Non-connective K-theory via universal invariants

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by

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Abstract. — In this article, we further the study of higher K-theory of dg categories via universal invariants, initiated in [34]. Our main result is the co-representability of non-connective K-theory by the base ring in the 'universal localizing motivator'. As an application, we obtain for free higher Chern characters, resp. higher trace maps, from negative K-theory to cyclic homology, resp. to topological Hochschild homology.

Résumé **(K-théorie négative via les invariants universels).** — Dans cet article, nous poursuivons l'étude de la K-théorie supérieure via une théorie des invariants universels, initiée dans [34]. Notre résultat principal est la coreprésentabilité de la K-théorie non connective par l'anneau de base dans le 'motivateur universel localisant'. En guise d'application, on obtient immédiatement de la sorte une construction des caractères de Chern, resp. des morphismes traces, de la K-théorie négative vers l'homologie cyclique, resp. vers l'homologie de Hochschild topologique.

Contents

Key words and phrases. — Negative K-theory, dg categories, Grothendieck derivators, higher Chern characters, higher trace maps, non-commutative algebraic geometry.

Introduction

Dg categories. — A *differential graded (=dg) category*, over a commutative base ring k, is a category enriched over cochain complexes of k-modules (morphisms sets are such complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} f \circ (dg)$. Dg categories enhance and solve many of the technical problems inherent to triangulated categories; see Keller's ICM-talk [24]. In *non-commutative algebraic geometry* in the sense of Bondal, Drinfeld, Kapranov, Kontsevich, Toën, Van den Bergh, ... [4] [5] [13] [14] [25] [26] [37], they are considered as dg-enhancements of (bounded) derived categories of quasi-coherent sheaves on a hypothetic non-commutative space.

Additive/Localizing invariants. — All the classical (functorial) invariants, such as algebraic K-theory, Hochschild homology, cyclic homology, and even topological Hochschild homology and topological cyclic homology (see [36, §10]), extend naturally from k-algebras to dg categories. In a "motivic spirit", in order to study *all* these classical invariants simultaneously, the notions of *additive* and *localizing* invariant have been introduced in [34]. These notions, that we now recall, make use of the language of Grothendieck derivators [21], a formalism which allows us to state and prove precise universal properties and to dispense with many of the technical problems one faces in using model categories; consult Appendix A.

Let $E : \mathsf{HO}(\mathsf{dgcat}) \to \mathbb{D}$ be a morphism of derivators, from the pointed derivator associated to the Morita model structure (see §2.5), to a strong triangulated derivator (in practice, $\mathbb{D}$ will be the derivator associated to a stable model category $\mathcal{M}$, and E will come from a functor $\mathsf{dgcat} \to \mathcal{M}$). We say that E is an *additive invariant* if it commutes with filtered homotopy colimits, preserves the terminal object and

add) sends the *split exact sequences* on the left (see 3.1)

$$\mathcal{A} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{B} \underset{P}{\overset{S}{\leftrightarrows}} \mathcal{C} \qquad\qquad [E(I)\ E(S)] : E(\mathcal{A}) \oplus E(\mathcal{C}) \xrightarrow{\sim} E(\mathcal{B})$$

to the direct sums on the right.

We say that E is a *localizing invariant* if it satisfies the same conditions as an additive invariant but with condition add) replaced by:

loc) sends the *exact sequences* on the left (see 3.1)

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C} \qquad\qquad E(\mathcal{A}) \xrightarrow{E(I)} E(\mathcal{B}) \xrightarrow{E(P)} E(\mathcal{C}) \longrightarrow E(\mathcal{A})[1]$$

to the distinguished triangles on the right.

In [34], the second named author have constructed the *universal additive invariant* (on the left) and the *universal localizing invariant* (on the right)

$$\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}} \qquad\qquad \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}} .$$

Roughly, every additive (resp. localizing) invariant $E : \mathsf{HO}(\mathsf{dgcat}) \to \mathbb{D}$ factors uniquely through $\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}$ (resp. $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}$); see Theorem 7.12. Notice that a localizing invariant is also an additive invariant, but the converse does not hold. Because of this universality property, which is a reminiscence of motives, $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ is called the *additive motivator* and $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ the *localizing motivator*. Recall that they are triangulated derivators.

Before going further, note that any triangulated derivator is canonically enriched over spectra; see §A.3. The spectra of morphisms in a triangulated derivator $\mathbb{D}$ we will be denoted by $\mathbb{R}\mathsf{Hom}(-,-)$. The main result from [34] is the following :

Theorem. — *([34, Thm. 15.10]) For every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(\mathcal{A})) \simeq K(\mathcal{A}),$$

where $\underline{k}$ is the dg category with a single object and k as the dg algebra of endomorphisms, and $K(\mathcal{A})$ is the Waldhausen's K-theory spectrum of $\mathcal{A}$ (see §2.6).

However, from the "motivic" point of view this co-representability result is not completely satisfactory, in the sense that all the classical invariants (except Waldhausen's K-theory) satisfy not only property add), but also the more subtle localizing property loc). Therefore, the base category $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$ of the localizing motivator is "morally" the category of *non-commutative motives*. The importance of the computation of the (spectra of) morphisms between two objects in the localizing motivator is by now obvious. Hence, a fundamental problem of the theory of non-commutative motives is to compute the localizing invariants (co-)represented by dg categories in $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$. In particular, we would like a co-representability theorem analogous to the preceding one, with $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ replaced by $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ and $K(\mathcal{A})$ replaced by the non-connective K-theory spectrum $I\!K(\mathcal{A})$ (see §2.6). The main result of this paper is a proof of the co-representability of non-connective K-theory by the base ring k :

Theorem. — *(Thm. 7.15) For every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})) \simeq I\!K(\mathcal{A}).$$

In particular, we obtain isomorphisms of abelian groups

$$\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k})[n], \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})) \simeq I\!K_n(\mathcal{A}),\ n \in \mathbb{Z}.$$

Negative K-theory goes back to the work of Bass, Karoubi, Pedersen-Weibel, Thomason-Trobaugh, Schlichting, ... [2] [23] [29] [30] [39] [32], and has been the source of many deep results "amalgamated" with somewhat ad-hoc constructions. This co-representability Theorem radically changes this state of affairs by offering, to the best of the authors knowledge, the first conceptual characterization of negative K-theory. It provides a completely new understanding of non-connective K-theory as the universal construction, with values in a stable context, which preserves filtered homotopy colimits and satisfies the localization property. It follows from this result and from the Yoneda Lemma that, for any localizing invariant $E : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathsf{Spt})$ with values in the triangulated derivator of spectra, the data of natural maps $I\!K(\mathcal{A}) \to E(\mathcal{A})$ are equivalent to the datum of a class in the stable homotopy group $E_0(k) = \pi_0(E(\underline{k}))$; see Theorem 8.1.

Applications. — Let

$$\mathbb{K}_n(-), HC_j(-), THH_j(-) : \mathsf{Ho}(\mathsf{dgcat}) \longrightarrow \mathsf{Mod}\text{-}\mathbb{Z}\,,\ n \in \mathbb{Z}, j \geq 0$$

be respectively, the n-th algebraic K-theory group functor, the j-th cyclic homology group functor and the j-th topological Hochschild homology group functor; see Section 8. The co-representability Theorem furnishes for free higher Chern characters and trace maps:

Theorem. — *(Thm. 8.4) We have the following canonical morphisms of abelian groups for every dg category $\mathcal{A}$:*

(i) *Higher Chern characters*

$$ch_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow HC_{n+2r}(\mathcal{A})\,,\ n \in \mathbb{Z}\,,\ r \geq 0\,,$$

such that $ch_{0,r} : \mathbb{K}_0(k) \longrightarrow HC_{2r}(k)$ sends $1 \in \mathbb{K}_0(k)$ to a generator of the k-module of rank one $HC_{2r}(k)$.

(ii) *When $k = \mathbb{Z}$, higher trace maps*

$$tr_n : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_n(\mathcal{A})\,, n \in \mathbb{Z}\,,$$

such that $tr_0 : \mathbb{K}_0(k) \longrightarrow THH_0(k)$ sends $1 \in \mathbb{K}_0(\mathbb{Z})$ to $1 \in THH_0(\mathbb{Z})$, and

$$tr_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_{n+2r-1}(\mathcal{A})\,, n \in \mathbb{Z}\,,\ r \geq 1\,,$$

such that $tr_{0,r} : \mathbb{K}_0(k) \longrightarrow THH_{2r-1}(k)$ sends $1 \in \mathbb{K}_0(\mathbb{Z})$ to a generator in the cyclic group $THH_{2r-1}(\mathbb{Z}) \simeq \mathbb{Z}/r\mathbb{Z}$.

(iii) *When $k = \mathbb{Z}/p\mathbb{Z}$, with p a prime number, higher trace maps*

$$tr_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_{n+2r}(\mathcal{A})\,, n, r \in \mathbb{Z}\,,$$

such that $tr_{0,r} : \mathbb{K}_0(k) \longrightarrow THH_{2r}(k)$ sends $1 \in \mathbb{K}_0(\mathbb{Z})$ to a generator in the cyclic group $THH_0(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.

Before explaining the main ideas behind the proof of the co-representability Theorem, we would like to emphasize that, as condition loc) is much more subtle than condition add), the tools and arguments used in the proof of co-representability result in the additive motivator, *are not available* when we work with the localizing motivator.

Strategy of the proof. — The proof is based on a rather "orthogonal" construction of the localizing motivator $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$. A crucial ingredient is the observation that we should start, not with the notion of Morita dg functor, but with the weaker notion of quasi-equiconic dg functor (which means that we want to work with pre-triangulated dg categories which are not necessarily idempotent complete; see §2.4). The reason for this is that negative K-theory is, in particular, an obstruction for the (triangulated) quotients to be karoubian (see [32]), which is a difficult aspect to see when we work with the notion of Morita dg functor. The other difficulty consists in understanding the localizing invariants associated to additive ones: Schlichting's construction of the non-connective K-theory spectrum is general enough to be applied to a wider class of functors than connective K-theory, and we would like to use it to understand as explicitely as possible $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ as a left Bousfield localization of $\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}$. One crucial step in Schlichting's construction is the study of a flasque resolution (of dg categories), obtained as a completion by countable sums. The problem with this completion functor is that it doesn't preserves filtered (homotopy) colimits. Our strategy consists in skirting this problem by considering invariants of dg categories which only preserve α-filtered homotopy colimits, for a big enough cardinal α. Through a careful analysis of Schlichting's construction, we prove a co-representability result in this wider setting, and then use the nice behaviour of non-connective K-theory to solve the problem with filtered homotopy colimits at the end.

Here is a more detailed account on the contents of the paper: in the first two Sections we recall some basic results and constructions concerning dg categories.

Let $\mathsf{Trdgcat}$ be the derivator associated with the quasi-equiconic model structure, α a regular cardinal and $\mathcal{A}$ a (fixed) dg category. In Section 3, we construct the *universal α-additive invariant*

$$\underline{\mathcal{U}}_\alpha^{\mathsf{add}} : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} \qquad \text{(see Theorem 3.2)}$$

(*i.e.* $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ sends quasi-equiconic dg functors to isomorphisms, preserves the terminal object and α-filtering homotopy colimits, satisfies condition add) and is universal for these properties). We then prove the identification of spectra

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})) \simeq \underline{K}(\mathcal{A}) \qquad \text{(see Theorem 3.7)}\,,$$

where $\underline{K}(\mathcal{A})$ denotes the Waldhausen's K-theory of the pre-triangulated dg category (associated to) $\mathcal{A}$. In Section 5, we introduce the class of *strict exact sequences* (see 5.1):

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A}\,,$$

where $\mathcal{B}$ is a pre-triangulated dg category, $\mathcal{A}$ is a *thick* dg subcategory of $\mathcal{B}$, and $\mathcal{B}/\mathcal{A}$ is the Drinfeld dg quotient [13] of $\mathcal{B}$ by $\mathcal{A}$. We localize the derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$ to force the strict exact sequences to become distinguished triangles, and obtain an homotopy colimit preserving morphism of triangulated derivators

$$\gamma_! : \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}\,.$$

Using Theorem 3.7 and Waldhausen's fibration Theorem we have:

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) \simeq \underline{K}(\mathcal{A}) \qquad \text{(see Theorem 5.7)}\,.$$

Then, we localize the derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ so that the Morita dg functors become invertible, which leads to a new homotopy colimit preserving morphism of triangulated derivators $l_! : \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} \to \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$. The derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$ is the analog of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$: in the case $\alpha = \aleph_0$, we even have, by definition, $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} = \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$.

In Section 6, we construct a set-up on $\mathsf{Trdgcat}$ (see Proposition 6.5) and relate it to Schlichting's set-up on Frobenius pairs; see Proposition 6.6. Using this set-up, we construct in Section 7 a morphism of derivators $V_l(-) : \mathsf{Trdgcat} \to \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$, for which, if α is big enough, we have:

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A})) \simeq I\!\!K(\mathcal{A}) \qquad \text{(see Proposition 7.5)}\,.$$

A key technical point is the fact that $V_l(-)$ is an α-localizing invariant; see Proposition 7.7. This allows us to prove the identification (still for α big enough):

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})\,) \simeq I\!K(\mathcal{A}) \qquad\qquad \text{(see Theorem 7.11)}\,.$$

In Proposition 7.14, we prove that the localizing motivator $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ can be obtained from $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$ by localizing it with respect to the the morphisms of shape

$$\underset{j\in J}{\mathrm{hocolim}}\, \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(D_j) \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})\,,$$

where $\mathcal{A}$ is a dg category, J is a filtered category, and $D : J \to \mathsf{dgcat}$ is a functor such that $\underset{j\in J}{\mathrm{hocolim}}\, D_j \simeq \mathcal{A}$. Let us now sum up the steps of the construction of the universal localizing motivator:

$$\begin{array}{ccccccc}
\mathsf{Trdgcat} & & \longrightarrow & & & & \mathsf{HO}(\mathsf{dgcat}) \\
\downarrow {\scriptstyle \underline{\mathcal{U}}_\alpha^{\mathsf{add}}} & & & & & & \downarrow {\scriptstyle \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}} \\
\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} & \xrightarrow{\gamma_!} & \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} & \xrightarrow{l_!} & \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} & \longrightarrow & \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}
\end{array}$$

Finally, using Theorem 7.11 and the fact that negative K-theory preserves filtered homotopy colimits, we prove the co-representability Theorem.

The last Section is devoted to the construction of Chern characters and trace maps. We have also included an Appendix containing several results on Grothendieck derivators, used throughout the article and which are also of independent interest.

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1. Preliminaries

1.1. Notations. — Throughout the article we work over a fixed commutative base ring k.

We will use freely the language and basic results of the theory of Grothendieck derivators. A short reminder of our favourite tools and notations can be found in the Appendix. If $\mathcal{M}$ is a model category, we denote by $\mathsf{HO}(\mathcal{M})$ its associated derivator; see §A.2. We will write e for the terminal category, so that, for a derivator $\mathbb{D}$, $\mathbb{D}(e)$ will be its underlying category (for instance, if $\mathbb{D} = \mathsf{HO}(\mathcal{M})$, $\mathbb{D}(e)$ is the usual homotopy category of the model category $\mathcal{M}$).

As any triangulated derivator $\mathbb{D}$ is canonically enriched over spectra (see §A.3), we will denote by $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X,Y)$ the spectrum of maps from X to Y in $\mathbb{D}$. If there is not any ambiguity in doing it, we will simplify the notations by writing $\mathbb{R}\mathsf{Hom}(X,Y) = \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X,Y)$.

Throughout the article the adjunctions are displayed vertically with the left (resp. right) adjoint on the left (resp. right) hand-side.

1.2. Triangulated categories. — For the basic notions concerning triangulated categories, consult Neeman's book [27]. We denote by Tri the category of triangulated categories. Recall that a sequence of triangulated categories

$$\mathcal{R} \xrightarrow{I} \mathcal{S} \xrightarrow{P} \mathcal{T}$$

is called *exact* if the composition is zero, the functor $\mathcal{R} \stackrel{I}{\to} \mathcal{S}$ is fully-faithful and the induced functor from the Verdier quotient $\mathcal{S}/\mathcal{R}$ to $\mathcal{T}$ is *cofinal*, *i.e.* it is fully faithful and every object in $\mathcal{T}$ is a direct summand of an object of $\mathcal{S}/\mathcal{R}$. An exact sequence of triangulated categories is called *split exact*, if there exists triangulated functors

$$\mathcal{S} \xrightarrow{R} \mathcal{R} \quad \text{and} \quad \mathcal{T} \xrightarrow{S} \mathcal{S}\,,$$

with R right adjoint to I, S right adjoint to P and $P \circ S = Id_{\mathcal{T}}$ and $R \circ I = Id_{\mathcal{R}}$ via the adjunction morphisms. Finally, recall from [1] the *idempotent completion* $\widetilde{\mathcal{T}}$ of a triangulated category $\mathcal{T}$. Notice that the idempotent completion of a split exact sequence is split exact.

