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Abstract

We describe three elementary models in three dimensional subelliptic geometry which corresponds to the three models of the Riemannian geometry (spheres, Euclidean spaces and Hyperbolic spaces) which are respectively the $SU(2)$, Heisenberg and $SL(2)$ groups. On those models, we prove parabolic Li-Yau inequalities on positive solutions of the heat equation. We use for that the Γ_2 techniques that we adapt to those elementary model spaces. The important feature developped here is that although the usual notion of Ricci curvature is meaningless (or more precisely leads to bounds of the form $-\infty$ for the Ricci curvature), we describe a parameter ρ which plays the same rôle than the lower bound on the Ricci curvature, and from which one deduces the same kind of results as one does in Riemannian geometry, like heat kernel upper bounds, Sobolev inequalities and diameter estimates.

1 Framework and Introduction

The estimation of heat kernel measures is a topic which had been under thorough investigation for the last thirty years at least. [12, 8] Among the many techniques developped for that, the famous Li-Yau parabolic inequality [12] is a very powerfull tool, which relies in Riemannian geometry bounds on the gradient on heat kernels to lower bounds on the Ricci curvature. More precisely, in it simplest form, it asserts that, if E is a smooth Riemmanian manifold with dimension n and non negative Ricci curvature, then if f is any positive solution of the heat equation

$$\partial_t f = \Delta f,$$

where Δ is the Laplace Beltrami operator of E, then, if $u = \log f$

$$\partial_t u \geq |\nabla u|^2 - \frac{n}{2}t.$$

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This is a very precise and powerful estimate. For the model case, which is here the Euclidean space $E = \mathbb{R}^n$ and when f is the heat kernel (that is the solution of the heat equation starting at time $t = 0$ from a Dirac mass), then this inequality is in fact an equality.
From this inequality, one may easily deduce Harnack inequalities and hence precise bounds on the heat kernel.
Many generalisations of this inequality have been developped, all of them including lower bounds on the Ricci tensor. In particular, it works for a general elliptic operator L under the assumption that it satisfies a curvature-dimension inequality $CD(\rho, n)$, which is the furthermost generalistion on the notion of lower bound on the Ricci curvature [6, 4].
In the non elliptic case, things appear to be infinitely more complicated. In particular, most of the hypoelliptic systems do not satisfy any $CD(\rho, n)$ inequality (any reasonable notion of lower bound on the Ricci tensor leads to the value $-\infty$). Nevertheless, some Li-Yau inequalities may be obtained [9].
In what follows, we shall use the Γ_2 techniques developped in [4] to produce these Li-Yau bounds. The method developped here works quite well on the simple models developped here (Heisenberg groups, $SU(2)$, $SL(2)$), but could be easily generalised to a larger class of hypoelliptic operators. We shall not try to present here the most general results, but concentrate for simplicty on the three model cases mentionned above. In fact, they should be thought of as the analogous of the model spaces of Riemannian geometry (Euclidean spaces, Spheres and Hyperbolic spaces).
In all what follows, given an elliptic second order operator L on a smooth manifold, with no constant term, one defines

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf)$$

which stands for $\nabla f \cdot \nabla g$ in the Riemannian case, and the curvature dimension inequality is defined from the operator Γ_2

$$\Gamma_2(f, f) = \frac{1}{2}(\Gamma(f, f) - 2\Gamma(f, Lf).$$

Then, L is said to satisfy a $CD(\rho, n)$ inequality if, for any smooth function f, one has

$$\Gamma_2(f, f) \geq \rho\Gamma(f, f) + \frac{1}{n}(Lf)^2.$$

The parabolic Li-Yau inequality is then described in terms of the quantity $|\nabla f|^2 = \Gamma(f, f)$ and the parameters ρ and n. For the Laplace Beltrami operator $L = \Delta$ on a smooth Riemannian manifold, this amounts to say that the dimension is at most n and that the Ricci curvature is bounded below by ρ. In the hypoelliptic models that we describe below, however, no such inequality holds (the best possible constant ρ is $-\infty$), but we shall produce some analoguous of the Li-Yau inequality through a parameter ρ which therefore plays the rôle of a substitute for the Ricci curvature.
In what follows we consider a three-dimensional Lie group $\mathbf{G}$ with Lie algebra $\mathfrak{g}$ and we assume that there is a basis $\{X, Y, Z\}$ of $\mathfrak{g}$ such that

$$[X, Y] = Z$$

$$[X, Z] = -\rho Y$$

$$[Y,Z]=\rho X$$

where $\rho\in\mathbb{R}$.

