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Mohammed Lemou, Florian Méhats, Pierre Raphael

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MOHAMMED LEMOU, FLORIAN MÉHATS, AND PIERRE RAPHAËL

Dedicated to the memory of our friend Naoufel Ben Abdallah

ABSTRACT. We consider the three dimensional gravitational Vlasov Poisson system which is a canonical model in astrophysics to describe the dynamics of galactic clusters. A well known conjecture [6] is the stability of spherical models which are nonincreasing radially symmetric steady states solutions. This conjecture was proved at the linear level by several authors in the continuation of the breakthrough work by Antonov [2] in 1961. In the previous work [29], we derived the stability of anisotropic models under *spherically symmetric perturbations* using fundamental monotonicity properties of the Hamiltonian under suitable generalized symmetric rearrangements first observed in the physics litterature [34, 12, 45, 1]. In this work, we show how this approach combined with a *new generalized* Antonov type coercivity property implies the orbital stability of spherical models under general perturbations.

1. Introduction and main results

1.1. The gravitational Vlasov Poisson system. We consider the three dimensional gravitational Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla \phi_f \cdot \nabla_v f = 0, & (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \\ f(t=0,x,v) = f_0(x,v) \geq 0, \end{cases} \tag{1.1}$$

where, throughout this paper,

$$\rho_f(x) = \int_{\mathbb{R}^3} f(x,v)\,dv \quad \text{and} \quad \phi_f(x) = -\frac{1}{4\pi|x|} * \rho_f \tag{1.2}$$

are the density and the gravitational Poisson field associated to f. This nonlinear transport equation is a well known model in astrophysics for the description of the mechanical state of a stellar system subject to its own gravity and the dynamics of galaxies, see for instance [6, 11].

The global Cauchy problem is solved in [33, 37, 39] where unique global classical solutions $f(t)$ in $\mathcal{C}^1_c$, the space of $\mathcal{C}^1$ compactly supported functions, are derived. Two fundamental properties of the nonlinear transport flow (1.1) are then first the preservation of the total Hamiltonian

$$\mathcal{H}(f(t)) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f(t,x,v)dxdv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \phi_f(t,x)|^2 dx = \mathcal{H}(f(0)), \tag{1.3}$$

and second the preservation of all the so-called Casimir functions: $\forall G \in \mathcal{C}^1([0,+\infty), \mathbb{R}^+)$ such that $G(0) = 0$,

$$\int_{\mathbb{R}^6} G(f(t,x,v))\,dxdv = \int_{\mathbb{R}^6} G(f_0(x,v))\,dxdv\,. \tag{1.4}$$

Equivalently, consider the distribution function associated to f:

$$\forall s \geq 0, \quad \mu_f(s) = \text{meas}\left\{(x,v) \in \mathbb{R}^6 : f(x,v) > s\right\}, \tag{1.5}$$

then (1.4) means the conservation law associated to nonlinear transportation:

$$\forall t \geq 0, \quad \mu_f(t) = \mu_{f_0}. \tag{1.6}$$

In this paper, we will deal with weak solutions in the natural energy space

$$\mathcal{E} = \left\{f \geq 0 \text{ with } f \in L^1 \cap L^\infty(\mathbb{R}^6) \text{ and } |v|^2 f \in L^1(\mathbb{R}^6)\right\}. \tag{1.7}$$

For all $f_0 \in \mathcal{E}$, (1.1) admits a weak solution $f(t)$, constructed for instance in [4, 22, 23], which is also a renormalized solution, see [7, 8]. Moreover, this solution still satisfies (1.4), belongs to $\mathcal{C}([0,+\infty), L^1(\mathbb{R}^6))$ and the energy conservation (1.3) is replaced by an inequality:

$$\forall t \geq 0, \quad \mathcal{H}(f(t)) \leq \mathcal{H}(f_0). \tag{1.8}$$

1.2. **Previous results.** Jean's theorem [5] gives a complete classification of radially symmetric steady state solutions to (1.1). Recall that radial symmetry in our setting means $f(x,v) \equiv f(|x|,|v|,x\cdot v)$. They are of the form

$$Q(x,v) = F(e,\ell)$$

where e,ℓ are respectively the microscopic energy and the kinetic momentum

$$e(x,v) = \frac{|v|^2}{2} + \phi_Q(x), \quad \ell = |x \wedge v|^2 \tag{1.9}$$

and are the *only* two invariants of the radially symmetric characteristic flow associated to the transport operator $\tau = v\cdot\nabla_x - \nabla\phi_Q\cdot\nabla_v$.

A canonical problem which has attracted a considerable amount of works both in the physical and the mathematical community is the question of the *nonlinear stability of steady states models.* The *linear* stability of all nonincreasing anisotropic models satisfying

$$\frac{\partial F}{\partial e} < 0 \tag{1.10}$$

is derived by Doremus, Baumann and Feix [10] (see also [13, 24, 41] for related works), following the pioneering work by Antonov in the 60's [2, 3]. This analysis is based on some coercivity properties of the linearized Hamiltonian under constraints formally arising from the linearization of the Casimir conservation laws (1.4), see Lynden-Bell [34], known as Antonov's coercivity property.

At the nonlinear level, the full orbital stability in the natural energy space $\mathcal{E}$ has been obtained for specific subclasses of steady states as a direct consequence of Lions' concentration compactness principle [31, 32], see [46, 14, 16, 17, 18, 9, 40, 26, 27, 28, 38]. This powerful strategy however only applies to specific models which are *global* minimizers of the Hamiltonian (1.3) under at most two Casimir type conservation laws, see [27, 28] for a more complete introduction.

A first attempt to treat the general case and use the full rigidity provided by the *continuum* of conservation laws (1.4) is proposed in [19], [15] where the first result of stability against radially symmetric perturbations is obtained for the King model $F(e) = (\exp(e_0 - e) - 1)_+$. The approach is based on Antonov's coercivity property and a direct linearization of the Hamiltonian near the King profile.

We proposed in [29] a different approach based on fine monotonicity properties of the Hamiltonian under suitable generalized symmetric rearrangements as first observed in pioneering breakthrough works in the physics litterature, see in particular Lynden-Bell [34], Gardner [12], Wiechen, Ziegler, Schindler [45], Aly [1]. This approach avoids the delicate step of linearization of the Hamiltonian and reduces the stability problem for the full distribution function f to *a minimization problem for a generalized energy involving the Poisson field* ϕ_f *only*. The main outcome is the *radial* stability of nondecreasing anisotropic models, proved in [29]:

Theorem 1.1 (Radial stability of nonincreasing anisotropic models, [29]). *Let* $Q(x,v)=F(e,\ell)$ *be a continuous, nonnegative compactly supported steady state solution to* (1.1). *Assume that* Q *is nonincreasing in the following sense: there exists* $e_0<0$ *such that* F *is* $\mathcal{C}^1$ *on* $\mathcal{O}=\{(e,\ell)\in\mathbb{R}\times\mathbb{R}_+ \,:\, F(e,\ell)>0\}\subset(-\infty,e_0)\times\mathbb{R}_+$ *and*

$$\frac{\partial F}{\partial e}<0 \quad on \quad \mathcal{O}.$$

Then Q *is stable in the energy norm by radially symmetric perturbations, ie: for all* $M>0$*, for all* $\varepsilon>0$*, there exists* $\eta>0$ *such that given* $f_0\in\mathcal{C}^1_c$ *radially symmetric with*

$$\|f_0-Q\|_{L^1}\leq\eta, \quad \|f_0\|_{L^\infty}\leq\|Q\|_{L^\infty}+M, \quad |\mathcal{H}(f_0)-\mathcal{H}(Q)|\leq\eta, \tag{1.11}$$

the corresponding global strong solution $f(t)$ *to* (1.1) *satisfies:*

$$\forall t\geq 0, \quad \|(1+|v|^2)(f(t)-Q)\|_{L^1}\leq\varepsilon. \tag{1.12}$$

1.3. **Statement of the result.** Our aim in this paper is to extend the stability result of Theorem 1.1 to the full set of non radial perturbations. Here we recall that the radial problem enjoys an additional rigidity because for $f(x.v)$ radially symmetric, the Casimir conservation laws (1.4) can be extended as follows: $\forall G(h,\ell)\geq 0$, $\mathcal{C}^1$ with $G(0,\ell)=0$,

$$\int_{\mathbb{R}^6} G(f(t,x,v),|x\wedge v|^2)dxdv=\int_{\mathbb{R}^6} G(f_0(x,v),|x\wedge v|^2)dxdv. \tag{1.13}$$

This additional conservation law is fundamental in the proof of Theorem 1.1, and at the linear level, it is intimately connected to Antonov's coercivity property which is essentially equivalent to the coercivity of the Hessian of the Hamiltonian (1.3) under the *full set of linearized constraints generated by* (1.13).

For the full non radial problem, (1.13) is lost. However, we claim that the strategy developped in [29] coupled with a *new generalized Antonov coercivity property* allows us to derive the classical conjecture of orbital stability of nonincreasing spherical models.

Theorem 1.2 (Orbital stability of spherical models). *Let* Q *be a continuous, nonnegative, non zero, compactly supported steady solution to* (1.1). *Assume that* Q *is a nonincreasing spherical model in the following sense: there exists a continuous function* $F:\mathbb{R}\to\mathbb{R}_+$ *such that*

$$\forall (x,v)\in\mathbb{R}^6, \quad Q(x,v)=F\left(\frac{|v|^2}{2}+\phi_Q(x)\right), \tag{1.14}$$

and there exists $e_0<0$ *such that* $F(e)=0$ *for* $e\geq e_0$*,* F *is* $\mathcal{C}^1$ *on* $(-\infty,e_0)$ *and*

$$F'<0 \quad on \quad (-\infty,e_0). \tag{1.15}$$

Then Q is orbitally stable in the energy norm by the flow (1.1)*: for all $M > 0$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, given $f_0 \in \mathcal{E}$ with*

$$\|f_0 - Q\|_{L^1} \leq \eta, \quad \mathcal{H}(f_0) \leq \mathcal{H}(Q) + \eta, \quad \|f_0\|_{L^\infty} \leq \|Q\|_{L^\infty} + M, \tag{1.16}$$

for any weak solution $f(t)$ to (1.1)*, there exists a translation shift $z(t)$ such that $\forall t \geq 0$,*

$$\|(1 + |v|^2)(f(t, x, v) - Q(x - z(t), v))\|_{L^1(\mathbb{R}^6)} \leq \varepsilon. \tag{1.17}$$

Comments on Theorem 1.2.

1. On the assumption on Q. Jean's theorem [5] ensures that the assumptions we make on Q are very general. Note that we allow F' to blow up on the boundary $e \to e_0$ which is known to happen for many standard models. We in particular extract from [6] two models of physical relevance which fit into our analysis:

– The generalized polytropic models:

$$F(e) = \sum_{0 \leq i,j \leq N} \alpha_{ij}(e_0 - e)_+^{q_i}, \quad 0 < q_i < \frac{7}{2}, \quad \alpha_{ij} \geq 0.$$

– The King model:

$$F(e) = (\exp(e_0 - e) - 1)_+ \quad \text{for some} \quad e_0 < 0.$$

2. Anisotropic models. Note that Theorem 1.2 deals with spherical models $Q = F(e)$ while the full class of anisotropic models $Q = F(e, \ell)$ is considered in Theorem 1.1. Let us insist that the orbital stability of all anisotropic models with respect to non radial perturbations *is not expected to hold in general* (see [6]) and nonradial instability mechanisms may happen induced by the non trivial dependence on kinetic momentum. We present a full non radial approach for spherical models only which is a canonical class, but which is likely not to be optimal. The derivation of sharp criterions of stability or instability for anisotropic models under non radial perturbations remains to be done.

3. Quantitative bounds. The proof of Theorem 1.2 will rely on a compactness argument, and one could ask for more quantitative bounds. Such bounds are available for the Poisson field and a consequence of our analysis is that for $f \in \mathcal{E}$ satisfying (1.16), we can find $z_f \in \mathbb{R}^3$ such that

$$\mathcal{H}(f) - \mathcal{H}(Q) + \|\phi_f\|_{L^\infty}\|f^* - Q^*\|_{L^1} \geq c_0\|\nabla\phi_f - \nabla\phi_Q(\cdot - z_f)\|_{L^2}^2$$

for some universal constant $c_0 > 0$, see (4.4), where f^* and Q^* denote respectively the usual symmetric decreasing rearrangements of f and Q, as defined in Lemma 2.3. The quantitative control of the full distribution function however seems to involve more subtle norms and would rely on weighted estimates for the bathtub principles, see (2.25). Such estimates were derived in the context of the incompressible 2D Euler in [36, 43], but they seem to be more involved in our case due to the nonlinear structure of the generalized symmetric rearrangement that we consider, see (1.19).

1.4. **Strategy of the proof.** Let us give a brief insight into the strategy of the proof of Theorem 1.2 which extends the approach introduced in [29].

Step 1. Monotonicity of the Hamiltonian under generalized symmetric rearrangements.

Let us define the Schwarz symmetrization of f as

$$f^*(s) = \inf\{\tau \geq 0 \quad : \quad \mu_f(\tau) \leq s\}, \tag{1.18}$$

where μ_f is defined by (1.5), which is the unique decreasing function on $\mathbb{R}_+$ with

$$\mu_f = \mu_{f^*}.$$

Given a potential ϕ in a suitable "Poisson field" class, we define the generalized symmetric nonincreasing rearrangement of f with respect to the microscopic energy $e = \frac{|v|^2}{2} + \phi(x)$ as the unique function of e which is equimeasurable to f, explicitely

$$f^{*\phi}(x,v) = f^* \circ a_\phi(e(x,v)), \quad a_\phi(e) = \text{meas}\{(x,v) \in \mathbb{R}^6, \quad \frac{|v|^2}{2} + \phi(x) < e\}. \tag{1.19}$$

Any nonincreasing spherical steady state solution to (1.1) is a fixed point of this transformation when generated by its own Poisson field:

$$Q^{*\phi_Q} = Q. \tag{1.20}$$

Moreover, the Hamiltonian (1.3) enjoys a nonlinear monotonicity property which was first observed in the physics litterature, see in particular Aly [1]:

$$\mathcal{H}(f) \geq \mathcal{H}(f^{*\phi_f}). \tag{1.21}$$

For perturbations which are equimeasurable to Q ie

$$f^* = Q^*, \tag{1.22}$$

we can more precisely lower bound the Hamiltonian by a functional which depends on the Poisson field only:

$$\mathcal{H}(f) - \mathcal{H}(Q) \geq \mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q) \tag{1.23}$$

where $\mathcal{J}$ can be interpreted as a generalized energy, [34]:

$$\mathcal{J}(\phi_f) = \mathcal{H}(Q^{*\phi_f}) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi_{Q^{*\phi_f}} - \nabla\phi_f|^2.$$

Step 2. Coercivity of the Hessian: a Poincaré inequality.

