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Abstract. We compute the K-theory of the comparison C^*-algebra associated to a manifold with corners. These comparison algebras are an example of the abstract pseudodifferential algebras introduced by Connes and Moscovici [12]. Our calculation is obtained by showing that the comparison algebras are a homomorphic image of a groupoid C^*-algebra. We then prove an index theorem with values in the K-theory groups of the comparison algebra.

Contents

Introduction

The work of Henri Moscovici encompasses many areas of mathematics, most notably Non-commutative Geometry, Group Representations, Geometry, and Abstract Analysis. His work on Non-commutative geometry, mostly joint works with Alain Connes, has lead to many breakthroughs in Index Theory and Operator Algebras, as well as to applications to other areas. We are happy to dedicate this paper to Henri Moscovici on the occasion of his 65th birthday.

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The problem studied in this paper pertains to the general program of understanding index theory on singular and non-compact spaces. On such spaces, the Fredholm property depends on more than the principal symbol, so cyclic cocycles are needed in order to obtain explicit index formulas. Moscovici has obtained many results in this direction, including [4, 6, 11, 12, 33, 34]. See also [5, 20, 8, 9, 38].

One of the central concepts in a recent paper by Connes and Moscovici, is that of an abstract algebra of pseudodifferential operators [12]. These algebras generalize similar algebras introduced earlier. In this paper, we would like to study certain natural C^*-algebras associated to non-compact Riemannian manifolds, applying in particular the point of view of the work of Connes and Moscovici mentioned above.

Let us now explain the framework of this paper. Let M_0 be a complete Riemannian manifold and let $\Delta = d^*d$ be the *positive* Laplace operator on M_0 associated to the metric. It is well known that Δ is essentially self-adjoint [13, 44] and the references therein, and hence we can define $\Lambda = (1 + \Delta)^{-1/2}$ using functional calculus. Let us also assume that a certain algebra $\mathcal{D} = \cup\mathcal{D}_n$ of differential operators is given on M_0, where $\mathcal{D}_n$ denotes the space of differential operators in $\mathcal{D}$ of degree at most n. Let us assume that $\Delta \in \mathcal{D}_2$ and that $L_n\Lambda^n$ defines a bounded operator on $L^2(M_0)$ for any $L_n \in \mathcal{D}_n$. Then the *comparison algebra* of M_0 (and $\mathcal{D}$) to be the C^*-algebra generated by the operators of the form $L_n\Lambda^n$ for any $L_n \in \mathcal{D}_n$. This definition is almost the same as the on in [14, 15], where the comparison algebra was defined as the C^*-algebra generated by all operators of the form $L_1\Lambda$ for any $L_1 \in \mathcal{D}_1$ *and all compact operators*. One of our results, Theorem 4, implies that the two definitions are the same for suitable M_0. Let us denote the comparison C^*-algebra of M_0 by $\mathfrak{A}(M_0)$ (the dependence on the algebra $\mathcal{D}$ will be implicit). The comparison algebra $\mathfrak{A}(M_0)$ is a convenient tool to study many analytic properties of differential operators on M_0, such as invertibility between Sobolev spaces, spectrum, compactness, the Fredholm property, and the index [10, 14, 19, 23, 25, 45]. For instance, the principal symbol of order zero pseudodifferential operators extends to a continuous map $\sigma_0 : \mathfrak{A}(M_0) \to C(S^*A)$ with kernel denoted $\mathfrak{A}_{-1}(M_0)$, where S^*A is a suitable compactification of the cosphere bundle of T^*M_0.

In this paper, we concentrate on the index properties of elliptic operators on a certain class of non-compact manifolds, called "manifolds with poly-cylindrical ends." Recall that a *manifold with poly-cylindrical ends* is, locally, a product of manifolds with cylindrical ends. Our index depends only on the principal symbol, so it takes values in the K-theory of the C^*-algebra $\mathfrak{A}_{-1}(M_0)$. Index calculations are sometimes necessary in applications, for instance in the study of Hartree's equation [21] and in the study of boundary value problems on polyhedral domain [26].