Example 1.1 ($\mathbf{SU}(2)$, $\rho=1$) *The Lie group $\mathbf{SU}(2)$ is the group of 2×2, complex, unitary matrices of determinant 1. Its Lie algebra $\mathfrak{su}(2)$ consists of 2×2, complex, skew-adjoint matrices of trace 0. A basis of $\mathfrak{su}(2)$ is formed by the Pauli matrices:*

$$X=\frac{1}{2}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\ Y=\frac{1}{2}\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix},Z=\frac{1}{2}\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},$$

for which the following relationships hold

$$[X,Y]=Z,\quad [X,Z]=-Y,\quad [Y,Z]=X. \tag{1.1}$$

Example 1.2 (Heisenberg group, $\rho=0$) *The Heisenberg group $\mathbb{H}$ is the group of 3×3 matrices:*

$$\begin{pmatrix} 1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix},\quad x,y,z\in\mathbb{R}.$$

The Lie algebra of $\mathbb{H}$ is spanned by the matrices

$$X=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ Y=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \text{ and } Z=\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

for which the following equalities hold

$$[X,Y]=Z,\ [X,Z]=[Y,Z]=0.$$

Example 1.3 ($\mathbf{SL}(2)$, $\rho=-1$) *The Lie group $\mathbf{SL}(2)$ is the group of 2×2, real matrices of determinant 1. Its Lie algebra $\mathfrak{sl}(2)$ consists of 2×2 matrices of trace 0. A basis of $\mathfrak{sl}(2)$ is formed by the matrices:*

$$X=\frac{1}{2}\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix},\ Y=\frac{1}{2}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\ Z=\frac{1}{2}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},$$

for which the following relationships hold

$$[X,Y]=Z,\quad [X,Z]=Y,\quad [Y,Z]=-X. \tag{1.2}$$

We consider on the Lie group $\mathbf{G}$ the subelliptic, left-invariant, second order differential operator

$$L=X^2+Y^2,$$

as well as the heat semigroup

$$P_t=e^{tL}.$$

We also set

$$\Gamma(f,f)=\frac{1}{2}(Lf^2-2fLf)=(Xf)^2+(Yf)^2,$$

and

$$\Gamma_2 = \frac{1}{2}(L\Gamma(f,f) - 2\Gamma(f,Lf)).$$

In the present setting,

$$\Gamma_2(f,f) = (X^2f)^2 + (Y^2f)^2 + \frac{1}{2}((XY+YX)f)^2 + \frac{1}{2}(Zf)^2 + \rho\Gamma(f,f) - 2(Xf)(YZf) + 2(Yf)(XZf). \tag{1.3}$$

The mist terms $-2(Xf)(YZf)+2(Yf)(XZf)$ prevents to find any lower bound on this quantity involving $\Gamma(f,f)$ and $(Lf)^2$ only, whence the absence of any $CD(\rho,n)$ inequality.

2 Li-Yau type estimates for the heat semigroup

The classical method of Li and Yau [12] consists in applying the maximum principle to a carefully chosen expression. The method developped in [4] is quite different. Considering a positive solution of the heat equation $\partial_t f = Lf$, and denoting $f \mapsto P_t f$ the associated heat kernel, one writes $u = \log f$ and look at the expression

$$\Phi(s) = P_s(f(t-s)\Gamma(u(t-s), u(t-s))),$$

defined for $0 < s < t$. Then, one obtains through the $CD(\rho,n)$ inequality a differential inequality

$$\Phi'(s) \geq (A\Phi(s) + B)^2 + C,$$

where A, B, C are expressions which are constant in t but may depend on the function f. Then, the parabolic Li-Yau inequality is obtained as a consequence of this differential inequality.
Here, we shall develop this method a bit further, looking at more complicated quantities like

$$P_s(f(t-s)(a(s)\Gamma(u(t-s), u(t-s)) + b(s)(Zu(t-s))^2)),$$

and try to get some differential inequality on it. The computations developed here are not restricted to Lie group, since we only use an generalized $CD(\rho,n)$ inequality. There are many hypoelliptic systems that may be treated under the same lines. The reason why we restrict ourselves to those model cases described previously are mainly for pedagogical reasons.
We have the following inequality, which is our technical starting point:

