pole of finite order, the order of the resulting singularity on the curvature is an invariant. Indeed,

$$K \sim -\frac{p(p+1)}{1-\sin^2\varphi} \sim -\frac{p(p+1)}{(\pi/2-\varphi)^2} \quad \text{as} \quad \varphi \to \pi/2$$

so the order is two, and K tends to $-\infty$ when φ tends to $\pi/2$. Under the same finiteness assumption, the curvature K_λ verifies

$$K_\lambda(\pi/2) \sim \frac{p}{1-\lambda} \to \infty \quad \text{as} \quad \lambda \to 1-$$

and the unfolding reveals the phenomenon of concentration of positive curvature at the equator, connected with the existence of conjugate points (see next section).

Remark 2.1. The finiteness assumption is essential in the previous result. If one considers $R = \exp(1/(1-X)^{2p})$, $p \geq 1$, which has an infinite order pole at $X = 1$, one readily gets

$$\begin{aligned} Q &\sim -\frac{p^2}{(1-X)^{2p+1}} \quad \text{as} \quad X \to 1, \\ Q_\lambda(1) &\sim \frac{p}{(1-\lambda)^{p+1}} \quad \text{as} \quad \lambda \to 1. \end{aligned}$$

The case of a first order pole (first example of §1) provides an alternative interpretation of the unfolding as a collapsing of the 2-sphere of revolution. Let g be the Clairaut-Liouville metric defined by $R = 1/(1-X)$, and note that for $0 \leq \lambda < 1$, $g_\lambda = (XR_\lambda \circ \sin^2)(\varphi)d\theta^2 + d\varphi^2$ is conformal to the restriction of the flat three-dimensional metric to an oblate ellipsoid of revolution with unit semi-major axis and semi-minor axis equal to $\mu = \sqrt{1-\lambda}$:

$$\begin{aligned} g_\lambda &= \frac{\sin^2\varphi}{1-\lambda\sin^2\varphi}d\theta^2 + d\varphi^2, \qquad (3) \\ &= (1-\lambda\sin^2\varphi)^{-1}g_{e,\sqrt{1-\lambda}} \qquad (4) \end{aligned}$$

where

$$g_{e,\mu} = \sin^2\varphi d\theta^2 + (\mu^2 + (1-\mu^2)\cos^2\varphi)d\varphi^2$$

is the announced canonical metric on the ellipsoid. Then, as λ goes to one, the semi-minor axis μ goes to zero, and we obtain the flat metric on a collapsed, flat ellipsoid, that is the flat metric on a two-sided Poincaré disk, $\mathbf{D}$. The unfolding thus connects the restriction of the flat metric on $\mathbf{S}^2$ to a metric conformal to

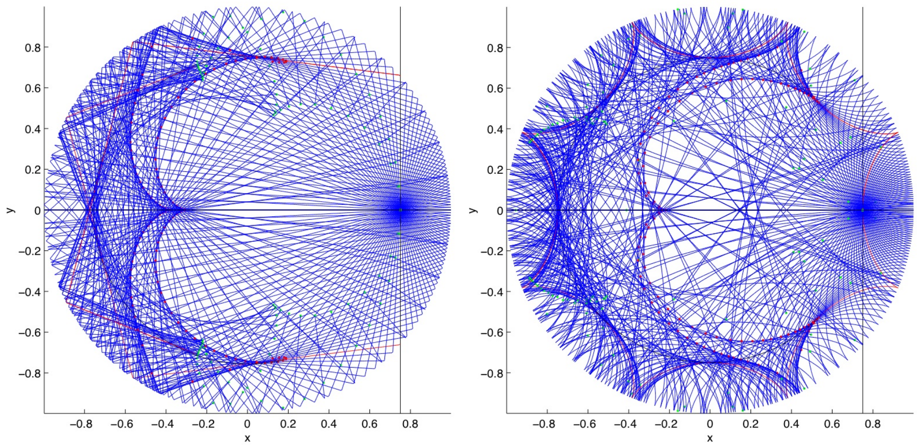

Figure 1: On the left, flat metric on the Poincaré disk. The conjugate locus formed by the enveloppe of geodesics reflecting on the boundary is also the astroid locus of the ellipsoid, the fourth cusp being implicitly on the other side of the disk. On the right, the picture for the same initial condition of the Clairaut-Liouville metric defined by $R = 1/(1-X)$. As for the Poincaré metric, contacts with the boundary are orthogonal.

the restriction of the flat metric to $\mathbf{D}$. Crossing the singularity at the equator is interpretated as going from one side of the disk to another, which can also be seen as generating reflections of the geodesics on its boundary. In particular, the conjugate locus for $\lambda = 1$ discussed in §3 is obtained on $\mathbf{D}$ as the enveloppe of geodesics reflecting on the boundary, that is as a caustic formed by the enveloppe of reflecting rays in the standard analogy from optics (see Figs. 1 and 2).