More precisely, let us first assume that the given manifold M_0 is the interior of the space of units of a groupoid Lie $\mathcal{G}$. Then the Lie groupoid structure of $\mathcal{G}$ gives rise to a natural algebra of differential operators $\mathcal{D}$ on M_0, such as in the case of singular foliations [10, 2]. In general, there will be no natural metric on M_0, and even if a metric is chosen on M_0, the associated Laplace operator $\Delta \notin \mathcal{D}$. However, if M_0 is a Lie manifold [1], then a natural class of metrics exists on M_0 and $\Delta \in \mathcal{D}$ for any metric in this class. Recall that M_0 is a *Lie manifold* if the tangent bundle TM_0 extends to a bundle $A \to M$ on a compactification M of M_0 to a manifold with corners such that the space of smooth section $\mathcal{V} := \Gamma(A)$ of A has a natural Lie algebra structure induced by the Lie bracket of vector fields and such that that the diffeomorphisms generated by vector fields in $\mathcal{V}$ preserve the faces of M [1].

We shall show that the comparison algebra of a Lie manifold M_0 identifies with a subalgebra of a homomorphic image of a groupoid (pseudodifferential) algebra. For any manifold with corners, we shall denote by $\mathcal{V}_M$ the Lie algebra of all vector fields tangent to all faces of M. Then $\mathcal{V}_M = \Gamma(A_M)$ for a unique (up to isomorphism) vector bundle $A_M \to M$. If the vector bundle $A \to M$ defining a Lie manifold M satisfies $A = A_M$, then we shall say that M_0, the interior of M, is a *manifold with poly-cylindrical ends*. In that case, we prove that $\mathfrak{A}(M_0)$ is (isomorphic to) the norm closure of $\Psi^0(\mathcal{G})$. We then use this result to compute the K-theory of the algebra $\mathfrak{A}_{-1}(M_0)$ and the index in $K_0(\mathfrak{A}_{-1}(M_0))$ of elliptic operators in the comparison algebra of a manifold with poly-cylindrical ends.

Let us explain in a little more detail our results. Let us assume that our algebra $\mathcal{D}$ of differential operators is generated by $\mathcal{C}^\infty(M)$ and $\mathcal{V}$. Let $D \in \mathcal{D}$, then the principal symbol of D extends to a symbol defined on A^*. Assume that D is elliptic, in the sense that its principal symbol is invertible on A^* outside the zero section. Then the K-theory six-term exact sequence applied to the tangent (or adiabatic) groupoid of $\mathcal{G}$ defines a map

$$\mathrm{ind}_a = \mathrm{ind}_a^M : K^0(A^*) \to K_0(\mathfrak{A}_{-1}(M_0)). \tag{1}$$

One of our main results is a computation of the groups $K_0(\mathfrak{A}_{-1}(M_0))$ and of the map ind_a in case M_0 is a manifold with poly-cylindrical ends.

Our calculation of the group $K_0(\mathfrak{A}_{-1}(M_0))$ is as follows. Consider an embedding $\iota : M \to X$ of manifolds with corners and let $\iota_!$ be the push-forward map in K-theory. Morita equivalence then gives rise to a morphism $\iota_* : K_0(C^*(M)) \to K_0(C^*(X))$. Then Theorem 7 states that the following diagram commutes

$$\begin{array}{ccc} K_0(C^*(M)) & \xrightarrow{\iota_*} & K_0(C^*(X)) \\ {\scriptstyle \mathrm{ind}_a^M}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{ind}_a^X} \\ K^0(A^*_M) & \xrightarrow{\iota_!} & K^0(A^*_X). \end{array} \tag{2}$$

If the manifold with corners X is such that the natural morphisms $\iota_* : K_0(C^*(M)) \to K_0(C^*(X))$ and $\mathrm{ind}_a^X : K^0(A^*_X) \to K_0(C^*(X))$ are isomorphisms, we are going to say that X is a *classifying space* for M. In that case, we can interpret the above diagram as a topological index theorem in the usual sense. We also obtain an identification of the groups $K_0(C^*(M))$ and of the map ind_a^M.