Proposition 2.1 *Let $f : \mathbf{G} \to \mathbb{R}_{\geq 0}$, $t > 0$ and $x \in \mathbf{G}$. For $0 \leq s \leq t$, consider the expressions*

$$\Phi_1(s) = P_s((P_{t-s}f)\Gamma(\ln P_{t-s}f))(x)$$

and

$$\Phi_2(s) = P_s((P_{t-s}f)(Z\ln P_{t-s}f)^2)(x).$$

Then, for every differentiable, non-negative and decreasing function $b : [0,t] \to \mathbb{R}$,

$$\left(-b'\Phi_1 + b\Phi_2\right)'(s) \geq -b'(s)\left(\left(\frac{b''(s)}{b'(s)} + 2\frac{b'(s)}{b(s)} + 2\rho\right)\mathcal{L}P_tf(x) - \frac{1}{4}\left(\frac{b''(s)}{b'(s)} + 2\frac{b'(s)}{b(s)} + 2\rho\right)^2 P_tf(x)\right).$$

Proof. We fix a positive function f, $t > 0$ and we perform all the following computations at a given point x.
With the same notations as Proposition 2.1, straightforward (but quite tedious) computations show that

$$\Phi_1'(s) = 2P_s((P_{t-s}f)\Gamma_2(\ln P_{t-s}f))$$

and

$$\Phi_2'(s) = 2P_s((P_{t-s}f)\Gamma(Z\ln P_{t-s}f)).$$

For the last equality we use the crucial facts that

$$[L, Z] = 0$$

and

$$X(f)Z(f)[X,Z](f) + Y(f)Z(f)[Y,Z](f) = 0.$$

Now, thanks to the Cauchy-Schwarz inequality, the expression (1.3), shows that for every $\lambda > 0$, and every smooth function g,

$$\Gamma_2(g) \geq \frac{1}{2}(\mathcal{L}g)^2 + \frac{1}{2}(Zg)^2 + \left(\rho - \frac{1}{\lambda}\right)\Gamma(g) - \lambda\Gamma(Zg).$$

We therefore obtain the following differential inequality

$$\Phi_1'(s) \geq P_s((P_{t-s}f)(\mathcal{L}\ln P_{t-s}f)^2) + \Phi_2(s) + \left(2\rho - \frac{2}{\lambda}\right)\Phi_1(s) - \lambda\Phi_2'(s).$$

We now have that for every $\gamma \in \mathbb{R}$,

$$(\mathcal{L}\ln P_{t-s}f)^2 \geq 2\gamma\mathcal{L}\ln P_{t-s}f - \gamma^2,$$

and

$$\mathcal{L}\ln P_{t-s}f = \frac{\mathcal{L}P_{t-s}f}{P_{t-s}f} - \frac{\Gamma(P_{t-s}f)}{(P_{t-s}f)^2}.$$

Thus, for every $\lambda > 0$ and every $\gamma \in \mathbb{R}$,

$$\Phi_1'(s) \geq \left(2\rho - \frac{2}{\lambda} - 2\gamma\right)\Phi_1(s) + \Phi_2(s) - 2\lambda\Phi_2'(s) + 2\gamma\mathcal{L}P_tf - \gamma^2P_tf.$$

Now for two functions a and b defined on the time interval $[0, t)$ with a positive, we have

$$(a\Phi_1 + b\Phi_2)' \geq \left(a' + (2\rho - \frac{2}{\lambda} - 2\gamma)a\right)\Phi_1 + (a + b')\Phi_2 + (-a\lambda + b)\Phi_2' + 2a\gamma LP_tf - a\gamma^2P_tf.$$

So, if b is a positive decreasing function on the time interval $[0, t)$, by choosing in the previous inequality

$$a = -b',$$

$$\lambda = -\frac{b}{b'}$$

and
$$\gamma = \frac{1}{2}\left(\frac{b''}{b'} + 2\frac{b'}{b} + 2\rho\right),$$
we get the desired result. $\square$