We finally obtain three conformal metrics on the Poincaré disk: The flat one, as the limit of the metric on the ellipsoid when the semi-minor axis vanishes,

$$g_{e,0} = dx^2 + dy^2$$

in Cartesian coordinates ($x+iy = \sin\varphi e^{i\theta}$), which has zero curvature, the metric g itself,

$$g = \frac{dx^2 + dy^2}{1-(x^2+y^2)}$$

which has negative curvature $K = -2/(1-(x^2+y^2))$ (going to $-\infty$ when coming to the boundary of the disk), and the usual Poincaré metric on the disk,

$$g_P = \frac{dx^2 + dy^2}{(1-(x^2+y^2))^2}$$

with constant negative curvature, $K = -1$. As for the Poincaré metric, the contact of geodesics of g with the boundary of the disk is orthogonal. This is actually the case for any metric $ds^2 = (dx^2+dy^2)/(1-(x^2+y^2))^p$, $p \geq 1$, since $ds^2 = (d\rho^2 + \rho^2 d\theta^2)/(1-\rho^2)^p$ in polar coordinates, so the associated Hamiltonian is

$$H = \frac{1}{2}(1-\rho^2)^p \left(p_\rho^2 + p_\theta^2/\rho^2\right).$$

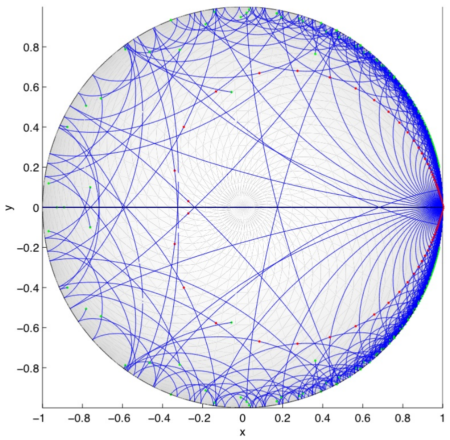

Figure 2: Geodesics of the Clairaut-Liouville metric defined by $R = 1/(1-X)$ for an initial condition on the boundary. See §3 for an analysis of the contact of the conjugate locus.

Restricting to $H = 1/2$, $p_\rho \sim 1/(1-\rho^2)^{p/2}$ when ρ tends to $1-$. If we set $q = x+iy = \rho e^{i\theta}$ and $d\tau = (1-\rho^2)^{p/2}dt$,

$$\begin{aligned} \frac{dq}{d\tau} &= (1-\rho^2)^{p/2}(p_\rho + ip_\theta/\rho)e^{i\theta}, \\ &\sim e^{i\theta} \quad \text{when} \quad \rho \to 1- \end{aligned}$$

and the contact with $\partial\mathbf{D}$ is indeed orthogonal.

3 Cut and conjugate loci

We finally address the question of optimality of Clairaut-Liouville metrics defined by a meromorphic function R not zero at the origin according to

$$g = (XR \circ \sin^2)(\varphi)d\theta^2 + d\varphi^2.$$

The analysis amounts to computing the cut and conjugate loci, and a first result in this direction (which applies to the two examples of §1) is provided by combining Proposition 2.1 with the sufficient conditions of [10].