We now linearize the functional $\mathcal{J}$ at ϕ_Q. The linear term drops thanks to the Euler-Lagrange equation (1.20) and the Hessian takes the following remarkable form

$$D^2\mathcal{J}(\phi_Q)(h,h) = \int_{\mathbb{R}^3} |\nabla h|^2 - \int_{\mathbb{R}^6} |F'(e)|(h - \Pi h)^2 dxdv \tag{1.24}$$

where Π, defined by (3.8), denotes after a suitable *phase space* change of variables the projection of h onto the functions which depend only on the microscopic energy e. A similar structure occured in [29] where the corresponding quadratic form was

$$\int_{\mathbb{R}^3} |\nabla h|^2 - \int_{\mathbb{R}^6} \left|\frac{\partial F}{\partial e}(e,\ell)\right| (h - \Pi_{e,\ell} h)^2 dxdv \tag{1.25}$$

and where $\Pi_{e,\ell}$ corresponds to the projection onto functions which depend on (e,ℓ) only (e and ℓ being defined by (1.9)). The strict coercivity of the quadratic form

(1.25) was then equivalent to Antonov's stability result, but this statement is no longer sufficient in our setting as (1.25) is lower bounded by (1.24).

We now claim the positivity of (1.24) for spherical models

$$D^2\mathcal{J}(\phi_Q)(h,h) = \int_{\mathbb{R}^3} |\nabla h|^2 - \int_{\mathbb{R}^6} |F'(e)|(h-\Pi h)^2 dxdv \geq 0, \tag{1.26}$$

and in fact the quadratic form is coercive up to the degeneracy induced by translation invariance[1]. For this, we reinterpret (1.26) as a generalized Poincaré inequality with sharp constant, and we claim that the classical approach developed by Hörmander [20, 21] for the proof of sharp weighted L^2 Poincaré inequalities:

$$d\mu = e^{-V(x)}dx, \quad \int_{\mathbb{R}^N} (f-\overline{f})^2 d\mu \lesssim \int_{\mathbb{R}^N} |\nabla f|^2 d\mu, \quad \overline{f} = \frac{\int_{\mathbb{R}^N} f d\mu}{\int_{\mathbb{R}^N} d\mu}$$

under the convexity assumption

$$\nabla^2 V \gtrsim 1 \tag{1.27}$$

can be adapted to our setting. In particular, the non trivial convexity property (1.27) appears in the setting of (1.26) as a consequence of the non linear structure of the steady state equation (1.20), see (3.48).

Step 3. Compactness up to translations.

The outcome of Step 2 is the variational characterization of Q, ϕ_Q respectively as the locally unique (up to translation shift) minimizers of the respectively constrained and unconstrained minimization problems

$$\inf_{f^*=Q^*} \mathcal{H}(f), \quad \inf \mathcal{J}(\phi).$$

More precisely, we will show that $\mathcal{J}(\phi) - \mathcal{J}(\phi_Q)$ controls the distance of ϕ to the manifold of translated Poisson fields $\phi_Q(\cdot + x)$, $x \in \mathbb{R}^3$, see Proposition 3.1.

From standard continuity arguments, the conservation law (1.6) and the inequality (1.8) ensure that Theorem 1.2 is now equivalent to the relative compactness in the energy space up to translation of generalized minimizing sequences:

$$f_n^* \to Q^* \ \text{ in } \ L^1 \ \text{ and } \ \limsup_{n\to+\infty} \mathcal{H}(f_n) \leq \mathcal{H}(Q).$$

A slight improvement of the lower bound (1.23) implies first the relative compactness up to translations

$$\nabla\phi_{f_n}(\cdot + x_n) \to \nabla\phi_Q \ \text{ in } \ L^2(\mathbb{R}^3).$$

The strong convergence in the energy norm of the full distribution function now follows from a further use of the extra terms in the monotonicity property (1.21) which yields:

$$\int (1+|v|^2)|f_n(x+x_n,v) - Q(x,v)|dxdv \to 0 \ \text{ as } \ n \to +\infty$$

and enables to conclude the proof of Theorem 1.2.

The paper is organized as follows. In section 2, we show how a suitable phase-space symmetrization allows to reduce the study of the Hamiltonian $\mathcal{H}$ to the study of a functional $\mathcal{J}$ which depends on the Poisson field ϕ_f only. In section 3, we show that ϕ_Q is a local minimizer of this new functional and that $\mathcal{J}(\phi) - \mathcal{J}(\phi_Q)$ controls the distance of ϕ to the manifold of translated functions $\phi_Q(\cdot + z)$, $z \in \mathbb{R}^3$,

[1] see Proposition 3.6 for precise statements

Proposition 3.1. In section 4, a sharp use of the monotonicity properties for both functionals $\mathcal{H}$ and $\mathcal{J}$ yields the compactness of the whole minimizing distribution functions. The proof of Theorem 1.2 then follows in section 5.

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2. Reduction to a functional of the gravitational potential

In this section, we introduce the notion of rearrangement with respect to a given Poisson type field, and show the monotonicity of the Hamiltonian under the corresponding transformation which allows to compare the minimization problem of $\mathcal{H}(f)$ under the constraint $f^* = Q^*$ to an unconstrained minimization problem on the Poisson field ϕ_f only. Our approach extends the one we developed in [29] to the case of non radial potentials, and most arguments are in fact simplified by the absence of kinetic momentum.

2.1. **Properties of Poisson fields.** Let us start with defining a suitable class of "Poisson type" potentials:

$$\mathcal{X} = \left\{ \phi \in \mathcal{C}(\mathbb{R}^3) \text{ such that } \phi \le 0, \ \lim_{|x|\to+\infty} \phi(x) = 0, \ \nabla\phi \in L^2(\mathbb{R}^3) \text{ and } m(\phi) > 0 \right\}$$

where

$$m(\phi) := \inf_{x\in\mathbb{R}^3} (1+|x|)|\phi(x)| \tag{2.1}$$

Notice that (2.1) implies:

$$\forall \phi, \tilde{\phi} \in \mathcal{X}, \quad \forall \lambda > 0, \quad m(\phi + \widetilde{\phi}) \ge m(\phi) + m(\widetilde{\phi}), \quad m(\lambda\phi) = \lambda m(\phi), \tag{2.2}$$

and thus $\mathcal{X}$ is convex. Moreover, there holds:

Lemma 2.1 (Properties of Poisson fields)**.** *Let $f \in \mathcal{E}$ nonzero and ϕ_f be its Poisson field given by* (1.2)*, then $\phi_f \in \mathcal{X}$.*

Proof. Let $f \in \mathcal{E}$, nonzero. From standard interpolation estimates, $\rho_f \in L^{5/3} \cap L^1$. Hence, by elliptic regularity, $\phi_f \in W^{2,5/3}_{loc}$, $\nabla\phi_f \in L^2(\mathbb{R}^3)$ and $\phi_f \in C^{0,\frac{1}{5}}$ by Sobolev embedding. Also $\phi_f \le 0$ and $\phi_f(x) \to 0$ as $|x| \to +\infty$ from (1.2). In particular, ϕ_f attains its infimum on $\mathbb{R}^3$ with

$$-\infty < \min \phi_f \le 0.$$

It remains to show that $m(\phi) > 0$ which follows from the existence of $C_f > 0$ such that:

$$\forall x \in \mathbb{R}^3, \quad \phi(x) \le -\frac{C_f}{1+|x|}. \tag{2.3}$$

Indeed, pick $R > 0$ such that

$$\int_{|y|<R} \rho_f(y)dy \ge \frac{\|f\|_{L^1}}{2} > 0,$$

and estimate for $|x|>R$:

$$-\phi(x)=\int_{\mathbb{R}^3}\frac{\rho_f(y)}{4\pi|x-y|}dy\geq\int_{|y|<R}\frac{\rho_f(y)}{4\pi(|x|+R)}dy\geq\frac{\|f\|_{L^1}}{8\pi(|x|+R)},$$

which yields (2.3). The proof of Lemma 2.1 is complete. □

Let us now associate to $\phi\in\mathcal{X}$ the following Jacobian function:

Lemma 2.2 (Properties of the Jacobian a_ϕ). *Let $\phi\in\mathcal{X}$. We define the Jacobian function $a_\phi:\mathbb{R}_-^*\to\mathbb{R}^+$ as:*

$$\forall e<0,\quad a_\phi(e)=\text{meas}\left\{(x,v)\in\mathbb{R}^6:\ \frac{|v|^2}{2}+\phi(x)<e\right\}.$$

Then:

(i) *There holds the explicit formula:*

$$\forall e<0,\quad a_\phi(e)=\frac{8\pi\sqrt{2}}{3}\int_{\mathbb{R}^3}(e-\phi(x))_+^{3/2}\,dx. \tag{2.4}$$

In particular, $a_\phi(e)=0$ for all $e<\min\phi$;

(ii) *a_ϕ is $\mathcal{C}^1$ on $(-\infty,0)$ and is a strictly increasing $\mathcal{C}^1$ diffeomorphism from $[\min\phi,0)$ onto $\mathbb{R}_+$.*

Proof. Let us prove (i). We have the inclusion

$$\left\{(x,v)\in\mathbb{R}^6:\ \frac{|v|^2}{2}+\phi(x)<e\right\}\subset\left\{(x,v)\in\mathbb{R}^6:\ \phi(x)<e\text{ and }|v|^2\leq 2(e-\min\phi)\right\}.$$

Let $e<0$. Since ϕ is continuous and goes to zero at the infinity, the set in the right-hand side is bounded in $\mathbb{R}^6$, thus $a_\phi(e)<+\infty$. The formula (2.4) now follows after passing to the spherical coordinates in velocity. We now prove (ii). Since, for all $e<0$, the set $\{x\in\mathbb{R}^3:\ \phi(x)<e\}$ is bounded, we may apply the dominated convergence theorem and get the continuity and differentiability of a_ϕ on $\mathbb{R}_-^*$, with

$$a_\phi'(e)=4\pi\sqrt{2}\int_{\mathbb{R}^3}(e-\phi(x))_+^{1/2}\,dx. \tag{2.5}$$

Hence a_ϕ' is nonnegative and clearly continuous. Moreover, if $a_\phi'(e)=0$ then $e-\phi(x)\leq 0$ for all $x\in\mathbb{R}^3$, which means that $e\leq\min\phi$. Therefore, if $e>\min\phi$, then $a_\phi'(e)>0$. It remains to prove that $\lim_{e\to 0-}a_\phi(e)=+\infty$. Since $\phi\in\mathcal{X}$, we have

$$a_\phi(e)\geq C\int_{\mathbb{R}^3}\left(e+\frac{m(\phi)}{1+|x|}\right)_+^{3/2}dx\to+\infty\quad\text{as}\quad e\to 0,$$

from $\int_{\mathbb{R}^3}\frac{dx}{(1+|x|)^{3/2}}=+\infty$, $m(\phi)>0$ and the monotone convergence theorem. This concludes the proof of Lemma 2.2. □

2.2. **Rearrangement with respect to the microscopic energy.** We introduce in this section the generalized rearrangement of f with respect to a Poisson field $\phi\in\mathcal{X}$. Let us start with recalling standard properties of the Schwarz symmetrization, [25, 30, 35].

Lemma 2.3 (Schwarz symmetrization or radial rearrangement). *Let $f \in L^1_+ \cap L^\infty$, then the Schwarz symmetrization f^* of f is the unique nonincreasing function on $\mathbb{R}_+$ such that f and f^* have the same distribution function:*

$$\forall s \geq 0, \quad \mu_f(s) = \mu_{f^*}(s)$$

with μ_f given by (1.5) *and μ_{f^*} defined analogously*[2]. *Equivalently, f^* is the pseudo-inverse of μ_f:*

$$\forall t \geq 0, \quad f^*(t) = \inf\{s \geq 0 : \mu_f(s) \leq t\}.$$

The following properties hold:
(i) $f^* \in L^1_+ \cap L^\infty$ with

$$f^*(0) = \|f\|_{L^\infty}, \quad \mathrm{Supp}(f^*) \subset [0, \mathrm{meas}(\mathrm{Supp}(f))];$$

(ii) for all $\beta \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\beta(0) = 0$,

$$\int_{\mathbb{R}_+} \beta(f^*(t))dt = \int_{\mathbb{R}^6} \beta(f(x,v))dxdv. \tag{2.6}$$

Observe that the above definition of f^* is equivalent to

$$\forall t \geq 0, \quad f^*(t) = \sup\{s \geq 0 : \mu_f(s) > t\},$$

with the convention that $f^*(t) = 0$ when the set $\{s \geq 0 : \mu_f(s) > t\}$ is empty. Note also that if f is continuous then f^* is continuous [42]. In particular, Q^* is continuous.