As a first corollary, by using the function
$$b(s) = (t-s)^\alpha, \quad \alpha > 2$$
and integrating from 0 to t we deduce

Corollary 2.2 *For all $\alpha > 2$, for every positive function f and $t > 0$,*
$$\Gamma(\ln P_t f) + \frac{t}{\alpha}(Z \ln P_t f)^2 \le \left(\frac{3\alpha - 1}{\alpha - 1} - \frac{2\rho t}{\alpha}\right)\frac{\mathcal{L}P_t f}{P_t f} + \frac{\rho^2 t}{\alpha} - \frac{\rho(3\alpha-1)}{\alpha-1} + \frac{(3\alpha-1)^2}{\alpha-2}\frac{1}{t}.$$

Observe that this takes a simpler form when $\rho \ge 0$, since then one can use proposition 2.1 with $\rho = 0$ and get

Corollary 2.3 *When $\rho \ge 0$, there exist constants A, B and C such that, with $u = \log(P_t f)$*
$$\partial_t u \ge A\Gamma(u) + Bt(Zu)^2 - \frac{C}{t}.$$

In particular, one gets $\partial_t u \ge -C/t$, which gives
$$P_t f \le t^{-C} P_1 f.$$

On the Heisenberg group, one sees that the behaviour of $P_t f$ when t goes to 0 is of order t^{-2} (a simple dilation argument shows that). Therefore, one sees that the optimal constant C in the previous inequality is $C = 2$. Unfortunately, it can be shown by some elementary considerations similar to those developped in the proof of corollary 2.5 that the best constant one may obtain from the previous proposition shall always produce a constant $C > 2$. This is a strong difference with the classical parabolic Li-Yau inequality where the inequality
$$\partial_t u \ge -\frac{n}{2t}$$
gives the right order of magnitude of the heat kernel near $t = 0$.

Now when $\rho > 0$, we easily get an exponential decay by using the function:
$$b(s) = \left(e^{-\frac{2\rho s}{3\alpha}} - e^{-\frac{2\rho t}{3\alpha}}\right)^\alpha, \quad \alpha > 2.$$

This writes:

Corollary 2.4 *For every $\alpha > 2$, for every positive function f, $x \in \mathbf{G}$ and $t > 0$,*
$$\Gamma(\ln P_t f)(x) + \frac{3}{2}\frac{1 - e^{-\frac{2\rho t}{3\alpha}}}{\rho}(Z \ln P_t f)^2(x) \le \frac{3\alpha-1}{\alpha-1}e^{-\frac{2\rho t}{3\alpha}}\frac{\mathcal{L}P_t f(x)}{P_t f(x)} + \frac{3}{2}\rho\frac{\left(1 - \frac{1}{3\alpha}\right)^2}{1 - \frac{2}{\alpha}}\frac{e^{-\frac{4\rho t}{3\alpha}}}{1 - e^{-\frac{2\rho t}{3\alpha}}}.$$

Moreover for $\rho > 0$ and t large, with more work we actually can do better.

Corollary 2.5 *Let us assume $\rho > 0$. There exist $t_0 > 0$ and $C > 0$, such that for any positive function f,*

$$|\partial_t \ln P_t f(x)| \leq C \exp(-\frac{\rho t}{3}), \quad t \geq t_0, x \in \mathbf{G}.$$

Proof. To make this proof we have to be more precise in the study of the differential inequality of Theorem 2.1. Start with this inequality and set $V(b) = -b^2 b'$ for b a positive decreasing function such that $b(t) = b'(t) = 0$. The constraints that the non negative function V on $[0, b_0]$ must satisfy are

$$t = \int_0^{b_0} \frac{x^2}{V(x)} dx$$

and

$$\left(\frac{V(x)}{x^2}\right)_{x=0} = 0.$$

We then get with $u_t = \ln P_t f$ and $a_0 = \frac{V(b_0)}{b_0^2}$

$$a_0 \Gamma(u_t) + b_0 (Z u_t)^2 \leq A \partial_t u_t + B,$$

where for any choice of such a function V, one has

$$A = \int_0^{b_0} \left(\frac{V'}{x^2} - 2\rho t\right) dx, \; B = \frac{1}{4} \int_0^{b_0} \left(\frac{V'}{x^2} - 2\rho t\right)^2 dx.$$

In this system, we see that changing $V(s)$ into $\frac{V(\lambda s)}{\lambda^3}$ and b_0 into $\frac{b_0}{\lambda}$ leaves t unchanged and multiply every constant a_0, A and B by $\frac{1}{\lambda}$. Therefore, we may assume that $b_0 = 1$ without any loss.