Let us now very briefly summarize the contents of the paper. In Section 1, we introduce comparison algebras and we show that they are closely related to groupoid algebras. We show that the groupoid C^*-algebra $\overline{\Psi^0(\mathcal{G}_M)}$ and the comparison C^*-algebra $\mathfrak{A}(M_0)$ are in fact isomorphic for M_0 a manifold with poly-cylindrical ends with compactification M. In Section 2, we recall the definition of the full C^*-analytic index using the tangent groupoid. In the process, we establish several technical results on tangent groupoids. Section 3 contains the main properties of the full C^*-analytic index. In this section, we also introduce the morphism j_* associated to an embedding of manifolds with corners j and we provide conditions for j_* and ind_a^M to be isomorphisms. We also discuss he compatibility of the full C^*-analytic index and of the shriek maps. This is then used to establish the equality of the full C^*-analytic and principal symbol topological index. Some of the proofs in this paper are only sketched. See [31] for full details.

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1. Groupoids and comparison algebras

We shall need to consider pseudodifferential operators on groupoids [32, 40]. For simplicity, *we shall assume from now on that all our manifolds with corners have embedded faces.*

1.1. **Pseudodifferential operators on groupoids.** Throughout this paper, we shall fix a Lie groupoid $\mathcal{G}$ with units M and Lie algebroid $A = A(\mathcal{G})$. Here M is allowed to have corners. To $\mathcal{G}$, there is associated the pseudodifferential calculus $\Psi^\infty(\mathcal{G})$, whose operators of order m form a linear space denoted $\Psi^m(\mathcal{G})$, $m \in \mathbb{R}$, see [32, 40]. For short, this calculus is defined as follows. Let $s : \mathcal{G} \to M$ be the source map and $\mathcal{G}_x = s^{-1}(x)$. Then $\Psi^m(\mathcal{G})$, $m \in \mathbb{Z}$, consists of smooth families of classical, order m pseudodifferential operators $(P_x \in \Psi^m(\mathcal{G}_x))$, $x \in M$, that are right invariant with respect to multiplication by elements of $\mathcal{G}$ and are "uniformly supported." To define what uniformly supported means, let us observe that the right invariance of the operators P_x implies that their distribution kernels K_{P_x} descend to a distribution $k_P \in I^m(\mathcal{G}, M)$ [29, 40]. Then the family $P = (P_x)$ is called *uniformly supported* if, by definition, k_P has compact support in $\mathcal{G}$.

We then have the following result [23, 30, 40].

Theorem 1. *The space $\Psi^\infty(\mathcal{G})$ is a filtered algebra, closed under adjoints, so that the usual principal symbol of pseudodifferential operators defines a surjective*

$$\sigma_{\mathcal{G}}^{(m)} : \Psi^m(\mathcal{G}) \to S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*),$$

with kernel $\Psi^{m-1}(\mathcal{G})$, for any $m \in \mathbb{Z}$.

1.2. **Comparison algebras.** We shall denote by π the natural action of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}^\infty_{\rm c}(M_0)$ or on its completions and by $\mathfrak{a}_{-\infty}$ the completion of $\pi(\Psi^{-\infty}(\mathcal{G}))$ acting on all Sobolev spaces $H^{-m}(M_0) \to H^m(M_0)$. Let us denote $\mathfrak{a}_n := \pi(\Psi^n(\mathcal{G})) + \mathfrak{a}_{-\infty}$. The following result was proved in [24] (see also [1, 25]).

Theorem 2. *The space $\mathfrak{a} := \pi(\Psi(\mathcal{G})) + \mathfrak{a}_{-\infty}$ is a filtered algebra by the pseudodifferential degree such that $\mathfrak{a}_0 := \pi(\Psi^0(\mathcal{G})) + \mathfrak{a}_{-\infty}$ consists of bounded operators, is closed under the adjoint, and is spectrally invariant. Moreover, $\mathfrak{a}_{-\infty}$ is a two-sided ideal of $\mathfrak{a}$ and*

$$\Lambda := (1 + \Delta)^{-1/2} \in \mathfrak{a}_{-1} := \pi(\Psi^{-1}(\mathcal{G})) + \mathfrak{a}_{-\infty}.$$

From the above theorem we obtain that the comparison algebra is a subalgebra of the norm closure of $\mathfrak{a}_0$.

Theorem 3. *Let M_0 be a Lie manifold. Then we have that*

$$\mathfrak{A}(M_0) \subset \overline{\mathfrak{a}_0} = \pi(\overline{\Psi^0(\mathcal{G})}).$$

Moreover, $\mathfrak{A}(M_0)$ contains all compact operators.