Proposition 3.1. *Let R be a meromorphic function not zero at the origin. Consider the unfolding $g_\lambda = (XR_\lambda \circ \sin^2)(\varphi)d\theta^2 + d\varphi^2$ of the Clairaut-Liouville metric defined by R, with $R_\lambda(X) = R(\lambda X)$. If $R'(0)$ is positive, for small enough nonnegative λ the cut locus of a point distinct from the poles is a subarc of the antipodal parallel of the point.*

Proof. Since $Q'_\lambda \sim 4\lambda(R'/R)(0) + O(\lambda^2)$, the curvature is monotone non-decreasing along half-meridians and the result follows from [10] main theorem. ☐

In the presence of a pole, however, there is no hope for the unfolding curvature to be monotone in the neighbourhood of $\lambda = 1$ as K tends to $-\infty$ when φ tends to $\pi/2$ whereas $K_\lambda(\pi/2)$ goes to $+\infty$ (case of an equatorial singularity, see

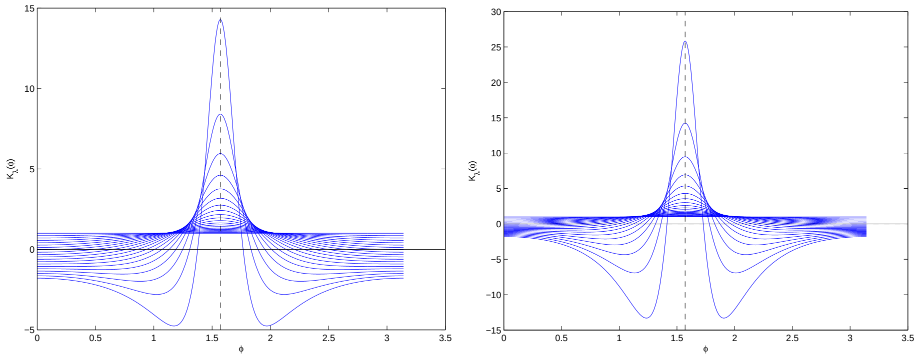

Figure 3: Curvature K_λ of the unfolding associated (i) to $R = 1/(1-X)$ (two-input orbit transfer of §1), (ii) to $R = (1/4)(1 + 2/(1-X) + 1/(1-X)^2)$ (single-input tangential orbit transfer of §1).

Fig. 3). Nevertheless, refined conditions are available that still ensure similar properties, holding for instance in the case of an order one pole.

Proposition 3.2. *[6] Consider the unfolding $g_\lambda = (XR_\lambda \circ \sin^2)(\varphi)d\theta^2 + d\varphi^2$ of the Clairaut-Liouville metric defined by $R = 1/(1-X)$. For $0 \leq \lambda < 1$, for each point distinct from the poles, the cut locus is a subarc of the antipodal parallel of the point and the conjugate locus has exactly four cusps.*

The limit case $\lambda = 1$ cannot be included in the analysis, at least regarding the conjugate locus as we shall see now.

Consider a meromorphic function with a pole of finite order, $p \geq 1$, at $X = 1$. The associated Clairaut-Liouville metric has a singularity of order $2p$ at the equator, and a local model is

$$ds^2 = dx^2 + \frac{dy^2}{x^{2p}}\cdot$$

The case $p = 1$ corresponds to the Heisenberg distribution, the case $p = 2$ to the flat Martinet one. We restrict the computation to geodesics issuing from the origin on the level set $H = 1/2$ of the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + x^{2p}p_y^2)$$

so that the initial adjoint state belongs to the union of the two lines $p_x = \pm 1$. We set $\lambda = p_y$ and restrict to positive λ by symmetry (the trivial geodesics $(\pm t, 0)$ being obtained for $\lambda = 0$). The coordinate x is then

$$x = \frac{1}{\sqrt[p]{\lambda}}q(t\sqrt[p]{\lambda}),$$

where q is defined through the quadrature

$$q^{-1} = \int_0^u \frac{dv}{\sqrt[p]{1 - v^{2p}}}\cdot$$

For $p = 1$, q is harmonic, elliptic for $p = 2$, and reciprocal to an hypergeometric function in general. More precisely,

$$q^{-1}(u) = u \cdot {}_2F_1(1/2, 1/(2p); 1 + 1/(2p); u^{2p})$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric series

$${}_2F_1(a, b; c; z) = \sum_{n \geq 0} \alpha_n \frac{z^n}{n!}\,,$$

and

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_{n+1}/\alpha_n &= (n + a)(n + b)/(n + c). \end{aligned}$$