2. Homotopy theories of dg categories

In this Section, we review the three homotopy theories of dg categories developped in [33, §1-2]. For a survey article on dg categories see [24]. Recall that a dg category (over our base ring k) is a category enriched over cochain complexes of k-modules in such a way that composition fulfills the Leibniz rule : $d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} f \circ (dg)$. We will denote by dgcat the category of small dg categories.

2.1. Dg cells. — Given a dg category $\mathcal{A}$ and two objects x and y, we denote by $\mathcal{A}(x,y)$ its complex of morphisms from x to y. Let $\underline{k}$ be the dg category with one object $*$ and such that $\underline{k}(*,*) := k$ (in degree zero). For $n \in \mathbb{Z}$, let S^n be the complex $k[n]$ (with k concentrated in degree n) and let D^n be the mapping cone on the identity of S^{n-1}. We denote by $\mathcal{S}(n)$ the dg category with two objects 1 et 2 such that $\mathcal{S}(n)(1,1) = k,\ \mathcal{S}(n)(2,2) = k,\ \mathcal{S}(n)(2,1) = 0,\ \mathcal{S}(n)(1,2) = S^n$ and composition given by multiplication. We denote by $\mathcal{D}(n)$ the dg category with two objects 3 and 4 such that $\mathcal{D}(n)(3,3) = k$, $\mathcal{D}(n)(4,4) = k$, $\mathcal{D}(n)(4,3) = 0$, $\mathcal{D}(n)(3,4) = D^n$ and with composition given by multiplication. Let $\iota(n) : \mathcal{S}(n-1) \to \mathcal{D}(n)$ be the dg functor that sends 1 to 3, 2 to 4 and S^{n-1} to D^n by the identity on k in degree $n-1$. We denote by I the set consisting of the dg functors $\{\iota(n)\}_{n\in\mathbb{Z}}$ and the dg functor $\emptyset \to \underline{k}$ (where the empty dg category $\emptyset$ is the initial one). This is the set of generating cofibrations of the three Quillen model structures on dgcat that we describe below. A dg category $\mathcal{A}$ is a *dg cell* if the map $\emptyset \to \mathcal{A}$ is transfinite composition of pushouts of elements of I. Given an infinite regular cardinal α [19, 10.1.10], we will say that a dg category $\mathcal{A}$ is an *α-small dg cell* if it is a dg cell and if $\mathcal{A}$ is α-small (which means that the functor $\mathsf{Hom}_{\mathsf{dgcat}}(\mathcal{A}, -)$ preserves α-filtered colimits).

2.2. Dg modules. — Let $\mathcal{A}$ be a small dg category. The *opposite dg category* $\mathcal{A}^{\mathsf{op}}$ of $\mathcal{A}$ has the same objects as $\mathcal{A}$ and its complexes of morphisms are defined by $\mathcal{A}^{\mathsf{op}}(x,y) = \mathcal{A}(y,x)$. Recall from [24, 3.1] that a *right dg $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(k)$, with values in the dg category $\mathcal{C}_{\mathsf{dg}}(k)$ of complexes of k-modules. We denote by $\mathcal{C}(\mathcal{A})$ (resp. by $\mathcal{C}_{\mathsf{dg}}(\mathcal{A})$) the category (resp. dg category) of dg $\mathcal{A}$-modules. Notice that $\mathcal{C}(\mathcal{A})$ is naturally endowed with a (cofibrantly generated) Quillen model structure whose weak equivalences and fibrations are defined objectwise. We denote by $\mathcal{D}(\mathcal{A})$ the *derived category of* $\mathcal{A}$, *i.e.* the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. Consider also the *Yoneda dg functor*

$$\begin{array}{rcl} \underline{h} : \mathcal{A} & \longrightarrow & \mathcal{C}_{\mathsf{dg}}(\mathcal{A}) \\ x & \mapsto & \mathcal{A}(-,x)\,, \end{array}$$

which sends an object x to the dg $\mathcal{A}$-module $\mathcal{A}(-,x)$ *represented by* x. Let $\mathsf{tri}(\mathcal{A})$ (resp. $\mathsf{perf}(\mathcal{A})$) be the smallest (resp. smallest thick) subcategory of $\mathcal{D}(\mathcal{A})$ which contains the right dg $\mathcal{A}$-modules $\underline{h}(x)$, with $x \in \mathcal{A}$. A dg functor $F : \mathcal{A} \to \mathcal{B}$ gives rise to a *restriction/extension of scalars* Quillen adjunction (on the left)

$$\begin{array}{ccc} \mathcal{C}(\mathcal{B}) & & \mathcal{D}(\mathcal{B}) \\ {\scriptstyle F_!}\uparrow\downarrow{\scriptstyle F^*} & & {\scriptstyle \mathbb{L}F_!}\uparrow\downarrow{\scriptstyle F^*} \\ \mathcal{C}(\mathcal{A}) & & \mathcal{D}(\mathcal{A})\,, \end{array}$$

which can be naturally derived (on the right). Moreover the derived extension of scalars functor $\mathbb{L}F_!$ restricts to two triangulated functors: $\mathsf{tri}(\mathcal{A}) \to \mathsf{tri}(\mathcal{B})$ and $\mathsf{perf}(\mathcal{A}) \to \mathsf{perf}(\mathcal{B})$.

2.3. Quasi-equivalences. — Let $\mathcal{A}$ be a dg category. The *homotopy category* $\mathsf{H}^0(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $\mathcal{A}$ and morphisms $\mathsf{H}^0(\mathcal{A})(x,y) = \mathrm{H}^0(\mathcal{A}(x,y))$.

Definition 2.1. — A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a *quasi-equivalence* if:

(i) for all objects $x, y \in \mathcal{A}$, the map

$$F(x,y) : \mathcal{A}(x,y) \longrightarrow \mathcal{B}(Fx, Fy)$$

is a quasi-isomorphism and

(ii) the induced functor $\mathsf{H}^0(F) : \mathsf{H}^0(\mathcal{A}) \to \mathsf{H}^0(\mathcal{B})$ is an equivalence.

By [33, Thm. 1.8] the category dgcat carries a (cofibrantly generated) model structure whose weak equivalences are the quasi-equivalences. We will denote by Hqe the homotopy theory hence obtained. With respect to this model structure all dg categories are fibrant. Moreover Hqe is naturally endowed with a derived tensor monoidal structure $- \otimes^{\mathbb{L}} -$, which admits an internal Hom-functor

$$\mathsf{rep}(-,-) : \mathsf{Hqe}^{\mathsf{op}} \times \mathsf{Hqe} \longrightarrow \mathsf{Hqe}\,.$$

See [37, Thm. 6.1] for details concerning the construction of the internal Hom-functor $\mathsf{rep}(-,-)$ (denoted by $\mathbb{R}\underline{\mathsf{Hom}}(-,-)$ in *loc. cit.*).

2.4. Quasi-equiconic dg functors. —

Definition 2.2. — A dg functor $F : \mathcal{A} \to \mathcal{B}$ is *quasi-equiconic* if the induced triangulated functor $\mathsf{tri}(\mathcal{A}) \to \mathsf{tri}(\mathcal{B})$ is an equivalence.

By [33, Thm. 2.2] the category dgcat carries a (cofibrantly generated) model structure whose weak equivalences are the quasi-equiconic dg functors. We will denote by Hec the homotopy theory hence obtained. We have then a well-defined functor

$$\begin{array}{rcl} \mathsf{tri} : \mathsf{Hec} & \longrightarrow & \mathsf{Tri} \\ \mathcal{A} & \mapsto & \mathsf{tri}(\mathcal{A})\,. \end{array}$$

The fibrant dg categories, which respect to the quasi-equiconic model structure, will be called *pre-triangulated*. By [33, Prop. 2.10] these correspond to the dg categories $\mathcal{A}$ for which the image of the Yoneda embedding $\mathsf{H}^0(\mathcal{A}) \hookrightarrow \mathcal{D}(\mathcal{A})$ is stable under (co)suspensions and cones, or equivalently for which we have an equivalence $\mathsf{H}^0(\mathcal{A}) \stackrel{\sim}{\to} \mathsf{tri}(\mathcal{A})$. Finally, the monoidal structure $-\otimes^{\mathbb{L}}-$ on Hqe descends to Hec and the internal Hom-functor $\mathsf{rep}(-,-)$ can be naturally derived

$$\mathsf{rep}_{tr}(-,-) : \mathsf{Hec}^{\mathsf{op}} \times \mathsf{Hec} \longrightarrow \mathsf{Hec}\,.$$

2.5. Morita dg functors. —

Definition 2.3. — A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a *Morita dg functor* if the derived extension of scalars functor $F_! : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is an equivalence.

By [33, Thm. 2.27] the category dgcat carries a (cofibrantly generated) model structure whose weak equivalences are the Morita dg functors. We will denote by Hmo the homotopy theory hence obtained. Since $\mathbb{L}F_!$ is an equivalence if and only if $\mathsf{perf}(\mathcal{A}) \to \mathsf{perf}(\mathcal{B})$ is an equivalence, we have a well-defined functor

$$\begin{array}{rcl} \mathsf{perf} : \mathsf{Hmo} & \longrightarrow & \mathsf{Tri} \\ \mathcal{A} & \mapsto & \mathsf{perf}(\mathcal{A})\,. \end{array}$$

The fibrant dg categories, which respect to the Morita model structure, will be called *Morita fibrant*. By [33, Prop. 2.34] these correspond to the dg categories $\mathcal{A}$ for which the image of the Yoneda embedding $\mathsf{H}^0(\mathcal{A}) \hookrightarrow \mathcal{D}(\mathcal{A})$ is stable under (co)suspensions, cones and direct factors, or equivalently for which we have an equivalence $\mathsf{H}^0(\mathcal{A}) \stackrel{\sim}{\to} \mathsf{perf}(\mathcal{A})$. Finally, the monoidal structure $-\otimes^{\mathbb{L}}-$ on Hec descends to Hmo and the internal Hom-functor $\mathsf{rep}_{tr}(-,-)$ can be naturally derived

$$\mathsf{rep}_{mor}(-,-) : \mathsf{Hmo}^{\mathsf{op}} \times \mathsf{Hmo} \longrightarrow \mathsf{Hmo}\,.$$

2.6. K-theory. — Let $\mathcal{A}$ be a dg category. We denote by $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$ (resp. by $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$) the full subcategory of $\mathcal{C}(\mathcal{A})$ formed by the right dg $\mathcal{A}$-modules which become isomorphic in $\mathcal{D}(\mathcal{A})$ to elements of $\mathsf{tri}(\mathcal{A})$ (resp. of $\mathsf{perf}(\mathcal{A})$), and which are moreover cofibrant. Notice that $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$ and $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$ are naturally endowed with a Waldhausen structure induced by the one on $\mathcal{C}(\mathcal{A})$; see [15, §3] or [11]. Moreover, if $F : \mathcal{A} \to \mathcal{B}$ is a dg functor, its extension of scalars functor $F_!$ restricts to two Waldhausen functors: $\mathsf{tri}^{\mathcal{W}}(\mathcal{A}) \to \mathsf{tri}^{\mathcal{W}}(\mathcal{B})$ and $\mathsf{perf}^{\mathcal{W}}(\mathcal{A}) \to \mathsf{perf}^{\mathcal{W}}(\mathcal{B})$. Notice that if F is a quasi-equiconic (resp. a Morita dg functor), the Waldhausen functor $\mathsf{tri}^{\mathcal{W}}(\mathcal{A}) \to$

$\mathsf{tri}^{\mathcal{W}}(\mathcal{B})$ (resp. $\mathsf{perf}^{\mathcal{W}}(\mathcal{A}) \to \mathsf{perf}^{\mathcal{W}}(\mathcal{B})$) is an equivalence. We denote by $\underline{K}(\mathcal{A})$ (resp. by $K(\mathcal{A})$) the Waldhausen K-theory spectrum [40] of $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$ (resp. of $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$). We have two well-defined functors:

$$\begin{array}{rcl} \underline{K} : \mathsf{Hec} & \longrightarrow & \mathsf{Ho}(\mathsf{Spt}) \\ \mathcal{A} & \mapsto & K(\mathsf{tri}^{\mathcal{W}}(\mathcal{A})) \end{array} \qquad \begin{array}{rcl} K : \mathsf{Hmo} & \longrightarrow & \mathsf{Ho}(\mathsf{Spt}) \\ \mathcal{A} & \mapsto & K(\mathsf{perf}^{\mathcal{W}}(\mathcal{A}))\,. \end{array}$$

Finally, we denote by $I\!\!K(\mathcal{A})$ the non-connective K-theory spectrum [32, §12.1] associated to the Frobenius category $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$ (see §6.2).

3. Additive invariants

In this Section, we re-prove the main Theorems of [34], using the quasi-equiconic model structure instead of the Morita model structure. Moreover, and in contrast with [34], we work with a fixed infinite regular cardinal α: we want to study additive invariants which preserve only α-filtered homotopy colimits of dg categories.

Definition 3.1. — A sequence in Hec (resp. in Hmo)

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C}$$

is called *exact* if the induced sequence of triangulated categories on the left (resp. on the right)

$$\mathsf{tri}(\mathcal{A}) \longrightarrow \mathsf{tri}(\mathcal{B}) \longrightarrow \mathsf{tri}(\mathcal{C}) \qquad \mathsf{perf}(\mathcal{A}) \longrightarrow \mathsf{perf}(\mathcal{B}) \longrightarrow \mathsf{perf}(\mathcal{C})$$

is exact. A *split exact sequence* in Hec (resp. in Hmo) is an exact sequence in Hec (resp. in Hmo), which is equivalent to one of the form

$$\mathcal{A} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{B} \underset{P}{\overset{S}{\leftrightarrows}} \mathcal{C}\,,$$

where I, P, R and S are dg functors, with R right adjoint to I, S right adjoint to P and $P \circ S = Id_{\mathcal{C}}$ and $R \circ I = Id_{\mathcal{A}}$ via the adjunction morphisms. Notice that a (split) exact sequence gives rise to a (split) exact sequence of triangulated categories (§1.2).

Let us denote by $\mathsf{Trdgcat}$ the derivator associated with the quasi-equiconic model structure on dgcat. In particular, we have $\mathsf{Trdgcat}(e) = \mathsf{Hec}$.

Theorem 3.2. — *There exists a morphism of derivators*

$$\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha} : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}\,,$$

with values in a strong triangulated derivator, which:

α-flt) *commutes with α-filtered homotopy colimits;*

$\underline{\text{p}}$) *sends the terminal object in* $\mathsf{Trdgcat}$ *to the terminal object in* $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ *and*

$\underline{\text{add}}$) *sends the split exact sequences in* Hec *to direct sums*

$$\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{A}) \oplus \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{C}) \xrightarrow{\sim} \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{B})\,.$$

Moreover $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ *is universal with respect to these properties, i.e. for every strong triangulated derivator* $\mathbb{D}$*, we have an equivalence of categories*

$$(\underline{\mathcal{U}}_\alpha^{\mathsf{add}})^* : \underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_\alpha^{\mathsf{add}}(\mathsf{Trdgcat}, \mathbb{D})\,,$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{Trdgcat}, \mathbb{D})$ *of morphisms of derivators which verify the above three conditions..*

Notation 3.3. — The objects of the category $\underline{\mathsf{Hom}}_\alpha^{\mathsf{add}}(\mathsf{Trdgcat}, \mathbb{D})$ will be called *α-additive invariants* and $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ the *universal α-additive invariant*.

Before proving Theorem 3.2, let us state a natural variation on a result of Toën and Vaquié [38, Prop. 2.2], which is verified by all three model structures on dgcat.

Let sSet be the category of simplicial sets and Spt the category of spectra [6]. Given a Quillen model category $\mathcal{M}$, we denote by $\mathsf{Ho}(\mathcal{M})$ its homotopy category [22] and by $\mathsf{Map}(-,-) : \mathsf{Ho}(\mathcal{M})^{\mathsf{op}} \times \mathsf{Ho}(\mathcal{M}) \to \mathsf{Ho}(\mathsf{sSet})$ its homotopy function complex [19, 17.4.1]. Let α be a infinite regular cardinal. An object X in $\mathcal{M}$ is said to be *homotopically α-small* if, for any α-filtered direct system $\{Y_j\}_{j\in J}$ in $\mathcal{M}$, the induced map

$$\underset{j\in J}{\mathrm{hocolim}}\, \mathsf{Map}(X, Y_j) \longrightarrow \mathsf{Map}(X, \underset{j\in J}{\mathrm{hocolim}}\, Y_j)$$

is an isomorphism in $\mathsf{Ho}(\mathsf{sSet})$.