Also, changing $V(s)$ into $cV(s)$ allows us to reduce to the case $t = 1$. So finally we have rephrased the problem as follows. For any non negative function V on $[0, 1]$ such that

$$\int_0^1 \frac{x^2}{V} dx = 1, \; (\frac{V(x)}{x^2})_{x=0} = 0,$$

and for any $u = \log P_t f$ with $f \geq 0$ one has

$$V(1)\Gamma(u) + t(Zu)^2 \leq (\alpha(V) - 2\rho t)\partial_t u + \frac{1}{4t}\left(\beta(V) - \alpha^2(V) + (\alpha(V) - 2\rho t)^2\right),$$

where

$$\alpha(V) = \int_0^1 \frac{V'}{x^2} dx, \; \beta(V) = \int_0^1 (\frac{V'}{x^2})^2 dx.$$

The preceding calculus is valid for any ρ. An easy integration by parts shows us the term $\alpha(V)$ is non negative whatever V is. But now for $\rho > 0$, observe that this time the term $\alpha(V) - 2\rho t$ can be made negative, and therefore we may get as in the elliptic case with strictly positive Ricci bound a universal upper bound on $|\partial_t u|$.

One has the obvious inequalities

$$\alpha(V) > V(1) - (\frac{V(x)}{x^2})_{x=0} + 8, \; \beta(V) > \alpha(V)^2,$$

and in the previous, no equality may occur (in the first one because then $\beta = \infty$ and in the second one because of the constraint on V.) The first inequality comes from

$$\int_0^1 \frac{V'}{x^2} dx = V(1) + 2 \int_0^1 \frac{V}{x^3} dx,$$

and

$$\int_0^1 \frac{V}{x^3} dx \int_0^1 \frac{x^2}{V} dx \geq (\int_0^1 \frac{dx}{\sqrt{x}})^2 = 4.$$

To make the term $\beta(V) - \alpha^2(V)$ small we are lead to choose $V = \lambda x^3$ on $[\epsilon, 1]$ and $V = \lambda \epsilon^{3-\gamma} x^\gamma$ on $[0, \epsilon]$, for some fixed $\gamma \in (5/2, 3)$. The constraint on V implies

$$\lambda = -\log \epsilon + \frac{1}{3-\gamma},$$

Meanwhile, we have

$$\alpha = \lambda(3 + 2\epsilon \frac{3-\gamma}{\gamma-2}),$$

and

$$\beta = \lambda^2(9 + \epsilon \frac{(15-\gamma)(3-\gamma)}{2\gamma-5}),$$

so that

$$\beta - \alpha^2 = \lambda^2 \epsilon \frac{(3-\gamma)^2}{\gamma-2} \left(\frac{\gamma+10}{2\gamma-5} + \epsilon \frac{4}{\gamma-2} \right)$$

By taking

$$\epsilon = \exp\left(-\frac{2\rho}{3} t + \frac{1}{3-\gamma} + R \right)$$

for t large enough to ensure $\varepsilon < 1$ one obtains

$$\alpha - 2\rho t \simeq -3R$$

and

$$\beta - \alpha^2 \simeq C t^2 \varepsilon \simeq C t^2 \exp(-\frac{2\rho t}{3})$$

With $R = ct \exp(-\frac{\rho}{3})$ the terms $(\alpha - 2\rho t)^2$ and $\beta - \alpha^2$ are of the same order and playing now with the sign of c, one gets

$$|\partial_t u| \leq C \exp(-\frac{t}{3}).$$

□

Interestingly, only from these estimates, we can deduce that for $\rho > 0$ the Lie group **G** has to be compact. (This is of course not new since the Lie algebra is that of a compact semi-simple Lie group). But also get an upper bound on the diameter similar to the classical upper bound of the Myers theorem, together with some precise informations on the Sobolev constants and the spectral gap. Those considerations in fact show that this parameter ρ may serve as a substitute of the Ricci lower bound for a Riemannian manifold. We proceed first by showing that in that case there is a spectral gap.

