Isoperimetry for spherically symmetric log-concave probability measures

Nolwen Huet

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Abstract

We prove an isoperimetric inequality for probability measures μ on $\mathbb{R}^n$ with density proportional to $\exp(-\phi(\lambda|x|))$, where $|x|$ is the euclidean norm on $\mathbb{R}^n$ and ϕ is a non-decreasing convex function. It applies in particular when $\phi(x) = x^\alpha$ with $\alpha \geq 1$. Under mild assumptions on ϕ, the inequality is dimension-free if λ is chosen such that the covariance of μ is the identity.

1 Introduction

In his paper [10], Bobkov studies the spectral gap for spherically symmetric probability measures μ on $\mathbb{R}^n$ with density

$$\frac{d\mu(x)}{dx} = \rho(|x|),$$

where ρ is log-concave. His main result can be stated as follows.

Theorem 1 (Bobkov [10])**.** *The best constant P_μ in the Poincaré inequality*

$$\mathrm{Var}_\mu(f) \leq P_\mu \int |\nabla f|^2 \, d\mu, \quad \forall f \text{ smooth}$$

satisfies

$$\frac{\mathrm{E}_\mu(|X|^2)}{n} \leq P_\mu \leq 12 \frac{\mathrm{E}_\mu(|X|^2)}{n}.$$

In particular, if μ is isotropic, we get

$$1 \leq P_\mu \leq 12,$$

which means a spectral gap not depending on n.

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Here "μ is isotropic" means that the covariance of μ is the identity. However, we already know from the spherically invariance of μ that the covariance is proportional to the identity. So in our case, the isotropy of μ reduces merely to $\mathrm{E}_\mu(|X|^2) = n$.

If we assume furthermore that μ itself is log-concave (see [11] for precisions about log-concave measures), that is to say that ρ is non-decreasing, then one can show an isoperimetric inequality for μ, thanks to a result of Ledoux [15] (generalized in [17] by E. Milman) bounding the Cheeger constant from below by the spectral gap.

Theorem 2. *There exists a universal constant $c > 0$ such that, for any $n \in \mathbb{N}$, all* log-concave *measures μ on $\mathbb{R}^n$* spherically symmetric *and* isotropic *satisfy the following isoperimetric inequality:*

$$\mathrm{Is}_\mu(a) \geq c\ a \wedge (1-a). \tag{1}$$

Here Is_μ denote the isoperimetric function of μ and $a \wedge b = \min(a, b)$. We need some notation to define Is_μ properly. Let A be a Borel set in $\mathbb{R}^n$. We define its ε-neighborhood by

$$A_\varepsilon = \{x \in X; d(x, A) \leq \varepsilon\}.$$

The boundary measure of A is

$$\mu^+(\partial A) = \liminf_{\varepsilon \to 0^+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}.$$

Now the isoperimetric function of μ is the largest function Is_μ on $[0, 1]$ such that for all Borel sets A,

$$\mu^+(\partial A) \geq \mathrm{Is}_\mu\big(\mu(A)\big).$$

The result of Bobkov answer the KLS-conjecture ([12]) in the particular case of spherically symmetric measures. This conjecture asserts that (1) is true for all log-concave and isotropic measures μ, with a universal constant c.

Our aim in this note is to sharpen Theorem 2 when ρ is "better" than log-concave. For instance, the Gaussian measure γ_n corresponding to $\rho(t) = (2\pi)^{-\frac{n}{2}} \exp -\frac{t^2}{2}$, is known to satisfy the log-Sobolev inequality and the following isoperimetric inequality:

$$\mathrm{Is}_{\gamma_n}(a) \geq c\ \big(a \wedge (1-a)\big) \sqrt{\log \frac{1}{a \wedge (1-a)}}$$

with constants not depending on n either. We can ask what happens for regimes between exponential and Gaussian or even beyond the Gaussian case. This idea has already be developed in [14, 2, 6, 4, 5] for product measures.

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex non-decreasing function of class $\mathcal{C}^2$ such that $\phi(0) = 0$. Then we consider the probability measure on $\mathbb{R}^n$

$$\mu_{n,\phi}(dx) = \frac{e^{-\phi(|x|)}\, dx}{Z_{n,\phi}}$$

and its associated radial measure on $[0, +\infty)$

$$\nu_{n,\phi}(dr) = |\mathbb{S}^{n-1}| \frac{r^{n-1} e^{-\phi(r)}\, dr}{Z_{n,\phi}}.$$

In the particular case $\phi(x) = \phi_\alpha(x) = x^\alpha$ with $\alpha \geq 1$, we note $\mu_{n,\alpha} = \mu_{n,\phi_\alpha}$ and $\nu_{n,\alpha} = \nu_{n,\phi_\alpha}$. We denote by σ_{n-1} the uniform probability measure on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. If X is a random variable of law $\mu_{n,\phi}$, then $|X|$ has the distribution $\nu_{n,\phi}$. Conversely, if r and θ are independent random variables whose distributions are respectively $\nu_{n,\phi}$ and σ_{n-1}, then $X = r\theta$ has the distribution $\mu_{n,\phi}$. In view of this representation, we will derive inequalities for $\mu_{n,\phi}$ from inequalities for $\nu_{n,\phi}$ and σ_{n-1}.

In the subgaussian case, following the results for $\mu_{1,\phi}^{\otimes n}$ from [5], we expect the isoperimetric function of $\mu_{n,\alpha}$ to be equal to a constant depending on n, times a symmetric function defined for $a \in [0, \frac{1}{2}]$ by

$$L_\alpha(a) = a\left(\log\frac{1}{a}\right)^{1-\frac{1}{\alpha}},$$

and more generally for $\mu_{n,\phi}$,

$$L_\phi(a) = \frac{a\log\frac{1}{a}}{\phi^{-1}\left(\log\frac{1}{a}\right)}.$$

Otherwise, since we are aiming at results which do not depend on n and because of the Central-Limit Theorem ([13]), we cannot expect better isoperimetric profile than the one of the Gaussian measure, proportional to

$$L_2(a) = a\sqrt{\log\frac{1}{a}}.$$

The point is to know the exact dependence in n of the constant in front of the term in a, and in particular to know whether we recover universal constants in the isotropic case. The main theorems of this paper are stated next.

Theorem 3. *There exists a universal constant $C > 0$ such that, for every $\alpha \geq 1$, for every $n \in \mathbb{N}^*$, and every $a \in [0, 1]$, it holds*

$$\mathrm{Is}_{\mu_{n,\alpha}}(a) \geq C n^{\frac{1}{2}-\frac{1}{\alpha}}\ \big(a \wedge (1-a)\big)\left(\log\frac{1}{a \wedge (1-a)}\right)^{1-\frac{1}{\alpha\wedge 2}}.$$

It can be seen as a corollary of the following more general theorem.

Theorem 4. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex non-decreasing function of class $\mathcal{C}^2$ such that $\phi(0) = 0$. If moreover we assume that*

i) $\sqrt{\phi}$ is concave, then for every $n \in \mathbb{N}^*$, and every $a \in [0, 1]$, it holds

$$\mathrm{Is}_{\mu_{n,\phi}}(a) \geq C\frac{\sqrt{n}}{\phi^{-1}(n)}\ \phi^{-1}(1)\frac{\big(a \wedge (1-a)\big)\log\frac{1}{a\wedge(1-a)}}{\phi^{-1}\left(\log\frac{1}{a\wedge(1-a)}\right)},$$

ii) $x \mapsto \sqrt{\phi(x)}/x$ is increasing, then for every $n \in \mathbb{N}^*$, and every $a \in [0, 1]$, it holds

$$\mathrm{Is}_{\mu_{n,\phi}}(a) \geq C\frac{\sqrt{n}}{\phi^{-1}(n)}\ \big(a \wedge (1-a)\big)\sqrt{\log\frac{1}{a \wedge (1-a)}},$$

where $C > 0$ is a universal constant.

Further hypotheses ensure the optimality of these bounds among products of functions of n and functions of a, and lead to dimension-free inequalities when normalizing measures to obtain isotropic ones. See Theorem 19 for more precise statement.