Proof. Let $L_n \in \mathcal{D}_n$. Since $\Lambda \in \mathfrak{a}_{-1}$ and $\mathcal{D}_n \subset \mathfrak{a}_n := \pi(\Psi^{-1}(\mathcal{G})) + \mathfrak{a}_{-\infty}$. It follows that $L_n\Lambda^n \in \mathfrak{a}_0$, and hence the result.

To show that $\mathfrak{A}(M_0)$ contains the subalgebra of compact operators, let us notice first that $e^{-t\Delta} \in \mathfrak{A}(M_0)$, since it can be written as a function of $\Lambda := (1+\Delta)^{-1/2}$. Since $\mathcal{C}_c^\infty(M_0) \subset \mathcal{C}^\infty(M)$, we have that $\phi e^{-t\Delta}\psi \subset \mathfrak{A}(M_0)$, and hence the later contains compact operators. To show that all compact operators are in $\mathfrak{A}(M_0)$, it is enough to show that $\mathfrak{A}(M_0)$ has no invariant subspaces. That is, it is enough to show that if $f, g \in L_2(M_0)$ are such that the inner product $(f, \phi e^{-t\Delta}\psi g)$ is zero for all $\phi, \psi \in \mathcal{C}_c^\infty(M_0)$, then either $f = 0$ or $g = 0$. Indeed, let us choose $\phi_n, \psi_n \in \mathcal{C}_c^\infty(M_0)$ such that $\phi_n f \to |f|$ and $\psi_n g \to |g|$ in $L^2(M_0)$. Then

$$(|f|, e^{-t\Delta}|g|) = \lim_{n\to\infty} (f, \phi_n e^{-t\Delta}\psi_n g) = 0,$$

which implies that either $f = 0$ or $g = 0$, since the heat kernel $e^{-t\Delta}$ has positive distribution kernel. □

Let M_0 be a manifold with poly-cylindrical ends and $\mathcal{G}$ be a groupoid such that $A(\mathcal{G}) = A_M$, where, we recall, $A_M \to M$ is a vector bundle such that $\mathcal{V}_M := \Gamma(A_M)$ consists of all smooth vector fields on M that are tangent to all the faces of M. A groupoid $\mathcal{G}$ with this property is said *to integrate* A_M, and is not unique. However, if the fibers of the source map $s : \mathcal{G} \to M$ are all connected and simply-connected, then $\mathcal{G}$ is unique (up to isomorphism) and will be denoted $\mathcal{G}_M$. For $\mathcal{G}_M$ the vector representation π is injective [25, 30]. We shall also denote by $C^*(M) = C^*(\mathcal{G}_M)$. Recall that $C^*(\mathcal{G}_M) = \overline{\Psi^{-1}(\mathcal{G})}$, [25, 30].

The algebra $\Psi(\mathcal{G}_M)$ was considered before by many authors, including [27, 42, 43]. For this algebra, we actually have equality in the above theorem.

Theorem 4. *Let M_0 be a manifold with poly-cylindrical ends. Then*

$$\mathfrak{A}(M_0) = \overline{\mathfrak{a}_0} \simeq \overline{\Psi^0(\mathcal{G}_M)}.$$

Proof. Let us recall that for manifolds with poly-cylindrical ends the vector representation π is injective on the norm closure $\overline{\Psi^0(\mathcal{G}_M)}$. We shall thus identify $\Psi^0(\mathcal{G}_M)$ with $\pi(\Psi^0(\mathcal{G}_M))$. Since the principal symbol map acting on both $\mathfrak{A}(M_0)$ and on $\overline{\Psi^0(\mathcal{G}_M)}$ has the same range, namely $C(S^*A_M)$, it is enough to show that $\mathfrak{A}_{-1}(M_0) = \overline{\Psi^{-1}(\mathcal{G})} = C^*(M)$.

Let us notice that we can consider families, so proving $\mathfrak{A}_{-1}(M_0) = C^*(M)$ is equivalent to proving $\mathfrak{A}_{-1}(M_0) \otimes C_0(X) = C^*(M) \otimes C_0(X)$. Moreover, the inclusion $\mathfrak{A}_{-1}(M_0) \subset C^*(M)$ is compatible with the natural representations of $C^*(M)$ associated to the faces of M, as seen from their construction in [28]. It is enough then to prove that we have isomorphisms on subquotients defined by these representations, which are all of the form $\mathfrak{A}_{-1}(X_0)\otimes C_0(X)$, for some lower dimensional manifolds. The proof proof finally reduces to show that $\mathcal{K}\subset\mathfrak{A}_{-1}(M_0)$, where $\mathcal{K}$ is the algebra of compact operators. For this we use Theorem 3 □

For manifolds with cylindrical ends (that is when M has no corners of codimension two or higher), this theorem was proved before in [16].