Proposition 2.6 *Let us assume $\rho > 0$. The spectrum of $-L$ lies in $\{0\} \cup [\frac{\rho}{3}, +\infty]$.*

Proof. We fix $x \in \mathbf{G}$ and denote by $p_t(x, \cdot)$ the heat kernel starting from x. We have for $t \geq t_0$,

$$\mid \partial_t \ln p_t(x, y) \mid \leq C \exp(-\frac{\rho t}{3}). \tag{2.4}$$

This shows us that $\ln p_t$ converges when $t \to \infty$. Let us call $\ln p_\infty$ this limit. Moreover, from Corollary 2.4, $\Gamma(\ln p_t)$ is bounded above by a constant $C(t)$ which goes to 0 when t goes to ∞. Since the oscillation between $\ln p_t(x, y_1)$ and $\ln p_t(x, y_2)$ is bounded above by $\sqrt{C(t)} d(y_1, y_2)$, for the associated Carnot-Caratheodory distance, which may be defined (see [2]) as

$$d(x, y) = \sup_{\{f, \Gamma(f,f) \leq 1\}} f(x) - f(y), \tag{2.5}$$

such that if $\Gamma(f, f) \leq C$, then $f(x) - f(y) \leq \sqrt{C} d(x, y)$.
In the limit, $\ln p_\infty(x, \cdot)$ is a constant. We deduce from this that the invariant measure μ is finite. We may then as well suppose that this measure is a probability, in which case $p_\infty = 1$. By integrating the inquality (2.4) from t to ∞ we therefore obtain for $t \geq t_0$:

$$\mid \ln p_t(x, y) \mid \leq C_2 \exp(-\frac{\rho t}{3})$$

and thus

$$\exp(-C_2 \exp(-\frac{\rho t}{3})) \leq p_t(x, y) \leq \exp(C_2 \exp(-\frac{\rho t}{3})).$$

This implies by the Cauchy-Schwarz inequality that for $f \in L^2(\mu)$ such that $\int f d\mu = 0$,

$$(P_t f)^2 \leq C_3 \exp(-\frac{2\rho t}{3}) \int f^2 d\mu.$$

For a symmetric Markov semigroup P_t, this is a standard fact (see [2] for example) that this is equivalent to say that the spectrum of $-L$ lies in $\{0\} \cup [\rho/3, \infty)$, or equivalently that we have a spectral gap inequality: for any function f in L^2 such that ∇f is in L^2, one has

$$\int f^2 \leq (\int f)^2 + \frac{3}{\rho} \int |\nabla f|^2 d\mu \tag{2.6}$$

$\square$

Remark 2.7 *It can be shown that the spectral gap is actually $\frac{\rho}{2}$ and not $\frac{\rho}{3}$.*

We can now conclude with a substitute of the Myers theorem:

Proposition 2.8 *Assume that $\rho > 0$, then the diameter of $\mathcal{L}$ for the Carnot-Caratheodory distance is finite.*

Proof. We are now going to prove a Sobolev inequality for the invariant measure μ. Indeed, for $0 < t \leq t_0$ we have

$$\partial_t \ln p_t \geq -C/t,$$

from which we get

$$\ln p_{t_0} - \ln p_t \geq -C \log(t_0/t),$$

and therefore

$$\ln p_t \leq A - C \log t$$

where A is a constant. This gives the ultracontractivity of the semigroup P_t with a polynomial bound t^{-C} when $t \to 0$.
Now it is a well known fact [13, 2] that this last property is equivalent to a Sobolev inequality

$$(\int f^{\frac{2C}{C-1}} d\mu)^{\frac{C-1}{C}} \leq A \int f^2 d\mu + B \int \|\nabla f\|^2 d\mu. \tag{2.7}$$

When we have both Sobolev inequality (2.7) and spectral gap inequality (2.6) then (see [2]) we have a tight Sobolev inequality, that is the Sobolev inequality (2.7) where $A = 1$.
In this situation, the diameter of E with respect to the distance defined in 2.5 is finite (see [5]), which concludes the proof.

□

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