Note that a straightforward application of Bobkov's inequality for log-concave measures (Theorem 7) leads to the good profile but with the wrong dimension dependent constant in front of the isoperimetric inequality. For instance, Lemma 4 of [2] and the computation of exponential moments imply the Theorem 3 with $n^{-\frac{1}{\alpha}}$ instead of $n^{\frac{1}{2}-\frac{1}{\alpha}}$.

We introduce in Section 2 the different hypotheses made on ϕ. Then we establish in Section 3 the isoperimetric inequality for the radial measure. The proof relies on an inequality for log-concave measures due to Bobkov and some estimates of probabilities of balls. Section 4 is devoted to the argument of tensorization which yields the isoperimetric inequality from the ones for the radial measure and the uniform probability measure on the sphere. A cut-off argument is needed to get rid of the case of large radius. This tensorization relies on a functional version of the inequality whose proof is postponed to Section 5. We combine the previous results in Section 6 to prove Theorem 4. Eventually, we discuss the isotropic case and the optimality of the inequalities in Section 7.

2 Hypotheses on ϕ

We make different assumptions on ϕ, corresponding to the different cases in Theorems 4, 6, and 19.

Hypotheses (H0) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing convex function of class $\mathcal{C}^2$ such that $\phi(0) = 0$.

Hypotheses (H1) ϕ satisfies (H0) and $x \mapsto \sqrt{\phi(x)}/x$ is non-increasing.

Hypotheses (H1') ϕ satisfies (H0) and $\sqrt{\phi}$ is concave.

Hypotheses (H2) ϕ satisfies (H0) and $x \mapsto \sqrt{\phi(x)}/x$ is non-decreasing.

Hypotheses (H2') ϕ satisfies (H2) and there exists $\alpha \geq 2$ such that $x \mapsto \phi(x)/x^\alpha$ is non-increasing.

The next lemma sums up some properties of ϕ under our assumptions.

Lemma 5.
- *Under (H0), it holds:*

  i. For all $t \geq 1$ and $x \geq 0$,
  $$\phi(tx) \geq t\phi(x).$$
  ii. For all $t \geq 1$ and $y \geq 0$,
  $$\phi^{-1}(ty) \leq t\phi^{-1}(y).$$
  iii. For all $x \geq 0$,
  $$x\phi'(x) \geq \phi(x).$$

- *Under (H1), it holds:*

  i. *For all $t \geq 1$ and $x \geq 0$,*
  $$t\phi(x) \leq \phi(tx) \leq t^2\phi(x).$$

  ii. *For all $t \geq 1$ and $y \geq 0$,*
  $$\sqrt{t}\phi^{-1}(y) \leq \phi^{-1}(ty) \leq t\phi^{-1}(y).$$

  iii. *For all $x \geq 0$,*
  $$\phi(x) \leq x\phi'(x) \leq 2\phi(x).$$

  iv. *For all $t \geq 1$ and $x \geq 0$,*
  $$\phi'(tx) \leq 2t\phi'(x).$$

- *Under (H2), it holds:*

  i. *For all $t \geq 1$ and $x \geq 0$,*
  $$\phi(tx) \geq t^2\phi(x).$$

  ii. *For all $t \geq 1$ and $y \geq 0$,*
  $$\phi^{-1}(ty) \leq \sqrt{t}\phi^{-1}(y).$$

  iii. *For all $x \geq 0$,*
  $$x\phi'(x) \geq 2\phi(x).$$

3 Isoperimetry for the radial measure $\nu_{n,\phi}$

In order to deal with $\mu_{n,\phi}$, a first step is to establish a similar isoperimetric inequality for its radial marginal.

Theorem 6. *There exists a universal constant $C > 0$ such that for every $n \in \mathbb{N}^*$, every $a \in [0, \frac{1}{2}]$, and every function ϕ,*

i) if ϕ satisfies (H1) then
$$\mathrm{Is}_{\nu_{n,\phi}}(a) \geq C\frac{\sqrt{n}}{\phi^{-1}(n)}\ \phi^{-1}(1)\frac{a\log\frac{1}{a}}{\phi^{-1}\left(\log\frac{1}{a}\right)}.$$

ii) if ϕ satisfies (H2) then
$$\mathrm{Is}_{\nu_{n,\phi}}(a) \geq C\frac{\sqrt{n}}{\phi^{-1}(n)}\ a\sqrt{\log\frac{1}{a}}.$$

As $\nu_{n,\phi}$ is a log-concave measure, we can apply the isoperimetric inequality shown by Bobkov in [9].

Theorem 7 (Bobkov [9]). *If μ is a log-concave measure on $\mathbb{R}^n$, then for all Borel sets A, for all $r > 0$, and for all $x_0 \in \mathbb{R}^n$,*

$$2r\mu^+(\partial A) \geq \mu(A) \log \frac{1}{\mu(A)} + \mu(A^{\complement}) \log \frac{1}{\mu(A^{\complement})} + \log \mu\{|x - x_0| \leq r\}, \qquad (2)$$

where $A^{\complement}$ denotes the complement of A.

One chooses r as small as possible but with $\mu\{|x-x_0| \leq r\}$ large enough, such that the sum of the two last terms is non-negative. This requires explicit estimates of probabilities of balls. In our case, we will use two different estimates valid for two ranges of r, leading to inequalities for two ranges of a.

The first lemma is due to Klartag [13]. The balls are centered at the maximum of density in order to capture a large fraction of the mass.

Lemma 8 (Klartag [13]). *Let $\nu(dr) = r^{n-1}\rho(r)\,dr$ be a probability measure on $\mathbb{R}^+$ with ρ a log-concave function of class $\mathcal{C}^2$. Let r_0 be the point where the density reaches its maximum. Then,*

$$\forall \delta \in [0, 1], \quad \nu\{|r - r_0| \geq \delta r_0\} \leq C_1 e^{-c_1 n \delta^2}$$

where $C_1 > 1$ and $0 < c_1 < 1$ are universal constants.

Bobkov's inequality combined with the latter lemma leads to the following proposition.

Proposition 9. *There exist two universal constants $c > 0$ and $C > 0$ such that for all functions ϕ satisfying (H0) and all n large enough to ensure $e^{-cn} < \frac{1}{2}$, it holds*

$$\forall a \in \left[e^{-cn}, \frac{1}{2}\right], \quad \mathrm{Is}_{\nu_{n,\phi}}(a) \geq C \frac{\sqrt{n}}{\phi^{-1}(n)}\; a\sqrt{\log \frac{1}{a}}.$$

Proof. Let C_1 and c_1 be the constants given by Lemma 8. Let $K > 0$ and set

$$\delta = \sqrt{\frac{K \log \frac{1}{a}}{c_1 n}}.$$

Choose $a \in \left[\exp\left(-\frac{c_1 n}{K}\right), \frac{1}{2}\right]$ and $K > \frac{\log C_1}{\log 2}$. It follows that $\delta \leq 1$ and $1 - C_1 a^K > 0$. Then Lemma 8 implies

$$(1 - a) \log \frac{1}{1 - a} + \log \nu_{n,\phi}\{|r - r_0| \leq \delta r_0\} \geq (1 - a) \log \frac{1}{1 - a} + \log(1 - C_1 a^K). \qquad (3)$$

The right-hand term of (3) cancels at 0 and is concave in a on $[0, \frac{1}{2}]$ if $K \geq 1$. Take K large enough such that it is also non-negative at $\frac{1}{2}$. Thus, by concavity, it is non-negative on $[0, \frac{1}{2}]$. So Bobkov's formula (2) yields

$$\mathrm{Is}_{\nu_{n,\phi}}(a) \geq \frac{1}{2}\sqrt{\frac{c_1 n}{K r_0^2}}\; a\sqrt{\log \frac{1}{a}}.$$

It remains to estimate the point r_0 where the density of $\nu_{n,\phi}$ reaches its maximum. The differentiation of the density leads to

$$r_0 \phi'(r_0) = n - 1.$$

By Lemma 5, $\phi(r_0) \leq n-1$. Thus

$$r_0 \leq \phi^{-1}(n).$$

□

Let us remark that under (H1), for all $a \leq \frac{1}{2}$,

$$\sqrt{\log \frac{1}{a}} \geq \sqrt{\log 2}\ \phi^{-1}(1) \frac{\log \frac{1}{a}}{\phi^{-1}\left(\log \frac{1}{a}\right)}.$$

So the latter proposition implies a stronger inequality that the one required under (H1), but only for large enough sets.