Given $\phi \in \mathcal{X}$, we now define the rearrangement of f with respect to the microscopic energy $\frac{|v|^2}{2} + \phi(x)$ as follows:

Lemma 2.4 (Symmetric rearrangement with respect to a microscopic energy). *Let $f \in \mathcal{E}$ and let $\phi \in \mathcal{X}$. Let f^* be the Schwarz rearrangement in $\mathbb{R}^6$ given by Lemma* 2.3. *We define the function*

$$f^{*\phi}(x,v) = \begin{cases} f^*\left(a_\phi\left(\frac{|v|^2}{2} + \phi(x)\right)\right) & \text{if} \quad \frac{|v|^2}{2} + \phi(x) < 0 \\ 0 & \text{if} \quad \frac{|v|^2}{2} + \phi(x) \geq 0 \end{cases} \tag{2.7}$$

on $\mathbb{R}^6$, where a_ϕ is defined by (2.4). *Then:*

(i) *$f^{*\phi}$ is equimeasurable with f, i.e.*

$$f^{*\phi} \in \mathrm{Eq}(f) = \{g \in L^1_+ \cap L^\infty \quad \text{with} \quad \mu_f = \mu_g\}. \tag{2.8}$$

(ii) *$f^{*\phi}$ belongs to the energy space, i.e. $f^{*\phi} \in \mathcal{E}$ with*

$$\int_{\mathbb{R}^6} \frac{|v|^2}{2} f^{*\phi} dxdv \leq C \|\nabla\phi\|_{L^2}^{4/3} \|f\|_{L^1}^{7/9} \|f\|_{L^\infty}^{2/9}. \tag{2.9}$$

Proof. Let us prove (i). The equimeasurability of f and $f^{*\phi}$ relies on the following elementary change of variable formula: let two nonnegative function $\alpha \in \mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\gamma \in L^1(\mathbb{R}_+)$, then

$$\int_{\frac{|v|^2}{2}+\phi(x)<0} \alpha\left(\frac{|v|^2}{2} + \phi(x)\right) \gamma\left(a_\phi\left(\frac{|v|^2}{2} + \phi(x)\right)\right) dxdv$$
$$= \int_{\min\phi}^0 \alpha(e)\gamma(a_\phi(e))a'_\phi(e)de = \int_0^{+\infty} \alpha\left(a_\phi^{-1}(s)\right)\gamma(s)ds. \tag{2.10}$$

[2]through the one dimensional Lebesgue measure

To obtain the first equality in (2.10), we pass to the spherical coordinates in velocity $u = |v|$ and perform the change of variable $e = \frac{u^2}{2} + \phi(x)$ in the integral of u:

$$\int_{\frac{|v|^2}{2}+\phi(x)<0} \alpha\left(\frac{|v|^2}{2}+\phi(x)\right)\gamma\left(a_\phi\left(\frac{|v|^2}{2}+\phi(x)\right)\right)dxdv$$

$$= 4\pi\sqrt{2}\int_{\mathbb{R}^3} dx \int_{\phi(x)}^0 \alpha(e)\gamma(a_\phi(e))\,(e-\phi(x))^{1/2}\,de.$$

$$= 4\pi\sqrt{2}\int_{\min\phi}^0 \alpha(e)\gamma(a_\phi(e))de\int_{\mathbb{R}^3}(e-\phi(x))_+^{1/2}\,dx.$$

We conclude thanks to the formula (2.5) of a'_ϕ. The second equality comes after the change of variable $s = a_\phi(e)$. Recall from Lemma 2.2 that a_ϕ is a $\mathcal{C}^1$ diffeomorphism from $[\min\phi, 0)$ onto $\mathbb{R}_+$.

Let $\beta \in \mathcal{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ such that $\beta(0) = 0$. From (2.10) and the definition (2.7), we get

$$\int_{\mathbb{R}^6}\beta\left(f^{*\phi}(x,v)\right)dxdv = \int_0^{+\infty}\beta(f^*(s))ds = \int_{\mathbb{R}^6}\beta(f(x,v))dxdv,$$

where we use (2.6). This proves that $f^{*\phi} \in \mathrm{Eq}(f)$.

Let us now prove (ii). From the equimeasurability of f and $f^{*\phi}$, we already deduce that

$$\|f\|_{L^1} = \|f^{*\phi}\|_{L^1}, \qquad \|f\|_{L^\infty} = \|f^{*\phi}\|_{L^\infty}. \tag{2.11}$$

Moreover, we have

$$\begin{aligned}\int_{\mathbb{R}^6}\frac{|v|^2}{2}f^{*\phi}dxdv &= \int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi(x)\right)f^{*\phi}dxdv - \int_{\mathbb{R}^6}\phi(x)f^{*\phi}dxdv\\ &\leq -\int_{\mathbb{R}^6}\phi(x)f^{*\phi}dxdv \leq \|\phi\|_{L^\infty}\|f^*\|_{L^1} < +\infty,\end{aligned}$$

where we used (2.7). More precisely:

$$\begin{aligned}\int_{\mathbb{R}^6}\frac{|v|^2}{2}f^{*\phi}dxdv &\leq -\int_{\mathbb{R}^6}\phi(x)f^{*\phi}dxdv = \int_{\mathbb{R}^3}\nabla\phi\cdot\nabla\phi_{f^{*\phi}}dx\\ &\leq C\|\nabla\phi\|_{L^2}\||v|^2f^{*\phi}\|_{L^1}^{1/4}\|f^{*\phi}\|_{L^1}^{7/12}\|f^{*\phi}\|_{L^\infty}^{1/6}\end{aligned}$$

where we used the Cauchy-Schwarz inequality and the following standard interpolation inequality: for all $g \in \mathcal{E}$,

$$\|\nabla\phi_g\|_{L^2}^2 \leq C\||v|^2g\|_{L^1}^{1/2}\|g\|_{L^1}^{7/6}\|g\|_{L^\infty}^{1/3}, \tag{2.12}$$

and (2.9) follows. This concludes the proof of Lemma 2.4. □

We end this subsection with an elementary lemma which will be useful in the sequel.

Lemma 2.5 (Pseudo inverse of $f^*\circ a_\phi$)**.** *Let $f \in \mathcal{E}$, nonzero, and $\phi \in \mathcal{X}$. We define the pseudo inverse of $f^*\circ a_\phi$ for $s \in (0, \|f\|_{L^\infty})$ as:*

$$(f^*\circ a_\phi)^{-1}(s) = \sup\{e \in [\min\phi, 0) : \quad f^*\circ a_\phi(e) > s\}. \tag{2.13}$$

Then $(f^*\circ a_\phi)^{-1}$ is a nonincreasing function from $(0, \|f\|_{L^\infty})$ to $[\min\phi, 0)$ and for all $(x, v) \in \mathbb{R}^6$ and $s \in (0, \|f\|_{L^\infty})$,

$$f^{*\phi}(x,v) > s \Longrightarrow \frac{|v|^2}{2} + \phi(x) \leq (f^*\circ a_\phi)^{-1}(s), \tag{2.14}$$

$$f^{*\phi}(x,v) \leq s \Longrightarrow \frac{|v|^2}{2} + \phi(x) \geq (f^* \circ a_\phi)^{-1}(s), \tag{2.15}$$

where $f^{*\phi}$ is defined by (2.7)*.*

Proof. Let $s \in (0, \|f\|_{L^\infty})$, then from $f^*(0) = \|f\|_{L^\infty}$, $f^*(t) \to 0$ as $t \to +\infty$ and Lemma 2.2,

$$\{e \in [\min \phi, 0): \quad f^* \circ a_\phi(e) > s\} \text{ is not empty} \tag{2.16}$$

and $(f^* \circ a_\phi)^{-1}(s)$ defined by (2.13) is strictly negative. The monotonicity of $(f^* \circ a_\phi)^{-1}$ follows from the monotonicity of f^* and a_ϕ. Assume that $f^{*\phi}(x,v) > s$, then from the definition (2.7), we have $\min \phi \leq \frac{|v|^2}{2} + \phi(x) < 0$. We also have $f^* \circ a_\phi(\frac{|v|^2}{2} + \phi(x)) > s$, therefore $\frac{|v|^2}{2} + \phi(x) \leq (f^* \circ a_\phi)^{-1}(s)$ from the definition (2.13). This proves (2.14). Assume now that $f^{*\phi}(x,v) \leq s$. Then, for all $e \in \{\tilde{e} \in [\min \phi, 0): \quad f^* \circ a_\phi(\tilde{e}) > s\}$ which is a non empty set, we have $\frac{|v|^2}{2} + \phi(x) > e$, and (2.15) is proved. □

2.3. **Spherical models are fixed points of the generalized rearrangement.** We now reinterpret the assumptions on Q in Theorem 1.2 and claim that spherical models are fixed points of the $f \to f^{*\phi_f}$ transformation[3].

Lemma 2.6 (Q is a fixed point of the $f^{*\phi_f}$ rearrangement)**.** *Let Q be a radially symmetric spherical models as in the assumptions of Theorem* 1.2*. Then we have*

$$F(e) = Q^* \circ a_{\phi_Q}(e), \quad \forall e \in [\phi_Q(0), 0), \qquad \text{and} \qquad Q^{*\phi_Q} = Q \quad \text{on } \mathbb{R}^6. \tag{2.17}$$

Proof. Observe first that, since the boundary of $\{Q(x,v) > 0\}$ is the level set $\frac{|v|^2}{2} + \phi_Q(x) = e_0$, we have $\mu_Q(0) = \text{meas }(\text{Supp}(Q))$. From the equimeasurability of Q and Q^*, we have

$$\begin{aligned} \mu_Q(F(e)) &= \text{meas}\left\{(x,v) \in \mathbb{R}^6, \ F\left(\frac{|v|^2}{2} + \phi_Q(x)\right) > F(e)\right\} \\ &= \text{meas}\{s \in \mathbb{R}_+^*, \ Q^*(s) > F(e)\}, \end{aligned}$$

for all $e \leq e_0$. Since F is strictly decreasing on $(-\infty, e_0]$, this is equivalent to

$$\mu_Q(F(e)) = a_{\phi_Q}(e) = \text{meas}\left\{s \in \mathbb{R}_+^*, \ Q^*(s) > F(e)\right\}, \qquad \forall e \leq e_0. \tag{2.18}$$

In particular $a_{\phi_Q}(e_0) = \text{meas}(\text{Supp}(Q)) > 0$, which implies that $\phi_Q(0) < e_0$. From (2.18) and the invertibility of both continuous functions F and a_{ϕ_Q} on $[\phi_Q(0), e_0]$, we deduce that μ_Q is continuous and one-to-one from $[0, F(\phi_Q(0))]$ to $[0, a_{\phi_Q}(e_0)]$. In particular, Q^* is the inverse of μ_Q on this interval (and not only its pseudo-inverse) and we have

$$Q^* \circ a_{\phi_Q}(e) = F(e), \qquad \forall e \in [\phi_Q(0), e_0]. \tag{2.19}$$

Identity (2.19) is still valid for $e_0 < e < 0$. Indeed, in this case, we have $F(e) = 0$, and $a_{\phi_Q}(e) > a_{\phi_Q}(e_0) = \text{meas}(\text{Supp}(Q))$, which implies that $Q^* \circ a_{\phi_Q}(e) = 0$. The first identity of (2.17) is then proved.

Now, the identity $Q^{*\phi_Q} = Q$ is a straightforward consequence of the first identity of (2.17). Indeed, we first observe that $\frac{|v|^2}{2} + \phi_Q(x) \geq \phi_Q(0)$. If $\frac{|v|^2}{2} + \phi_Q(x) \geq 0$ then $F\left(\frac{|v|^2}{2} + \phi_Q(x)\right) = 0$ and $Q^{*\phi_Q}(x,v) = 0$ from the definitions of F and $Q^{*\phi_Q}$.

[3]Note that this is essentially a characterization of spherical models

If $\frac{|v|^2}{2}+\phi_Q(x)<0$, then we apply the first identity to $e=\frac{|v|^2}{2}+\phi_Q(x)$ and get the desired equality. The proof of Lemma 2.6 is complete. □

2.4. **Monotonicity of the Hamiltonian under symmetric rearrangement.** We are now in position to derive the monotonicity of the Hamiltonian under the generalized rearrangement which is the first key to our analysis and was already observed in the physics litterature, see [1] and references therein. Given $f\in\mathcal{E}\setminus\{0\}$, by Lemma 2.1 we have $\phi_f\in\mathcal{X}$ and we will note to ease notation:

$$\widehat{f}=f^{*\phi_f}. \tag{2.20}$$

Given $\phi\in\mathcal{X}$, we define the functional

$$\mathcal{J}_{f^*}(\phi)=\mathcal{H}(f^{*\phi})+\frac{1}{2}\|\nabla\phi-\nabla\phi_{f^{*\phi}}\|_{L^2}^2 \tag{2.21}$$

which is well defined from Proposition 2.4. We claim:

Proposition 2.7 (Monotonicity of the Hamiltonian under the $f^{*\phi_f}$ rearrangement)**.** *Let $f\in\mathcal{E}\setminus\{0\}$ and $\widehat{f}$ given by* (2.20)*, then:*

$$\mathcal{H}(f)\geq\mathcal{J}_{f^*}(\phi_f)\geq\mathcal{H}(\widehat{f}). \tag{2.22}$$

Moreover, $\mathcal{H}(f)=\mathcal{H}(\widehat{f})$ if and only if $f=\widehat{f}$.

Proof. First compute for all $(f,g)\in\mathcal{E}$:

$$\begin{aligned}
\mathcal{H}(f) &= \frac{1}{2}\int_{\mathbb{R}^6}|v|^2 f-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi_f|^2\\
&= \int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f\right)(f-g)+\frac{1}{2}\int_{\mathbb{R}^6}|v|^2 g+\int_{\mathbb{R}^3}\phi_f g+\frac{1}{2}\int|\nabla\phi_f|^2\\
&= \mathcal{H}(g)+\frac{1}{2}\|\nabla\phi_f-\nabla\phi_g\|_{L^2}^2+\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f(x)\right)(f-g).
\end{aligned} \tag{2.23}$$

Replacing g by $\widehat{f}=f^{*\phi_f}$ yields from (2.21):

$$\mathcal{H}(f)=\mathcal{J}_{f^*}(\phi_f)+\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f(x)\right)(f-f^{*\phi_f})\,dxdv, \tag{2.24}$$

and hence (2.7) follows from

$$\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f(x)\right)(f-\widehat{f})\,dxdv\geq 0, \tag{2.25}$$

with equality if and only if $f=\widehat{f}$. The proof of (2.25) is reminiscent from the standard inequality for symmetric rearrangement, see [30]:

$$\int_{\mathbb{R}^6}|x|f^*\leq\int_{\mathbb{R}^6}|x|f.$$

Indeed, use the layer cake representation

$$f(x,v)=\int_{t=0}^{\|f\|_{L^\infty}}\mathbb{1}_{t<f(x,v)}dt$$

and Fubini to derive:

$$
\begin{aligned}
&\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f\right)(f-\widehat{f})\,dxdv\\
&\quad=\int_{t=0}^{\|f\|_{L^\infty}}dt\int_{\mathbb{R}^6}\left(\mathbb{1}_{t<f(x,v)}-\mathbb{1}_{t<\widehat{f}(x,v)}\right)\left(\frac{|v|^2}{2}+\phi_f\right)dxdv\\
&\quad=\int_{t=0}^{\|f\|_{L^\infty}}dt\int_{\mathbb{R}^6}\left(\mathbb{1}_{\widehat{f}(x,v)\le t<f(x,v)}-\mathbb{1}_{f(x,v)\le t<\widehat{f}(x,v)}\right)\left(\frac{|v|^2}{2}+\phi_f\right)dxdv\\
&\quad=\int_{t=0}^{\|f\|_{L^\infty}}dt\left(\int_{S_1(t)}\left(\frac{|v|^2}{2}+\phi_f\right)dxdv-\int_{S_2(t)}\left(\frac{|v|^2}{2}+\phi_f\right)dxdv\right) \qquad (2.26)
\end{aligned}
$$

with

$$S_1(t)=\{\widehat{f}(x,v)\le t<f(x,v)\},\qquad S_2(t)=\{f(x,v)\le t<\widehat{f}(x,v)\}.$$