2. The analytic index

2.1. **The adiabatic and tangent groupoids.** For the definition and study of the full C^*-analytic index, we shall need the adiabatic and tangent groupoids associated to a differentiable groupoid $\mathcal{G}$. We now recall their definition.

Let $\mathcal{G}$ be a Lie groupoid with space of units M. We construct both the *adiabatic groupoid* ${}^{ad}\mathcal{G}$ and the *tangent groupoid* ${}^{T}\mathcal{G}$ [10, 22, 23, 32, 41]. The space of units of ${}^{ad}\mathcal{G}$ is $M\times[0,\infty)$ and the tangent groupoid ${}^{T}\mathcal{G}$ will be defined as the restriction of ${}^{ad}\mathcal{G}$ to $M\times[0,1]$. The underlying set of the groupoid ${}^{ad}\mathcal{G}$ is the disjoint union:

$${}^{ad}\mathcal{G} = A(\mathcal{G})\times\{0\}\cup\mathcal{G}\times(0,\infty).$$

We endow $A(\mathcal{G})\times\{0\}$ with the structure of commutative bundle of Lie groups induced by its vector bundle structure. We endow $\mathcal{G}\times(0,\infty)$ with the product groupoid structure. Then the groupoid operations of ${}^{ad}\mathcal{G}$ are such that $A(\mathcal{G})\times\{0\}$ and $\mathcal{G}\times(0,\infty)$ are subgroupoids with the induced structure. Now let us endow ${}^{ad}\mathcal{G}$ with a differentiable structure. To do so, it is enough to specify $A({}^{ad}\mathcal{G})$, since its knowledge completely determines the differentiable structure of ${}^{ad}\mathcal{G}$ [39]. Then

$$\Gamma(A({}^{ad}\mathcal{G})) = t\Gamma(A(\mathcal{G}\times[0,\infty))). \tag{3}$$

More precisely, consider the product groupoid $\mathcal{G}\times[0,\infty)$ with pointwise operations. Then a section $X\in\Gamma(A(\mathcal{G}\times[0,\infty)))$ can be identified with a smooth function $[0,\infty)\ni t\to X(t)\in\Gamma(A(\mathcal{G}))$. We then require $\Gamma(A({}^{ad}\mathcal{G}))=\{tX(t)\}$, with $X\in\Gamma(A(\mathcal{G}\times[0,\infty)))$.

It is easy to show that

Lemma 1. *Let $\mathcal{H}=\mathcal{G}\times\mathbb{R}^n$, as above. We have that $C^*(\mathcal{H}_{ad})\simeq C^*(\mathcal{G}_{ad})\otimes\mathcal{C}_0(\mathbb{R}^n)$ and that $C^*({}^{T}\mathcal{H})\simeq C^*({}^{T}\mathcal{G})\otimes\mathcal{C}_0(\mathbb{R}^n)$, the tensor product being the (complete, maximal) C^*–tensor product.*

2.2. **The full C^*-analytic index.** For each $t\in[0,1]$, $M\times\{t\}$ is a closed invariant subset of $M\times[0,\infty)$ for the adiabatic and tangent groupoids, and hence we obtain an *evaluation morphism*

$$e_t : C^*({}^{T}\mathcal{G})\to C^*({}^{T}\mathcal{G}_{M\times\{t\}}),$$

and, in particular, an exact sequence

$$0 \to C^*({}^T\mathcal{G}_{M\times(0,1]}) \to C^*({}^T\mathcal{G}) \xrightarrow{e_0} C^*(A(\mathcal{G})) \to 0. \tag{4}$$

Since $K_*(C^*({}^T\mathcal{G}_{M\times(0,1]})) = K_*(C^*(\mathcal{G}) \otimes \mathcal{C}_0((0,1]) = 0$, the evaluation map e_0 is induces an isomorphism in K-theory.