Proposition 3.4. — *Let $\mathcal{M}$ be a cellular Quillen model category, with I a set of generating cofibration. If the (co)domains of the elements of I are cofibrant, α-small and homotopically α-small, then :*

(i) *A α-filtered colimit of trivial fibration is a trivial fibration.*
(ii) *For any α-filtered direct system $\{X_j\}_{j\in J}$ in $\mathcal{M}$, the natural morphism*

$$\underset{j\in J}{\mathrm{hocolim}}\, X_j \longrightarrow \underset{j\in J}{\mathrm{colim}}\, X_j$$

is an isomorphism in $\mathsf{Ho}(\mathcal{M})$.
(iii) *Any object X in $\mathcal{M}$ is equivalent to a α-filtered colimit of α-small I-cell objects.*
(iv) *A object X in $\mathcal{M}$ is homotopically α-small presented if and only if it is equivalent to a retraction in* $\mathsf{Ho}(\mathcal{M})$ *of a α-small I-cell object.*

The proof of this Proposition is completely similar to the proof of [38, Prop. 2.2], and so we omit it.

Proof. — (of Theorem 3.2) The construction on $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$ is analogous to the construction of the additive motivator $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ [34, 15.1]. The only difference is that we start with the quasi-equiconic model structure instead of the Morita model structure and we consider α-filtered homotopy colimits instead of filtered homotopy colimits.

Let us now guide the reader throughout the several steps of its construction. The quasi-equiconic model structure satisfies all the hypothesis of Proposition 3.4. Therefore we can construct, as in [34, §5], the universal morphism of derivators which preserves α-filtered homotopy colimits

$$\mathbb{R}\underline{h} : \mathsf{Trdgcat} \longrightarrow \mathsf{L}_\Sigma \mathsf{Hot}_{\mathsf{dgcat}_\alpha}\,.$$

Although the derivator $\mathsf{Trdgcat}$ is not pointed (the canonical dg functor $\emptyset \to 0$ is not quasi-equiconic), we can still construct a localization morphism

$$\Phi : \mathsf{L}_{\Sigma}\mathsf{Hot}_{\mathsf{dgcat}_\alpha} \longrightarrow \mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha} ,$$

by a procedure analogous to the one of [34, §6]. We obtain the following universal property: for every pointed derivator $\mathbb{D}$, the composition $\Phi \circ \mathbb{R}\underline{h}$ induces (as in [34, Prop. 6.1]) an equivalence of categories

$$(\Phi \circ \mathbb{R}\underline{h})^* : \underline{\mathsf{Hom}}_!(\mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\alpha\text{-}\mathsf{flt},\underline{\mathsf{p}}}(\mathsf{Trdgcat}, \mathbb{D}) .$$

Now, since Sections [34, §12-13] can be entirely re-written using the quasi-equiconic model structure instead of the Morita model structure, we can localize $\mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha}$ as in [34, 14.5]. We obtain then a localization morphism

$$\mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha} \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{unst}}_\alpha ,$$

with values in a "Unstable Motivator". Finally, we stabilize $\underline{\mathrm{Mot}}^{\mathsf{unst}}_\alpha$, as in [34, §8], and obtain the universal α-additive motivator $\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$. The morphism $\underline{\mathcal{U}}^{\mathsf{add}}_\alpha$ is given by the composition

$$\mathsf{Trdgcat} \overset{\Phi\circ\mathbb{R}\underline{h}}{\longrightarrow} \mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha} \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{unst}}_\alpha \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha .$$

The proof of the universality of $\underline{\mathcal{U}}^{\mathsf{add}}_\alpha$, with respect to the properties of Theorem 3.2, is now entirely analogous to the one of [34, Thm. 15.4]. $\surd$

Proposition 3.5. — *The set of objects $\{\underline{\mathcal{U}}^{\mathsf{add}}_\alpha(\mathcal{B})[n] \mid n \in \mathbb{Z}\}$, with $\mathcal{B}$ a α-small dg cell, forms a set of compact generators of $\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha(e)$.*

Proof. — Recall from the proof of Theorem 3.2, the construction of $\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$ and $\underline{\mathcal{U}}^{\mathsf{add}}_\alpha$. Notice that by construction the objects $(\Phi \circ \mathbb{R}\underline{h})(\mathcal{B})$, with $\mathcal{B}$ a α-small dg cell, form a set of homotopically finitely presented generators of $\mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha}$. Since $\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$ is obtained from $\mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha}$ using the operations of stabilization and left Bousfield localization with respect to a set of morphisms whose (co)domains are homotopically finitely presented, [34, Lemma 8.2] and [34, Lemma 7.1] allow us to conclude the proof. $\surd$

Remark 3.6. — Notice that, for a triangulated derivator $\mathbb{D}$, and for an object X of $\mathbb{D}(e)$, the morphism $\mathbb{R}\mathsf{Hom}(X, -)$ preserves small sums if and only if X is compact in the triangulated category $\mathbb{D}(e)$ (which means that the functors $\mathsf{Hom}_{\mathbb{D}(e)}(X[n], -)$ preserve small sums for any integer n). Moreover, as the morphism $\mathbb{R}\mathsf{Hom}(X, -)$ preserves finite homotopy (co-)limits for any object X, the morphism $\mathbb{R}\mathsf{Hom}(X, -)$ preserves small sums if and only if it preserves small homotopy colimits. Hence X is compact in $\mathbb{D}$ (in the sense that $\mathbb{R}\mathsf{Hom}(X, -)$ preserves filtered homotopy colimits) if and only if it is compact in the triangulated category $\mathbb{D}(e)$, which is also equivalent to the property that $\mathbb{R}\mathsf{Hom}(X, -)$ preserves arbitrary small homotopy colimits. We will use freely this last characterization in the sequel of these notes.

In particular, Proposition 3.5 can be restated by saying that $\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}^{\mathsf{add}}_\alpha(\mathcal{B}), -)$ preserves homotopy colimits for any α-small dg cell $\mathcal{B}$.

Theorem 3.7. — *For any dg categories $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{B}$ a α-small dg cell, we have a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_{\alpha}^{\mathsf{add}}(\mathcal{B}), \underline{\mathcal{U}}_{\alpha}^{\mathsf{add}}(\mathcal{A})) \simeq \underline{K}(\mathsf{rep}_{tr}(\mathcal{B}, \mathcal{A})) .$$

Proof. — The proof of this co-representability Theorem is entirely analogous to the one of [34, Thm. 15.10]. Simply replace $\mathsf{rep}_{mor}(-,-)$ by $\mathsf{rep}_{tr}(-,-)$ and $K(-)$ by $\underline{K}(-)$ in (the proofs of) [34, Prop. 14.11], [34, Thm. 15.9] and [34, Thm. 15.10]. $\surd$

4. Idempotent completion

Let us denote by

$$(-)^{\wedge} : \mathsf{dgcat} \longrightarrow \mathsf{dgcat} \qquad \mathcal{A} \mapsto \mathcal{A}^{\wedge}$$

the Morita fibrant resolution functor, obtained by the small object argument [19, §10.5.14], using the generating trivial cofibrations of the Morita model structure. By [33, Prop. 2.27], the functor $(-)^{\wedge}$ preserves quasi-equiconic dg functors and so it gives rise to a morphism of derivators

$$(-)^{\wedge} : \mathsf{Trdgcat} \longrightarrow \mathsf{Trdgcat} .$$

Notice that by construction, we have also a 2-morphism $\mathrm{Id} \Rightarrow (-)^{\wedge}$ of derivators.

Proposition 4.1. — *The morphism $(-)^{\wedge}$ preserves :*

(i) *α-filtered homotopy colimits,*
(ii) *the terminal object and*
(iii) *split exact sequences (3.1).*

Proof. — Condition (i) follows from Proposition 3.4 (ii) and the fact that the (co)domains of the generating trivial cofibrations of the Morita model structure are α-small. Condition (ii) is clear. In what concerns condition (iii), notice that we have a commutative diagram (up to isomorphism)

$$\begin{array}{ccc} \mathsf{Hec} & \xrightarrow{(-)^{\wedge}} & \mathsf{Hec} \\ {\scriptstyle \mathsf{tri}}\downarrow & & \downarrow{\scriptstyle \mathsf{tri}} \\ \mathsf{Tri} & \xrightarrow[\widetilde{(-)}]{} & \mathsf{Tri} . \end{array}$$

Since the idempotent completion functor $\widetilde{(-)}$ preserves split exact sequences, the proof is finished. $\surd$

By Proposition 4.1, the composition

$$\mathsf{Trdgcat} \stackrel{(-)^{\wedge}}{\longrightarrow} \mathsf{Trdgcat} \stackrel{\underline{\mathcal{U}}_{\alpha}^{\mathsf{add}}}{\longrightarrow} \underline{\mathrm{Mot}}_{\alpha}^{\mathsf{add}}$$

is a α-additive invariant (3.3). Therefore, by Theorem 3.2 we obtain an induced morphism of derivators and a 2-morphism

$$(-)^{\wedge} : \underline{\mathrm{Mot}}_{\alpha}^{\mathsf{add}} \longrightarrow \underline{\mathrm{Mot}}_{\alpha}^{\mathsf{add}} \qquad \mathrm{Id} \Rightarrow (-)^{\wedge} ,$$

such that $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})^\wedge \simeq \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A}^\wedge)$ for every dg category $\mathcal{A}$.

5. Localizing invariants

In this Section, we work with a fixed infinite regular cardinal α.

5.1. Waldhausen exact sequences. —

Definition 5.1. — Let $\mathcal{B}$ be a pre-triangulated dg category. A *thick* dg subcategory of $\mathcal{B}$ is a full dg subcategory $\mathcal{A}$ of $\mathcal{B}$ such that the induced functor $\mathsf{H}^0(\mathcal{A}) \to \mathsf{H}^0(\mathcal{B})$ turns $\mathsf{H}^0(\mathcal{A})$ into a thick subcategory of $\mathsf{H}^0(\mathcal{B})$ (which means that $\mathcal{A}$ is pre-triangulated and that any object of $\mathsf{H}^0(\mathcal{B})$ which is a direct factor of an object of $\mathsf{H}^0(\mathcal{A})$ is an object of $\mathsf{H}^0(\mathcal{A})$). A *strict exact sequence* is a diagram of shape

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A}\,,$$

in which $\mathcal{B}$ is a pre-triangulated dg category, $\mathcal{A}$ is a thick dg subcategory of $\mathcal{B}$, and $\mathcal{B}/\mathcal{A}$ is the Drinfeld dg quotient [13] of $\mathcal{B}$ by $\mathcal{A}$.

Remark 5.2. — Any split short exact sequence is a strict exact sequence.

Notation 5.3. — We fix once for all a *set* $\mathcal{E}$ of representatives of homotopy α-small thick inclusions, by which we mean that $\mathcal{E}$ is a set of full inclusions of dg categories $\mathcal{A} \to \mathcal{B}$ with the following properties:

(i) For any inclusion $\mathcal{A} \to \mathcal{B}$ in $\mathcal{E}$, the dg category $\mathcal{B}$ is pre-triangulated, and $\mathcal{A}$ is thick in $\mathcal{B}$.

(ii) For any inclusion $\mathcal{A} \to \mathcal{B}$ in $\mathcal{E}$, there exists an α-small dg cell $\mathcal{B}_0$ (2.1) and a quasi-equiconic dg functor $\mathcal{B}_0 \to \mathcal{B}$.

(iii) For any α-small dg cell $\mathcal{B}_0$, there exists a quasi-equiconic dg functor $\mathcal{B}_0 \to \mathcal{B}$, with $\mathcal{B}$ cofibrant and pre-triangulated, such that any inclusion $\mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ thick in $\mathcal{B}$ is in $\mathcal{E}$.

Proposition 5.4. — *Let $\mathcal{B}$ be a pre-triangulated dg category, and $\mathcal{A}$ a thick dg subcategory of $\mathcal{B}$. Then there exists an α-filtered direct system $\{\mathcal{A}_j \xrightarrow{\epsilon_j} \mathcal{B}_j\}_{j\in J}$ of dg functors, such that for any index j, we have a commutative diagram of dg categories of shape*

$$\begin{array}{ccc} \mathcal{A}'_j & \longrightarrow & \mathcal{B}'_j \\ \downarrow & & \downarrow \\ \mathcal{A}_j & \longrightarrow & \mathcal{B}_j\,, \end{array}$$

in which the vertical maps are quasi-equivalences, the map $\mathcal{A}'_j \to \mathcal{B}'_j$ belongs to $\mathcal{E}$, and moreover there exists an isomorphism in the homotopy category of arrows between dg categories (with respect to the quasi-equiconic model structure) of shape

$$\operatorname*{hocolim}_{j\in J} \{\mathcal{A}_j \xrightarrow{\epsilon_j} \mathcal{B}_j\} \xrightarrow{\sim} (\mathcal{A} \longrightarrow \mathcal{B})\,.$$

Proof. — By Proposition 3.4, there exists an α-filtered direct system of α-small dg cells $\{\mathcal{B}''_j\}_{j\in J}$ in Hec, such that

$$\underset{j\in J}{\mathrm{hocolim}}\, \mathcal{B}''_j \stackrel{\sim}{\longrightarrow} \mathcal{B}\,.$$

By taking a termwise fibrant replacement $\{\mathcal{B}_j\}_{j\in J}$ of the diagram $\{\mathcal{B}''_j\}_{j\in J}$, with respect to the quasi-equiconic model category structure, we obtain a α-filtered diagram of pre-triangulated dg categories, and so an isomorphism in Hec of shape

$$\underset{j\in J}{\mathrm{hocolim}}\, \mathcal{B}_j \stackrel{\sim}{\longrightarrow} \mathcal{B}\,.$$

Using the following fiber products

$$\begin{array}{ccc} \mathcal{A}_j & \xrightarrow{\epsilon_j} & \mathcal{B}_j \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B}\,, \end{array}$$

we construct an α-filtered direct system $\{\mathcal{A}_j \stackrel{\epsilon_j}{\to} \mathcal{B}_j\}_{j\in J}$ such that the map

$$\underset{j\in J}{\mathrm{hocolim}}\, \mathcal{A}_j \stackrel{\sim}{\longrightarrow} \mathcal{A}$$

is an isomorphism in Hec. To prove this last assertion, we consider the commutative diagram of triangulated categories

$$\begin{array}{ccccc} \mathrm{colim}\, \mathsf{H}^0(\mathcal{A}_j) & = & \mathsf{H}^0(\mathrm{hocolim}\, \mathcal{A}_j) & \longrightarrow & \mathsf{H}^0(\mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{colim}\, \mathsf{H}^0(\mathcal{B}_j) & = & \mathsf{H}^0(\mathrm{hocolim}\, \mathcal{B}_j) & \longrightarrow & \mathsf{H}^0(\mathcal{B})\,, \end{array}$$

and use the fact that thick inclusions are stable under filtered colimits in the category of triangulated categories.

Now, for each index j, there exists a quasi-equiconic dg functor $\mathcal{B}''_j \to \mathcal{B}'_j$ with $\mathcal{B}'_j$ cofibrant and pre-triangulated, such that any thick inclusion into $\mathcal{B}'_j$ is in $\mathcal{E}$. As $\mathcal{B}'_j$ and $\mathcal{B}_j$ are isomorphic in Hec, and as $\mathcal{B}'_j$ is cofibrant and $\mathcal{B}_j$ is pre-triangulated, we get a quasi-equivalence $\mathcal{B}'_j \to \mathcal{B}_j$. In conclusion, we obtain pullback squares

$$\begin{array}{ccc} \mathcal{A}'_j & \longrightarrow & \mathcal{B}'_j \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{A}_j & \longrightarrow & \mathcal{B}_j \end{array}$$

in which the vertical maps are quasi-equivalences, and the map $\mathcal{A}'_j \to \mathcal{B}'_j$ belongs to $\mathcal{E}$. $\surd$

The derivator $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ admits a (left proper and cellular) Quillen model and so we can consider its left Bousfield localization with respect to the set of maps

(5.1.1) $$\Theta_\epsilon : \mathsf{cone}[\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\epsilon) : \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{A}) \longrightarrow \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{B})\,] \longrightarrow \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{B}/\mathcal{A})\,,$$

where $\epsilon : \mathcal{A} \to \mathcal{B}$ belongs to $\mathcal{E}$ (see 5.3). We obtain then a new derivator $\underline{\mathrm{Mot}}^{\mathsf{wloc}}_{\alpha}$ (which admits also a left proper and cellular Quillen model) and an adjunction

$$\begin{array}{c}\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha} \\ {\scriptstyle\gamma_!}\downarrow \uparrow{\scriptstyle\gamma^*} \\ \underline{\mathrm{Mot}}^{\mathsf{wloc}}_{\alpha}\,.\end{array}$$

The morphism $\gamma_!$ preserves homotopy colimits and sends the maps Θ_ϵ to isomorphisms, and it is universal with respect to these properties; see A.2.