To cope with smaller sets, we need another estimate for balls with greater radius.

Lemma 10. *Let ϕ be a function satisfying (H0) and $n \in \mathbb{N}^*$. Then for all $r \geq \phi^{-1}(2n)$,*

$$\nu_{n,\phi}\{(r,+\infty)\} \leq F_{n,\phi}(r) = \left(\frac{er}{\phi^{-1}(n)}\right)^n e^{-\phi(r)} \leq 1.$$

Note that this tail bound gives estimates of probability of balls centered at 0 for $\nu_{n,\phi}$, but also for $\mu_{n,\phi}$ since

$$\nu_{n,\phi}\{(r,+\infty)\} = \mu_{n,\phi}\{|x| \geq r\}.$$

This lemma can thereby be used to derive isoperimetric inequalities from Bobkov's formula for both measures.

Proof. The main tool is integration by part.

$$\begin{aligned}
\int_r^{+\infty} t^{n-1} e^{-\phi(t)}\, dt &= \int_r^{+\infty} \frac{t^{n-1}}{\phi'(t)}\ \phi'(t) e^{-\phi(t)}\, dt \\
&= \frac{r^{n-1}}{\phi'(r)} e^{-\phi(r)} + \int_r^{+\infty} \left[\frac{n-1}{t\phi'(t)} - \frac{\phi''(t)}{\left(\phi'(t)\right)^2}\right]\ t^{n-1} e^{-\phi(t)}\, dt \\
&\leq \frac{r^{n-1}}{\phi'(r)} e^{-\phi(r)} + \int_r^{+\infty} \frac{n-1}{t\phi'(t)}\ t^{n-1} e^{-\phi(t)}\, dt.
\end{aligned}$$

If $t \geq r \geq \phi^{-1}(2n) \geq \phi^{-1}\big(2(n-1)\big)$, then $t\phi'(t) \geq 2(n-1)$. So the last integral in the above inequality is less than $\frac{1}{2}\int_r^{+\infty} t^{n-1} e^{-\phi(t)}\, dt$. Moreover $r\phi'(r) \geq 2n$. Hence

$$\int_r^{+\infty} t^{n-1} e^{-\phi(t)}\, dt \leq 2 \frac{r^{n-1}}{\phi'(r)} e^{-\phi(r)} \leq \frac{r^n}{n} e^{-\phi(r)}.$$

It remains to deal with the normalization constant which makes $\nu_{n,\phi}$ a probability measure:

$$\begin{aligned}
\int_0^{+\infty} n t^{n-1} e^{-\phi(t)}\, dt &\geq \int_0^{\phi^{-1}(n)} n t^{n-1} e^{-\phi(t)}\, dt \\
&\geq e^{-n} \int_0^{\phi^{-1}(n)} n t^{n-1}\, dt = \left(\frac{\phi^{-1}(n)}{e}\right)^n.
\end{aligned}$$

Putting all together, we get the desired bound on the tail of $\nu_{n,\phi}$:

$$\nu_{n,\phi}\{(r,+\infty)\} = \frac{\int_r^{+\infty} t^{n-1}e^{-\phi(t)}\,dt}{\int_0^{+\infty} t^{n-1}e^{-\phi(t)}\,dt} \leq \left(\frac{er}{\phi^{-1}(n)}\right)^n e^{-\phi(r)}.$$

Then one can show that the bound is non-increasing for $r \geq \phi^{-1}(n)$ and is equal to 1 for $r = \phi^{-1}(n)$. $\square$

Then, we show an isoperimetric inequality simultaneously for $\mu_{n,\phi}$ and $\nu_{n,\phi}$ in the range of small sets.

Proposition 11. *For every $c > 0$, there exists $C > 0$ such that for all functions ϕ satisfying the hypotheses of (H0),*

$$\forall a \in \left[0, e^{-cn} \wedge \frac{1}{2}\right], \quad \mathrm{Is}_\mu(a) \geq C\, \frac{a \log \frac{1}{a}}{\phi^{-1}\left(\log \frac{1}{a}\right)},$$

where μ stands for $\mu_{n,\phi}$ or $\nu_{n,\phi}$.

Note that this is worth showing the result for every $c > 0$. Indeed, to prove Theorem 6, we combine this result with Proposition 9 where this constant is already fixed but unknown.

Proof. As before, we start from (2) and set $r(a) = \phi^{-1}\left(K \log \frac{1}{a}\right)$, where K is a constant large enough to ensure

$$Kc \geq 2, \tag{4}$$

$$K - 1 \geq \frac{1}{c}, \tag{5}$$

$$eKc \exp\left(-(K-1)c\right) \leq \frac{1}{2}. \tag{6}$$

By Lemma 5, $r \leq K\phi^{-1}\left(\log \frac{1}{a}\right)$, as $K > 1$. So the result is deduced from Bobkov's inequality (2) provided that

$$(1-a)\log \frac{1}{1-a} + \log \mu_{n,\alpha}\{|x| \leq r\} \geq 0. \tag{7}$$

Now, by concavity,

$$\forall x \in \left[0, \frac{1}{2}\right], \quad (1-x)\log \frac{1}{1-x} \geq \log 2\ x, \quad \text{and} \quad \log(1-x) \geq -2\log 2\ x.$$

So, for all $a \in [0, \frac{1}{2}]$,

$$(1-a)\log \frac{1}{1-a} + \log \mu_{n,\alpha}\{|x| \leq r\} \geq \log 2 \Big(a - 2F_{n,\phi}(r)\Big) \geq 0,$$

as soon as

$$r \geq \phi^{-1}(2n) \quad \text{and} \quad F_{n,\phi}(r) \leq \frac{a}{2}.$$

Assume that $a \leq \exp(-cn) \wedge \frac{1}{2}$. Then $r \geq \phi^{-1}(Kcn) \geq \phi^{-1}(2n)$ by (4). Let us define the function G by

$$G(a) = \frac{F_{n,\phi}\big(r(a)\big)}{a}.$$

Then (7) holds as soon as $G(a) \leq \frac{1}{2}$. To handle this, it is easier to look on G as a function of r. We know that $a = \exp\left(-\frac{\phi(r)}{K}\right)$. So

$$G(a) = \left(\frac{er}{\phi^{-1}(n)}\right)^n \exp\left(-\phi(r)\Big(1 - \frac{1}{K}\Big)\right).$$

This function is non-increasing in r when

$$r\phi'(r) \geq \frac{n}{1 - \frac{1}{K}}.$$

This is the case if $r \geq \phi^{-1}\left(\frac{Kn}{K-1}\right)$. Moreover $\phi^{-1}(Kcn) \geq \phi^{-1}\left(\frac{Kn}{K-1}\right)$ by (5). Thus, when $a \leq \exp(-cn)$,

$$G(a) \leq G\big(\exp(-cn)\big) \leq \Big[eKc\exp\big(-(K-1)c\big)\Big]^n \leq \frac{1}{2^n} \leq \frac{1}{2}.$$

□

Under (H1), this result is again stronger than the one required since then

$$1 \geq \frac{\sqrt{n}}{\phi^{-1}(n)}\phi^{-1}(1).$$

We could also derive the required inequality under (H2), but with $\sqrt{n}/\phi^{-1}(cn)$ instead of $\sqrt{n}/\phi^{-1}(n)$. So we prefer to prove it directly, following the above proof.

Proposition 12. *For every $c > 0$, there exists $C > 0$ such that for all functions ϕ satisfying (H2),*

$$\forall a \in \left[0, e^{-cn} \wedge \frac{1}{2}\right], \quad \mathrm{Is}_\mu(a) \geq C\frac{\sqrt{n}}{\phi^{-1}(n)}\, a\sqrt{\log\frac{1}{a}},$$

where μ stands for $\mu_{n,\phi}$ or $\nu_{n,\phi}$.