The C^*-algebra $C^*(A(\mathcal{G}))$ is commutative and we have $C^*(A(\mathcal{G})) \simeq \mathcal{C}_0(A^*(\mathcal{G}))$. Therefore $K_*(C^*(A(\mathcal{G}))) = K^*(A^*(\mathcal{G}))$. In turn, this isomorphism allows us to define the *full C^*-analytic index* ind_a as the composition map

$$\mathrm{ind}_a^{\mathcal{G}} = e_1 \circ e_0^{-1} : K^*(A^*(\mathcal{G})) \to K_*(C^*(\mathcal{G})), \tag{5}$$

where $e_1 : C^*({}^T\mathcal{G}) \to C^*({}^T\mathcal{G}_{M\times\{1\}}) = C^*(\mathcal{G})$ is defined by the restriction map to $M \times \{1\}$. The definition of the full C^*-analytic index gives the following.

Proposition 1. *Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid $\pi : A(\mathcal{G}) \to M$. Also, let $N \subset F \subset M$ be a closed, invariant subset which is an embedded submanifold of a face F of M. Then the full C^*-analytic index defines a morphism of the six-term exact sequences associated to the pair $(A^*(\mathcal{G}), \pi^{-1}(N))$ and to the ideal $C^*(\mathcal{G}_{N^c}) \subset C^*(\mathcal{G})$, $N^c := M \smallsetminus N$*

$$\begin{array}{ccccccc}
K^0(\pi^{-1}(N^c)) & \longrightarrow & K^0(A^*(\mathcal{G})) & \longrightarrow & K^0(\pi^{-1}(N)) & \longrightarrow & K^1(\pi^{-1}(N^c)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^0(C^*(\mathcal{G}_{N^c})) & \longrightarrow & K^0(C^*(\mathcal{G})) & \longrightarrow & K^0(C^*(\mathcal{G}_N)) & \longrightarrow & K^1(C^*(\mathcal{G}_{N^c}))
\end{array}$$

Proof. The six-term, periodic long exact sequence in K-theory associated to the pair $(A^*(\mathcal{G}), \pi^{-1}(N))$ is naturally isomorphic to the six-term exact sequence in K-theory associated to the pair $\mathcal{C}_0(A^*_{M\smallsetminus N}) \subset \mathcal{C}_0(A^*(\mathcal{G}))$. The result follows from the naturality of the six-term exact sequence in K-theory and the definition of the full C^*-analytic index (6). □

Recall that for M a smooth manifold with corners with embedded faces, we have denoted $A(\mathcal{G}_M) = A_M$ and $C^*(M) := C^*(\mathcal{G}_M)$. Then the full C^*-analytic index becomes the desired map

$$\mathrm{ind}_a^M : K^*(A^*_M) \to K_*(C^*(M)). \tag{6}$$

See [17, 18] for more properties of the analytic index.

Remark 5. Assume M has no corners (or boundary). Then $\mathcal{G}_M = M \times M$ is the product groupoid and hence $\Psi^\infty(\mathcal{G}_M) = \Psi^\infty(M)$. In particular, $C^*(M) := C^*(\mathcal{G}_M) \simeq \mathcal{K}$, the algebra of compact operators on M. In this case $K_0(C^*(M)) = \mathbb{Z}$, and ind_a is precisely the analytic index as introduced by [3]. This construction holds also for the case when M is not compact, if one uses pseudodifferential operators of order zero that are "multiplication at infinity," as in [7].

3. Properties of the full C^*-analytic index

The following proposition is an important step in the proof of our index theorem, Theorem 8.

Proposition 2. *Let X be a manifold with embedded faces such that each open face of X is diffeomorphic to a Euclidean space. Then the full C^*-analytic index*

$$\mathrm{ind}_a^X : K^*(A_X^*) \to K_*(C^*(X)),$$

defined in Equation (6)*, is an isomorphism.*

Proof. The proof is by induction on the number of faces of X using Proposition 1, the six-term exact sequence in K-theory, and the Five Lemma in homological algebra. □

Remark 6. The above proposition can be regarded as a Baum–Connes isomorphism for manifolds with corners.