Observe from $\widehat{f}\in \mathrm{Eq}(f)$ that:

$$\text{for a.e. } t>0,\qquad \mathrm{meas}(S_1(t))=\mathrm{meas}(S_2(t)). \qquad (2.27)$$

We thus conclude from (2.14) and (2.27): $\forall t\in(0,\|f\|_{L^\infty})$,

$$\int_{S_2(t)}\left(\frac{|v|^2}{2}+\phi_f(x)\right)dxdv\le \mathrm{meas}(S_2(t))(f^*\circ a_{\phi_f})^{-1}(t)=\int_{S_1(t)}(f^*\circ a_{\phi_f})^{-1}(t)dxdv.$$

Injecting this into (2.26) together with (2.15) yields:

$$
\begin{aligned}
&\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f\right)(f-\widehat{f})\,dxdv\ge\\
&\qquad\int_0^{\|f\|_{L^\infty}}dt\int_{S_1(t)}\left(\frac{|v|^2}{2}+\phi_f(x)-(f^*\circ a_{\phi_f})^{-1}(t)\right)dxdv\ge 0
\end{aligned}
$$

and (2.25) is proved. We also have the analogous inequality for $S_2(t)$:

$$
\begin{aligned}
&\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f\right)(f-\widehat{f})\,dxdv\ge\\
&\qquad\int_0^{\|f\|_{L^\infty}}dt\int_{S_2(t)}\left((f^*\circ a_{\phi_f})^{-1}(t)-\frac{|v|^2}{2}-\phi_f(x)\right)dxdv\ge 0.
\end{aligned}
$$

Let us now study the case of equality in (2.25). If

$$\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_f(x)\right)(f-\widehat{f})\,dxdv=0,$$

the above chain of equalities implies that for a.e $t>0$, either $\mathrm{meas}(S_1(t))=\mathrm{meas}(S_2(t))=0$ or, a.e. $(x_1,v_1)\in S_1(t)$, a.e $(x_2,v_2)\in S_2(t)$,

$$\frac{|v_1|^2}{2}+\phi_f(x_1)=(f^*\circ a_{\phi_f})^{-1}(t)=\frac{|v_2|^2}{2}+\phi_f(x_2).$$

The last assertion contradicts the fact that $\widehat{f}(x_1,v_1)\le t<\widehat{f}(x_2,v_2)$. Therefore, for a.e $t\in(0,\|f\|_{L^\infty})$, we have $\mathrm{meas}(S_1(t))=\mathrm{meas}(S_2(t))=0$. On the other hand, $\|f\|_{L^\infty}=\|f^*\|_{L^\infty}$ and hence $\mathrm{meas}(S_1(t))=\mathrm{meas}(S_2(t))=0$ for $t>\|f\|_{L^\infty}$. Hence $\mathrm{meas}(S_1(t))=\mathrm{meas}(S_2(t))=0$ for a.e. $t>0$, which implies $f=\widehat{f}$. This concludes the proof of Proposition 2.7. □

3. Study of the reduced functional $\mathcal{J}$

In this section, we focus onto the functional on $\mathcal{X}$:

$$\mathcal{J}(\phi) = \mathcal{J}_{Q^*}(\phi) = \mathcal{H}(Q^{*\phi}) + \frac{1}{2}\|\nabla\phi - \nabla\phi_{Q^{*\phi}}\|_{L^2}^2 \tag{3.1}$$

We claim that locally near ϕ_Q, $\mathcal{J}(\phi) - \mathcal{J}(\phi_Q)$ is equivalent to the distance of ϕ to the manifold of translated Poisson fields $\phi_Q(\cdot + x)$, $x \in \mathbb{R}^3$.

Proposition 3.1 (Coercive behavior of $\mathcal{J}$ near ϕ_Q)**.** *There exist universal constants $c_0, \delta_0 > 0$ and a continuous map $\phi \to z_\phi$ from $(\dot{H}^1, \|\cdot\|_{\dot{H}^1}) \to \mathbb{R}^3$ such that the following holds true. Let $\phi \in \mathcal{X}$ with*

$$\inf_{z\in\mathbb{R}^3} (\|\phi - \phi_Q(\cdot - z)\|_{L^\infty} + \|\nabla\phi - \nabla\phi_Q(\cdot - z)\|_{L^2}) < \delta_0, \tag{3.2}$$

then:

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_Q) \geq c_0\|\nabla\phi - \nabla\phi_Q(\cdot - z_\phi)\|_{L^2}^2. \tag{3.3}$$

This section will be devoted to the proof of Proposition 3.1 which relies first on the second order Taylor expansion of $\mathcal{J}$ at ϕ_Q, Proposition 3.3, and then on the coercivity of the Hessian which is the second main key to our analysis, Proposition 3.6, and corresponds to a generalized Antonov's coercivity property.

3.1. Differentiability of $\mathcal{J}$.

Our aim in this section is to prove the differentiability of $\mathcal{J}$ at ϕ_Q and to compute the first two derivatives.

Let us start with differentiability properties of the function $\phi \mapsto a_\phi$ defined in Lemma 2.2, see Appendix A for the proof.

Lemma 3.2 (Continuity and differentiability properties of $\phi \mapsto a_\phi$)**.** *Let $\phi, \widetilde{\phi} \in \mathcal{X}$ and let $h = \phi - \widetilde{\phi}$. Then the following holds.*
(i) The function $(\lambda, e) \mapsto a_{\phi+\lambda h}(e)$ is a $\mathcal{C}^1$ function on $[0,1] \times \mathbb{R}_-^*$ and we have

$$\frac{\partial}{\partial\lambda} a_{\phi+\lambda h}(e) = -4\pi\sqrt{2}\int_{\mathbb{R}^3} (e - \phi(x) - \lambda h(x))_+^{1/2} h(x)dx. \tag{3.4}$$

(ii) Let $s \in \mathbb{R}_+^*$. Then the function $\lambda \mapsto a_{\phi+\lambda h}^{-1}(s)$ is differentiable on $[0,1]$ and we have

$$\frac{\partial}{\partial\lambda} a_{\phi+\lambda h}^{-1}(s) = \frac{\displaystyle\int_{\mathbb{R}^3} (a_{\phi+\lambda h}^{-1}(s) - \phi(x) - \lambda h(x))_+^{1/2} h(x)dx}{\displaystyle\int_{\mathbb{R}^3} (a_{\phi+\lambda h}^{-1}(s) - \phi(x) - \lambda h(x))_+^{1/2} dx}. \tag{3.5}$$

We are now in position to differentiate the functional $\mathcal{J}$.

Proposition 3.3 (Differentiability of $\mathcal{J}$)**.** *The functional $\mathcal{J}$ defined by* (3.1) *on $\mathcal{X}$ satisfies the following properties.*
(i) Differentiability of $\mathcal{J}$. *Let $\phi, \widetilde{\phi} \in \mathcal{X}$, then the function*

$$\lambda \mapsto \mathcal{J}(\phi + \lambda(\widetilde{\phi} - \phi))$$

is twice differentiable on $[0,1]$.
(ii) Taylor expansion of $\mathcal{J}$ near ϕ_Q. *There holds the Taylor expansion near ϕ_Q: $\forall\phi \in \mathcal{X}$,*

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_Q) = \frac{1}{2}D^2\mathcal{J}(\phi_Q)(\phi - \phi_Q, \phi - \phi_Q) + \eta\left(\|\phi - \phi_Q\|_{L^\infty}\right)\|\nabla\phi - \nabla\phi_Q\|_{L^2}^2 \tag{3.6}$$

where

$$\eta(\delta) \to 0 \text{ as } \delta \to 0.$$

Moreover, the second derivative of $\mathcal{J}$ at ϕ_Q in the direction h is given by

$$D^2\mathcal{J}(\phi_Q)(h,h) \tag{3.7}$$
$$= \int_{\mathbb{R}^3} |\nabla h|^2\,dx - \int_{\mathbb{R}^6} \left|F'\left(\frac{|v|^2}{2} + \phi_Q(x)\right)\right| (h(x) - \Pi h(x,v))^2\,dxdv\,,$$

where Πh is the projector:

$$\Pi h(x,v) = \frac{\displaystyle\int_{\mathbb{R}^3} \left(\frac{|v|^2}{2} + \phi_Q(x) - \phi_Q(y)\right)_+^{1/2} h(y)dy}{\displaystyle\int_{\mathbb{R}^3} \left(\frac{|v|^2}{2} + \phi_Q(x) - \phi_Q(y)\right)_+^{1/2} dy}. \tag{3.8}$$

Remark 3.4. The projector Πh given by (3.8) should be understood as the projector onto the functions which depend only on the microscopic energy $e(x,v) = \frac{|v|^2}{2} + \phi_Q(x)$.

Proof. Let us decompose $\mathcal{J}$ into a kinetic and a potential part:

$$\mathcal{J}(\phi) = \mathcal{J}_{Q^*}(\phi) = \mathcal{H}(Q^{*\phi}) + \frac{1}{2}\|\nabla\phi - \nabla\phi_{Q^{*\phi}}\|^2 = \frac{1}{2}\int |\nabla\phi|^2 dx + \mathcal{J}_0(\phi) \tag{3.9}$$

with

$$\mathcal{J}_0(\phi) = \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi(x)\right) Q^{*\phi}(x,v)\,dxdv. \tag{3.10}$$

Note from Proposition 2.4 that $Q^{*\phi} \in \mathcal{E}$ and is supported in $\frac{|v|^2}{2} + \phi < 0$, thus

$$-\infty < \mathcal{J}_0(\phi) \leq 0.$$

Let $\phi, \widetilde{\phi} \in \mathcal{X}$ and let $h = \widetilde{\phi} - \phi$. We shall differentiate with respect to λ the function $\mathcal{J}_0(\phi + \lambda h)$.

Step 1. First derivative of $\mathcal{J}_0$.

Introduce the following primitive of Q^*:

$$G(s) = \int_0^s Q^*(\sigma)d\sigma, \tag{3.11}$$

which is a uniformly bounded $\mathcal{C}^1$ function with bounded derivative, since by assumption Q (thus Q^*) is continuous and compactly supported. We first transform the expression (3.10) of $\mathcal{J}_0$. By making the change of variable in velocity $e = \frac{|v|^2}{2} + \phi$ and using (2.5), we get

$$\begin{aligned}\mathcal{J}_0(\phi) &= \int_{\min\phi}^0 eQ^*\left(a_\phi(e)\right) a_\phi'(e)\,de = \int_{\min\phi}^0 e\left(G \circ a_\phi\right)'(e)\,de \\ &= \left[eG(a_\phi(e))\right]_{\min\phi}^0 - \int_{\min\phi}^0 G\left(a_\phi(e)\right)de = -\int_{-\infty}^0 G\left(a_\phi(e)\right)de.\end{aligned}$$

Note that the boundary term is dropped thanks to the definition (3.11) and the following properties:

$$a_\phi(\min\phi) = 0, \quad \lim_{e\to 0-} a_\phi(e) = +\infty, \quad \int_0^{+\infty} Q^*(\sigma)d\sigma = \|Q\|_{L^1} < +\infty.$$

In order to differentiate $\mathcal{J}_0(\phi+\lambda h)$ with respect to λ, we now use (3.4) and the $\mathcal{C}^1$ smoothness of G to derive: $\forall e<0$

$$\frac{\partial}{\partial\lambda}G(a_{\phi+\lambda h}(e)) = -4\pi\sqrt{2}\,Q^*\left(a_{\phi+\lambda h}(e)\right)\int_{\mathbb{R}^3}(e-\phi(x)-\lambda h(x))_+^{1/2}h(x)dx\,.$$

Recall that we have Supp $(Q^*)=[0,L_0]$, with

$$L_0 = \text{meas (Supp } Q) < +\infty.$$

Hence, from Lemma 3.2 *(i)*, we deduce that there exists $e_1<e_2<0$ such that

$$\left\{(\lambda,e)\in[0,1]\times\mathbb{R}_-^*\,:\,a_{\phi+\lambda h}(e)\in \mathrm{Supp}(Q^*)\right\}\subset[0,1]\times[e_1,e_2]. \tag{3.12}$$

Moreover, we have the following uniform bound: for all (λ,e),

$$\begin{aligned}\left|\frac{\partial}{\partial\lambda}G(a_{\phi+\lambda h}(e))\right| &\leq 4\pi\sqrt{2}\|Q^*\|_{L^\infty}\int_{\mathbb{R}^3}(e_2-(1-\lambda)\phi(x)-\lambda\widetilde{\phi}(x))_+^{1/2}h(x)dx\\ &\leq 4\pi\sqrt{2}\|Q^*\|_{L^\infty}\int_{\mathbb{R}^3}(e_2-\phi(x)-\widetilde{\phi}(x))_+^{1/2}h(x)dx<+\infty.\end{aligned}$$

Therefore, Lebesgue's derivation theorem ensures:

$$\begin{aligned}\frac{\partial}{\partial\lambda}\mathcal{J}_0(\phi+\lambda h) &= -\frac{\partial}{\partial\lambda}\int_{-\infty}^0 G\left(a_{\phi+\lambda h}(e)\right)de\\ &= 4\pi\sqrt{2}\int_{-\infty}^0\int_{\mathbb{R}^3}Q^*\left(a_{\phi+\lambda h}(e)\right)(e-\phi(x)-\lambda h(x))_+^{1/2}h(x)\,dxde. \end{aligned}\tag{3.13}$$

Step 2. Second derivative of $\mathcal{J}_0$.

Let us now compute the second derivative of $\mathcal{J}_0(\phi+\lambda h)$ with respect to λ. First, an integration by parts of (3.13) with respect to the variable e gives

$$\frac{\partial}{\partial\lambda}\mathcal{J}_0(\phi+\lambda h) = -\frac{8\pi\sqrt{2}}{3}\int_{-\infty}^0\int_{\mathbb{R}^3}Q^{*\prime}\left(a_{\phi+\lambda h}(e)\right)a_{\phi+\lambda h}'(e)(e-\phi(x)-\lambda h(x))_+^{3/2}h(x)\,dxde.$$

Applying the change of variable $s=a_{\phi+\lambda h}(e)$, we obtain

$$\begin{aligned}\frac{\partial}{\partial\lambda}\mathcal{J}_0(\phi+\lambda h) &= -\frac{8\pi\sqrt{2}}{3}\int_0^{L_0}ds\,Q^{*\prime}(s)\int_{\mathbb{R}^3}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{3/2}h(x)dx\\ &= -\frac{8\pi\sqrt{2}}{3}\int_0^{L_0}\int_{\mathbb{R}^3}Q^{*\prime}(s)g(\lambda,x,s)h(x)dsdx,\end{aligned}\tag{3.14}$$

with

$$g(\lambda,x,s)=(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{3/2}.$$

Recall that, by (3.12), the quantity $e=a_{\phi+\lambda h}^{-1}(s)$ can be restricted to some interval $[e_1,e_2]$ in this integral, with $e_1<e_2<0$. Moreover, as in Step 1 of the proof of Lemma 3.2, one deduces from the decay at the infinity of ϕ and $\widetilde{\phi}$ that the domain

$$\left\{x\in\mathbb{R}^3\,:\,\phi(x)+\lambda h(x)\right\}\leq e_2$$

is bounded independently of λ. Therefore, the variable x in the integral (3.14) can be restricted to a bounded domain.