Theorem 5.5. — *The composition morphism*

$$\underline{\mathcal{U}}^{\mathsf{wloc}}_{\alpha} : \mathsf{Trdgcat} \xrightarrow{\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}} \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha} \xrightarrow{\gamma_!} \underline{\mathrm{Mot}}^{\mathsf{wloc}}_{\alpha}$$

has the following properties:

α-flt) *it commutes with α-filtered homotopy colimits;*

p) *it preserves the terminal object;*

wloc) *it sends strict exact sequences (5.1) to distinguished triangles in $\underline{\mathrm{Mot}}^{\mathsf{wloc}}_{\alpha}$.*

Moreover $\underline{\mathcal{U}}^{\mathsf{wloc}}_{\alpha}$ is universal with respect to these properties, i.e. for every strong triangulated derivator $\mathbb{D}$, we have an equivalence of categories

$$(\underline{\mathcal{U}}^{\mathsf{wloc}}_{\alpha})^* : \underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}^{\mathsf{wloc}}_{\alpha}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}^{\mathsf{wloc}}_{\alpha}(\mathsf{Trdgcat}, \mathbb{D})\,,$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{Trdgcat}, \mathbb{D})$ of morphisms of derivators which verify the above three conditions..

Proof. — Since $\gamma_!$ preserves homotopy colimits, the morphism $\underline{\mathcal{U}}^{\mathsf{wloc}}_{\alpha}$ satisfies conditions α-flt), p) and add) stated in Theorem 3.2. Condition wloc) follows from Proposition 5.4 and the fact that $\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}$ commutes with α-filtered homotopy colimits.. Finally, the universality of $\underline{\mathcal{U}}^{\mathsf{wloc}}_{\alpha}$ is a consequence of Theorem 3.2 and Definition A.2 (notice that, as any split exact sequence is a strict exact sequence, under condition p), condition add) is implied by condition wloc)). $\surd$

Let us now prove a general result, which will be used in the proofs of Theorems 5.7 and 7.15.

Proposition 5.6. — *Let $\mathbb{D}$ be a stable derivator, S a set of morphisms in the base category $\mathbb{D}(e)$ and X a compact object in $\mathbb{D}(e)$. Let us consider the left Bousfield localization of $\mathbb{D}$ with respect to S (see A.2):*

$$\begin{array}{c}\mathbb{D} \\ {\scriptstyle\gamma_!}\downarrow \uparrow{\scriptstyle\gamma^*} \\ \mathsf{L}_S\mathbb{D}\,.\end{array}$$

If the functor $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, -) : \mathbb{D}(e) \to \mathsf{Ho}(\mathsf{Spt})$ sends the elements of S to isomorphisms, then $\gamma_!(X)$ is compact in $\mathsf{L}_S\mathbb{D}(e)$ and for every object $M \in \mathbb{D}(e)$, we have a *natural weak equivalence of spectra*

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, M) \simeq \mathbb{R}\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}}(\gamma_!(X), \gamma_!(M))\,.$$

Proof. — Let us start by showing the above weak equivalence of spectra. By [34, Prop. 4.1], $\mathsf{L}_S\mathbb{D}(e)$ is the localization of $\mathbb{D}(e)$ with respect to the smallest class W of maps in $\mathbb{D}(e)$, which contains the class S, has the two-out-of-three property, and which is closed under homotopy colimits. Since X is compact and the functor $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, -)$ sends the elements of S to isomorphisms, we conclude that it also sends all the elements of W to isomorphisms. Finally, since for every object $M \in \mathbb{D}(e)$, the co-unit adjunction morphism $M \to \gamma^*\gamma_!(M)$ belongs to W, we obtain the following weak equivalences

$$\mathbb{R}\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}}(\gamma_!(X), \gamma_!(M)) \simeq \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, \gamma^*\gamma_!(M)) \simeq \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, M)\,.$$

We now show that $\gamma_!(X)$ is compact in $\mathsf{L}_S\mathbb{D}(e)$. Notice that for every object $N \in \mathsf{L}_S\mathbb{D}(e)$, the unit adjunction morphism $\gamma_!\gamma^*(N) \xrightarrow{\sim} N$ is an isomorphism. Let $(N_i)_{i\geq 0}$ be a family of objects in $\mathsf{L}_S\mathbb{D}(e)$. We have the following equivalences

$$\begin{aligned}
\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X), \bigoplus_i N_i) & \simeq \mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X), \bigoplus_i \gamma_!\gamma^*(N_i)) \\
& \simeq \mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X), \gamma_!(\bigoplus_i \gamma^*(N_i))) \\
& \simeq \mathsf{Hom}_{\mathbb{D}(e)}(X, \bigoplus_i \gamma^*(N_i)) \\
& \simeq \bigoplus_i \mathsf{Hom}_{\mathbb{D}(e)}(X, \gamma^*(N_i)) \\
& \simeq \bigoplus_i \mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X), \gamma_!\gamma^*(N_i)) \\
& \simeq \bigoplus_i \mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X), N_i)
\end{aligned}$$

and so the proof is finished. $\surd$

***Theorem 5.7* (Waldhausen Localization Theorem).** — *The object $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k})$ is compact in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$ and for every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Hom}(\,\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})\,) \simeq \underline{K}(\mathcal{A})\,.$$

Proof. — The proof consists on verifying the conditions of Proposition 5.6, with $\mathbb{D} = \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$, $S = \{\Theta_\epsilon \,|\, \epsilon \in \mathcal{E}\}$ (see 5.1.1), $X = \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k})$ and $M = \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})$. Since by Proposition 3.5, $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k})$ is compact in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}(e)$, it is enough to show that the functor

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}), -) : \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}(e) \longrightarrow \mathsf{Ho}(\mathsf{Spt})$$

sends the elements of S to isomorphisms. For this, consider the following diagram

$$\begin{array}{ccccc}
\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A}) & \xrightarrow{\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\epsilon)} & \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}) & \longrightarrow & \mathsf{cone}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\epsilon)) \\
\| & & \| & & \downarrow{\scriptstyle \Theta_\epsilon} \\
\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A}) & \xrightarrow[\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\epsilon)]{} & \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}) & \longrightarrow & \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}/\mathcal{A})\,.
\end{array}$$

By virtue of Theorem 3.7 and Proposition 3.5, the functor $\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}), -)$, applied to the above diagram, gives the following one

$$\begin{array}{ccccc} \underline{K}(\mathcal{A}) & \xrightarrow{\underline{K}(\epsilon)} & \underline{K}(\mathcal{B}) & \longrightarrow & \mathsf{cone}(\underline{K}(\epsilon)) \\ \| & & \| & & \downarrow \\ \underline{K}(\mathcal{A}) & \xrightarrow[\underline{K}(\epsilon)]{} & \underline{K}(\mathcal{B}) & \longrightarrow & \underline{K}(\mathcal{B}/\mathcal{A}) \,, \end{array}$$

where the upper line is a homotopy cofiber sequence of spectra. Now, consider the following sequence of Waldhausen categories

$$\mathsf{tri}^{\mathcal{W}}(\mathcal{A}) \longrightarrow \mathsf{tri}^{\mathcal{W}}(\mathcal{B}) \longrightarrow \mathsf{tri}^{\mathcal{W}}(\mathcal{B}/\mathcal{A}) \,.$$

Let us denote by $v\mathsf{tri}^{\mathcal{W}}(\mathcal{B})$ the Waldhausen category with the same cofibrations as $\mathsf{tri}^{\mathcal{W}}(\mathcal{B})$, but whose weak equivalences are the maps with cone in $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$. Since $\mathsf{tri}(\mathcal{A})$ is thick in $\mathsf{tri}(\mathcal{B})$, we conclude by Waldhausen's fibration Theorem [40, Thm. 1.6.4] (applied to the inclusion $\mathsf{tri}^{\mathcal{W}}(\mathcal{B}) \subset v\mathsf{tri}^{\mathcal{W}}(\mathcal{B})$) that the lower line of the above diagram is a homotopy fiber sequence of spectra. Since the (homotopy) category of spectra is stable, we conclude that the right vertical map is a weak equivalence of spectra. Finally, since by Theorem 3.7, we have a natural isomorphism in the stable homotopy category of spectra

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})) \simeq \underline{K}(\mathcal{A}) \,,$$

the proof is finished. $\surd$

5.2. Morita invariance. —

Notation 5.8. — Let $\mathcal{S}$ be the set of morphisms in Hec of the form $\mathcal{B} \to \mathcal{B}^\wedge$, with $\mathcal{B}$ a α-small dg cell.

Proposition 5.9. — *For any dg category $\mathcal{A}$, there exists an α-filtered direct system $\{\mathcal{B}_j \to \mathcal{B}_j^\wedge\}_{j\in J}$ of elements of $\mathcal{S}$, such that*

$$\underset{j\in J}{\mathrm{hocolim}}\, \{\mathcal{B}_j \longrightarrow \mathcal{B}_j^\wedge\} \stackrel{\sim}{\longrightarrow} (\mathcal{A} \longrightarrow \mathcal{A}^\wedge) \,.$$

Proof. — By Proposition 3.4 (iii), there exits an α-filtered direct system $\{\mathcal{B}_j\}_{j\in J}$ of α-small dg cells such that

$$\underset{j\in J}{\mathrm{hocolim}}\, \mathcal{B}_j \stackrel{\sim}{\longrightarrow} \mathcal{A} \,.$$

Since by Proposition 4.1 (i), the functor $(-)^\wedge$ preserves α-filtered homotopy colimits, this achieves the proof. $\surd$

Since the derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ admits a (left proper and cellular) Quillen model, we can localize it with respect to the image of the set $\mathcal{S}$ under $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}$. We obtain then a new derivator $\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$ and an adjunction

$$\begin{array}{c} \underline{\mathrm{Mot}}^{\mathsf{wloc}}_\alpha \\ l_! \downarrow \ \uparrow l^* \\ \underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha\,. \end{array}$$

Proposition 5.10. — *The composition*

$$\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha : \mathsf{Trdgcat} \xrightarrow{\underline{\mathcal{U}}^{\mathsf{wloc}}_\alpha} \underline{\mathrm{Mot}}^{\mathsf{wloc}}_\alpha \xrightarrow{l_!} \underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$$

sends the Morita dg functors to isomorphisms.

Proof. — Observe first that $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ sends the maps of shape $\mathcal{A} \to \mathcal{A}^\wedge$ to isomorphisms: this follows from Proposition 5.9 and from the fact that $\underline{\mathcal{U}}^{\mathsf{wloc}}_\alpha$ commutes with α-filtered homotopy colimits. Since $F : \mathcal{A} \to \mathcal{B}$ is a Morita dg functor if and only if $F^\wedge : \mathcal{A}^\wedge \to \mathcal{B}^\wedge$ is a quasi-equivalence, this ends the proof. $\surd$

By Proposition 5.10, $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ descends to the derivator $\mathsf{HO}(\mathsf{dgcat})$ associated with the Morita model structure on dgcat.

Theorem 5.11. — *The morphism*

$$\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$$

α-flt) *commutes with α-filtered homotopy colimits;*

$\underline{\mathrm{p}}$) *preserves the terminal object and*

loc) *sends the exact sequences in* Hmo *(3.1) to distinguished triangles in* $\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$.

Moreover $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ *is universal with respect to these properties, i.e. for every strong triangulated derivator* $\mathbb{D}$*, we have an equivalence of categories*

$$(\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha)^* : \underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}^{\mathsf{loc}}_\alpha(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})\,,$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ *of morphisms of derivators which verify the above three conditions.*

Notation 5.12. — The objects of the category $\underline{\mathsf{Hom}}^{\mathsf{loc}}_\alpha(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ will be called *α-localizing invariants* and $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ the *universal α-localizing invariant.*

Proof. — Since $l_!$ preserves homotopy colimits, the morphism $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ satisfies conditions α-flt) and $\underline{\mathrm{p}}$). In what concerns condition loc), we can suppose by [24, Thm. 4.11] that we have a exact sequence in Hmo of the form

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{B}/\mathcal{A}\,.$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{I} & \mathcal{B} & \longrightarrow & \mathcal{B}/\mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}^\wedge & \xrightarrow[I^\wedge]{} & \mathcal{B}^\wedge & \longrightarrow & \mathcal{B}^\wedge/\mathcal{A}^\wedge\,, \end{array}$$

where the right vertical map is the induced one. Since the first two vertical maps are Morita dg functors, the right vertical one is also a Morita dg functor. Furthermore, since the lower line is a strict exact sequence, Proposition 5.10 and Theorem 5.5 imply that $\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}$ satisfies condition loc). The universality of $\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}$ is now a clear consequence of Theorem 5.5 and Proposition 5.10. $\surd$

6. Schlichting's set-up

In this Section we built a set-up [32, §2.2] in Hec, that will be used in Section 7 to construct the negative K-theory (spectrum). In Proposition 6.6, we relate it with Schlichting's set-up on Frobenius pairs.

6.1. Set-up. — We start by constructing a infinite sums completion functor. Recall from [35, §6], the Quillen adjunctions (where we have taken $\alpha = \aleph_1$)

$$\begin{array}{c}\mathsf{dgcat}_{ex,\aleph_1}\\ F_1 \uparrow \downarrow U_1\\ T_{\aleph_1}\text{-}\mathsf{alg}\\ F \uparrow \downarrow U\\ \mathsf{dgcat}\,,\end{array}$$

where dgcat is endowed with the quasi-equivalence model structure. Consider the composed functor

$$\mathsf{dgcat} \xrightarrow{(-)^{\mathsf{pre\text{-}tr}}} \mathsf{dgcat} \xrightarrow{F_1 \circ F} \mathsf{dgcat}_{ex,\aleph_1} \xrightarrow{U \circ U_1} \mathsf{dgcat}\,,$$

where $(-)^{\mathsf{pre\text{-}tr}}$ denotes Bondal-Kapranov's pre-triangulated envelope [4].

Lemma 6.1. — *The above composed functor maps quasi-equiconic dg functors between cofibrant dg categories to quasi-equivalences.*

Proof. — Notice first that by [33, Lemma 2.13], $(-)^{\mathsf{pre\text{-}tr}}$ is a pre-triangulated resolution functor. Therefore by [33, Prop. 2.14], $(-)^{\mathsf{pre\text{-}tr}}$ maps quasi-equiconic dg functors (between cofibrant dg categories) to quasi-equivalences (between cofibrant dg categories). Since $F \circ F_1$ is a left Quillen functor, quasi-equivalences between cofibrant dg categories are mapped to weak equivalences in $\mathsf{dgcat}_{ex,\aleph_1}$. Finally, the composed functor $U \circ U_1$ maps weak equivalences to quasi-equivalences, and so the proof is finished. $\surd$

By Lemma 6.1 and [19, Prop. 8.4.8], the above composed functor admits a total left derived functor (with respect to the quasi-equiconic model structure), which we denote by

$$\mathcal{F}(-) : \mathsf{Hec} \longrightarrow \mathsf{Hec}\,.$$

Notice that by construction, we have also a natural transformation $\mathrm{Id} \Rightarrow \mathcal{F}(-)$.

Lemma 6.2. — *The functor $\mathcal{F}(-)$ gives rise to a morphism of derivators*

$$\mathcal{F}(-) : \mathsf{Trdgcat} \longrightarrow \mathsf{Trdgcat}\,,$$

which preserves $\aleph_1$-filtered homotopy colimits.

Proof. — This follows from the fact that $(-)^{\mathsf{pre\text{-}tr}}$ and $U \circ U_1$ preserve $\aleph_1$-filtered homotopy colimits. The case of $(-)^{\mathsf{pre\text{-}tr}}$ is clear. In what concerns $U \circ U_1$ see [35, Prop. 6.7]. $\surd$

Proposition 6.3. — *Let $\mathcal{A}$ be a dg category.*

(i) *The dg category $\mathcal{F}(\mathcal{A})$ has countable sums.*
(ii) *The dg category $\mathcal{F}(\mathcal{A})$ is Morita fibrant.*
(iii) *The dg functor $\mathcal{A} \to \mathcal{F}(\mathcal{A})$ induces a fully faithful triangulated functor $\mathsf{tri}(\mathcal{A}) \to \mathsf{tri}(\mathcal{F}(\mathcal{A})) \simeq \mathsf{H}^0(\mathcal{F}(\mathcal{A}))$. Moreover the image of $\mathsf{H}^0(\mathcal{A})$ in $\mathsf{H}^0(\mathcal{F}(\mathcal{A}))$ forms a set of compact generators.*

Proof. — Condition (i) follows from [35, Prop. 3.8]. By [35, 6.4] we observe that $\mathcal{F}(\mathcal{A})$ is pre-triangulated (2.4). Since in any triangulated category with countable sums (for instance $\mathsf{H}^0(\mathcal{F}(\mathcal{A}))$) every idempotent splits [27, Prop. 1.6.8], we conclude that $\mathcal{F}(\mathcal{A})$ is Morita fibrant (2.5). Finally, condition (iii) follows from the construction of $\mathcal{F}(-)$. $\surd$

Remark 6.4. — By Proposition 6.3 (iii) and [32, §3.1, Lemma 2], we have an equivalence

$$\widetilde{\mathsf{tri}(\mathcal{A})} \xrightarrow{\sim} \mathsf{H}^0(\mathcal{F}(\mathcal{A}))_c\,,$$

between the idempotent completion of $\mathsf{tri}(\mathcal{A})$ and the triangulated subcategory of compact objects in $\mathsf{H}^0(\mathcal{F}(\mathcal{A}))$.