The reciprocal of q is hence equal to

$$q^{-1}(u) = \sum_{n \geq 0} \alpha_n \frac{u^{2np+1}}{n!}$$

with $\alpha_n = (a)_n(b)_n/(c)_n$ for $a = 1/2$, $b = 1/(2p)$ and $c = 1 + 1/(2p)$, the notation $(a)_n$ standing for the Pochhammer symbol

$$(a)_n = a(a + 1) \cdots (a + n - 1) = \Gamma(a + n)/\Gamma(a).$$

Here,

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_1 &= \frac{1}{2}\frac{1}{2p+1}\,, \\ \alpha_2 &= \frac{3}{4}\frac{2p+1}{8p^2+6p+1}\,, \\ \alpha_3 &= \frac{15}{8}\frac{8p^2+6p+1}{48p^3+44p^2+12p+1}\,, \\ &\ldots \end{aligned}$$

which gives the usual Taylor series of the reciprocal of the sine function for $p = 1$: $\operatorname{asin} u = u + u^3/6 + 3x^5/40 + 5x^7/112...$

Eventually, $\dot{y} = \lambda x^{2p}$, so

$$y = \frac{1}{(\sqrt[p]{\lambda})^{p+1}} r(t\sqrt[p]{\lambda}),$$

where r is defined by a second quadrature,

$$r = \int_0^t q^{2p} ds.$$

In the Heisenberg case ($p = 1$), one has

$$\begin{aligned} q &= \sin t, \\ r &= t/2 - (1/4)\sin 2t. \end{aligned}$$

In the flat Martinet case ($p = 2$), one has (see [1])

$$\begin{aligned} q &= -\mathrm{cn}(t\sqrt{2} + K), \\ r &= (3\sqrt{2})^{-1}(t\sqrt{2} + 2\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}(t\sqrt{2} + K)), \end{aligned}$$

where cn, sn and dn are the Jacobi elliptic functions, here for the parameter $k = 1/\sqrt{2}$. The constant K is the complete elliptic integral of the first kind for the same value of the parameter,

$$K = \int_0^1 \frac{dv}{\sqrt{1-v^2}\sqrt{1-k^2v^2}}\cdot$$

Returning to the general case, $p \geq 1$, symmetry reasons imply that nontrivial minimizing geodesics emanating from the origin intersect on the y-axis so that the cut locus is the axis minus the origin, provided conjugate points are located after these intersections with the axis. The conjugate locus is the set of first critical values of the exponential mapping on $H = 1/2$,

$$\exp_t(\lambda) = \exp_{(0,0),t}(\lambda) = (x(t,\lambda), y(t,\lambda)).$$

Let s_p denote the first positive root of

$$qr' - (p+1)q'r = 0, \tag{5}$$

the first conjugate times are readily $t_\lambda = s_p/\sqrt[p]{\lambda}$. Since the conjugate locus can obviously not be contained into one of the two axes, the following holds.

Lemma 3.1. *The conjugate locus at the origin of $ds^2 = dx^2 + dy^2/x^{2p}$ is the set $y = \pm C_p x^{p+1}$ minus the origin where*

$$C_p = \frac{1}{p+1}\sqrt{\frac{q^{2p}(s_p)}{1-q^{2p}(s_p)}}$$

is nonzero, s_p being the first positive root of (5), q being defined by the quadrature

$$q^{-1} = \int_0^u \frac{dv}{\sqrt[p]{1-v^{2p}}}\cdot$$

The invariants C_p can be estimated numerically, and one has

$$\begin{aligned} C_1 &\simeq 2.246, \\ C_2 &\simeq 1.245, \\ C_3 &\simeq 0.863, \\ &\ldots \end{aligned}$$

while the corresponding conjugate loci are portrayed Fig. 4.

As a byproduct of the previous analysis, we get the following result.

Proposition 3.3. *Let R be a meromorphic function with a pole of finite order p at $X = 1$. The conjugate locus of a point on the equator of the associated Clairaut-Liouville metric has two contacts of order $p + 1$ with the meridian passing through the point.*

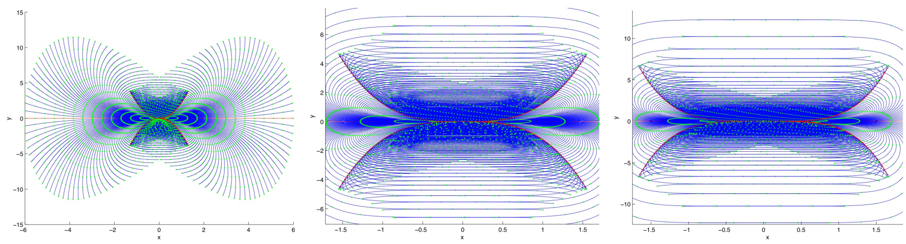

Figure 4: Geodesics and conjugate loci of at the origin of $ds^2 = dx^2 + dy^2/x^{2p}$ for $p = 1$, 2 and 3 (from left to right). Computation provided using the `cotcot` code (`www.n7.fr/apo/cotcot`).