Let us differentiate (3.14) with respect to λ. From (3.5), one gets

$$\begin{aligned}\frac{\partial}{\partial\lambda}g(\lambda,x,s) &= -\frac{3}{2}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{1/2}h(x)\\ &+\frac{3}{2}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{1/2}\frac{\int_{\mathbb{R}^3}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{1/2}h(x)\,dx}{\int_{\mathbb{R}^3}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{1/2}\,dx}\end{aligned}$$

with the uniform estimate: for all $s\in[0,L_0]$, $\lambda\in[0,1]$, $x\in\mathbb{R}^3$

$$\left|\frac{\partial}{\partial\lambda}g(\lambda,x,s)\right|\le 3(e_2+|\min\phi|+|\min\widetilde{\phi}|)_+^{1/2}\|h\|_{L^\infty}. \tag{3.15}$$

Since the function $s\mapsto Q^*(s)$ is monotone decreasing from $\|Q\|_{L^\infty}$ to 0, the function $Q^{*\prime}$ belongs to $L^1(0,L_0)$, and hence the uniform domination (3.15) allows us to apply Lebesgue's derivation theorem and get:

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}\mathcal{J}_0(\phi+\lambda h) &= 4\pi\sqrt{2}\int_0^{L_0}ds\,Q^{*\prime}(s)\int_{\mathbb{R}^3}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{1/2}(h(x))^2dx\\ &-4\pi\sqrt{2}\int_0^{L_0}ds\,Q^{*\prime}(s)\frac{\left(\int_{\mathbb{R}^3}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{1/2}h(x)\,dx\right)^2}{\int_{\mathbb{R}^3}(a_{\phi+\lambda h}^{-1}(s)-\phi(x)-\lambda h(x))_+^{1/2}\,dx}.\end{aligned} \tag{3.16}$$

Step 3. Identification of the first and second derivative of $\mathcal{J}$ at ϕ_Q.
Let $\phi\in\mathcal{X}$ and $h=\phi-\phi_Q$. We claim that

$$D\mathcal{J}(\phi_Q)(h)=0. \tag{3.17}$$

Indeed, first remark from (3.9) that

$$D\mathcal{J}(\phi_Q)(h)=D\mathcal{J}_0(\phi_Q)(h)+\int_{\mathbb{R}^3}\nabla\phi_Q\cdot\nabla h\,dx. \tag{3.18}$$

Next, by (3.13) and (2.17):

$$\begin{aligned}D\mathcal{J}_0(\phi_Q)(h) &= 4\pi\sqrt{2}\int_{-\infty}^0\int_{\mathbb{R}^3}Q^*\left(a_{\phi_Q}(e)\right)(e-\phi_Q(x))_+^{1/2}h(x)\,dxde\\ &= 4\pi\sqrt{2}\int_{-\infty}^0\int_{\mathbb{R}^3}F(e)(e-\phi_Q(x))_+^{1/2}h(x)\,dxde.\end{aligned}$$

Applying the change of variable $e\mapsto u=\sqrt{2(e-\phi_Q(x))}$, it comes

$$\begin{aligned}D\mathcal{J}_0(\phi_Q)(h) &= 4\pi\int_0^{+\infty}\int_{\mathbb{R}^3}F\left(\frac{u^2}{2}+\phi_Q(x)\right)h(x)u^2dudx\\ &= \int_{\mathbb{R}^6}Q(x,v)h(x)\,dxdv,\end{aligned}$$

where we used the expression (1.14) of Q. Hence, from the Poisson equation, we deduce after an integration by parts that

$$D\mathcal{J}_0(\phi_Q)(h)=-\int_{\mathbb{R}^3}\nabla\phi_Q\cdot\nabla h\,dx,$$

which together with (3.18) implies (3.17).

Let us now identify the right second derivative of $\mathcal{J}$ at ϕ_Q. We have

$$D^2\mathcal{J}(\phi_Q)(h,h) = D^2\mathcal{J}_0(\phi_Q)(h,h) + \int_{\mathbb{R}^3} |\nabla h|^2\,dx \tag{3.19}$$

and, by (3.16),

$$D^2\mathcal{J}_0(\phi_Q)(h,h) = 4\pi\sqrt{2}\int_0^{L_0} ds\, Q^{*\prime}(s)\int_{\mathbb{R}^3}(a_{\phi_Q}^{-1}(s)-\phi_Q(x))_+^{1/2}(h(x))^2 dx$$
$$-4\pi\sqrt{2}\int_0^{L_0} ds\, Q^{*\prime}(s)\frac{\left(\int_{\mathbb{R}^3}(a_{\phi_Q}^{-1}(s)-\phi_Q(x))_+^{1/2}h(x)\,dx\right)^2}{\int_{\mathbb{R}^3}(a_{\phi_Q}^{-1}(s)-\phi_Q(x))_+^{1/2}\,dx}.$$

Using first the change of variable $s\mapsto e = a_{\phi_Q}^{-1}(s)$, (2.5) and $F = Q^*\circ a_{\phi_Q}$, we get

$$\begin{aligned} D^2\mathcal{J}_0(\phi_Q)(h,h) &= 4\pi\sqrt{2}\int_{-\infty}^0 de\, F'(e)\int_{\mathbb{R}^3}(e-\phi_Q(x))_+^{1/2}(h(x))^2 dx \\ &\quad -4\pi\sqrt{2}\int_{-\infty}^0 F'(e)\frac{\left(\int_{\mathbb{R}^3}(e-\phi_Q(x))_+^{1/2}h(x)\,dx\right)^2}{\int_{\mathbb{R}^3}(e-\phi_Q(x))_+^{1/2}\,dx}. \end{aligned}$$

We next apply the change of variable $e\mapsto u = \sqrt{2(e-\phi_Q(x))}$ to get

$$\begin{aligned} D^2\mathcal{J}_0(\phi_Q)(h,h) &= \int_{\mathbb{R}^6} F'(e)(h(x))^2\,dxdv - \int_{\mathbb{R}^6} F'(e)h(x)\Pi h(e)dxdv \\ &= -\int_{\mathbb{R}^6}|F'(e)|(h(x)-\Pi h(e))^2 dxdv, \end{aligned}$$

where we used the shorthand notation $e = \frac{|v|^2}{2}+\phi_Q(x)$ and the fact that Π given by (3.8) is the projector onto the functions which depend only on e. This together with (3.19) concludes the proof of (3.7).

Step 4. Proof of the Taylor expansion (3.6).

Let $\phi\in\mathcal{X}$ and $h=\phi-\phi_Q$. We first deduce from the fact that $\mathcal{J}(\phi_Q+\lambda h)$ is twice differentiable with respect to λ that

$$\mathcal{J}(\phi_Q+h)-\mathcal{J}(\phi_Q) = \int_0^1(1-\lambda)\frac{\partial^2}{\partial\lambda^2}\mathcal{J}(\phi_Q+\lambda h)\,d\lambda$$

and hence:

$$\begin{aligned} &\mathcal{J}(\phi_Q+h)-\mathcal{J}(\phi_Q)-\frac{1}{2}D^2\mathcal{J}(\phi_Q)(h,h) \qquad (3.20)\\ =\ & \int_0^1(1-\lambda)\left(D^2\mathcal{J}(\phi_Q+\lambda h)-D^2\mathcal{J}(\phi_Q)\right)(h,h)\,d\lambda \\ =\ & \|\nabla h\|_{L^2}^2\int_0^1(1-\lambda)\left(D^2\mathcal{J}_0(\phi_Q+\lambda h)-D^2\mathcal{J}_0(\phi_Q)\right)\left(\frac{h}{\|\nabla h\|_{L^2}},\frac{h}{\|\nabla h\|_{L^2}}\right)d\lambda. \end{aligned}$$

We now claim the following continuity property:

$$\sup_{\lambda\in[0,1]}\ \sup_{\|\nabla\widetilde{h}\|_{L^2}=1}\left|\left(D^2\mathcal{J}_0(\phi_Q+\lambda(\phi-\phi_Q))-D^2\mathcal{J}_0(\phi_Q)\right)(\widetilde{h},\widetilde{h})\right|\to 0 \tag{3.21}$$

as $\|\phi-\phi_Q\|_{L^\infty}\to 0$. Assume (3.21), then

$$\lim_{\|h\|_{L^\infty}\to 0}\int_0^1(1-\lambda)\left(D^2\mathcal{J}_0(\phi_Q+\lambda h)-D^2\mathcal{J}_0(\phi_Q)\right)\left(\frac{h}{\|\nabla h\|_{L^2}},\frac{h}{\|\nabla h\|_{L^2}}\right)d\lambda=0$$

and (3.20) now yields the Taylor expansion (3.6).
Proof of (3.21). We argue by contradiction and assume that there exists $\varepsilon>0$, H_n, $\widetilde{h}_n$ and $\lambda_n\in[0,1]$ such that

$$\|H_n\|_{L^\infty}\leq\frac{1}{n},\quad \|\nabla\widetilde{h}_n\|_{L^2}=1\,, \tag{3.22}$$

and

$$\left|D^2\mathcal{J}_0(\phi_Q+\lambda_nH_n)(\widetilde{h}_n,\widetilde{h}_n)-D^2\mathcal{J}_0(\phi_Q)(\widetilde{h}_n,\widetilde{h}_n)\right|>\varepsilon. \tag{3.23}$$

We denote $h_n=\lambda_nH_n$. Recall from (3.16):

$$\begin{aligned}
&D^2\mathcal{J}_0(\phi_Q+h_n)(\widetilde{h}_n,\widetilde{h}_n)=\\
&\quad=4\pi\sqrt{2}\int_0^{L_0}ds\,Q^{*\prime}(s)\int(a^{-1}_{\phi_Q+h_n}(s)-(\phi_Q+h_n)(x))_+^{1/2}(\widetilde{h}_n(x))^2\,dx\\
&\quad-4\pi\sqrt{2}\int_0^{L_0}ds\,Q^{*\prime}(s)\frac{\left(\int(a^{-1}_{\phi_Q+h_n}(s)-(\phi_Q+h_n)(x))_+^{1/2}\widetilde{h}_n(x)\,dx\right)^2}{\int(a^{-1}_{\phi_Q+h_n}(s)-(\phi_Q+h_n(x))_+^{1/2}\,dx}.
\end{aligned} \tag{3.24}$$

Let us analyze the sequence $e_n=a^{-1}_{\phi_Q+h_n}(s)$. We claim that:

$$\forall s\in(0,L_0),\quad \lim_{n\to+\infty}e_n=a^{-1}_{\phi_Q}(s). \tag{3.25}$$

Indeed, we observe

$$s=a_{\phi_Q+h_n}(e_n)\geq\frac{8\pi\sqrt{2}}{3}\int(e_n-\phi_Q(x)-\|h_n\|_{L^\infty})_+^{3/2}\,dx\geq a_{\phi_Q}\left(e_n-\frac{1}{n}\right)$$

which yields $e_n\leq a^{-1}_{\phi_Q}(s)+1/n$. Similarly, we also have $e_n\geq a^{-1}_{\phi_Q}(s)-1/n$ and (3.25) follows.

Let us now pass to the limit in (3.24). Note first that the domain of integration in x of these integrals is uniformly bounded as $n\to+\infty$. Indeed, the set of integration is

$$D_n(e_n):=\left\{x\in\mathbb{R}^3:\ \phi_Q(x)+h_n(x)<e_n\right\}\subset\left\{x\in\mathbb{R}^3:\ \phi_Q(x)\leq e_n+1/n\right\},$$

which is bounded for n large enough, since $e_n\leq a^{-1}_{\phi_Q}(L_0)+1/n\leq\frac{1}{2}a^{-1}_{\phi_Q}(L_0)$, and the continuous function ϕ_Q converges to zero at infinity.

Now the local compactness of the Sobolev embedding $\dot{H}^1\hookrightarrow L^p_{loc}$ for $1\leq p<6$ implies that there exists $\widetilde{h}\in\dot{H}^1_{rad}$ such that –up to a subsequence–

$$\widetilde{h}_n\to\widetilde{h}\ \text{ in }\ L^2_{loc}\ \text{ as }\ n\to+\infty. \tag{3.26}$$

Hence, for all $s \in (0, L_0)$ and for $i = 0, 1, 2$, (3.25), (3.26) ensure:

$$\begin{aligned}\int_{\mathbb{R}^3} &(a^{-1}_{\phi_Q+h_n}(s) - (\phi_Q + h_n)(x))_+^{1/2} (\widetilde{h}_n(x))^i \, dx \\ &= \int_{|x|\le R} (a^{-1}_{\phi_Q+h_n}(s) - (\phi_Q + h_n)(x))_+^{1/2} (\widetilde{h}_n(x))^i \, dx \\ &\to \int_{\mathbb{R}^3} (a^{-1}_{\phi_Q}(s) - \phi_Q(x))_+^{1/2} (\widetilde{h}(x))^i \, dx \quad \text{as} \quad n \to +\infty.\end{aligned}$$

Moreover, by Cauchy-Schwarz and $a^{-1}_{\phi_Q+h_n}(s) \le 0$:

$$\begin{aligned}\frac{\left(\int (a^{-1}_{\phi_Q+h_n}(s) - \phi_Q - h_n)_+^{1/2} \widetilde{h}_n \, dx\right)^2}{\int (a^{-1}_{\phi_Q+h_n}(s) - \phi - h_n)_+^{1/2} \, dx} &\le \int (a^{-1}_{\phi_Q+h_n}(s) - \phi_Q - h_n)_+^{1/2} (\widetilde{h}_n)^2 \, dx \\ &\lesssim \int_{|x|\le R} (\|\phi_Q\|_{L^\infty} + \|h_n\|_{L^\infty})^{1/2} (\widetilde{h}_n)^2 dx \lesssim 1.\end{aligned}$$

Recall now that the function $Q^{*\prime}$ is L^1 on $[0, L_0]$, since Q^* is decreasing and bounded. Therefore, Lebesgue's convergence theorem applied to (3.24) yields:

$$D^2 \mathcal{J}_0(\phi_Q + h_n)(\widetilde{h}_n, \widetilde{h}_n) \to D^2 \mathcal{J}_0(\phi_Q)(\widetilde{h}, \widetilde{h}).$$

A similar argument gives:

$$D^2 \mathcal{J}_0(\phi_Q)(\widetilde{h}_n, \widetilde{h}_n) \to D^2 \mathcal{J}_0(\phi_Q)(\widetilde{h}, \widetilde{h}) \tag{3.27}$$

as $n \to +\infty$. This contradicts (3.23) and concludes the proof of (3.21).