Proposition 3. *Let $\iota : M \to X$ be a closed embedding of manifold with corners. Assume that, for each open face F of X, the intersection $F \cap M$ is a non-empty open face of M and that every open face of M is obtained in this way. Then $K_*(C^*(M)) \to K_*(C^*(X))$ is an isomorphism denoted ι_*.*

Proof. Recall from [35] that two locally compact groupoids G and H are *equivalent* provided there exists a topological space Ω and two continuous, surjective open maps $r : \Omega \to \mathcal{G}^{(0)}$ and $d : \Omega \to H^{(0)}$ together with a left (respectively right) action of G (respectively H) on Ω with respect to r (respectively d), such that r (respectively d) is a principal fibration of structural groupoid H (respectively G). An important theorem of Muhly–Renault–Williams states that if G and H are equivalent, then $K_*(C^*(G)) \simeq K_*(C^*(H))$ [35].

Our result then follows from the fact that $\Omega := r^{-1}(M)$ establishes the desired equivalence between $\mathcal{G}_M$ and $\mathcal{G}_X$. □

We can now prove a part of our principal symbol topological index theorem, Theorem 8, involving an embedding $\iota : M \to X$ of our manifold with corners M into another manifold with corners X. This theorem amounts to the fact that the diagram (2) is commutative. In order to prove this, we shall first consider a tubular neighborhood

$$M \overset{k}{\hookrightarrow} U \overset{j}{\hookrightarrow} X \tag{7}$$

of M in X, so that $\iota = j \circ k$. The diagram (2) is then decomposed into the two diagrams below, and hence the proof of the commutativity of the diagram (2) reduces to the proof of the commutativity of the two diagrams below, whose morphisms are defined as follows: the morphism k_* is defined by Proposition 3 and the morphis j_* is defined by the inclusion of algebras. The morphism ι_* is defined by $\iota_* = j_* \circ k_*$. Finally, the morphism $k_!$ is the push-forward morphism.

Let us now turn our attention to the following diagram:

$$\begin{array}{ccccc} K_*(C^*(M)) & \xrightarrow{k_*} & K_*(C^*(U)) & \xrightarrow{j_*} & K_*(C^*(X)) \\ {\scriptstyle \mathrm{ind}_a^M}\uparrow & & {\scriptstyle \mathrm{ind}_a^U}\uparrow & & \uparrow{\scriptstyle \mathrm{ind}_a^X} \\ K^*(A^*_M) & \xrightarrow{k_!} & K^*(A^*_U) & \xrightarrow{j_!} & K^*(A^*_X). \end{array} \tag{8}$$

The commutativity of the left diagram is part of the following proposition, which is the most technical part of the proof. Its proof is obtained by integrating a Lie algebroid obtained as a double deformation of a tangent space.

Proposition 4. *Let $\pi : U \to M$ be a vector bundle over a manifold with corners M and let $k : M \to U$ be the "zero section" embedding. Then the following diagram commutes:*

$$\begin{array}{ccc} K_*(C^*(M)) & \underset{\simeq}{\xrightarrow{k_*}} & K_*(C^*(U)) \\ {\scriptstyle \mathrm{ind}_a^M}\uparrow & & {\scriptstyle \mathrm{ind}_a^U}\uparrow \\ K^*(A^*_M) & \underset{k_!}{\xrightarrow{\simeq}} & K^*(A^*_U) \end{array} \tag{9}$$

The commutativity of the second square in the Diagram 8 follows from the naturality of the tangent groupoid construction.

Proposition 5. *Let $j : U \to X$ be the inclusion of the open subset U. Then the diagram below commutes:*

$$\begin{array}{ccc} K_*(C^*(U)) & \xrightarrow{j_*} & K_*(C^*(X)) \\ {\scriptstyle \mathrm{ind}_a^U}\uparrow & & \uparrow{\scriptstyle \mathrm{ind}_a^X} \\ K^*(A^*_U) & \underset{\simeq}{\xrightarrow{j_*}} & K^*(A^*_X). \end{array}$$

As explained above, the previous two propositions give

Theorem 7. *Let* $M \overset{\iota}{\to} X$ *be a closed embedding of manifolds with corners. Then the diagram*

$$\begin{array}{ccc} K_*(C^*(M))) & \xrightarrow{\iota_*} & K_*(C^*(X)) \\ \uparrow{\scriptstyle \mathrm{ind}_a^M} & & {\scriptstyle \mathrm{ind}_a^X}\uparrow \\ K^*(A_M^*) & \xrightarrow{\iota_!} & K^*(A_X^*) \end{array} \tag{10}$$

is commutative.