Corollary 3.1. *Consider the unfolding $g_\lambda = (XR_\lambda \circ \sin^2)(\varphi)d\theta^2 + d\varphi^2$ of the Clairaut-Liouville metric defined by $R = 1/(1-X)$. For a point on the equator, the two cusps of the conjugate locus on the equator bifurcate for $\lambda = 1$ into order two contacts with the meridian through the point.*

Numerical simulations (see Fig. 5) indicate that the cut and conjugate loci for the Clairaut-Liouville metrics defined respectively by $R = 1/(1-X)$ and $R = (1/4)(1 + 2/(1-X) + 1/(1-X)^2)$ are as follows: Simple antipodal cut and 4-cusp conjugate locus for points outside the equator and the poles (no bifurcation), cut equal to the equator minus the point and 2-cusp conjugate locus with two contacts of order $p + 1$ ($p = 1$ or 2, respectively) with the meridian passing through the point when it is on the equator.

An interesting question is to know whether it is possible to extend the results of [6, 10] so as to treat such cases. Among the assumptions to be required on the metric, one may for instance consider the fact that in the two aforementioned examples, despite the singularity, curvatures remain monotone functions along half-meridians, decreasing towards $-\infty$. Indeed, in the order one case,

$$Q = -\frac{2}{1-X}\,, \quad Q' = -\frac{2}{(1-X)^2}\,,$$

and in the order two case,

$$Q = -\frac{(1+X)(4-X)}{(2-X)(1-X)}\,, \quad Q' = -\frac{6(3-2X)}{(2-X)^2(1-X)^2}\cdot$$

Another point is to be able, as in the order one case, to include in the analysis the unfolding of such metrics though the sense of variations changes at least twice.

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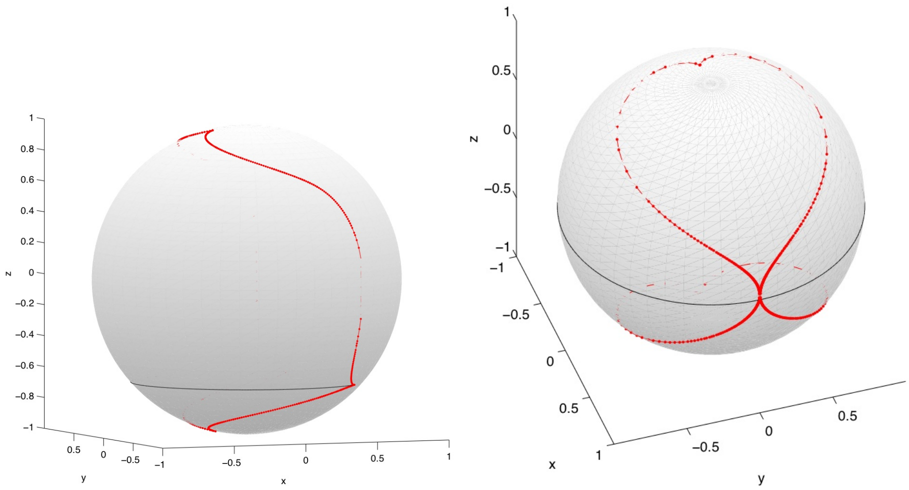

Figure 5: Conjugate loci of the Clairaut-Liouville metric defined by $R = 1/(1 - X)$. Left, for an initial condition outside the equator: The simple antipodal cut and the 4-cusp conjugate locus are observed. Right, for an initial condition on the equator: The cut locus is the equator minus the point itself, and two cusps of the conjugate locus are still observed while the other two bifurcate into contacts of order two with the meridian at the starting point. Computation provided using the `cotcot` code (`www.n7.fr/apo/cotcot`).

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