The proof of Proposition 3.3 is complete. □

Remark 3.5. We have proved in this last Step 4 that for all sequence $\widetilde{h}_n$ bounded in $\dot{H}^1$, after extraction of a subsequence, we have the strong convergence (3.27). Hence the quadratic form $D^2 \mathcal{J}_0(\phi_Q)$ is compact on $\dot{H}^1$.

3.2. **A new Antonov type inequality.** We now turn to the second key of our analysis which is a generalization of the celebrated Antonov's stability property –see Proposition 4.1 in [29] for a precise statement–:

Proposition 3.6 (Generalized Antonov's stability property)**.** *Let Q satisfy the assumptions of Theorem* 1.2 *and consider the linear operator generated by the Hessian* (3.7)*:*

$$\mathcal{L}h = -\Delta h - \int_{\mathbb{R}^3} |F'(e)|(h - \Pi h) dv.$$

Then $\mathcal{L}$ is a compact perturbation of the Laplacian operator on $\dot{H}^1$ and is positive:

$$\forall h \in \dot{H}^1, \quad (\mathcal{L}h, h) = D^2 \mathcal{J}(\phi_Q)(h, h) \ge 0. \tag{3.28}$$

Moreover,

$$Ker(\mathcal{L}) = \{h \in \dot{H}^1 \ \ \textit{with} \ \ \mathcal{L}h = 0\} = Span(\partial_{x_i} \phi_Q)_{1 \le i \le 3}.$$

In particular, there exists $c_0 > 0$ such that

$$\forall h \in \dot{H}^1, \quad (\mathcal{L}h, h) \ge c_0 \|\nabla h\|_{L^2}^2 - \frac{1}{c_0} \sum_{i=1}^3 \left(\int_{\mathbb{R}^3} h \Delta(\partial_{x_i} \phi_Q) \right)^2. \tag{3.29}$$

Remark 3.7. The fact that the kernel is completely explicit and purely generated by the symmetry group is remarkable and reminiscent from similar statements in dispersive equations, see Weinstein [44], the coercivity on the radial sector being always the most delicate problem.

Proof of Proposition 3.6

Step 1. Positivity away from radial modes.

Let $h \in \dot{H}^1_{rad}$, and let us introduce the projection of h onto the radial sector

$$h_0(r) = \frac{1}{4\pi}\int_{\mathbb{S}^2} h(r\sigma)d\sigma,$$

where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$ and $d\sigma$ denotes the surface measure on $\mathbb{S}^2$ induced by the Lebesgue measure. We have the decomposition

$$h = h_0 + h_1, \quad h_0 \in \dot{H}^1_{rad}, \quad h_1 \in (\dot{H}^1_{rad})^{\perp}.$$

The angular integration in (3.8) ensures $\Pi h_1 = 0$ and thus

$$(\mathcal{L}h, h) = (\mathcal{L}h_0, h_0) + \int_{\mathbb{R}^3} |\nabla h_1|^2 - \int_{\mathbb{R}^3} V_Q\, h_1^2$$

with

$$V_Q(r) = \int_{\mathbb{R}^3} |F'(e)|dv = 4\pi\sqrt{2}\int_{\phi_Q(0)}^0 |F'(e)|\,(e-\phi_Q(r))_+^{1/2}\,de,$$

where we applied the change of variable $e = \frac{|v|^2}{2} + \phi_Q(r)$. Since $F'(e) < 0$ and F is bounded on $[\phi_Q(0), 0]$, the function $|F'|$ belongs to L^1. Therefore by dominated convergence, the function V_Q is continuous. Moreover, since $F(e) = 0$ for $e \geq e_0$ and since ϕ_Q is strictly increasing, we have:

$$\mathrm{Supp}(V_Q) = [0, (\phi_Q)^{-1}(e_0)].$$

Hence, V_Q being continuous and compactly supported, the Schrödinger operator $-\Delta - V_Q$ is a compact perturbation of the Laplacian on $\dot{H}^1$. Observe that ϕ_Q' (and also $\partial_{x_i}\phi_Q$ for $i = 1, \cdots, 3$) belongs to $\dot{H}^1(\mathbb{R}^3)$. Translating the ϕ_Q equation yields:

$$\Delta\phi_Q(x+x_0) = \rho_Q(x+x_0) = \frac{8\pi\sqrt{2}}{3}\int_{-\infty}^0 |F'(e)|(e-\phi_Q(x+x_0))_+^{3/2}de$$

and differentiating this relation with respect to x_0 yields at $x_0 = 0$:

$$\mathcal{L}(\nabla\phi_Q) = 0. \tag{3.30}$$

We now claim from standard argument that this implies the positivity of $\mathcal{L}$ away from radial modes, see [44] for related statements:

$$\forall h \in \left(\dot{H}^1_{rad}\right)^{\perp}, \quad (\mathcal{L}h, h) \geq 0, \tag{3.31}$$

and

$$\{h \in (\dot{H}^1_{rad})^{\perp} \ \text{ with } \ \mathcal{L}h = 0\} = \mathrm{Span}(\partial_{x_i}\phi_Q)_{1\leq i\leq 3}, \tag{3.32}$$

Let us briefly recall the argument. Let us decompose $h \in (\dot{H}^1_{rad})^{\perp}$ into spherical harmonics,

$$h = \sum_{k\geq 1}\sum_j h_{k,j} Y_{k,j}(\hat{x})$$

where $\hat{x} = \frac{x}{r}$ is the spherical variable and $-\Delta_{\mathbb{S}^2} Y_{k,j} = \lambda_k Y_{k,j}$. Then the radiality of V_Q ensures the orthogonal decomposition

$$\mathcal{L}h = \sum_{k\geq 1}\sum_j A_k h_{k,j}$$

with

$$A_k = -\partial_r^2 - \frac{2}{r}\partial_r + \frac{\lambda_k}{r^2} - V_Q(r), \quad \lambda_k = k(k+1). \tag{3.33}$$

For $k = 1$, we have $\nabla\phi_Q = \phi_Q'(r)\hat{x}$ and (3.30) implies $A_1\phi_Q' = 0$. Since $\phi_Q' > 0$ for $r > 0$, $\phi_Q' \in \dot{H}^1$, ϕ_Q' is from standard Sturm Liouville results the ground state of A_1 which is thus positive with kernel on $\dot{H}^1_{rad}$ spanned by ϕ_Q'. Now (3.33) ensures that A_k is definite positive on $\dot{H}^1_{rad}$ for $k \geq 2$ and (3.31), (3.32) follow.

Step 2. Coercivity away from radial modes.

We now claim:

$$\forall h \in (\dot{H}^1_{rad})^\perp, \quad (\mathcal{L}h, h) \geq c_1\|\nabla h\|^2_{L^2} - \frac{1}{c_1}\sum_{i=1}^3\left(\int_{\mathbb{R}^3} h\Delta(\partial_{x_i}\phi_Q)\right)^2 \tag{3.34}$$

for some universal constant $c_1 > 0$. Let us briefly recall the argument which is standard. From (3.31),

$$I = \inf\left\{(\mathcal{L}h, h), \quad h \in (\dot{H}^1_{rad})^\perp, \quad \int_{\mathbb{R}^3} V_Q h^2 = 1, \quad \int_{\mathbb{R}^3} h\Delta(\partial_{x_i}\phi_Q) = 0\right\} \geq 0.$$

We argue by contradiction and assume $I = 0$, then there exists a sequence $h_n \in (\dot{H}^1_{rad})^\perp$ with

$$\int_{\mathbb{R}^3} V_Q h_n^2 = 1, \quad \int_{\mathbb{R}^3} |\nabla h_n|^2 - \int_{\mathbb{R}^3} V_Q h_n^2 \leq \frac{1}{n}, \quad \int_{\mathbb{R}^3} h_n\Delta(\partial_{x_i}\phi_Q) = 0.$$

From Sobolev embeddings, $h_n \to h$ in L^p_{loc}, $1 \leq p < 6$, up to a subsequence. Moreover, from (3.30), $\Delta(\partial_{x_i}\phi_Q)$ is compactly supported and in L^2 from which passing to the limit yields

$$(\mathcal{L}h, h) \leq 0, \quad \int_{\mathbb{R}^3} h\Delta(\partial_{x_i}\phi_Q) = 0, \quad \int_{\mathbb{R}^3} V_Q h^2 = 1 \tag{3.35}$$

and hence $h \neq 0$ attains the infimum. From Lagrange multipier theory, we thus can find $(\lambda_i)_{1\leq i\leq 3}$ with

$$\mathcal{L}h = \lambda_0 V_Q h + \sum_{i=1}^3 \lambda_i\Delta(\partial_{x_i}\phi_Q).$$

Taking the inner product with h yields $\lambda_0 = 0$, then with $\partial_{x_i}\phi_Q$ yields $\lambda_i = 0$, and thus $\mathcal{L}h = 0$. From (3.32), $h \in \text{Span}(\partial_{x_i}\phi_Q)_{1\leq i\leq 3}$, but this contradicts the orthogonality relations (3.35), and (3.34) follows.

Step 3. Strategy: Hörmander's proof of Poincaré inequality.

The relative compactness of $\mathcal{L}$ with respect to Δ in $\dot{H}^1$ follows from Remark 3.5. It thus remains to prove (3.29) which from (3.34) and the Fredholm alternative is equivalent to:

$$\forall h \in \dot{H}^1_{rad}, \quad h \neq 0, \quad (\mathcal{L}h, h) > 0. \tag{3.36}$$

Our main observation is now from (3.7) that (3.36) is nothing but a *Poincaré inequality with sharp constant*, and we now claim that we can adapt the celebrated proof by Hörmander [20, 21] to our setting. Hörmander's approach involves two key steps: the introduction of a self-adjoint operator adapted to the projection involved, and a suitable convexity property. The operator will be given by

$$Tf(e,r) = \frac{1}{r^2\sqrt{2(e-\phi_Q(r))}}\partial_r f \tag{3.37}$$

which essentially satisfies the requirement

$$\Pi h = 0 \ \text{ implies } \ h \in \mathrm{Im}(T),$$

and the convexity will correspond to the lower bound:

$$-\frac{T^2 g}{g} \geq \frac{3}{(r\sqrt{2(e-\phi_Q(r))})^4}\left(\rho_Q(r) + \frac{\phi_Q'(r)}{r}\right) \tag{3.38}$$

with

$$g(r,e) = \left(r\sqrt{2(e-\phi_Q(r))}\right)^3.$$

Note that the original proof of Antonov's stability criterion can be revisited as well using the transport operator $\tau = v\cdot\nabla_x - \nabla_x\phi_Q\cdot\nabla_v$ in the radial case as differential operator and whose image can be realized in the radial setting as the kernel of the *full* projection including the kinetic momentum ℓ, see [19], [29] for more details.

Step 4. Integration by parts.

Recalling that $\phi_Q(r)$ is strictly increasing, for all $e \in (\phi_Q(0), 0)$, we shall denote

$$r(e) = \phi_Q^{-1}(e).$$

Let $h \in \dot{H}^1_{rad}$ non zero. Let

$$\begin{aligned}\mathcal{U} &= \{(r,e): \quad r>0, \ \ e\in(\phi_Q(0),0), \ \ e-\phi_Q(r)>0\}\\ &= \{(r,e): \quad e\in(\phi_Q(0),0), \ \ r\in(0,r(e))\}.\end{aligned}$$

Given $\varepsilon > 0$, we let $0 \leq \chi_\varepsilon(e) \leq 1$ be a smooth cut off function such that

$$\mathrm{Supp}(\chi_\varepsilon) \subset (\phi_Q(0)+\varepsilon, e_0-\varepsilon), \ \ \chi_\varepsilon \equiv 1 \ \text{ on } \ [\phi_Q(0)+2\varepsilon, e_0-2\varepsilon].$$

We let

$$r_\varepsilon = r(\phi_Q(0)+\varepsilon).$$

Observe that

$$\int_0^{r(e)} \sqrt{e-\phi_Q(\tau)}\,\tau^2 d\tau \geq c_\varepsilon > 0 \ \text{ on } \ \mathrm{Supp}(\chi_\varepsilon),$$

and hence the radial interpolation estimate

$$\|\sqrt{r}h(r)\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\nabla h\|_{L^2(\mathbb{R}^3)} \tag{3.39}$$

and (3.8) ensure:

$$|\Pi h(e)| \leq C_\varepsilon \ \text{ on } \ \mathrm{Supp}(\chi_\varepsilon). \tag{3.40}$$

Let us then define on $\widetilde{\mathcal{U}} = \mathcal{U} \cap (0, r(e_0)) \times (\phi_Q(0), e_0)$:

$$f(r,e) = \int_0^r (h(\tau) - \Pi h(e))\sqrt{2(e-\phi_Q(\tau))}\,\tau^2 d\tau. \tag{3.41}$$

Then f is $\mathcal{C}^1$ with respect to the variable $r>0$ with

$$Tf = h - \Pi h \tag{3.42}$$

on $\widetilde{\mathcal{U}}$, where T is given by (3.37). Moreover, (3.39) and (3.40) yield the bound at the origin: $\forall e \in \mathrm{Supp}(\chi_\varepsilon)$,

$$|f(r,e)| \leq C_\varepsilon r^{5/2}, \tag{3.43}$$

and from (3.8) we get

$$f\left(r(e),e\right) = \sqrt{2}\int_0^{+\infty}(h(\tau)-\Pi h(e))(e-\phi_Q(\tau))_+^{1/2}\,\tau^2 d\tau = 0.$$

Hence, near the boundary $r=r(e)$, we estimate using (3.40): $\forall r \geq r_\varepsilon, \quad \forall e \in \mathrm{Supp}\chi_\varepsilon$,

$$|f(r,e)| = \left|\int_r^{r(e)}(h(\tau)-\Pi h(e))\sqrt{2(e-\phi_Q(\tau))}\tau^2 d\tau\right| \leq C_\varepsilon (e-\phi_Q(r))^{3/2}, \tag{3.44}$$

where we used $e-\phi_Q(r) \sim C(r(e)-r)$ deduced from $\phi_Q'(r) \gtrsim \phi_Q'(r_\varepsilon) > 0$.