Proof. We set

$$r(a) = \sqrt{\frac{K\big(\phi^{-1}(n)\big)^2}{n}\log\frac{1}{a}}, \tag{8}$$

where K is a constant large enough to verify

$$Kc \geq 2,$$

$$K - 1 \geq \frac{1}{2c},$$

$$e\sqrt{Kc}\exp\left(-(K-1)c\right) \leq \frac{1}{2}.$$

Assume that $a \leq \exp(-cn) \wedge \frac{1}{2}$, then

$$r \geq \sqrt{Kc}\,\phi^{-1}(n) \geq \sqrt{\frac{Kc}{2}}\phi^{-1}(2n) \geq \phi^{-1}(2n).$$

So we can use the estimate from Lemma 10. Consider as before

$$G(a) = \frac{F_{n,\phi}\big(r(a)\big)}{a}.$$

Then, as explained in the proof of Proposition 11, Bobkov's formula (2) yields the required isoperimetric inequality as soon as

$$G(a) \leq \frac{1}{2}.$$

From (8), we deduce

$$a = \exp\left(-\frac{nr^2}{K\big(\phi^{-1}(n)\big)^2}\right).$$

So if we express G as a function of r,

$$G(a) = \left(\frac{er}{\phi^{-1}(n)}\right)^n \exp\left(-\phi(r) + \frac{nr^2}{K\big(\phi^{-1}(n)\big)^2}\right).$$

The derivative $\partial_r G^{\frac{1}{n}}$ is of the same sign as

$$1 + \frac{2r^2}{K\big(\phi^{-1}(n)\big)^2} - \frac{r\phi'(r)}{n}.$$

Under hypothesis (H2), $r\phi'(r) \geq 2\phi(r) \geq 2n\,(r/\phi^{-1}(n))^2$ as soon as $r \geq \phi^{-1}(n)$. Thus, when $r \geq \sqrt{Kc}\,\phi^{-1}(n)$,

$$1 + \frac{2r^2}{K\big(\phi^{-1}(n)\big)^2} - \frac{r\phi'(r)}{n} \leq 1 + \frac{2r^2}{\big(\phi^{-1}(n)\big)^2}\left(\frac{1}{K} - 1\right) \leq 1 + 2Kc\left(\frac{1}{K} - 1\right) \leq 0,$$

since $\frac{1}{K} - 1 < 0$. So G is non-increasing in r when $r \geq \sqrt{Kc}\,\phi^{-1}(n)$, and for all $a \leq \exp(-cn)$, it holds

$$G(a) \leq \left(e\sqrt{Kc}\right)^n \exp\left(cn - \phi\Big(\sqrt{Kc}\,\phi^{-1}(n)\Big)\right) \leq \left[e\sqrt{Kc}\exp\left(-(K-1)c\right)\right]^n \leq \frac{1}{2}.$$

We have again used Hypothesis (H2) which ensures $\phi\Big(\sqrt{Kc}\,\phi^{-1}(n)\Big) \geq Kcn$. $\square$

Combining Proposition 9 for big sets, and Proposition 11 or Proposition 12 for small sets yields Theorem 6.

4 Tensorization and cut-off argument

We derive the isoperimetric inequality for $\mu_{n,\alpha}$ by tensorization from the ones for the radial measure and the uniform probability measure on the sphere, following the idea of the proof by Bobkov of Theorem 1. For that purpose, we need a functional version of our isoperimetric inequality. In [8] and [3], the authors give conditions so that isoperimetric inequalities translate into functional inequalities. Actually this works in our setting as explained in Section 5.

Let $\kappa > 0$. Let $J : [0,1] \to \mathbb{R}^+$ be a continuous convex function symmetric with respect to 1/2, with J(0)=J(1)=0, and such that the following property holds : for any measure μ on $\mathbb{R}^d$ and constant $C \geq 0$, if

$$\mathrm{Is}_\mu \geq CJ,$$

then for all smooth functions $f : \mathbb{R}^d \to [0,1]$,

$$\kappa J\left(\int f\,d\mu\right) \leq \int J(f)\,d\mu + \frac{1}{C}\int |\nabla f|\,d\mu.$$

Remark. Ideally, one would expect $\kappa = 1$. For instance the latter inequality implies the former one and is tight for constant functions only in the case $\kappa = 1$. However this does not matter here as we tensorize only once.

For such profiles J, we can show the following proposition.

Proposition 13. *Let μ be a measure on $\mathbb{R}^n$ with radial measure ν. Assume that there exists positive constants C_ν and $C_{\sigma_{n-1}}$ such that*

$$\mathrm{Is}_\nu \geq C_\nu J \quad \text{and} \quad \mathrm{Is}_{\sigma_{n-1}} \geq C_{\sigma_{n-1}} J.$$

There exist $\kappa_1, \kappa_2 > 0$ depending only on κ such that, for every $n \in \mathbb{N}^*$, for every $r_2 > r_1 > 0$ and a such that

$$r_2 - r_1 \geq \frac{1}{C_\nu J(\frac{1}{2})}, \tag{9}$$

$$\kappa_1\,\nu\{[r_1, +\infty)\} \leq a \leq \frac{1}{2}, \tag{10}$$

it holds

$$\mathrm{Is}_\mu(a) \geq \kappa_2 \min\left(C_\nu, \frac{C_{\sigma_{n-1}}}{r_2}\right) J(a).$$

Proof. Let $f : \mathbb{R}^n \to [0,1]$ be a smooth function. We recall some facts on radial and spherical differentiation. If we define g on $\mathbb{R}^+ \times \mathbb{S}^{n-1}$ by $g(r,\theta) = f(r\theta)$, then the partial derivatives of g can be computed as follows:

$$\partial_r g = \langle \nabla f, \theta \rangle,$$
$$\nabla_\theta g = r\,\Pi_{\theta^\perp}(\nabla f),$$

where $\Pi_{\theta^\perp}$ is the orthogonal projection on $\theta^\perp$. Hence,

$$\nabla f = \partial_r g\, \theta + \frac{1}{r}\nabla_\theta g,$$
$$|\nabla f|^2 = |\partial_r g|^2 + \frac{1}{r^2}|\nabla_\theta g|^2.$$

First, we apply the functional inequality for σ_{n-1} to the function F defined on $\mathbb{S}^{n-1}$ by

$$F(\theta) = \int f(r\theta)\, d\nu(r).$$

As $\int F\, d\sigma_{n-1} = \int f\, d\mu$, this yields

$$\kappa J\left(\int f\, d\mu\right) \le \int J(F)\, d\sigma_{n-1} + \frac{1}{C_{\sigma_{n-1}}}\int |\nabla_{\mathbb{S}^{n-1}} F|\, d\sigma_{n-1}.$$

On one hand,

$$\nabla_{\mathbb{S}^{n-1}} F(\theta) = \int r\, \Pi_{\theta^\perp}(\nabla f)(r\theta)\, d\nu(r).$$

On the other hand, we can use the inequality for ν to bound $J(F)$. Indeed, for all $\theta \in \mathbb{S}^{n-1}$,

$$\kappa J\big(F(\theta)\big) \le \int J\big(f(r\theta)\big)\, d\nu(r) + \frac{1}{C_\nu}\int |\partial_r f(r\theta)|\, d\nu(r).$$

Putting all together,

$$\kappa^2 J\left(\int f\, d\mu\right) \le \int J(f)\, d\mu + \frac{1}{C_\nu}\int |\partial_r f|\, d\mu + \frac{\kappa}{C_{\sigma_{n-1}}}\int |x|\ |\Pi_{\theta^\perp}(\nabla f)|\ d\mu(x). \quad (11)$$