4. A topological index theorem

Motivated by Theorem 7 and by the results of Section 3 (see Propositions 2 and 3) we introduce the following definition.

Definition 1. *A* strong classifying manifold X_M *of* M *is a compact manifold with corners* X_M*, together with a closed embedding* $\iota : M \to X_M$ *with the following properties:*

(i) *each open face of* X_M *is diffeomorphic to a Euclidean space,*
(ii) $F \to F \cap M$ *induces a bijection between the open faces of* X_M *and* M*.*

Note that if $M \subset X_M$ are as in the above definition, then each face of M is the transverse intersection of M with a face of X_M.

Proposition 6. *Let* M *be a manifold with embedded faces with embedded faces, and* $\iota : M \hookrightarrow X_M$ *be a strong classifying space of* M*. Then the maps* ι_* *and* ind_a^X *of Theorem 7 are isomorphisms. That is, a strong classifying space for* M *is a classifying space for* M*.*

Proof. This was proved in Propositions 2 and 3. □

Let $\iota : M \to X_M$ be a classifying space for M. The above proposition then allows us to define (see diagram 10)

$$\mathrm{ind}_t^M := \iota_*^{-1} \circ \mathrm{ind}_a^X \circ \iota_! : K^*(A_M^*) \to K_*(C^*(M)).$$

If M is a smooth compact manifold (so, in particular, $\partial M = \emptyset$), then $C^*(M) = \mathcal{K}$, the algebra of compact operators on $L^2(M)$ and hence $K_0(C^*(M)) = \mathbb{Z}$. Any embedding $\iota : M \hookrightarrow \mathbb{R}^N$ will then be a classifying space for M. Moreover, for $X = \mathbb{R}^n$, the map $\iota_*^{-1} \circ \mathrm{ind}_a^X : K^*(TX) \to \mathbb{Z}$ is the inverse of $j_! : K^0(pt) \to K^0(T\mathbb{R}^N)$ and hence $\mathrm{ind}_t^{\mathbb{R}^N} = (j_!)^{-1}\iota_!$, which is the definition of the topological index from [3]. In view of this fact, we shall also call the map ind_t^M *the topological index* associated to M. Theorem 7 then gives the following result:

Theorem 8. *Let M be a manifold with corners and A_M and $C^*(M)$ be the Lie algebroid and the C^*-algebra associated to M. Then the principal topological index map ind_t^M depends only on M, that is, it is independent of the classifying space X_M, and we have*

$$\mathrm{ind}_t^M = \mathrm{ind}_a^M : K^*(A_M^*) \to K_*(C^*(M)).$$

If M is a smooth compact manifold (*no boundary*), this recovers the Atiyah-Singer index theorem on the equality of the full C^*-analytic and principal symbol topological index [3].

5. K-theory of comparison algebras

The isomorphism $K_*(C^*(M)) \simeq K^*(X_M)$ provides us with a way of determining the groups $K_*(C^*(M))$. In particular, we have completed the determination of the K-theory groups of the comparison algebra $\mathfrak{A}_{-1}(M_0) = C^*(M)$, if the interior of M is a endowed with a metric making it a manifold with poly-cylindrical ends.

Theorem 9. *Let M be a manifold with corners and embedded faces. We have $K_*(C^*(M)) \simeq K^*(X_M)$. Moreover $K_j(C^*(M)) \otimes \mathbb{Q} \simeq \mathbb{Q}^{p_j}$, where p_j is the number of faces of M of dimension $\equiv j$ modulo 2.*

The last part of the above theorem is proved by showing that the Atiyah-Hirzebruch spectral sequence of X_M collapses at E^2. This is part of a joint work with Etienne Fieux.

It is not difficult to construct a classifying manifold X_M of M [31], that is, a manifold such that $\iota_* : K_0(C^*(M)) \to K_0(C^*(X))$ and $\mathrm{ind}_a^X : K^0(A_X^*) \to K_0(C^*(X))$ are isomorphisms. Let us assume M is compact with embedded faces. The space X_M is obtained from an embedding $X \to [0,\infty)^N$ for some large N, and then by removing suitable hyperplanes from the boundary of $[0,\infty)^N$ such that each face of M is the transverse intersection of M and of a face of X_M. See also [36, 37].

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