We can then associate to every dg category $\mathcal{A}$, an exact sequence

$$\mathcal{A} \longrightarrow \mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{A}) := \mathcal{F}(\mathcal{A})/\mathcal{A}\,,$$

where $\mathcal{F}(\mathcal{A})/\mathcal{A}$ denotes the Drinfeld's dg quotient [13]. In this way, we obtain also a functor

$$\mathcal{S}(-) : \mathsf{Hec} \longrightarrow \mathsf{Hec}\,.$$

Proposition 6.5. — *The category* Hec *endowed with $\mathcal{F}(-)$, $\mathcal{S}(-)$ and with the functor*

$$\begin{array}{rcl} \mathsf{tri} : \mathsf{Hec} & \longrightarrow & \mathsf{Tri} \\ \mathcal{A} & \mapsto & \mathsf{tri}(\mathcal{A})\,, \end{array}$$

satisfies the hypothesis of Schlichting's set-up [32, Section 2.2].

Proof. — By construction, we have a exact sequence

$$\mathcal{A} \longrightarrow \mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{A})$$

for every $\mathcal{A} \in \mathsf{Hec}$. Moreover, since $\mathcal{F}(\mathcal{A})$ has countable sums the Grothendieck group $I\!K_0(\mathcal{F}(\mathcal{A}))$ is trivial. It remains to show that $\mathcal{F}(-)$ and $\mathcal{S}(-)$ preserve exact sequences. For this, consider the following diagram in Hec

$$
\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{I} & \mathcal{B} & \xrightarrow{P} & \mathcal{C} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\mathcal{A}) & \xrightarrow{\mathcal{F}(I)} & \mathcal{F}(\mathcal{B}) & \xrightarrow{\mathcal{F}(P)} & \mathcal{F}(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}(\mathcal{A}) & \xrightarrow[\mathcal{S}(I)]{} & \mathcal{S}(\mathcal{B}) & \xrightarrow[\mathcal{S}(P)]{} & \mathcal{S}(\mathcal{C}) \,,
\end{array}
$$

where the upper line is a exact sequence. Notice that by Remark 6.4, the triangulated functors $\mathsf{tri}(\mathcal{F}(I))$ and $\mathsf{tri}(\mathcal{F}(P))$ preserve compact objects. Moreover, since these functors preserve also countable sums [35, 6.4], [32, §3.1, Corollary 3] implies that the middle line is also a exact sequence. Finally, by [32, §3.1, Lemma 3] and Remark 6.4, we conclude that the lower line is also a exact sequence. $\surd$

6.2. Relation with Frobenius pairs. — Let us denote by $(\mathrm{Frob}; \mathcal{F}_{\mathcal{M}}, \mathcal{S}_{\mathcal{M}})$ the Schlichting's set-up on the category of Frobenius pairs. See [32, §5] for details. Recall from [32, §4.3, Definition 6] the *derived category* functor

$$
\begin{array}{rcl}
D : \mathrm{Frob} & \longrightarrow & \mathsf{Tri} \\
\mathbf{A} = (\mathcal{A}, \mathcal{A}_0) & \mapsto & \mathcal{D}\mathbf{A} = \underline{\mathcal{A}}/\underline{\mathcal{A}_0} \,,
\end{array}
$$

where $\underline{\mathcal{A}}/\underline{\mathcal{A}_0}$ denotes the Verdier's quotient of the stable categories. Notice that we have a natural functor

$$
\begin{array}{rcl}
\mathbf{E} : \mathsf{Hec} & \longrightarrow & \mathrm{Frob} \\
\mathcal{A} & \mapsto & (E(\mathcal{A}), E(\mathcal{A})\text{-prinj}) \,,
\end{array}
$$

where $E(\mathcal{A}) = \mathsf{perf}^{\mathcal{W}}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$ is the Frobenius category associated to the cofibrant right $\mathcal{A}$-modules, which become isomorphic in $\mathcal{D}(\mathcal{A})$ to elements of $\mathsf{tri}(\mathcal{A})$, and $E(\mathcal{A})$-prinj is a the full subcategory of projective-injective objects of $E(\mathcal{A})$. Since

$$
\underline{E(\mathcal{A})}/\underline{E(\mathcal{A})\text{-prinj}} \simeq \mathsf{tri}(\mathcal{A}) \,,
$$

we have a (up to equivalence) commutative diagram

$$
\begin{array}{ccc}
\mathsf{Hec} & \xrightarrow{\mathbf{E}} & \mathrm{Frob} \\
{\scriptstyle \mathsf{tri}}\downarrow & \swarrow{\scriptstyle D} & \\
\mathsf{Tri} & & .
\end{array}
$$

Proposition 6.6. — *The natural functor*

$$
\mathbf{E} : (\mathsf{Hec}; \mathcal{F}, \mathcal{S}) \longrightarrow (\mathrm{Frob}; \mathcal{F}_{\mathcal{M}}, \mathcal{S}_{\mathcal{M}})
$$

is such that $\mathbf{E} \circ \mathcal{F}$ *and* $\mathbf{E} \circ S$ *are weakly isomorphic to* $\mathcal{F}_{\mathcal{M}} \circ \mathbf{E}$ *and* $\mathcal{S}_{\mathcal{M}} \circ \mathbf{E}$.

Proof. — We start by showing that $\mathbf{E} \circ \mathcal{F}$ and $\mathcal{F}_{\mathcal{M}} \circ \mathbf{E}$ are weakly isomorphic. Let $\mathcal{A}$ be a (pre-triangulated) dg category and $\mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})$ the left Bousfiled localization of $\mathcal{C}(\mathcal{F}\mathcal{A})$, with respect to the set

$$S := \{ \bigoplus_{i \in I} \underline{h}(x_i) \longrightarrow \underline{h}(\bigoplus_{i\in I} x_i) \,|\, x_i \in \mathcal{F}\mathcal{A}\,,\, |I| < \aleph_1 \,\}\,.$$

We now introduce an "intermediate" Frobenius pair $\mathbf{E}'(\mathcal{F}\mathcal{A}) := (E'(\mathcal{F}\mathcal{A}), E'(\mathcal{F}\mathcal{A})_0)$. The Frobenius category $E'(\mathcal{F}\mathcal{A}) \subset \mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})$ consists of the cofibrant right $\mathcal{F}\mathcal{A}$-modules, which become isomorphic in $\mathsf{Ho}(\mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A}))$ to representable ones, and $E'(\mathcal{F}\mathcal{A})_0$ consists of the right $\mathcal{F}\mathcal{A}$-modules M whose morphism $M \to 0$ to the terminal object is a S-equivalence. Notice that, as any representable right $\mathcal{F}\mathcal{A}$-module is S-local in $\mathcal{C}(\mathcal{F}\mathcal{A})$, we have a natural weak equivalence $\mathbf{E}(\mathcal{F}\mathcal{A}) \overset{\sim}{\to} \mathbf{E}'(\mathcal{F}\mathcal{A})$ of Frobenius pairs. Consider the following (solid) commutative diagram

$$\begin{array}{ccccc}
 & & \overset{\text{colim}}{\curvearrowright} & & \\
\mathcal{F}_{\mathcal{M}}(E(\mathcal{A})) & \longleftarrow & E(\mathcal{A}) & \hookrightarrow & \mathcal{C}(\mathcal{A}) \\
 & & \downarrow & & \downarrow \\
 & \searrow{\scriptstyle \Phi} & E(\mathcal{F}\mathcal{A}) & \hookrightarrow & \mathcal{C}(\mathcal{F}\mathcal{A}) \\
 & & \downarrow{\scriptstyle \sim} & & \downarrow \\
 & & E'(\mathcal{F}\mathcal{A}) & \hookrightarrow & \mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})\,,
\end{array}$$

where Φ is the composition $\mathcal{F}_{\mathcal{M}}(E(\mathcal{A})) \overset{\text{colim}}{\to} \mathcal{C}(\mathcal{A}) \to \mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})$. We obtain then the following commutative diagram of Frobenius pairs

$$\begin{array}{ccc}
\mathbf{E}(\mathcal{A}) & \longrightarrow & \mathbf{E}(\mathcal{F}\mathcal{A}) \\
\downarrow & & \downarrow{\scriptstyle \sim} \\
\mathcal{F}_{\mathcal{M}}(\mathbf{E}(\mathcal{A})) & \longrightarrow & \mathbf{E}'(\mathcal{F}\mathcal{A})\,.
\end{array}$$

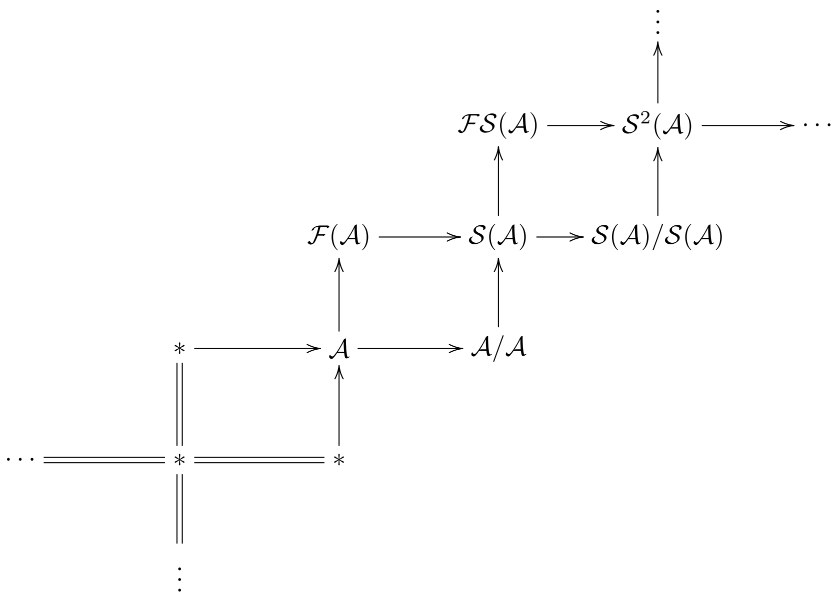
By [32, §5.2, Proposition 1] and Proposition 6.3 (iii) the triangulated categories $\mathcal{D}\mathcal{F}_{\mathcal{M}}(\mathbf{E}(\mathcal{A}))$ and $\mathsf{tri}(\mathcal{F}(\mathcal{A})) \simeq \mathcal{D}\mathbf{E}(\mathcal{F}\mathcal{A})$ (and so $\mathcal{D}\mathbf{E}'(\mathcal{F}(\mathcal{A}))$ are compactly generated by $\mathsf{tri}(\mathcal{A})$. Therefore, the lower map in the above square is also a weak equivalence of Frobenius pairs. In conclusion, we obtain a (two steps) zig-zag of weak equivalences relating $\mathcal{F}_{\mathcal{M}}(\mathbf{E}(\mathcal{A}))$ and $\mathbf{E}(\mathcal{F}\mathcal{A})$. We now show that $\mathbf{E} \circ \mathcal{S}$ and $\mathcal{S}_{\mathcal{M}} \circ \mathbf{E}$ are weakly isomorphic. For this, notice that we have the following induced zig-zag of weak equivalences of Frobenius pairs relating $\mathcal{S}_{\mathcal{M}}(\mathbf{E}(\mathcal{A}))$ and $\mathbf{E}(\mathcal{S}(\mathcal{A}))$:

$$\begin{array}{ccccc}
 & & (E(\mathcal{F}\mathcal{A}), S_0\mathbf{E}(\mathcal{F}\mathcal{A})) & \overset{\sim}{\longrightarrow} & \mathbf{E}(\mathcal{S}(\mathcal{A})) \\
 & & \downarrow{\scriptstyle \sim} & & \\
\mathcal{S}_{\mathcal{M}}(\mathbf{E}(\mathcal{A})) & \overset{\sim}{\longrightarrow} & (E'(\mathcal{F}\mathcal{A}), S_0\mathbf{E}'(\mathcal{F}\mathcal{A})) & & ,
\end{array}$$

where $S_0\mathbf{E}(\mathcal{F}\mathcal{A})$ (resp. $S_0\mathbf{E}'(\mathcal{F}\mathcal{A})$) is the full subcategory of $E(\mathcal{F}\mathcal{A})$ (resp. of $E'(\mathcal{F}\mathcal{A})$) of objects sent to zero in $\mathcal{D}\mathbf{E}(\mathcal{F}\mathcal{A})/\mathcal{D}\mathbf{E}(\mathcal{A})$ (resp. in $\mathcal{D}\mathbf{E}'(\mathcal{F}\mathcal{A})/\mathcal{D}\mathbf{E}(\mathcal{A})$). $\surd$

7. Negative K-theory

7.1. Construction. — Let $\mathbf{V} = W^{\mathsf{op}}$, where W is the poset $\{(i,j) \,|\, |i-j| \leq 1\} \subset \mathbb{Z} \times \mathbb{Z}$ considered as a small category (using the product partial order of $\mathbb{Z} \times \mathbb{Z}$). We can construct for every dg category $\mathcal{A}$, a diagram $\mathrm{Dia}(\mathcal{A}) \in \mathsf{Trdgcat}(\mathbf{V})$ as follows



Notice that the assignment $\mathcal{A} \mapsto \mathrm{Dia}(\mathcal{A})$ is functorial. We obtain then a morphism of derivators

$$\mathrm{Dia}(-) : \mathsf{Trdgcat} \longrightarrow \mathsf{Trdgcat}_{\mathbf{V}} .$$

Lemma 7.1. — *Let $\mathcal{A}$ be a (pre-triangulated) dg category with countable sums. Then for any dg category $\mathcal{B}$, $\mathsf{rep}_{tr}(\mathcal{B}, \mathcal{A})$ has countable sums.*

Proof. — Recall that $\mathcal{A}$ has countable sums if and only if the diagonal functor

$$\mathcal{A} \longrightarrow \prod_{\mathbb{N}} \mathcal{A}$$

has a left adjoint. Since for any dg category $\mathcal{B}$, $\mathsf{rep}_{tr}(\mathcal{B}, -)$ is a 2-functor which moreover preserves derived products, the proof is finished. $\surd$

Proposition 7.2. — *Let $\mathcal{A}$ be a dg category and $n \geq 0$. Then $\mathcal{S}^n(\mathcal{A})/\mathcal{S}^n(\mathcal{A})$ and $\mathcal{F}\mathcal{S}^n(\mathcal{A})$ become trivial after application of $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$.*

Proof. — Since $\mathcal{S}^n(\mathcal{A})/\mathcal{S}^n(\mathcal{A})$ is clearly isomorphic to the terminal object in Hec and $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ preserves the terminal object, $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{S}^n(\mathcal{A})/\mathcal{S}^n(\mathcal{A}))$ is trivial. In what concerns $\mathcal{FS}^n(\mathcal{A})$, it is enough by Proposition 3.5, to show that for every α-small dg cell $\mathcal{B}$, the spectrum

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}), \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{FS}^n(\mathcal{A}))) \simeq \underline{K}\mathsf{rep}_{tr}(\mathcal{B}, \mathcal{FS}^n(\mathcal{A}))$$

is (homotopically) trivial. Since, by Proposition 6.3 (i), $\mathcal{FS}^n(\mathcal{A})$ has countable sums, Lemma 7.1 implies that $\mathsf{rep}_{tr}(\mathcal{B}, \mathcal{FS}^n(\mathcal{A}))$ has the same property, which in turns implies, by additivity, that the identity of $\underline{K}\mathsf{rep}_{tr}(\mathcal{B}, \mathcal{FS}^n(\mathcal{A}))$ is homotopic to zero. $\surd$

Notation 7.3. — Let $V(-)$ be the composed morphism

$$\mathsf{Trdgcat} \overset{\mathrm{Dia}(-)^\wedge}{\longrightarrow} \mathsf{Trdgcat}_{\mathbf{V}} \overset{\underline{\mathcal{U}}_\alpha^{\mathsf{add}}}{\longrightarrow} (\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}})_{\mathbf{V}} \,,$$

where $\mathrm{Dia}(-)^\wedge$ is obtained from $\mathrm{Dia}(-)$ by applying $(-)^\wedge$ objectwise. By Proposition 7.2, for any dg category $\mathcal{A}$, $V(\mathcal{A})$ is a spectrum in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$; see §A.3. We define $V_a(\mathcal{A}) = \Omega^\infty V(\mathcal{A})$ as the infinite loop object associated to $V(\mathcal{A})$. This construction defines a morphism of derivators

$$V_a : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} \,.$$

We also define V_l as the composition $\gamma_! \circ V_a$.

$$V_l : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} \,.$$

Remark 7.4. — For any dg category $\mathcal{A}$, we have by definition

$$V_a(\mathcal{A}) = \Omega^\infty V(\mathcal{A}) \simeq \underset{n\geq 0}{\mathrm{hocolim}}\, \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \,.$$

Moreover, since we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Spec}(\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}) & \xrightarrow{\mathsf{Spec}(\gamma_!)} & \mathsf{Spec}(\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}) \\ \Omega^\infty \downarrow & & \downarrow \Omega^\infty \\ \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} & \xrightarrow[\gamma_!]{} & \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} \,, \end{array}$$

$V_l(\mathcal{A})$ can be described as

$$V_l(\mathcal{A}) = \underset{n\geq 0}{\mathrm{hocolim}}\, \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \,.$$

Notice also that we have a natural 2-morphism of derivators $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(-) \Rightarrow V_a(-)$.