We now integrate by parts from (3.42) using the cancellations at the boundary of $\mathcal{U}$ given by (3.43), (3.44) and the bounds (3.40), (3.39):

$$\begin{aligned}
&\int_{\mathbb{R}^6}\chi_\varepsilon|F'(e)|(h-\Pi h)^2 dxdv\\
&= 16\pi^2\int_{\phi_Q(0)}^0 \chi_\varepsilon|F'(e)|de\int_0^{r(e)}(h-\Pi h)Tf\sqrt{2(e-\phi_Q(r)}\,r^2dr\\
&= 16\pi^2\int_{\phi_Q(0)}^0 \chi_\varepsilon|F'(e)|de\int_0^{r(e)}(h-\Pi h)\partial_r f dr\\
&= -16\pi^2\int_{\widetilde{\mathcal{U}}}\chi_\varepsilon|F'(e)|f\partial_r h\, dedr.
\end{aligned}$$

We now use Cauchy-Schwarz together with the identity

$$\rho_Q(r) = \frac{8\pi\sqrt{2}}{3}\int_{\phi_Q(0)}^0 |F'(e)|(e-\phi_Q(r))_+^{3/2}de$$

to estimate:

$$\begin{aligned}
&\int_{\mathbb{R}^6}\chi_\varepsilon|F'(e)|(h-\Pi h)^2 dxdv\\
&\leq (4\pi)^{3/2}\|\nabla h\|_{L^2(\mathbb{R}^3)}\left(\int_0^{r(e_0)}\frac{dr}{r^2}\left(\int_{\phi_Q(r)}^{e_0}\chi_\varepsilon|F'(e)|f de\right)^2\right)^{1/2}\\
&\leq (4\pi)^{3/2}\|\nabla h\|_{L^2(\mathbb{R}^3)}\left[\frac{3}{8\pi\sqrt{2}}\int_0^{r(e_0)}\frac{\rho_Q(r)}{r^2}dr\int_{\phi_Q(r)}^{e_0}\chi_\varepsilon|F'(e)|\frac{f^2}{(e-\phi_Q(r))^{3/2}}de\right]^{1/2}\\
&= \|\nabla h\|_{L^2(\mathbb{R}^3)}\left(3\int\chi_\varepsilon\rho_Q(r)\frac{f^2}{r^4(\sqrt{2(e-\phi_Q)})^4}|F'(e)|dxdv\right)^{1/2}.
\end{aligned} \tag{3.45}$$

We now claim the following Hardy type control:

$$3\int \chi_\varepsilon \left(\rho_Q + \frac{\phi_Q'}{r}\right) \frac{f^2}{r^4(\sqrt{2(e-\phi_Q)})^4}|F'(e)|dxdv \leq \int \chi_\varepsilon |F'(e)|(Tf)^2 dxdv. \tag{3.46}$$

Assume (3.46), then (3.42) and (3.45) yield:

$$\int_{\mathbb{R}^6} \chi_\varepsilon |F'(e)|(h-\Pi h)^2 dxdv + 3\int \chi_\varepsilon \frac{\phi_Q'}{r} \frac{f^2}{r^4(\sqrt{2(e-\phi_Q)})^4}|F'(e)|dxdv \leq \int_{\mathbb{R}^3} |\nabla h|^2 dx.$$

Letting $\varepsilon \to 0$ now yields $(\mathcal{L}h, h) \geq 0$. Moreover $(\mathcal{L}h, h) = 0$ implies $f = 0$ in $\tilde{\mathcal{U}}$, thus $h(r) = \Pi h(e)$ on $\tilde{\mathcal{U}}$ and $(\mathcal{L}h, h) = \int |\nabla h|^2 = 0$ and thus h is zero. This concludes the proof of (3.36).

Step 5. Hardy type control.

The Hardy control (3.46) is a consequence of the convexity estimate (3.38). Indeed, let g be a given smooth function in $\tilde{\mathcal{U}}$, let $f = qg$ and compute:

$$\begin{aligned}
(Tf)^2 &= (gTq + qTg)^2 = g^2(Tq)^2 + g(Tg)T(q^2) + q^2(Tg)^2 \\
&= g^2(Tq)^2 + T(q^2 g(Tg)) - q^2((Tg)^2 + gT^2 g) + q^2(Tg)^2 \\
&\geq T(q^2 gTg) - q^2 gT^2 g = T(q^2 gTg) - \frac{T^2 g}{g} f^2.
\end{aligned} \tag{3.47}$$

We now look for g such that

$$-\frac{T^2 g}{g} \geq \frac{3}{\left[r\sqrt{2(e-\phi_Q)}\right]^4}\left(\rho_Q + \frac{\phi_Q'}{r}\right). \tag{3.48}$$

Let

$$u = \sqrt{2(e-\phi_Q)} \text{ so that } Tg(r,u) = \frac{\partial_r g}{r^2 u} - \frac{\phi_Q'}{r^2 u^2}\partial_u g,$$

and thus:

$$\begin{aligned}
T^2 g &= \frac{1}{r^4 u^2}\left[\partial^2_{rr} g - \frac{2}{r}\partial_r g\right] - \frac{\rho_Q}{r^4 u^3}\partial_u g + \frac{\phi_Q'}{r^4 u^4}\left[\partial_r g + 4\frac{u}{r}\partial_u g - 2u\partial^2_{ru} g\right] \\
&+ \frac{(\phi_Q')^2}{r^4 u^5}\left[u\partial^2_{uu} g - 2\partial_u g\right],
\end{aligned}$$

where we used the Poisson equation satisfied by ϕ_Q. The choice $g = r^3 u^3$ yields:

$$-\frac{T^2 g}{g} = \frac{3}{r^4 u^4}\left(\rho_Q + \frac{\phi_Q'}{r}\right).$$

Injecting this into (3.47) and integrating on $\tilde{\mathcal{U}}$ yields:

$$\begin{aligned}
\int \chi_\varepsilon |F'(e)|(Tf)^2 dxdv &\geq 3\int \chi_\varepsilon \left(\rho_Q + \frac{\phi_Q'}{r}\right)\frac{f^2}{r^4(\sqrt{2(e-\phi_Q)})^4}|F'(e)|dxdv \\
&+ \int \chi_\varepsilon |F'(e)| T\left(f^2 \frac{Tg}{g}\right) dxdv.
\end{aligned}$$

The bounds (3.43), (3.44) now justify the integration by parts

$$\int \chi_\varepsilon |F'(e)| T\left(f^2 \frac{Tg}{g}\right) dxdv = 16\pi^2 \int_{\phi_Q(0)}^{e_0} \chi_\varepsilon |F'(e)| de \int_0^{r(e)} \partial_r \left(f^2 \frac{Tg}{g}\right) dr = 0$$

and (3.46) follows. This concludes the proof of Proposition 3.6. □

3.3. **Proof of Proposition 3.1.** We are now in position to conclude the proof of Proposition 3.1 which is a classical consequence of modulation theory coupled with the coercivity estimate (3.29).

Step 1. Implicit function theorem

Given $\alpha > 0$, let $U_\alpha = \{\phi \in \dot{H}^1(\mathbb{R}^3); \quad \|\nabla\phi - \nabla\phi_Q\|_{L^2} < \alpha\}$, and for $\phi \in \dot{H}^1$, $z \in \mathbb{R}^3$, define

$$\varepsilon_z(x) = \phi(x+z) - \phi_Q(x). \tag{3.49}$$

We claim that there exists $\overline{\alpha} > 0$, a neighbourhood V of the origin in $\mathbb{R}^3$ and a unique C^1 map $U_{\overline{\alpha}} \to V$ such that if $\phi \in U_{\overline{\alpha}}$, there is a unique $z \in V$ such that ε_z defined as in (3.49) satisfies

$$\forall 1 \le i \le 3, \qquad \int_{\mathbb{R}^3} \varepsilon_z \Delta(\partial_{x_i}\phi_Q) dx = 0. \tag{3.50}$$

Moreover, there exists a constant $C > 0$ such that if $u \in U_{\overline{\alpha}}$, then

$$|z| + \|\nabla\varepsilon_z\|_{L^2} \le C\|\nabla\phi - \nabla\phi_Q\|_{L^2}. \tag{3.51}$$

Indeed, we define the following functionals of (ϕ, z):

$$\mathcal{F}_i(\phi, z) = \int_{\mathbb{R}^3} \varepsilon_z \Delta(\partial_{x_i}\phi_Q) dx, \quad 1 \le i \le 3$$

and obtain at the point $(\phi, z) = (\phi_Q, 0)$,

$$\frac{\partial \mathcal{F}_i}{\partial z_j} = -\delta_{ij} \|\nabla \partial_{x_i}\phi_Q\|_{L^2}^2.$$

The Jacobian of the above functional is $-\Pi_{i=1}^3 \|\nabla\partial_{x_i}\phi_Q\|_{L^2}^2 < 0$, hence the implicit function theorem ensures the existence of $\overline{\alpha} > 0$, a neighbourhood V of the origin in $\mathbb{R}^3$ and a unique C^1 map $U_{\overline{\alpha}} \to V$ such that (3.50) holds.

Step 2. Conclusion

Let $\phi \in \mathcal{X}$ with

$$\inf_{z\in\mathbb{R}^3} (\|\phi - \phi_Q(\cdot - z)\|_{L^\infty} + \|\nabla\phi - \nabla\phi_Q(\cdot - z)\|_{L^2}) < \delta_0$$

for some small enough $\delta_0 > 0$ to be chosen later. Then there exists z_1 such that

$$\|\phi - \phi_Q(\cdot - z_1)\|_{L^\infty} + \|\nabla\phi - \nabla\phi_Q(\cdot - z_1)\|_{L^2} < 2\delta_0. \tag{3.52}$$

For $\delta_0 \le \frac{\overline{\alpha}}{2}$ small enough, we may apply Step 1 to $\phi(x + z_1)$ and find $z_2 \in \mathbb{R}^3$, $\varepsilon \in \dot{H}^1$ satisfiying the orthogonality conditions (3.50) and the smallness (3.51) such that $\phi(x + z_1) = (\phi_Q + \varepsilon)(x - z_2)$, or equivalently

$$\phi(x) = (\phi_Q + \varepsilon)(x - z_\phi), \quad z_\phi = z_1 + z_2. \tag{3.53}$$

In fact, for δ_0 small enough, a shift z_ϕ satisfying (3.53), the orthogonality conditions (3.50) and the smallness condition (3.51), is unique. This is a simple consequence of the uniqueness of the pair (z_2, ε_{z_2}) in Step 1. The continuity of the map $\phi \to z_\phi$ from $(\dot{H}^1, \|\cdot\|_{\dot{H}^1}) \to \mathbb{R}^3$ then follows. Moreover, from (3.51), (3.52):

$$\begin{aligned}
\|\varepsilon\|_{L^\infty} &= \|\phi(x+z_1+z_2) - \phi_Q(x)\|_{L^\infty} \\
&\leq \|\phi(x+z_1+z_2) - \phi_Q(x+z_2)\|_{L^\infty} + \|\phi_Q(x+z_2) - \phi_Q(x)\|_{L^\infty} \\
&\leq \|\phi(x+z_1) - \phi_Q(x)\|_{L^\infty} + C|z_2| \\
&\leq \|\phi(x+z_1) - \phi_Q(x)\|_{L^\infty} + C\|\nabla\phi(x+z_1) - \nabla\phi_Q(x)\|_{L^2} \leq C\delta_0.
\end{aligned}$$

Provided δ_0 small enough, we may now apply the Taylor expansion (3.6) together with the coercivity (3.29) and the orthogonality conditions (3.50), and obtain from the translation invariance of $\mathcal{J}$:

$$\begin{aligned}
\mathcal{J}(\phi) - \mathcal{J}(\phi_Q) &= \mathcal{J}(\phi_Q + \varepsilon) - \mathcal{J}(\phi_Q) \geq c_0\|\nabla\varepsilon\|_{L^2}^2 - \eta(\|\varepsilon\|_{L^\infty})\|\nabla\varepsilon\|_{L^2}^2 \geq \frac{c_0}{2}\|\nabla\varepsilon\|_{L^2}^2 \\
&\geq \frac{c_0}{2}\|\nabla\phi - \nabla\phi_Q(\cdot - z_\phi)\|_{L^2}^2.
\end{aligned}$$

This concludes the proof of Proposition 3.1.

4. **Compactness of local minimizing sequences of the Hamiltonian**

The aim of this section is to prove the following compactness result which is the heart of the proof of Theorem 1.2.

Proposition 4.1 (Compactness of local minimizing sequences)**.** *Let $\delta_0 > 0$ be as in Proposition 3.1. Let $\phi \to z_\phi$ the continuous map from $(\dot{H}^1, \|\cdot\|_{\dot{H}^1}) \to \mathbb{R}^3$ build in Proposition 3.1. Let f_n be a sequence of functions of $\mathcal{E}$, bounded in L^∞, such that*

$$\inf_{z\in\mathbb{R}^3} \left(\|\phi_{f_n} - \phi_Q(\cdot - z)\|_{L^\infty} + \|\nabla\phi_{f_n} - \nabla\phi_Q(\cdot - z)\|_{L^2}\right) < \delta_0, \tag{4.1}$$

and

$$\limsup_{n\to+\infty} \mathcal{H}(f_n) \leq \mathcal{H}(Q), \qquad f_n^* \to Q^* \text{ in } L^1(\mathbb{R}_+) \quad \text{as } n \to +\infty. \tag{4.2}$$

Then

$$\int (1+|v|^2)|f_n - Q(x - z_{\phi_{f_n}})| \to 0 \quad \text{as} \quad n \to +\infty. \tag{4.3}$$

Proof. Step 1: *Compactness of the potential*

We first claim the following quantitative lower bound which generalizes the monotonicity formula (2.22): let $f \in \mathcal{E}$ such that ϕ_f satisfies (3.2), let z_{ϕ_f} given by Proposition 3.1, then

$$\mathcal{H}(f) - \mathcal{H}(Q) + \|\phi_f\|_{L^\infty}\|f^* - Q^*\|_{L^1} \geq c_0\|\nabla\phi_f - \nabla\phi_Q(\cdot - z_{\phi_f})\|_{L^2}^2. \tag{4.4}$$

Indeed,

$$\mathcal{H}(f) - \mathcal{H}(Q) \geq \mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_Q) = \mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_f) + \mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q), \tag{4.5}$$

where we have used that $\mathcal{H}(Q) = \mathcal{J}(\phi_Q)$. Now, we recall that

$$\mathcal{J}_{f^*}(\phi) = \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi\right) f^{*\phi}(x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx.,$$

and deduce from the change of variables formula (2.10) that

$$\mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_f) = \int_0^{+\infty} a_{\phi_f}^{-1}(s)\left(f^*(s) - Q^*(s)\right) ds.$$

Since $|a_{\phi_f}^{-1}(s)| \leq -\min\,\phi_f = \|\phi_f\|_{L^\infty}$, we have

$$\mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_f) \geq -\|\phi_f\|_{L^\infty}\|f^* - Q^*\|_{L^1}.$$

Inserting this estimate into (4.5) and using Proposition (4.4) yields (4.4) .