We would like to get $|x|$ out of the last integral. As it is not bounded, we use a cut-off argument similar to the one in Sodin's article [19], while simpler in our case. Heuristically, we use the fact that on "a set of large measure", $|x|$ is almost constant, close to its expectation for instance. Let us introduce a cut-off function $h(r\theta) = h_1(r)$ with

$$h_1 = \begin{cases} 1 & \text{on } [0, r_1) \\ \dfrac{r_2 - r}{r_2 - r_1} & \text{on } [r_1, r_2] \\ 0 & \text{on } (r_2, +\infty) \end{cases}$$

with $0 < r_1 < r_2$ to be chosen later (typically of the same order as $\mathrm{E}_\mu|X|$). It holds

$$\nabla(fh) = h\nabla f + f\nabla h,$$

thus

$$|\partial_r(fh)| \le |\partial_r f| + ||f||_\infty |\partial_r h|,$$
$$|\Pi_{\theta^\perp}\big(\nabla(fh)\big)| \le h\, |\Pi_{\theta^\perp}(\nabla f)|.$$

As $h = 0$ if $|x| > r_2$,

$$\int |x| \left|\Pi_{\theta^\perp}\big(\nabla(fh)\big)\right| \, d\mu(x) \leq r_2 \int |\Pi_{\theta^\perp}(\nabla f)| \, d\mu(x).$$

Besides, we can bound the derivative of h so that

$$\int |\partial_r h| \, d\mu \leq \frac{\nu\big([r_1, r_2]\big)}{r_2 - r_1}.$$

Finally, Inequality (11) applied to fh yields

$$\begin{aligned}
\kappa^2 J\left(\int fh \, d\mu\right) - \int J(fh) \, d\mu - \frac{||f||_\infty \nu\big([r_1, r_2]\big)}{C_\nu(r_2 - r_1)} \\
&\leq \max\left(\frac{1}{C_\nu}, \frac{\kappa r_2}{C_{\sigma_{n-1}}}\right) \left(\int |\partial_r f| + |\Pi_{\theta^\perp}(\nabla f)| \, d\mu\right) \\
&\leq \sqrt{2} \max\left(\frac{1}{C_\nu}, \frac{\kappa r_2}{C_{\sigma_{n-1}}}\right) \int |\nabla f| \, d\mu. \qquad (12)
\end{aligned}$$

Hence we have almost the functional inequality for f and μ with an additional term that we expect to be negligible. It is easier to look at functions approximating characteristic functions to go back from fh to f in the left hand term.

Let $A \subset \mathbb{R}^n$ be a closed set of measure $a \leq \frac{1}{2}$. Let $K > 0$ and $t \in (0, 1)$ constants to be chosen later. Assume the following constraints on r_1, r_2, and a:

$$\begin{aligned}
C_\nu(r_2 - r_1) &\geq K, \\
\nu\{[r_1, +\infty)\} &\leq ta.
\end{aligned}$$

Then it holds

$$\begin{aligned}
&\mu\{\mathbb{1}_A h = 1\} \geq \mu\big(A \setminus \{h < 1\}\big) \geq (1 - t)a, \\
&\mu\{\mathbb{1}_A h > 0\} \leq a \leq \frac{1}{2}.
\end{aligned}$$

As J is non-decreasing on $(0, \frac{1}{2})$, concave, and $J(0) = 0$,

$$J\left(\int \mathbb{1}_A h \, d\mu\right) \geq J\big((1 - t)a\big) \geq (1 - t)J(a).$$

Besides J cancels at 0 and 1, and reaches its maximum at $\frac{1}{2}$, so

$$\begin{aligned}
\int J(\mathbb{1}_A h) \, d\mu &\leq J(\tfrac{1}{2}) \ \mu\{0 < \mathbb{1}_A h < 1\} \\
&\leq J(\tfrac{1}{2})\Big(\mu\{\mathbb{1}_A h > 0\} - \mu\{\mathbb{1}_A h = 1\}\Big) \\
&\leq J(\tfrac{1}{2}) \, ta.
\end{aligned}$$

As for the third term of (12), it is bounded by

$$\frac{\nu\big([r_1, r_2]\big)}{C_\nu(r_2 - r_1)} \leq \frac{ta}{K}.$$

For $\varepsilon > 0$, we approximate $\mathbb{1}_A$ by a smooth function $f_\varepsilon : \mathbb{R}^n \to [0,1]$ with $f_\varepsilon = 1$ on A and $f_\varepsilon = 0$ outside A_ε. Then we apply (12) to f_ε and let ε to 0, taking advantage of the continuity of J:

$$\sqrt{2}\max\Big(\frac{1}{C_\nu}, \frac{\kappa r_2}{C_{\sigma_{n-1}}}\Big)\mu^+(\partial A) \geq \kappa^2(1-t)J(a) - \Big(J(\tfrac{1}{2}) + \frac{1}{K}\Big)\, ta.$$

Now by concavity, $J(a) \geq 2J(\frac{1}{2})a$ on $\big[0, \frac{1}{2}\big]$. Hence

$$\begin{aligned}\sqrt{2}\max\Big(\frac{1}{C_\nu}, \frac{\kappa r_2}{C_{\sigma_{n-1}}}\Big)\mu^+(\partial A) &\geq \Big(\kappa^2(1-t) - \frac{J(\frac{1}{2}) + \frac{1}{K}}{2J(\frac{1}{2})}t\Big) J(a)\\ &= \Big(\kappa^2 - t\Big(\kappa^2 + \frac{1}{2} + \frac{1}{2KJ(\frac{1}{2})}\Big)\Big) J(a).\end{aligned}$$

Taking for instance $K = \big(J(\frac{1}{2})\big)^{-1}$ and $t = \kappa^2/(2(\kappa^2+1))$ yields a non-trivial result.

Note that looking at closed sets was not a real restriction. Indeed, if $\liminf_{\varepsilon \to 0^+} \mu(A_\varepsilon) - \mu(A) > 0$ then $\mu^+(\partial A) = +\infty$. □

5 Getting functional inequalities

To apply Proposition 13 to our case, we need to know how to pass from an isoperimetric inequality to a functional inequality. Actually we can approximate L_ϕ by an other profile satisfying the hypotheses made in Section 4, assuming furthermore that $\sqrt{\phi}$ is concave.

This new profile appears to be the isoperimetric function $\mathrm{Is}_{\mu_{1,\phi}}$ of $\mu_{1,\phi}$, denoted by I_ϕ henceforth.

Lemma 14. *There exist universal constants $d_1 > 0$ and $d_2 > 0$ such that for all ϕ satisfying (H1'),*

$$d_1 I_\phi \leq L_\phi \leq d_2 I_\phi.$$

The second inequality is a consequence of Proposition 2.3 from [18] by Milman and Sodin, up to the uniform estimation of the normalizing constant of $\mu_{1\phi}$. However we give a self-contained proof of Lemma 14 at the end of this section for completeness. In the next lemma, I_ϕ is shown to satisfy the required properties.

Lemma 15. *Let ϕ satisfying (H1').*

i) The function I_ϕ is continuous and concave on $[0,1]$, symmetric with respect to $1/2$, and $I_\phi(0) = I_\phi(1) = 0$.

ii) Let μ be a measure on $\mathbb{R}^d$ and $C \geq 0$. If

$$\mathrm{Is}_\mu \geq C I_\phi,$$

then for all smooth functions $f : \mathbb{R}^d \to [0,1]$,

$$\kappa I_\phi\Big(\int f\, d\mu\Big) \leq \int I_\phi(f)\, d\mu + \frac{1}{C}\int |\nabla f|\, d\mu,$$

where $\kappa > 0$ is a universal constant.