Proposition 7.5. — *For any dg category $\mathcal{A}$, we have a natural natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A})) \simeq I\!\!K(\mathcal{A}) \,.$$

Proof. — Using Proposition 6.6 it is easy to check that we have a weak equivalence of spectra :

$$I\!\!K(\mathcal{A}) \simeq \operatorname*{hocolim}_{n\geq 0} K(\mathcal{S}^n(\mathcal{A}))[-n]\,.$$

As we also have a weak equivalence of spectra

$$\operatorname*{hocolim}_{n\geq 0} K(\mathcal{S}^n(\mathcal{A}))[-n] \simeq \operatorname*{hocolim}_{n\geq 0} K(\mathcal{S}^n(\mathcal{A})^\wedge)[-n]$$

(see the proof of [32, §12.1, Thm. 8]). We deduce from this the following computations.

$$\begin{array}{rcl} I\!\!K(\mathcal{A}) & \simeq & \operatorname*{hocolim}_{n\geq 0} K(\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \\ & \simeq & \operatorname*{hocolim}_{n\geq 0} \underline{K}\,\mathsf{rep}_{tr}(\underline{k}, \mathcal{S}^n(\mathcal{A})^\wedge)[-n] \\ (7.1.1) \qquad & \simeq & \operatorname*{hocolim}_{n\geq 0} \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{A})^\wedge))[-n] \\ (7.1.2) \qquad & \simeq & \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A}))\,. \end{array}$$

Equivalence (7.1.1) comes from Theorem 5.7, and equivalence (7.1.2) from the compactness of $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(k)$. $\surd$

7.2. Co-representability in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$. —

Proposition 7.6. — *The morphism $V_l(-) : \mathsf{Trdgcat} \to \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ sends Morita dg functors to isomorphisms.*

Proof. — As in the proof of Theorem 5.11, it is enough to show that $V_l(-)$ sends the maps of shape $\mathcal{A} \to \mathcal{A}^\wedge$ to isomorphisms. For this consider the following diagram

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{F}(\mathcal{A}) & \longrightarrow & \mathcal{S}(\mathcal{A}) := \mathcal{F}(\mathcal{A})/\mathcal{A} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(P)} & & \downarrow{\scriptstyle \mathcal{S}(P)} \\ \mathcal{A}^\wedge & \longrightarrow & \mathcal{F}(\mathcal{A}^\wedge) & \longrightarrow & \mathcal{S}(\mathcal{A}^\wedge) := \mathcal{F}(\mathcal{A}^\wedge)/\mathcal{A}^\wedge\,. \end{array}$$

By Proposition 6.3, $\mathcal{F}(P)$ is an isomorphism. Observe that the induced morphism $\mathcal{S}(P)$, between the Drinfeld dg quotients, is also an isomorphism. Finally using the description of $V_l(-)$ of Remark 7.4, the proof is finished. $\surd$

Notice that, by Proposition 7.6, the morphism $V_l(-)$ descends to $\mathsf{HO}(\mathsf{dgcat})$.

Proposition 7.7. — *Assume that $\alpha \geq \aleph_1$. The morphism of derivators*

$$V_l(-) : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$$

is an α-localizing invariant (5.12).

Proof. — We start by showing condition α-flt). Since, by Lemma 6.2, the functor $\mathcal{F}(-)$ preserves α-filtered colimits, so does (by construction) the functor $\mathcal{S}^n(-)$, $n \geq 0$. Now, Proposition 4.1 (i), Theorem 5.5 and the classical Fubini rule on homotopy colimits imply the claim. Condition $\underline{\mathrm{p}}$) is clear. In what concerns condition loc), let

$$0 \longrightarrow \mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C} \longrightarrow 0$$

be an exact sequence in Hmo. Consider the following diagram in $\mathsf{Trdgcat}(\mathbf{V})$

$$\begin{array}{ccccc} \mathrm{Dia}(\mathcal{A})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{B})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge := \mathrm{Dia}(\mathcal{A})^\wedge/\mathrm{Dia}(\mathcal{B})^\wedge \\ \| & & \| & & \downarrow \\ \mathrm{Dia}(\mathcal{A})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{B})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{C})^\wedge \,, \end{array}$$

where $\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge$ is obtained by applying the Drinfeld's dg quotient objectwise. Since the upper line belongs is objectwise a strict exact sequence, we obtain a distinguished triangle

$$V_l(\mathcal{A}) \longrightarrow V_l(\mathcal{B}) \longrightarrow \Omega^\infty \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) \longrightarrow V_l(\mathcal{A})[1]$$

in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$. We have the following explicit description of $\Omega^\infty \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge)$:

$$\Omega^\infty \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) = \underset{n}{\mathrm{hocolim}}\, \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{B})^\wedge/\mathcal{S}^n(\mathcal{A})^\wedge)[-n]\,.$$

We now show that the morphism

$$D : \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge)$$

becomes an isomorphism after application of the infinite loop functor Ω^∞. For this, consider the following solid diagram

$$\begin{array}{ccccc} \mathcal{S}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{A})^\wedge \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}^n(\mathcal{B})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{B})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{B})^\wedge \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}^n(\mathcal{B})^\wedge/\mathcal{S}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{B})^\wedge/\mathcal{FS}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{B})^\wedge/\mathcal{S}^{n+1}(\mathcal{A})^\wedge \\ \downarrow & \overset{\psi_n}{\nearrow} & \downarrow{\scriptstyle \phi} & & \downarrow \\ \mathcal{S}^n(\mathcal{C})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{C})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{C})^\wedge \,, \end{array}$$

where the composition in each line is trivial. Since the dg functor $\mathcal{FS}^n(\mathcal{A})^\wedge \to \mathcal{FS}^n(\mathcal{B})^\wedge$ preserves countable sums, [32, §3.1, Thm. 2] implies that $\mathcal{FS}^n(\mathcal{B})^\wedge/\mathcal{FS}^n(\mathcal{A})^\wedge$ is Morita fibrant and so ϕ is an isomorphism. We obtain then a well defined morphism ψ_n, which induces an interpolation map

$$\Psi_n : \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{C})^\wedge)[-n] \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^{n+1}(\mathcal{B})^\wedge/\mathcal{S}^{n+1}(\mathcal{A})^\wedge)[-(n+1)]\,.$$

Notice that we have commutative diagrams

$$\begin{array}{ccc} \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge)_n[-n] & \longrightarrow & \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge)_{n+1}[-(n+1)] \\ {\scriptstyle D_n[-n]}\downarrow & \overset{\Psi_n}{\nearrow} & \downarrow{\scriptstyle D_{n+1}[-(n+1)]} \\ \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge)_n[-n] & \longrightarrow & \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge)_{n+1}[-(n+1)]\,. \end{array}$$

The existence of such a set of interpolation maps $\{\Psi_n\}_{n\in\mathbb{N}}$ implies that the map

$$\operatorname*{hocolim}_n \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{B})^\wedge/\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \stackrel{\sim}{\longrightarrow} \operatorname*{hocolim}_n \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{C})^\wedge)[-n]$$

is an isomorphism; see Lemma A.7. In other words, the induced morphism

$$\Omega^\infty(D) : \Omega^\infty\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) \stackrel{\sim}{\longrightarrow} \Omega^\infty\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge) = V_l(\mathcal{C})$$

is an isomorphism. $\surd$

Corollary 7.8. — *(of Proposition 7.7) If $\alpha \geq \aleph_1$, since $V_l(-)$ is an α-localizing invariant, Theorem 5.11 implies the existence of a morphism of derivators (which commutes with homotopy colimits)*

$$\mathsf{Loc} : \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$$

such that $\mathsf{Loc}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})) \simeq V_l(\mathcal{A})$, for every dg category $\mathcal{A}$.

Proposition 7.9. — *If $\alpha \geq \aleph_1$, the two morphisms of derivators*

$$\mathsf{Loc}\,,\ l^* : \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$$

are canonically isomorphic.

Proof. — Consider the composed morphism $L := \mathsf{Loc} \circ l_!$. Notice that, by virtue of Theorem 5.11 and of Proposition 7.6, the 2-morphism of derivators $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}} \Rightarrow V_l(-)$ can be naturally extended to a 2-morphism $\eta : \mathrm{Id} \Rightarrow L$. We show first that the couple (L,η) defines a left Bousfield localization of the category $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$, *i.e.* we prove that the natural transformations $L\eta$ and η_L are equal isomorphisms. Since by construction, L commutes with homotopy colimits, Theorem 5.11 implies that it is enough to show it for the objects of the form $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})$, with $\mathcal{A}$ a (pre-triangulated) dg category. Notice that, by Proposition 7.6, we have $V_l(\mathcal{A}) \simeq V_l(\mathcal{A}^\wedge)$ and Proposition 7.7, applied recursively to the exact sequences

$$\mathcal{S}^n(\mathcal{A}) \longrightarrow \mathcal{F}\mathcal{S}^n(\mathcal{A}) \longrightarrow \mathcal{S}^{n+1}(\mathcal{A}),\ n \geq 0\,,$$

shows us that we have $V_l(\mathcal{S}^n(\mathcal{A})) \simeq V_l(\mathcal{A})[n]$. This implies the following isomorphisms

$$\begin{aligned} L^2(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) &= L(V_l(\mathcal{A})) \\ &\simeq \operatorname*{hocolim}_{n\geq 0} V_l(\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \\ &\simeq \operatorname*{hocolim}_{n\geq 0} V_l(\mathcal{S}^n(\mathcal{A}))[-n] \\ &\simeq \operatorname*{hocolim}_{n\geq 0} (V_l(\mathcal{A})[n])[-n] \\ &\simeq V_l(\mathcal{A})\,. \end{aligned}$$

More precisely, a careful description of the above isomorphisms shows that the morphisms

$$\eta_{L(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A}))}, L(\eta_{\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})}) : L(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) \longrightarrow L^2(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A}))$$

become equal isomorphisms after composing with the canonical isomorphism

$$L^2(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) \stackrel{\sim}{\longrightarrow} \operatorname*{hocolim}_{m,n\geq 0} (\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})[m+n])[-m-n]\,.$$

Hence $\eta_{L(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A}))} = L(\eta_{\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})})$ is an isomorphism, which proves that (L, η) defines a left Bousfield localization of the category $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$.

The general formalism of left Bousfield localizations makes that we are now reduced to prove the following property : a morphism of $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$ becomes an isomorphism after applying the functor L if and only if it becomes an isomorphism after applying the functor $l_!$. For this purpose, it is even sufficient to prove that the induced morphism $l_!\eta : l_! \Rightarrow l_!(L)$ is an isomorphism. Once again it is enough to show it for the objects of the form $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})$. By virtue of Theorem 5.11, the spectrum $\{\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{A})^\wedge)\}_{n\geq 0}$ becomes an Ω-spectrum in $\mathsf{St}(\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}})$ after application of $l_!$. Therefore, we have an isomorphism $\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A}) \simeq l_! V_l(\mathcal{A})$. $\surd$

Remark 7.10. — Since the morphism of derivators Loc preserves homotopy colimits, Proposition 7.9 and Theorem 5.7 imply that $\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(k)$ is compact in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}(e)$.

Theorem 7.11. — *If $\alpha \geq \aleph_1$, then for every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})) \simeq I\!K(\mathcal{A}) .$$

Proof. — We have the following isomorphisms.

$$\begin{aligned} \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})) &\simeq \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), l^*(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A}))) \\ &\simeq \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), \mathsf{Loc}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A}))) &(7.2.1)\\ &\simeq \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A})) &(7.2.2)\\ &\simeq I\!K(\mathcal{A}) &(7.2.3)\end{aligned}$$

Equivalence (7.2.1) comes from Proposition 7.9, equivalence (7.2.2) follows from Corollary 7.8, and equivalence (7.2.3) is Proposition 7.5. $\surd$

7.3. Filtered homotopy colimits. — Recall from [34, §10], the construction of the *universal localizing invariant*

$$\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}} .$$

Theorem 7.12. — *([34, Thm. 10.5]) The morphism $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}$:*

flt) *commutes with filtered homotopy colimits;*

p) *preserves the terminal object and*

loc) *sends the exact sequences in* Hmo *(3.1) to distinguished triangles in* $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$.

Moreover $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}$ is universal with respect to these properties, i.e. for every strong triangulated derivator $\mathbb{D}$, we have an equivalence of categories

$$(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}})^* : \underline{\mathsf{Hom}}_!(\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\mathsf{dg}}^{\mathsf{loc}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}) ,$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ *of morphisms of derivators which verify the above three conditions.*

Notation 7.13. — The objects of the category $\underline{\mathsf{Hom}}_{\mathsf{dg}}^{\mathsf{loc}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ will be called *localizing invariants.*

Note that $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}} = \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\aleph_0}$, so that Theorem 7.12 is a particular case of Theorem 5.11 for $\alpha = \aleph_0$.

We will consider now a fixed infinite regular cardinal $\alpha \geq \aleph_1$. By Theorem 5.11, we have a unique morphism of derivators (which preserves all homotopy colimits)

$$t_! : \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha} \longrightarrow \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}} ,$$

such that $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} = t_! \circ \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}$.

Proposition 7.14. — *The morphism $t_! : \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha} \to \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ describes the derivator $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ as the left Bousfiled localization of the derivator $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$ by all maps of shape*

$$\underset{j\in J}{hocolim}\, \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(D_j) \longrightarrow \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\mathcal{A})\,, \tag{7.3.1}$$

where $\mathcal{A}$ is a dg category, J is a filtered category, and $D : J \to \mathsf{dgcat}$ is a functor such that $\underset{j\in J}{hocolim}\, D_j \simeq \mathcal{A}$ in Hmo. Moreover, the morphism $t_!$ has a fully faithful right adjoint.

Proof. — The first assertion follows directly from Theorem 7.12 and from the universal property of left Bousfield localizations of derivators (A.2).

In order to prove the second assertion, we start by replacing $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ (resp. $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$) by $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$ (resp. by $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$). We will describe the derivator $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$ as the left Bousfield localization of $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ by an explicit small set of maps, namely, the set T of maps of shape

$$\underset{j\in J}{\mathrm{hocolim}}\, \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(D_j) \longrightarrow \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{A})\,, \tag{7.3.2}$$

where $\mathcal{A}$ is an α-small dg cell, J is a filtered category, and $D : J \to \mathsf{dgcat}$ is a functor with values in homotopically finitely presented dg categories, such that $\underset{j\in J}{\mathrm{hocolim}}\, D_j \simeq \mathcal{A}$ in Hec.

Notice that, by construction of $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$, for any triangulated strong derivator $\mathbb{D}$, there is a canonical equivalence of categories between $\underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}, \mathbb{D})$ and the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{dgcat}_{\alpha}, \mathbb{D}) = \mathbb{D}(\mathsf{dgcat}^{\mathsf{op}}_{\alpha})$ which consists of morphisms $\mathsf{dgcat}_{\alpha} \to \mathbb{D}$ which preserve the terminal object, send Morita equivalences to isomorphisms, and send split exact sequences of α-small dg cells to split distinguished triangles in $\mathbb{D}$.