Proof. Let us first remark that $\mu_{1,\phi}$ is an even log-concave probability measure on the real line. Hence half-lines solve the isoperimetric problem and we can express explicitly I_ϕ (see e.g. [7]). Let $f_\phi : x \mapsto \frac{e^{-\phi(|x|)}}{Z_\phi}$ be the density of $\mu_{1,\phi}$, $F_\phi(x) = \mu_{1,\phi}\big\{(-\infty, x)\big\}$ its cumulative distribution function, and $G_\phi(x) = \mu_{1,\phi}\big\{(x, +\infty)\big\}$. Then

$$I_\phi = f_\phi \circ F_\phi^{-1} = f_\phi \circ G_\phi^{-1}$$

and the properties stated in i) are clearly satisfied. Besides the transfer principle emphasized by Barthe in [2] holds : if $\mathrm{Is}_\mu \geq cI_\phi$ then μ satisfies essentially the same functional inequalities as $\mu_{1,\phi}$. As a consequence, it remains to establish that for all smooth functions $f : \mathbb{R} \to [0, 1]$,

$$\kappa I_\phi \left(\int f \, d\mu_{1,\phi} \right) \leq \int I_\phi(f) \, d\mu_{1,\phi} + \int |f'| \, d\mu_{1,\phi}.$$

Now, applying the 2-dimensional isoperimetric inequality to the set

$$\big\{(x, y) \in \mathbb{R}^2 ; y \leq F_\phi^{-1}(f(x))\big\},$$

one can show (see e.g. [3]) that

$$\mathrm{Is}_{\mu_{1,\phi}^{\otimes 2}} \left(\int f \, d\mu_{1,\phi} \right) \leq \int I_\phi(f) \, d\mu_{1,\phi} + \int |f'| \, d\mu_{1,\phi}.$$

So, ii) is shown if there exists a universal $\kappa > 0$ such that

$$\mathrm{Is}_{\mu_{1,\phi}^{\otimes 2}} \geq \kappa I_\phi.$$

Actually, a stronger dimension-free inequality holds and is stated in the next lemma.

Lemma 16. *There exists $\kappa > 0$ such that for all ϕ satisfying (H1') and all n,*

$$\mathrm{Is}_{\mu_{1,\phi}^{\otimes n}} \geq \kappa I_\phi.$$

Barthe, Roberto, and Cattiaux prove it in [5] without verifying the universality of κ. However one can check that their constant can be uniformly controlled for every ϕ satisfying (H1'), by using the same estimates for G_ϕ and Z_ϕ as in the proof of Proposition Lemma 14. Indeed, a Beckner inequality is shown to hold with a constant uniform in ϕ, thanks to their explicit bound. It tensorizes and implies a super-Poincaré with a constant uniform in n and ϕ, which translates into the isoperimetric inequality of Lemma 16.

One can also check the simple criterion given by E. Milman in [16] for a tensorization result. As the function defined by

$$t \mapsto \frac{L_\phi(t)}{L_2(t)} = \frac{\sqrt{\log \frac{1}{t}}}{\phi^{-1}\left(\log \frac{1}{t}\right)}$$

is non-decreasing under (H1) — all the more under (H1') — then by Lemma 14 there exists a universal constant $D > 0$ such that

$$\forall 0 < t \leq s \leq \frac{1}{2}, \quad \frac{I_\phi(t)}{L_2(t)} \leq D \frac{I_\phi(s)}{L_2(s)}.$$

This also implies Lemma 16. So, up to the proof of Lemma 14, we are done. ☐

Proof of Lemma 14. We can restrict ourselves to the case $\phi(1) = 1$. Indeed if we set $\phi_\lambda(x) = \phi(\lambda x)$, one can show $L_{\phi_\lambda} = \lambda L_\phi$ and $I_{\phi_\lambda} = \lambda I_\phi$. This hypothesis ensures that $1 \le \phi'(1) \le 2$ and also that

$$t^2 \le \phi(t) \le t \text{ on } [0,1] \quad \text{ and } \quad t \le \phi(t) \le t^2 \text{ on } [1,+\infty).$$

Let $r \ge 0$. By integration by part,

$$\int_r^{+\infty} e^{-\phi} = \frac{e^{-\phi(r)}}{\phi'(r)} - \int_r^{+\infty} \frac{\phi''}{(\phi')^2} e^{-\phi}.$$

By the properties of ϕ and especially as $(\sqrt{\phi})'' \le 0$,

$$0 \le \int_r^{+\infty} \frac{\phi''}{(\phi')^2} e^{-\phi} \le \int_r^{+\infty} \frac{e^{-\phi}}{2\phi} = \frac{e^{-\phi(r)}}{2\phi(r)\phi'(r)} - \int_r^{+\infty} \frac{(\phi')^2 + \phi\phi''}{2(\phi\phi')^2} e^{-\phi} \le \frac{e^{-\phi(r)}}{2\phi(r)\phi'(r)}.$$

Hence

$$\frac{e^{-\phi(r)}}{\phi'(r)} \left(1 - \frac{1}{2\phi(r)}\right) \le \int_r^{+\infty} e^{-\phi} \le \frac{e^{-\phi(r)}}{\phi'(r)}.$$

In particular, if $r \ge 1$,

$$\frac{e^{-\phi(r)}}{2\phi'(r)} \le \int_r^{+\infty} e^{-\phi} \le \frac{e^{-\phi(r)}}{\phi'(r)}. \tag{13}$$

Now let us estimate the normalizing constant for $\mu_{1,\phi}$, denoted by Z_ϕ.

$$Z_\phi = 2\int_0^{+\infty} e^{-\phi} = 2\left(\int_0^1 e^{-\phi} + \int_1^{+\infty} e^{-\phi}\right) \le 2(1 + e^{-1}).$$

Moreover

$$\int_0^1 e^{-\phi} \ge \int_0^1 e^{-x}\, dx = 1 - e^{-1},$$

so

$$Z_\phi \ge 2(1 - e^{-1}) > 1.$$

By symmetry, we consider the case $a \in \left[0, \frac{1}{2}\right]$. We set $a = G_\phi(r) = \int_r^{+\infty} \frac{e^{-\phi}}{Z_\phi}$. It follows that $r \ge 0$. Then, to prove the lemma, we need only to compare

$$\frac{e^{-\phi(r)}}{Z_\phi} \qquad \text{with} \qquad L_\phi\Big(G_\phi(r)\Big) = G_\phi(r) \frac{\log \frac{1}{G_\phi(r)}}{\phi^{-1}\left(\log \frac{1}{G_\phi(r)}\right)}.$$

Recall that $\phi(t) \le t\phi'(t) \le 2\phi(t)$ under (H1) so that

$$\frac{1}{2}\phi' \circ \phi^{-1}(x) \le \frac{x}{\phi^{-1}(x)} \le \phi' \circ \phi^{-1}(x)$$

and

$$\frac{G_\phi(r)}{2}\, \phi' \circ \phi^{-1}\left(\log \frac{1}{G_\phi(r)}\right) \le L_\phi\Big(G_\phi(r)\Big) \le G_\phi(r)\, \phi' \circ \phi^{-1}\left(\log \frac{1}{G_\phi(r)}\right).$$

Assume first that $r \geq 1$ so that (13) holds. On one hand,

$$\begin{aligned}
L_\phi\Big(G_\phi(r)\Big) &\geq \frac{G_\phi(r)}{2}\,\phi'\circ\phi^{-1}\left(\log\frac{1}{G_\phi(r)}\right)\\
&\geq \frac{e^{-\phi(r)}}{4Z_\phi\phi'(r)}\phi'\circ\phi^{-1}\Big(\log\big(Z_\phi\phi'(r)e^{\phi(r)}\big)\Big)\\
&\geq \frac{e^{-\phi(r)}}{4Z_\phi} \quad \text{since } Z_\phi\phi'(r)\geq 1 \text{ and } \phi'\circ\phi^{-1} \text{ is non-decreasing.}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
L_\phi\Big(G_\phi(r)\Big) &\leq G_\phi(r)\,\phi'\circ\phi^{-1}\left(\log\frac{1}{G_\phi(r)}\right)\\
&\leq \frac{e^{-\phi(r)}}{Z_\phi\phi'(r)}\phi'\circ\phi^{-1}\Big(\log\big(2Z_\phi\phi'(r)e^{\phi(r)}\big)\Big).
\end{aligned}$$

One can show that

$$\begin{aligned}
2Z_\phi\phi'(r)e^{\phi(r)} &\leq 2Z_\phi r\phi'(r)e^{\phi(r)} \leq 4Z_\phi\phi(r)e^{\phi(r)} \leq \phi(4Z_\phi r)e^{\phi(r)}\\
&\leq e^{\phi(4Z_\phi r)}e^{\phi(r)} \leq e^{2\phi(4Z_\phi r)} \leq e^{\phi(8Z_\phi r)}.
\end{aligned}$$