The map $\mathsf{dgcat}_{\alpha} \to \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ corresponding to the identity of $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ can be restricted to finite dg cells (*i.e.* $\aleph_0$-small dg cells), hence defines a canonical map $\mathsf{dgcat}_f \to \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ which preserves the terminal object, send Morita equivalences to isomorphisms, and send split exact sequences of finite dg cells to direct sums. In other words, it defines a canonical homotopy colimit preserving map

$$\varphi : \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0} \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}\,.$$

Let $loc_T : \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha} \to \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$ be the left Bousfield localization of $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ by T. By definiton of loc_T, we get the following essentially commutative diagram of derivators

$$
\begin{array}{ccccccc}
\mathsf{dgcat}_{\alpha} & \xrightarrow{\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}} & \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha} & & \xrightarrow{\quad loc_T \quad} & & \\
\downarrow & & & & & & \searrow \\
\mathsf{dgcat} & \xrightarrow[\underline{\mathcal{U}}^{\mathsf{add}}_{\aleph_0}]{} & \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0} & \xrightarrow[\varphi]{} & \underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha} & \xrightarrow[loc_T]{} & L_T\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}\,,
\end{array}
$$

from which we deduce that the morphism $loc_T \circ \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}$ preserves filtered homotopy colimits. This implies that the canonical map (obtained from Theorem 3.2 and from the universal property of the left Bousfield localization by T)

$$L_T\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha} \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$$

is an equivalence of derivators. This in turns implies that $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}} = \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\aleph_0}$ is the left Bousfield localization of $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$ by the set of maps of shape

$$\underset{j\in J}{\mathrm{hocolim}}\, \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(D_j) \longrightarrow \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\mathcal{A})\,, \tag{7.3.3}$$

where $\mathcal{A}$ is an α-small dg cell, J is a filtered category, and $D : J \to \mathsf{dgcat}$ is a functor with values in homotopically finitely presented dg categories, such that $\underset{j\in J}{\mathrm{hocolim}}\, D_j \simeq \mathcal{A}$ in Hmo. In particular, $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ is obtained from $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$ as a left Bousfield localization by a small set of maps, hence comes from a left Bousfield localization at the level of the underlying model categories (see §A.2), which implies that the canonical map $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha} \to \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ has a fully faithful right adjoint[(1)]. $\surd$

We obtain then an adjunction

$$
\begin{array}{c}
\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha} \\
{\scriptstyle t_!}\downarrow\ \uparrow{\scriptstyle t^*} \\
\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}\,.
\end{array}
$$

7.4. Co-representability in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$. —

Theorem 7.15. — *The object $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k})$ is compact in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$, and, for every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Hom}(\,\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k}), \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{A})\,) \simeq I\!\!K(\mathcal{A})\,.$$

---

(1) The existence of this fully faithful right adjoint can be otained directly from the general theory of left Bousfield localizations and from the fact that any homotopy colimit preserving morphism of derivators which are obtained from combinatorial model categories has a right adjoint; see [31]. However, our proof is much more explicit!

Proof. — The proof consists on verifying the conditions of Proposition 5.6, with $\mathbb{D} = \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$, S the set of maps of shape (7.3.3), $X = \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\underline{k})$ and $M = \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\mathcal{A})$. Since by Remark 7.10, $\underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\underline{k})$ is compact in $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}(e)$, it is enough to show that the functor

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\underline{k}), -) : \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}(e) \longrightarrow \mathsf{Ho}(\mathsf{Spt})$$

sends the elements of S to isomorphisms. Notice that this follows from the fact that negative K-theory preserves filtered homotopy colimits (see [32, §7, Lemma 6]) and that, by Theorem 7.11, we have a natural isomorphism in the stable homotopy category of spectra

$$\mathbb{R}\mathsf{Hom}(\,\underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\underline{k}), \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\mathcal{A})\,) \simeq I\!\!K(\mathcal{A})\,,$$

for every dg category $\mathcal{A}$. $\surd$

Corollary 7.16. — *We have natural isomorphisms of abelian groups*

$$\mathsf{Hom}(\,\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k})[n], \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{A})) \simeq I\!\!K_n(\mathcal{A})\,,\;\; n \in \mathbb{Z}\,.$$

8. Higher Chern characters

In this Section we show how our co-representability Theorem 7.15 furnishes us for free higher Chern characters and higher trace maps; see Theorem 8.4. Recall that, throughout the article, we have been working over a commutative base ring k.

Consider a localizing invariant (7.13)

$$E : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})\,.$$

We also have a localizing invariant associated to non-connective K-theory (§2.6) :

$$I\!\!K : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})\,.$$

We will use Theorem 7.15 and its Corollary to understand the natural transformations from $I\!\!K$ to E.

Since E and $I\!\!K$ are localizing invariants, by virtue of Theorem 7.12, they descend to morphisms of derivators

$$\overline{E}, \overline{I\!\!K} : \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}} \longrightarrow \mathsf{HO}(\mathsf{Spt})$$

which commute with homotopy colimits, such that $\overline{E} \circ \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} = E$ and $\overline{I\!\!K} \circ \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} = I\!\!K$.

Theorem 8.1. — *There is a canonical bijection between the following sets.*

(i) *The set of* 2*-morphisms of derivators* $\overline{I\!\!K} \Rightarrow \overline{E}$.
(ii) *The set of* 2*-morphisms of derivators* $I\!\!K \Rightarrow E$.
(iii) *The set* $E_0(\underline{k}) := \pi_0(E(\underline{k}))$.

Proof. — The bijection between (i) and (ii) comes directly from Theorem 7.12. Given a natural transformation $c : I\!\!K \Rightarrow E$, we have in particular a map

$$c(k) : I\!\!K_0(k) = \pi_0 I\!\!K(k) \longrightarrow \pi_0 E(\underline{k}) = E_0(\underline{k})\,,$$

which furnishes us an element $c(k)(1)$ in $E_0(\underline{k})$. On the other hand, given $x \in E_0(\underline{k})$, seen as a map $x : S^0 \to E(\underline{k})$ in the stable homotopy category of spectra, we obtain, for any small category I, and any functor $\mathcal{A}$ from I^{op} to dgcat, a morphism

$$c_x(\mathcal{A}) : I\!\!K(\mathcal{A}) \longrightarrow E(\mathcal{A})$$

in the stable homotopy category of presheaves of spectra on I. The map $c_x(\mathcal{A})$ is obtained as follows: by virtue of Theorem 7.15, we have

$$I\!\!K(\mathcal{A}) \simeq \mathbb{R}\mathsf{Hom}(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k}), \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{A})\,)\,,$$

while we obviously have the identification

$$E(\mathcal{A}) \simeq \mathbb{R}\mathsf{Hom}(S^0, E(\mathcal{A}))\,.$$

Hence the composition

$$\mathbb{R}\mathsf{Hom}(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k}), \mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{A})\,) \longrightarrow \mathbb{R}\mathsf{Hom}(E(\underline{k}), E(\mathcal{A})) \xrightarrow{x^*} \mathbb{R}\mathsf{Hom}(S^0, E(\mathcal{A}))$$

defines a map $c_x(\mathcal{A})$ in the stable homotopy category of presheaves of spectra on I (here, we have considered S^0 and $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k})$ as objects in $\mathsf{Ho}(\mathsf{Spt})$ and $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ respectively). This construction is 2-functorial, and thus induces a 2-morphism of derivators $I\!\!K \Rightarrow E$, hence a 2-morphism of derivators $\overline{I\!\!K} \Rightarrow \overline{E}$.

The fact that we obtain a bijection between (b) and (c) follows now directly from the enriched version of the Yoneda Lemma (recall that the functors $\overline{I\!\!K}$ and $\overline{E}$ are enriched over spectra; see §A.3). $\surd$

Remark 8.2. — As $\overline{I\!\!K}$ and $\overline{E}$ are enriched over spectra, there is a spectra $\mathbb{R}\mathsf{Nat}(\overline{I\!\!K}, \overline{E})$ of natural transformations. Theorem 7.15 and the enriched Yoneda Lemma leads to an enriched version of the preceding Theorem (as it was suggested in its proof):

Corollary 8.3. — *There is a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Nat}(\overline{I\!\!K}, \overline{E}) \simeq E(\underline{k})\,.$$

Proof. — Theorem 7.15 implies that the natural map

$$\mathbb{R}\mathsf{Nat}(\overline{I\!\!K}, \overline{E}) \longrightarrow \overline{E}(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k})) = E(\underline{k})$$

induced by the unit $1 \in I\!\!K(k)$ leads to a bijection

$$\pi_0 \mathbb{R}\mathsf{Nat}(\overline{I\!\!K}, \overline{E}) \simeq \pi_0 E(\underline{k})$$

(as $\pi_0 \mathbb{R}\mathsf{Nat}(\overline{I\!\!K}, \overline{E})$ is the set of 2-morphisms $\overline{I\!\!K} \Rightarrow \overline{E}$). By applying this to the morphism of derivators $E(-)[-n] : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathsf{Spt})$ (*i.e.* E composed with the suspension functor iterated n times), one deduces that the map

$$\pi_n \mathbb{R}\mathsf{Nat}(\overline{I\!\!K}, \overline{E}) \longrightarrow \pi_n E(\underline{k})$$

is bijective for any integer n. $\surd$

As an illustration, let us consider

$$I\!K_n(-) : \mathsf{Ho}(\mathsf{dgcat}) \longrightarrow \mathsf{Mod}\text{-}\mathbb{Z}\,,\ n \in \mathbb{Z}\,,$$

the n-th algebraic K-theory group functor [32, §12]

$$HC_j(-) : \mathsf{Ho}(\mathsf{dgcat}) \longrightarrow \mathsf{Mod}\text{-}\mathbb{Z}\,,\ j \geq 0\,,$$

the j-th cyclic homology group functor [34, Thm. 10.7] and

$$THH_j(-) : \mathsf{Ho}(\mathsf{dgcat}) \longrightarrow \mathsf{Mod}\text{-}\mathbb{Z}\,,\ j \geq 0\,,$$

the j-th topological Hochschild homology group functor [36, §11].

Theorem 8.4. — *We have the following functorial morphisms of abelian groups for every dg category $\mathcal{A}$:*

(i) *Higher Chern characters*

$$ch_{n,r} : I\!K_n(\mathcal{A}) \longrightarrow HC_{n+2r}(\mathcal{A})\,,\ n \in \mathbb{Z}\,,\ r \geq 0\,,$$

such that $ch_{0,r} : I\!K_0(k) \longrightarrow HC_{2r}(k)$ sends $1 \in I\!K_0(k)$ to a generator of the k-module of rank one $HC_{2r}(k)$.

(ii) *When $k = \mathbb{Z}$, higher trace maps*

$$tr_n : I\!K_n(\mathcal{A}) \longrightarrow THH_n(\mathcal{A})\,, n \in \mathbb{Z}\,,$$

such that $tr_0 : I\!K_0(k) \longrightarrow THH_0(k)$ sends $1 \in I\!K_0(\mathbb{Z})$ to $1 \in THH_0(\mathbb{Z})$, and

$$tr_{n,r} : I\!K_n(\mathcal{A}) \longrightarrow THH_{n+2r-1}(\mathcal{A})\,, n \in \mathbb{Z}\,,\ r \geq 1\,,$$

such that $tr_{0,r} : I\!K_0(k) \longrightarrow THH_{2r-1}(k)$ sends $1 \in I\!K_0(\mathbb{Z})$ to a generator in the cyclic group $THH_{2r-1}(\mathbb{Z}) \simeq \mathbb{Z}/r\mathbb{Z}$.

(iii) *When $k = \mathbb{Z}/p\mathbb{Z}$, with p a prime number, higher trace maps*

$$tr_{n,r} : I\!K_n(\mathcal{A}) \longrightarrow THH_{n+2r}(\mathcal{A})\,, n, r \in \mathbb{Z}\,,$$

such that $tr_{0,r} : I\!K_0(k) \longrightarrow THH_{2r}(k)$ sends $1 \in I\!K_0(\mathbb{Z})$ to a generator in the cyclic group $THH_0(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.

Proof. — (i) Recall from [34, Thm. 10.7], that we have a *mixed complex* localizing invariant

$$C : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\Lambda\text{-}\mathsf{Mod})\,.$$

This defines a cyclic homology functor

$$HC : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathcal{D}(k)\,,$$

where $HC(\mathcal{A}) = k \otimes^{\mathbb{L}}_{\Lambda} C(\mathcal{A})$, and where $\mathcal{D}(k)$ denotes the triangulated derivator associated to the model category of unbounded complexes of k-modules (localized by quasi-isomorphisms). We thus have the following formula (see Lemma A.6) :

$$\begin{aligned} HC_j(\mathcal{A}) &= \mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A})[-j]) \\ &\simeq \mathsf{Hom}_{\mathcal{D}(k)}((S^0 \otimes k, HC(\mathcal{A})[-j]) \\ &\simeq \mathsf{Hom}_{\mathsf{Ho(Spt)}}(S^0, \mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A})[-j])) \\ &\simeq \pi_0(\mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A})[-j]))\,. \end{aligned}$$

Since

$$HC_*(k) \simeq k[u], \ |u| = 2\,,$$

we conclude from Theorem 8.1 applied to $E = \mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(-))[-2r]$, that, for each integer $r \geq 0$, the element $u^r \in HC_{2r}(k)$ defines a natural map

$$I\!\!K(\mathcal{A}) \longrightarrow \mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A}))[-2r]\,.$$

(ii) By Blumberg-Mandell's localization Theorem [3, Thm. 6.1] and the connection between dg and spectral categories developped in [36], we have a localizing invariant

$$THH : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})\,.$$

Thanks to Bökstedt [16, 0.2.3], we have the following calculation

$$THH_j(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad j=0 \\ \mathbb{Z}/r\mathbb{Z} & \text{if} \quad j = 2r-1, \ r \geq 1 \\ 0 & \qquad \text{otherwise,} \end{cases}$$

hence the canonical generators of $\mathbb{Z}$ and $\mathbb{Z}/r\mathbb{Z}$ furnishes us the higher trace maps by applying Theorem 8.1 to $E = THH(-)[-n]$ with $n = 0$, or with $n = 2r-1$ and $r \geq 1$.

(iii) The proof is the same as the preceding one : over the ring $\mathbb{Z}/p\mathbb{Z}$, we have the following calculation (see [16, 0.2.3])

$$THH_j(\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

Hence the canonical generator $\mathbb{Z}/p\mathbb{Z}$ furnishes us the higher trace maps by applying Theorem 8.1 to $E = THH(-)[-2r]$.

$\surd$

A
Grothendieck derivators

A.1. Notations. — The original reference for derivators is Grothendieck's manuscript [21] and Heller's monograph [17] on homotopy theories. See also [28, 8, 9, 12]. A derivator $\mathbb{D}$ consists of a strict contravariant 2-functor from the 2-category of small categories to the 2-category of categories

$$\mathbb{D} : \mathsf{Cat}^{\mathsf{op}} \longrightarrow \mathsf{CAT},$$

subject to certain conditions, the main ones being that for any functor between small categories $u : X \to Y$, the *inverse image functor*

$$u^* = \mathbb{D}(u) : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$$

has a left adjoint, called the *homological direct image functor*,

$$u_! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)\,,$$

as well as right adjoint, called the *cohomological direct image functor*

$$u_! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)\,.$$

See [8] for details. The essential example to keep in mind is the derivator $\mathbb{D} = \mathsf{HO}(\mathcal{M})$ associated to a (complete and cocomplete) Quillen model category $\mathcal{M}$ and defined for every small category X by

$$\mathsf{HO}(\mathcal{M})\ (X) = \mathsf{Ho}\big(\mathsf{Fun}(X^{\mathsf{op}}, \mathcal{M})\big)\,,$$

where $\mathsf{Fun}(X^{\mathsf{op}}, \mathcal{M})$ is the category of presheaves on X with values in $\mathcal{M}$; see [8, Thm. 6.11]. We denote by e the 1-point category with one object and one (identity) morphism. Heuristically, the category $\mathbb{D}(e)$ is the basic "derived" category under consideration in the derivator $\mathbb{D}$. For instance, if $\mathbb{D} = \mathsf{HO}(\mathcal{M})$ then $\mathbb{D}(e) = \mathsf{Ho}(\mathcal{M})$ is the usual homotopy category of $\mathcal{M}$.

Definition A.1. — We now recall some technical properties of derivators:

(i) A derivator $\mathbb{D}$ is called *strong* if for every finite free category X and every small category Y, the natural functor $\mathbb{D}(X \times Y) \longrightarrow \mathsf{Fun}(X^{\mathsf{op}}, \mathbb{D}(Y))$ is full and essentially surjective. See [18] for details.

(ii) A derivator $\mathbb{D}$ is called *regular* if in $\mathbb{D}$, sequential homotopy colimits commute with finite products and homotopy pullbacks. See also [18] for details.

(iii) A derivator $\mathbb{D}$ is *pointed* if for any closed immersion $i : Z \to X$ in Cat the cohomological direct image functor $i_* : \mathbb{D}(Z) \longrightarrow \mathbb{D}(X)$ has a right adjoint, and if moreover and dually, for any open immersion $j : U \to X$ the homological direct image functor $j_! : \mathbb{D}(U) \longrightarrow \mathbb{D}(X)$ has a left adjoint. See [12, 1.13] for details.