Thus

$$\frac{G_\phi(r)\log\frac{1}{G_\phi(r)}}{\phi^{-1}\left(\log\frac{1}{G_\phi(r)}\right)} \leq \frac{e^{-\phi(r)}}{Z_\phi\phi'(r)}\phi'(8Z_\phi r) \leq 32(1+e^{-1})\frac{e^{-\phi(r)}}{Z_\phi}.$$

Now assume that $r \leq 1$. Let us remark that $L_\phi : a \mapsto \frac{a\log\frac{1}{a}}{\phi^{-1}\left(\log\frac{1}{a}\right)}$ is non-decreasing on $[0,\frac{1}{2}]$. Indeed, ${L_\phi}'$ is of the same sign as

$$(x-1)\phi^{-1}(x)\phi'\circ\phi^{-1}(x)+x, \qquad \text{with } x=\log\frac{1}{a}.$$

If $x>1$,

$$(x-1)\phi^{-1}(x)\phi'\circ\phi^{-1}(x)+x \geq (x-1)x+x \geq 0.$$

Else $x\in[\log 2, 1]$ and

$$(x-1)\phi^{-1}(x)\phi'\circ\phi^{-1}(x)+x \geq (x-1)2x+x = x(2x-1)\geq 0.$$

So

$$\begin{aligned}
L_\phi\Big(G_\phi(r)\Big) \geq L_\phi\Big(G_\phi(1)\Big) &\geq \frac{G_\phi(1)}{2}\,\phi'\circ\phi^{-1}\left(\log\frac{1}{G_\phi(1)}\right)\\
&\geq \frac{e^{-\phi(1)}}{4Z_\phi\phi'(1)}\phi'\circ\phi^{-1}\Big(\log\big(Z_\phi\phi'(1)e^{\phi(1)}\big)\Big)\\
&\geq \frac{e^{-1}}{4Z_\phi} \geq \frac{e^{-1}}{4}\,\frac{e^{-\phi(r)}}{Z_\phi}.
\end{aligned}$$

Similarly for the lower bound,

$$
\begin{aligned}
L_\phi\Big(G_\phi(r)\Big) \le L_\phi\Big(G_\phi(0)\Big) &\le G_\phi(0)\, \phi' \circ \phi^{-1} \left(\log \frac{1}{G_\phi(0)}\right) \\
&\le \frac{1}{2}\phi' \circ \phi^{-1}(\log 2) \le 1 \le \frac{2(1+e^{-1})}{e^{-1}} \, \frac{e^{-\phi(r)}}{Z_\phi}.
\end{aligned}
$$

□

6 Isoperimetry for $\mu_{n,\phi}$

Now we can apply Proposition 13 to $\mu_{n,\phi}$ with $J = I_\phi$ when ϕ satisfies (H1') or $J = \mathrm{Is}_\gamma$ the Gaussian isoperimetric function when ϕ satisfies (H2). Indeed by Theorem 6 and Lemma 14,

$$
\mathrm{Is}_{\nu_{n,\phi}} \ge C_{\nu_{n,\phi}} J
$$

with $C_{\nu_{n,\phi}} = C\phi^{-1}(1)\frac{\sqrt{n}}{\phi^{-1}(n)}$ under (H1) and $C_{\nu_{n,\phi}} = C\frac{\sqrt{n}}{\phi^{-1}(n)}$ under (H2), where $C > 0$ is a universal constant. As for the sphere, it is known that σ_{n-1} satisfies Gaussian isoperimetry with a constant of order $\sqrt{n}$, e.g. by a curvature-dimension criterion (cf [1]). That means that for every $a \le \frac{1}{2}$ and every ϕ satisfying (H1),

$$
\begin{aligned}
\mathrm{Is}_{\sigma_{n-1}}(a) &\ge C\sqrt{n}\,\mathrm{Is}_\gamma(a) \\
&\ge CK\sqrt{n}\, a\sqrt{\log \frac{1}{a}} \\
&\ge CK\sqrt{\log 2}\sqrt{n}\, \phi^{-1}(1)\frac{\log \frac{1}{a}}{\phi^{-1}\left(\log \frac{1}{a}\right)}
\end{aligned}
$$

where $K > 0$ is a universal constant.

Proposition 17. *For every $c > 0$, there exists $C > 0$ such that if $e^{-cn} < \frac{1}{2}$, then for every function ϕ,*

i) if ϕ satisfies (H1') then

$$
\forall a \in \left[e^{-cn}, \frac{1}{2}\right], \quad \mathrm{Is}_{\mu_{n,\phi}}(a) \ge C\frac{\sqrt{n}}{\phi^{-1}(n)}\, \phi^{-1}(1)\frac{a\log\frac{1}{a}}{\phi^{-1}\left(\log\frac{1}{a}\right)}.
$$

ii) if ϕ satisfies (H2) then

$$
\forall a \in \left[e^{-cn}, \frac{1}{2}\right], \quad \mathrm{Is}_{\mu_{n,\phi}}(a) \ge C\frac{\sqrt{n}}{\phi^{-1}(n)}\, a\sqrt{\log\frac{1}{a}}.
$$

Proof. We only prove i). We can restrict ourselves to the case $\phi(1) = 1$. Let κ be the constant coming from Lemma 15, then let κ_1 and κ_2 be the corresponding constants given by Proposition 13. Set c_1 large enough to ensure

$$
c_1 \ge 2,
$$

$$\max(\kappa_1, 1)ec_1e^{-c_1} \leq e^{-c}.$$

If we take $r_1 = \phi^{-1}(c_1 n)$, then we know by Lemma 10 that

$$\kappa_1 \nu_{n,\phi}\{(r_1, +\infty)\} \leq \kappa_1 \left[ec_1e^{-c_1}\right]^n \leq e^{-cn}.$$

Here we use that $\phi^{-1}(c_1 n) \leq c_1 \phi^{-1}(n)$. So for all ϕ, for all n, and all $a \in \left[e^{-cn}, \frac{1}{2}\right]$, Condition (10) holds, i.e.

$$\kappa_1\, \nu_{n,\phi}\{[r_1, +\infty)\} \leq a \leq \frac{1}{2}.$$

Now there exists a universal $C > 0$ such that $C_{\nu_{n,\phi}} \geq C\frac{\sqrt{n}}{\phi^{-1}(n)}$ by Theorem 6 and Lemma 15 as explained at the beginning of the section (recall that here $\phi^{-1}(1) = 1$). So, if we set $r_2 = (1 + \frac{1}{CI_\phi(\frac{1}{2})})\phi^{-1}(c_1 n)$, then Condition (9) is also satisfied, i.e.

$$r_2 - r_1 \geq \frac{1}{C_{\nu_{n,\phi}} I_\phi(\frac{1}{2})}.$$

Thus Proposition 13 yields

$$\mathrm{Is}_{\mu_{n,\phi}}(a) \geq \kappa_2 \min\left(C_{\nu_{n,\phi}}, \frac{C_{\sigma_{n-1}}}{r_2}\right) I_\phi(a).$$

Besides, there exists a universal $d > 0$ such that $I_\phi \geq d\, L_\phi$ according to Lemma 14. In particular,

$$I_\phi\left(\frac{1}{2}\right) \geq d\, L_\phi\left(\frac{1}{2}\right) = \frac{d \log 2}{2\phi^{-1}(\log 2)} \geq \frac{d\sqrt{\log 2}}{2}.$$

We can deduce an upper bound for r_2. Finally, we have established

$$\mathrm{Is}_{\mu_{n,\phi}}(a) \geq \kappa_2 C d \min\left(1, \left[c_1\left(1 + \frac{2}{Cd\sqrt{\log 2}}\right)\right]^{-1}\right) \frac{\sqrt{n}}{\phi^{-1}(n)}\ L_\phi(a).$$

□

Therefore we have proved Theorem 4 at least for a large enough. We complete the proof with Proposition 11 or Proposition 12 for smaller sets.

7 Optimality and the isotropic case

One can ask whether the isoperimetric inequalities obtained are optimal at least up to universal constants, and whether we recover dimension-free results in the case of isotropic measures.

We consider only bounds for the isoperimetric profile constructed as product of a function of n times a function of a. When ϕ satisfies (H1'), inequalities of Theorem 4 are optimal in a for $n = 1$, according to Lemma 14. In the supergaussian case, the central limit theorem for convex bodies of Klartag (see [13]), in the simpler case of spherically symmetric distributions, ensures that we cannot find a profile bounding from below $\mathrm{Is}_{\mu_{n,\phi}}$ for all n, better than the Gaussian one (times a constant depending possibly on n). Else we should have concentration properties stronger than Gaussian. However by Klartag's theorem, there exists a sequence of positive number $\varepsilon_n \to 0$ such that for every Borel set $A \subset \mathbb{R}$ and every $r > 0$,

$$1 - \mu_{n,\phi}\left(\left(A \times \mathbb{R}^{n-1}\right)_r\right) \geq 1 - \gamma(A_r) - \varepsilon_n,$$

where γ denotes the standard normal distribution. Thus we cannot have a rate of concentration valid for all n better than the Gaussian one.

So optimal inequalities should be of the type

$$\forall a \in \left[0, \frac{1}{2}\right], \quad \mathrm{Is}_{\mu_{n,\phi}}(a) \geq C_{\mu_{n,\phi}}(n)\ \phi^{-1}(1) \frac{a \log \frac{1}{a}}{\phi^{-1}\left(\log \frac{1}{a}\right)} \qquad \text{under (H1')},$$

$$\geq C_{\mu_{n,\phi}}(n)\ a \sqrt{\log \frac{1}{a}} \qquad \text{under (H2)}.$$

This implies

$$\forall a \in \left[0, \frac{1}{2}\right], \quad \mathrm{Is}_{\mu_{n,\phi}}(a) \geq c\ C_{\mu_{n,\phi}}(n)\ a,$$

where $c > 0$ is universal. Now Poincaré inequalities are equivalent up to universal constants to Cheeger inequalities (see [17]), so the optimal constant in n should be

$$C_{\mu_{n,\phi}}(n) = C \sqrt{\frac{n}{\mathrm{E}_{\mu_{n,\phi}}\left(|X|^2\right)}},$$

in view of Theorem 1, with $C > 0$ a universal constant.

Thus, the two questions raised at the beginning of the section appear to be connected to the same property, namely $\mathrm{E}_{\mu_{n,\phi}}\left(|X|^2\right) \simeq \left(\phi^{-1}(n)\right)^2$. Undoubtedly, this must be quite standard, nevertheless we state and prove the next lemma for completeness.

Lemma 18. *i) Let ϕ be a function satisfying (H0). Define $r_n(\phi)$ the point where the density of the radial measure $\nu_{n,\phi}$ reaches its maximum, and $\mathrm{E}_{\mu_{n,\phi}}|X|^2$ the second moment of $\mu_{n\phi}$. For every $M > 1$, there exists $n_0 \in \mathbb{N}$ not depending on ϕ such that, for all $n \geq n_0$,*

$$\frac{1}{M}\sqrt{\mathrm{E}_{\mu_{n,\phi}}|X|^2} \leq r_n(\phi) \leq M \sqrt{\mathrm{E}_{\mu_{n,\phi}}|X|^2}.$$

ii) Besides, if there exists $\alpha \geq 1$ such that $x \mapsto \phi(x)/x^\alpha$ is non-increasing, then

$$\phi^{-1}(n) \geq r_n(\phi) \geq e^{-\frac{1}{e}} \phi^{-1}(n).$$

Proof. To prove the first point, we can assume that $\mu_{n,\phi}$ is isotropic, that is to say that $\mathrm{E}_{\mu_{n,\phi}}\left(|X|^2\right) = n$. Let X be a random variable with distribution $\mu_{n,\phi}$. In the following, we denote by $\mathbb{P}$, E, and Var the corresponding probability, esperance, and variance. Let $\delta \in (0, 1)$. In view of Lemma 8, there exist universal constants $c > 0$ and $C > 0$ such that

$$\mathbb{P}\left\{\left|r_n(\phi) - |X|\right| \geq \delta r_n(\phi)\right\} \leq C e^{-cn\delta^2}.$$

On the other hand, Bobkov proved in [10] the following upper bound for the variance of $|X|$ to establish Theorem 1:

$$\mathrm{Var}|X| \leq \frac{\left(\mathrm{E}|X|\right)^2}{n},$$

which can also be reformulate

$$n\mathrm{E}|X|^2 \leq (n+1)\left(\mathrm{E}|X|\right)^2.$$

Then

$$\begin{aligned}\mathrm{E}\left(\sqrt{\mathrm{E}|X|^2} - |X|\right)^2 &= 2\sqrt{\mathrm{E}|X|^2}\left(\sqrt{\mathrm{E}|X|^2} - \mathrm{E}|X|\right)\\ &\leq 2\left(\sqrt{1+\frac{1}{n}} - 1\right)\sqrt{\mathrm{E}|X|^2}\mathrm{E}|X| \leq \frac{\mathrm{E}|X|^2}{n}.\end{aligned}$$

So, by Chebychev's inequality it holds for all $t > 0$:

$$\mathbb{P}\left\{\left|\sqrt{\mathrm{E}|X|^2} - |X|\right| \geq t\sqrt{\mathrm{E}|X|^2}\right\} \leq \frac{1}{nt^2}.$$

Fix $\delta \in (0,1)$, and choose n large enough to ensure $Ce^{-cn\delta^2} + 1/n\delta^2 < 1$. Then there exist $x > 0$ such that $|r_n(\phi) - x| \leq \delta r_n(\phi)$ and $|\sqrt{\mathrm{E}|X|^2} - x| \leq \delta\sqrt{\mathrm{E}|X|^2}$. It follows

$$\frac{1-\delta}{1+\delta}\sqrt{n} \leq r_n(\phi) \leq \frac{1+\delta}{1-\delta}\sqrt{n}.$$

Now, $r_n(\phi)$ satisfies $r_n(\phi)\phi'\big(r_n(\phi)\big) = n-1$. Therefore, as already mentioned, (H0) ensures that $r_n(\phi) \leq \phi^{-1}(n)$. Assume moreover the existence of $\alpha \geq 1$ such that $x \mapsto \phi(x)/x^\alpha$ is non-increasing. Then

$$r_n(\phi) \geq \phi^{-1}\left(\frac{n-1}{\alpha}\right) \geq \phi^{-1}\left(\frac{n}{2\alpha}\right) \geq \left(\frac{1}{2\alpha}\right)^{\frac{1}{2\alpha}}\phi^{-1}(n) \geq e^{-\frac{1}{e}}\phi^{-1}(n).$$

□

Eventually, we can state the following theorem.

Theorem 19.

- *If ϕ satisfies $(H1')$ or if ϕ satisfies $(H2')$, then the inequality proved in Theorem 4 is optimal.*
- *For any $n \in \mathbb{N}$, let us choose $\lambda > 0$ such that μ_{n,ϕ_λ} is isotropic, when replacing ϕ by $\phi_\lambda : x \mapsto \phi(\lambda x)$. Then it holds a dimension-free isoperimetric inequality. More precisely, there exist a universal $C > 0$ and a universal $n_0 \in \mathbb{N}$ such that*

  i) if ϕ satisfies $(H1')$ then

  $$\forall a \in \left[0, \frac{1}{2}\right], \quad \forall n \geq n_0, \quad \mathrm{Is}_{\mu_{n,\phi_\lambda}}(a) \geq C\ \phi^{-1}(1)\frac{a\log\frac{1}{a}}{\phi^{-1}\left(\log\frac{1}{a}\right)};$$

  ii) if ϕ satisfies $(H2')$, then

  $$\forall a \in \left[0, \frac{1}{2}\right], \quad \forall n \geq n_0, \quad \mathrm{Is}_{\mu_{n,\phi_\lambda}}(a) \geq C\ a\sqrt{\log\frac{1}{a}}.$$

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