(iv) A derivator $\mathbb{D}$ is called *triangulated* or *stable* if it is pointed and if every global commutative square in $\mathbb{D}$ is cartesian exactly when it is cocartesian. See [12, 1.15] for details.

A strong derivator is the same thing as a small homotopy theory in the sense of Heller [18]. By [10, Prop. 2.15], if $\mathcal{M}$ is a Quillen model category, its associated derivator $\mathsf{HO}(\mathcal{M})$ is strong. Moreover, if sequential homotopy colimits commute with finite products and homotopy pullbacks in $\mathcal{M}$, the associated derivator $\mathsf{HO}(\mathcal{M})$ is regular. Notice also that if $\mathcal{M}$ is pointed, then so is $\mathsf{HO}(\mathcal{M})$. Finally, a pointed model category $\mathcal{M}$ is stable if and only if its associated derivator $\mathsf{HO}(\mathcal{M})$ is triangulated. Let $\mathbb{D}$ and $\mathbb{D}'$ be derivators. We denote by $\underline{\mathsf{Hom}}(\mathbb{D}, \mathbb{D}')$ the category of all morphisms of derivators. Its morphisms will be called 2-morphisms of derivators. Finally we denote by $\underline{\mathsf{Hom}}_!(\mathbb{D}, \mathbb{D}')$ the category of morphisms of derivators which commute with homotopy colimits; see [8, 9]. For instance, any colimit (resp. limit) preserving left (resp. right) Quillen functor induces a morphism of derivators which preserves homotopy colimits (resp. limits); see [8, Prop. 6.12].

If $\mathbb{D}$ is a derivator and if A is a small category, we denote by $\mathbb{D}_A$ the derivator defined by $\mathbb{D}_A(X) = \mathbb{D}(A \times X)$. One can think of $\mathbb{D}_A$ as the derivator of presheaves on A with values in $\mathbb{D}$.

A.2. Left Bousfield localization. — Let $\mathbb{D}$ be a derivator and S a class of morphisms in the base category $\mathbb{D}(e)$.

Definition A.2. — The derivator $\mathbb{D}$ admits a *left Bousfield localization* with respect to S if there exists a morphism of derivators

$$\gamma : \mathbb{D} \longrightarrow \mathsf{L}_S\mathbb{D}\,,$$

which commutes with homotopy colimits, sends the elements of S to isomorphisms in $\mathsf{L}_S\mathbb{D}(e)$ and satisfies the following universal property: for every derivator $\mathbb{D}'$ the morphism γ induces an equivalence of categories

$$\gamma^* : \underline{\mathsf{Hom}}_!(\mathsf{L}_S\mathbb{D}, \mathbb{D}') \xrightarrow{\sim} \underline{\mathsf{Hom}}_{!,S}(\mathbb{D}, \mathbb{D}')\,,$$

where $\underline{\mathsf{Hom}}_{!,S}(\mathbb{D}, \mathbb{D}')$ denotes the category of morphisms of derivators which commute with homotopy colimits and send the elements of S to isomorphisms in $\mathbb{D}'(e)$.

Let $\mathcal{M}$ be a left proper, cellular model category and $\mathsf{L}_S\mathcal{M}$ its left Bousfield localization [19, 4.1.1] with respect to a set of morphisms S in $\mathsf{Ho}(\mathcal{M})$, *i.e.* to perform the localization we choose in $\mathcal{M}$ a representative of each element of S. Then, by [34, Thm. 4.4], the induced morphism of derivators $\mathsf{HO}(\mathcal{M}) \to \mathsf{HO}(\mathsf{L}_S\mathcal{M})$ is a left Bousfield localization of derivators with respect to S. In this situation, we have a natural adjunction of derivators

$$\begin{array}{c}\mathsf{HO}(\mathcal{M})\\ \uparrow\downarrow\\ \mathsf{HO}(\mathsf{L}_S\mathcal{M})\,.\end{array}$$

By [34, Lemma 4.3], the Bousfield localization $\mathsf{L}_S\mathbb{D}$ of a *triangulated* derivator $\mathbb{D}$ remains triangulated as long as S is stable under the desuspension functor $-[-1]$. Such stable left Bousfield localizations are moreover canonically enriched over spectra (see below).

A.3. Stabilization and spectral enrichment. — Let $\mathbb{D}$ be a regular pointed strong derivator. In [18], Heller constructed the universal morphism to a triangulated strong derivator

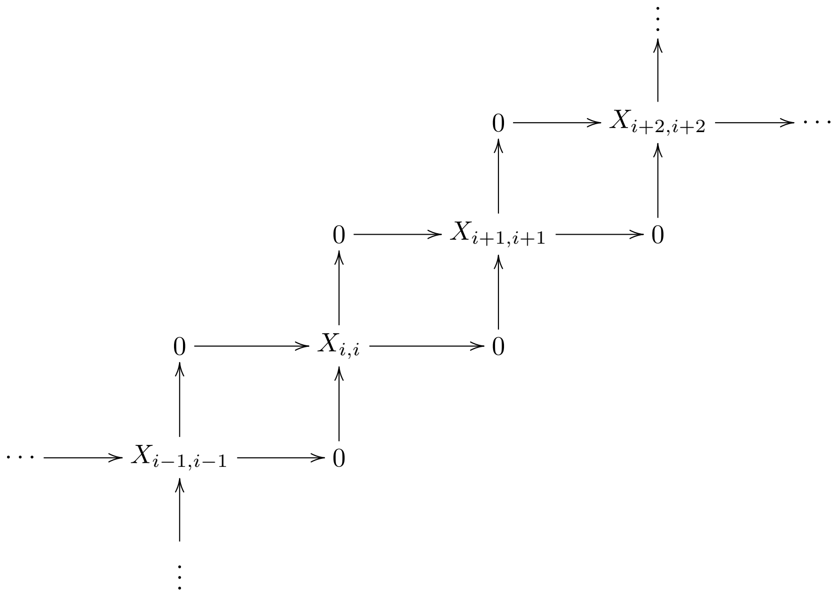
$$\mathsf{stab} : \mathbb{D} \longrightarrow \mathsf{St}(\mathbb{D})\,,$$

in the sense of the following Theorem.

***Theorem A.3* (Heller [18]).** — *Let $\mathbb{T}$ be a triangulated strong derivator. Then the morphism* stab *induces an equivalence of categories*

$$\mathsf{stab}^* : \underline{\mathsf{Hom}}_!(\mathsf{St}(\mathbb{D}), \mathbb{T}) \longrightarrow \underline{\mathsf{Hom}}_!(\mathbb{D}, \mathbb{T})\,.$$

The derivator $\mathsf{St}(\mathbb{D})$ is described as follows. Let $\mathbf{V} = W^{\mathsf{op}}$, where W is the poset $\{(i,j)\,|\,|i-j| \leq 1\} \subset \mathbb{Z} \times \mathbb{Z}$ considered as a small category. A *spectrum* in $\mathbb{D}$ is an object X in $\mathbb{D}(\mathbf{V})$ such that $X_{i,j} \simeq 0$ for $i \neq j$. This defines a derivator $\mathsf{Spec}(\mathbb{D})$ (as a full subderivator of $\mathbb{D}_{\mathbf{V}}$). Spectra are diagrams of the following shape:



In particular, we get maps $X_{i,i} \to \Omega(X_{i+1,i+1})$, $i \in \mathbb{Z}$. The derivator $\mathsf{St}(\mathbb{D})$ is obtained as the full subderivator of $\mathsf{Spec}(\mathbb{D})$ which consists of Ω-spectra, *i.e.* spectra X such that the maps $X_{i,i} \to \Omega(X_{i+1,i+1})$ are all isomorphisms.

There is a morphism of derivators, called the *infinite loop functor*,

$$\Omega^\infty : \mathsf{Spec}(\mathbb{D}) \longrightarrow \mathbb{D}$$

defined by

$$\Omega^\infty(X) = \operatorname*{hocolim}_n \Omega^n(X_{n,n})$$

(where Ω^n denotes the loop space functor iterated n times). There is also a shift morphism

$$(-) < n >: \mathsf{Spec}(\mathbb{D}) \longrightarrow \mathsf{Spec}(\mathbb{D})$$

defined by $(X < n >)_{i,j} = X_{n+i,n+j}$ for $n \in \mathbb{Z}$.

The derivator $\mathsf{St}(\mathbb{D})$ can be described as the left Bousfield localization of $\mathsf{Spec}(\mathbb{D})$ by the maps $X \to Y$ which induce isomorphisms $\Omega^\infty(X < n >) \simeq \Omega^\infty(Y < n >)$ for any integer n; see [18]. Note that, for an Ω-spectrum X, we have a canonical isomorphism: $\Omega^\infty(X < n >) \simeq X_{n,n}$.

For a small category A, let Hot_A (resp. $\mathsf{Hot}_{\bullet,A}$) be the derivator associated to the projective model category structure on the category of simplicial presheaves (resp. of pointed simplicial presheaves) on A. Then the homotopy colimit preserving morphism

$$\mathsf{Hot}_A \longrightarrow \mathsf{Hot}_{\bullet,A} \ , \quad X \longmapsto X_+ = X \amalg *$$

is the universal one with target a pointed regular strong derivators; see [9, Prop. 4.17]. Denote by Spt_A the stable Quillen model category of presheaves of spectra on A, endowed with the projective model structure. The infinite suspension functor defines a homotopy colimit preserving morphism

$$\Sigma^\infty : \mathsf{Hot}_{\bullet,A} \longrightarrow \mathsf{HO}(\mathsf{Spt}_A)\,,$$

and as $\mathsf{HO}(\mathsf{Spt}_A)$ is a triangulated strong derivator, it induces a unique homotopy colimit preserving morphism

$$\mathsf{St}(\mathsf{Hot}_{\bullet,A}) \longrightarrow \mathsf{HO}(\mathsf{Spt}_A)$$

whose composition with $\mathsf{stab} : \mathsf{Hot}_{\bullet,A} \to \mathsf{St}(\mathsf{Hot}_A)$ is the infinite suspension functor. A particular case of [33, Thm. 3.31] gives:

Theorem A.4. — *The canonical morphism $\mathsf{St}(\mathsf{Hot}_{\bullet,A}) \to \mathsf{HO}(\mathsf{Spt}_A)$ is an equivalence of derivators. As a consequence, the map $\Sigma^\infty : \mathsf{Hot}_{\bullet,A} \to \mathsf{HO}(\mathsf{Spt}_A)$ is the universal homotopy colimit preserving morphism from $\mathsf{Hot}_{\bullet,A}$ to a triangulated derivator.*

The composition of the Yoneda embedding $A \to \mathsf{Hot}_A$ with the infinite suspension functor gives a canonical morphism

$$h : A \longrightarrow \mathsf{HO}(\mathsf{Spt}_A)\,.$$

A combination of the preceding Theorem and of [9, Corollary 4.19] leads to the following statement.

Theorem A.5. — *For any triangulated derivator $\mathbb{D}$, the functor*

$$h^* : \underline{\mathsf{Hom}}_!(\mathsf{HO}(\mathsf{Spt}_A), \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}(A, \mathbb{D}) \simeq \mathbb{D}(A^{\mathsf{op}})$$

is an equivalence of categories.

Hence, given any object X in $\mathbb{D}(e)$, there is a unique homotopy colimit preserving morphism of triangulated derivators

$$\mathsf{HO}(\mathsf{Spt}) \longrightarrow \mathbb{D}\,, \quad E \longmapsto E \otimes X$$

which sends the stable 0-sphere to X. This defines a canonical action of $\mathsf{HO}(\mathsf{Spt})$ on $\mathbb{D}$; see [9, Thm. 5.22].

Lemma A.6. — *For any small category A, the functor*

$$\mathsf{Ho}(\mathsf{Spt}_A) = \mathsf{HO}(\mathsf{Spt})(A) \longrightarrow \mathbb{D}(A)\,, \quad E \longmapsto E \otimes X$$

has a right adjoint

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, -) : \mathbb{D}(A) \longrightarrow \mathsf{HO}(\mathsf{Spt})(A)\,.$$

Proof. — This follows from the Brown representability Theorem applied to the compactly generated triangulated category $\mathsf{Ho}(\mathsf{Spt}_A)$; see [27, Theorem 8.4.4]. $\surd$

Using Theorem A.5, one sees easily that the functors of Lemma A.6 define a morphism

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, -) : \mathbb{D} \longrightarrow \mathsf{HO}(\mathsf{Spt})$$

which is right adjoint to the morphism $(-) \otimes X$. In particular, we have the formula

$$\mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, Y)) \simeq \mathsf{Hom}_{\mathbb{D}(e)}(E \otimes X, Y)$$

for any spectrum E and any objects X and Y in $\mathbb{D}(e)$.

This enrichment in spectra is compatible with homotopy colimit preserving morphisms of triangulated derivators (see [9, Thm. 5.22]) : if $\Phi : \mathbb{D} \to \mathbb{D}'$ is a homotopy colimit preserving morphism of triangulated derivators, then for any spectrum E and any object X of $\mathbb{D}$, we have a canonical coherent isomorphism

$$E \otimes \Phi(X) \simeq \Phi(E \otimes X)\,.$$

As a consequence, if moreover Φ has a right adjoint Ψ, then we have canonical isomorphisms in the stable homotopy category of spectra :

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}'}(\Phi(X), Y) \simeq \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, \Psi(Y))\,.$$

Indeed, to construct such an isomorphism, it is sufficient to construct a natural isomorphism of abelian groups, for any spectrum E :

$$\mathsf{Hom}_{\mathsf{Ho(Spt)}}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}'}(\Phi(X), Y)) \simeq \mathsf{Hom}_{\mathsf{Ho(Spt)}}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, \Psi(Y)))\,.$$

Such an isomorphism is obtained by the following computations :

$$\begin{aligned}\mathsf{Hom}_{\mathsf{Ho(Spt)}}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}'}(\Phi(X), Y)) &\simeq \mathsf{Hom}_{\mathbb{D}'}(E \otimes \Phi(X), Y)\\ &\simeq \mathsf{Hom}_{\mathbb{D}'}(\Phi(X \otimes E), Y)\\ &\simeq \mathsf{Hom}_{\mathbb{D}}(X \otimes E, \Psi(Y))\\ &\simeq \mathsf{Hom}_{\mathsf{Ho(Spt)}}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, \Psi(Y)))\,.\end{aligned}$$

A.4. A cofinality argument. —

Lemma A.7. — *Let $\mathbb{D}$ a strong triangulated derivator, and let $D_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathbb{D}(\mathbb{N}^{\mathsf{op}})$. If there exists a family of interpolation maps $\{\Psi_n\}_{n\in\mathbb{N}}$, making the diagram*

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ {\scriptstyle D_0}\downarrow & \nearrow_{\Psi_0} & {\scriptstyle D_1}\downarrow & \nearrow_{\Psi_1} & {\scriptstyle D_2}\downarrow & \nearrow_{\Psi_2} & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots \end{array}$$

commutative in $\mathbb{D}(e)$, then the induced morphism

$$\operatorname{hocolim} D_\bullet : \operatorname*{hocolim}_n X_n \longrightarrow \operatorname*{hocolim}_n Y_n$$

is an isomorphism.

Proof. — Given an object $F_\bullet$ in $\mathbb{D}(\mathbb{N}^{\mathsf{op}})$, we can consider the *weak homotopy colimit* of $F_\bullet$

$$F' = \mathsf{cone}\big(\mathbf{1} - \mathbf{s} : \bigoplus_n F_n \longrightarrow \bigoplus_n F_n\big)\,,$$

where $\mathbf{s}$ is the morphism induced by the maps $F_n \to F_{n+1}$. The object F' is called the sequential colimit of F' in [20, §2.2]. For any object A of $\mathbb{D}(e)$, we then have a Milnor short exact sequence

$$0 \longrightarrow \lim_n{}^1 \mathsf{Hom}_{\mathbb{D}(e)}(F_n[1], A) \longrightarrow \mathsf{Hom}_{\mathbb{D}(e)}(F', A) \longrightarrow \lim_n \mathsf{Hom}_{\mathbb{D}(e)}(F_n, A) \longrightarrow 0\,.$$

As the morphism $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(-, A)$ preserves homotopy limits, we deduce from [7, Thm. IX.3.1] that we also have a Milnor short exact sequence

$$0 \to \lim_n{}^1 \mathsf{Hom}_{\mathbb{D}(e)}(F_n[1], A) \to \mathsf{Hom}_{\mathbb{D}(e)}(\underset{n}{\mathrm{hocolim}}\, F_n, A) \to \lim_n \mathsf{Hom}_{\mathbb{D}(e)}(F_n, A) \to 0\,.$$

We deduce from this that there exists a (non unique) isomorphism

$$\underset{n}{\mathrm{hocolim}}\, F_n \simeq F'\,.$$

The proof now follows easily from a cofinality argument for weak homotopy colimits. See for instance [27, Lemma 1.7.1]. $\surd$

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