Let us now consider a sequence $f_n \in \mathcal{E}$ satisfying the assumptions of Proposition 4.1, then (4.4) applied to f_n ensures:

$$\|\nabla\phi_{f_n}(.+z_{\phi_{f_n}}) - \nabla\phi_Q\|_{L^2} \to 0, \qquad \text{as } n \to \infty. \tag{4.6}$$

Step 2: Strong convergence of f_n to Q

To ease notations, we shall still denote by f_n the translated function $f_n(.+z_{\phi_{f_n}}, v)$. We then observe the identity:

$$\mathcal{H}(f_n) - \mathcal{H}(Q) + \frac{1}{2}\|\nabla\phi_{f_n} - \nabla\phi_Q\|_{L^2}^2 = \int_{\mathbb{R}^6}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)(f_n - Q)dxdv \tag{4.7}$$

which implies, from (4.2) and (4.6), that

$$\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)(f_n - Q)dxdv \to 0, \qquad \text{as } n \to \infty. \tag{4.8}$$

Now, we observe from the change of variables (2.10) that

$$\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)(Q - f_n^{*\phi_Q})dxdv = \int_0^{+\infty} a_{\phi_Q}^{-1}(s)\left(Q^*(s) - f_n^*(s)\right)ds.$$

Since $|a_{\phi_Q}^{-1}(s)| \leq -\phi_Q(0)$ we get

$$\left|\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)(Q - f_n^{*\phi_Q})dxdv\right| \leq |\phi_Q(0)|\|Q^* - f_n^*\|_{L^1},$$

which implies from (4.2) that

$$\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)(Q - f_n^{*\phi_Q})dxdv \to 0, \qquad \text{as } n \to \infty. \tag{4.9}$$

Summing (4.8) and (4.9) yields

$$T_n = \int_{\mathbb{R}^6}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)(f_n - f_n^{*\phi_Q})dxdv \to 0, \qquad \text{as } n \to \infty. \tag{4.10}$$

We now argue as in the proof of (2.25), and write (4.10) in the following equivalent form

$$T_n = \int_{t=0}^{+\infty} dt\left(\int_{S_1^n(t)}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)dxdv - \int_{S_2^n(t)}\left(\frac{|v|^2}{2} + \phi_Q(x)\right)dxdv\right) \to 0, \tag{4.11}$$

where

$$S_1^n(t) = \{(x,v) \in \mathbb{R}^6, f_n^{*\phi_Q}(x,v) \leq t < f_n(x,v)\},$$
$$S_2^n(t) = \{(x,v) \in \mathbb{R}^6, f_n(x,v) \leq t < f_n^{*\phi_Q}((x,v)\}.$$

From (2.15), we have

$$\frac{|v|^2}{2} + \phi_Q(x) \geq (f_n^* \circ a_{\phi_Q})^{-1}(t), \qquad \forall (x,v) \in S_1^n(t).$$

Thus

$$T_n \geq \int_{t=0}^{+\infty} dt \left(\int_{S_1^n(t)} (f_n^* \circ a_{\phi_Q})^{-1}(t) dx dv - \int_{S_2^n(t)} \left(\frac{|v|^2}{2} + \phi_Q(x) \right) dx dv \right). \quad (4.12)$$

As a consequence of the equimeasurability of $f_n^{*\phi_Q}$ and f_n, we know that

$$\mathrm{meas}(S_1^n(t)) = \mathrm{meas}(S_2^n(t)),$$

and then (4.12) gives:

$$T_n \geq \int_{t=0}^{+\infty} dt \int_{S_2^n(t)} \left[(f_n^* \circ a_{\phi_Q})^{-1}(t) - \left(\frac{|v|^2}{2} + \phi_Q(x) \right) \right] dx dv. \quad (4.13)$$

From (2.14), we have

$$(f_n^* \circ a_{\phi_Q})^{-1}(t) \geq \frac{|v|^2}{2} + \phi_Q(x), \qquad \forall (x,v) \in S_2^n(t)$$

Thus, from (4.10) and (4.13), we get

$$A_n = \left[(f_n^* \circ a_{\phi_Q})^{-1}(t) - \left(\frac{|v|^2}{2} + \phi_Q(x) \right) \right] \mathbb{1}_{S_2^n(t)}(x,v) \to 0 \quad (4.14)$$

as $n \to +\infty$, for almost every $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ (up to a subsequence). We now claim that this implies

$$B_n = \left[(Q^* \circ a_{\phi_Q})^{-1}(t) - \left(\frac{|v|^2}{2} + \phi_Q(x) \right) \right] \mathbb{1}_{\overline{S}_2^n(t)}(x,v) \to 0, \quad (4.15)$$

as $n \to +\infty$, for almost every $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, where

$$\overline{S}_2^n(t) = \{(x,v) \in \mathbb{R}^6, f_n(x,v) \leq t < Q(x,v)\}.$$

To prove (4.15), we write

$$S_2^n = \left(S_2^n \backslash \overline{S}_2^n \right) \cup \left(S_2^n \cap \overline{S}_2^n \right), \qquad \overline{S}_2^n = \left(\overline{S}_2^n \backslash S_2^n \right) \cup \left(S_2^n \cap \overline{S}_2^n \right),$$

and get

$$\begin{aligned} A_n - B_n \;=\; & \left(\frac{|v|^2}{2} + \phi_Q(x) - (Q^* \circ a_{\phi_Q})^{-1}(t) \right) \mathbb{1}_{\overline{S}_2^n(t) \backslash S_2^n(t)} \\ + & \left((f_n^* \circ a_{\phi_Q})^{-1}(t) - \frac{|v|^2}{2} - \phi_Q(x) \right) \mathbb{1}_{S_2^n(t) \backslash \overline{S}_2^n(t)} \\ + & \left[(f_n^* \circ a_{\phi_Q})^{-1}(t) - (Q^* \circ a_{\phi_Q})^{-1}(t) \right] \mathbb{1}_{S_2^n(t) \cap \overline{S}_2^n(t)} . \end{aligned} \quad (4.16)$$

We shall now examine the behavior of each of these terms when $n \to \infty$. We first observe that for all $g, h \in L^1(\mathbb{R}^6)$ with $g \geq 0$, $h \geq 0$, we have

$$\int_0^{+\infty} \mathrm{meas}\,\{g \leq t < h\}\, dt = \int_{\mathbb{R}^6} (h-g)_+ dx dv \leq \|g - h\|_{L^1}, \quad (4.17)$$

and thus from (4.2):

$$\int_0^{+\infty} \mathrm{meas}(S_2^n(t) \backslash \overline{S}_2^n(t)) dt \leq \|f_n^{*\phi_Q} - Q\|_{L^1} = \|f_n^* - Q^*\|_{L^1} \quad \to 0.$$

Using in addition the estimate

$$\left| (f_n^* \circ a_{\phi_Q})^{(-1)}(t) \right| \leq |\phi_Q(0)|,$$

we deduce that the first two terms of the decomposition (4.16) go to 0 almost everywhere when n goes to infinity. We now treat the third term and show that, for all (t,x,v),

$$\liminf_{n\to\infty}\left[(f_n^*\circ a_{\phi_Q})^{-1}(t)-(Q^*\circ a_{\phi_Q})^{-1}(t)\right]\mathbb{1}_{S_2^n(t)\cap\overline{S}_2^n(t)}\geq 0. \tag{4.18}$$

To prove (4.18), we first use the strong L^1 convergence (4.2) to get

$$\forall e\in(\phi_Q(0),0)\backslash A,\quad f_n^*(a_{\phi_Q}(e))\to Q^*(a_{\phi_Q}(e)), \tag{4.19}$$

where A is a zero-measure set in $\mathbb{R}$, and claim that the monotonicity of f_n^* in e and the *continuity* of Q^* in e ensure:

$$\forall e\in(\phi_Q(0),0),\quad f_n^*(a_{\phi_Q}(e))\to Q^*(a_{\phi_Q}(e)). \tag{4.20}$$

Indeed, let $e\in(\phi_Q(0),0)$, and $(x_p,y_p)\in(\phi_Q(0),0)\backslash A$ such that $x_p\leq e\leq y_p$ and $x_p\to e$, $y_p\to e$. As $f_n^*\circ a_{\phi_Q}$ is decreasing, we have

$$f_n^*\circ a_{\phi_Q}(y_p)\leq f_n^*\circ a_{\phi_Q}(e)\leq f_n^*\circ a_{\phi_Q}(x_p).$$

From (4.19) we then get

$$Q^*(a_{\phi_Q}(y_p))\leq\liminf_{n\to\infty}f_n^*\circ a_{\phi_Q}(e)\leq\limsup_{n\to\infty}f_n^*\circ a_{\phi_Q}(e)\leq Q^*(a_{\phi_Q}(x_p)).$$

Now we pass to the limit $p\to\infty$ and use the continuity of $Q^*\circ a_{\phi_Q}$ to get the claim (4.20).

Now, we turn back to the proof of (4.18) and fix (t,x,v). Take then any e such that

$$\phi_Q(0)<e<0,\quad\text{and}\quad Q^*(a_{\phi_Q}(e))>t, \tag{4.21}$$

which implies from (4.20):

$$f_n^*(a_{\phi_Q}(e))>t,$$

for n large enough. Using the definition of the pseudo inverse given in Lemma 2.5, we then obtain $e\leq(f_n^*\circ a_{\phi_Q})^{-1}(t)$ for n large enough, and hence

$$e\leq\liminf_{n\to\infty}(f_n^*\circ a_{\phi_Q})^{-1}(t).$$

Since this equality holds for all e satisfying (4.21), we conclude from the definition of the pseudo inverse $(Q^*\circ a_{\phi_Q})^{-1}(t)$ that

$$\liminf_{n\to\infty}(f_n^*\circ a_{\phi_Q})^{-1}(t)\geq(Q^*\circ a_{\phi_Q})^{-1}(t),$$

which yields (4.18).

We now turn to the decomposition (4.16) and get from (4.18)

$$\liminf(A_n-B_n)\geq 0,\quad\text{for almost all }(t,x,v).$$

Finally, observing that $B_n\geq 0$ and using (4.14), we conclude that (4.15) holds true. Observe now that

$$t<Q(x,v)\ \text{ implies }\ Q(x,v)=F\left(\frac{|v|^2}{2}+\phi_Q(x)\right)>t.$$

By the assumptions of Theorem 1.2, $e\to F(e)$ is continuous and strictly decreasing with respect to $e=\frac{|v|^2}{2}+\phi_Q(x)$ for $(x,v)\in\{Q>0\}$, and thus:

$$t<Q(x,v)\ \text{ implies }\ (Q^*\circ a_{\phi_Q})^{(-1)}(t)-\frac{|v|^2}{2}-\phi_Q(x)>0.$$

We then deduce from (4.15) and from $\overline{S}_2^n(t) = \{(x,v) : f_n(x,v) \le t < Q(x,v)\}$ that, up to a subsequence extraction,

$$\mathbb{1}_{\{f_n \le t < Q\}} \quad \to 0, \quad \text{as} \quad n \to \infty,$$

for almost every $(t,x,v) \in \mathbb{R}_+^* \times \mathbb{R}^6$. Now from $\mathbb{1}_{\{f_n \le t < Q\}} \le \mathbb{1}_{\{t<Q\}}$ and

$$\int_0^\infty \int_{\mathbb{R}^6} \mathbb{1}_{\{t<Q\}} dxdvdt = \|Q\|_{L^1} < +\infty.$$

we may apply the dominated convergence theorem to conclude:

$$\int_0^\infty \int_{\mathbb{R}^6} \mathbb{1}_{\{f_n \le t < Q\}} dxdvdt \to 0 \quad \text{as} \quad n \to \infty.$$

Injecting this into (4.17) yields

$$\int_{\mathbb{R}^6} (Q - f_n)_+ dxdv \to 0 \quad \text{as} \quad n \to \infty. \tag{4.22}$$

Now we claim that, using $f_n^* \to Q^*$ in L^1, this implies

$$\int_{\mathbb{R}^6} (f_n - Q)_+ dxdv \to 0 \quad \text{as} \quad n \to \infty. \tag{4.23}$$

Indeed, we write

$$\begin{aligned}
\int_{\mathbb{R}^6} (f_n - Q)_+ dxdv &\le \int_{\mathbb{R}^6} (f_n - f_n^{*\phi_Q})_+ dxdv + \int_{\mathbb{R}^6} (f_n^{*\phi_Q} - Q)_+ dxdv \\
&\le \int_0^{+\infty} \text{meas}\left\{ f_n^{*\phi_Q} \le t < f_n \right\} dt + \|f_n^{*\phi_Q} - Q\|_{L^1} \\
&= \int_0^{+\infty} \text{meas}\left\{ f_n \le t < f_n^{*\phi_Q} \right\} dt + \|f_n^* - Q^*\|_{L^1} \\
&= \int_{\mathbb{R}^6} (f_n^{*\phi_Q} - f_n)_+ dxdv + \|f_n^* - Q^*\|_{L^1} \\
&\le \int_{\mathbb{R}^6} (Q - f_n)_+ dxdv + \int_{\mathbb{R}^6} (f_n^{*\phi_Q} - Q)_+ dxdv + \|f_n^* - Q^*\|_{L^1} \\
&\le \int_{\mathbb{R}^6} (Q - f_n)_+ dxdv + 2\|f_n^* - Q^*\|_{L^1}
\end{aligned}$$

where we repeatedly used (4.17) and the fact that $f_n^{*\phi_Q} \in \text{Eq}(f_n)$ implies

$$\forall t > 0, \quad \text{meas}\left\{ f_n^{*\phi_Q} \le t < f_n \right\} = \text{meas}\left\{ f_n \le t < f_n^{*\phi_Q} \right\}.$$

As $f_n^* \to Q^*$ in L^1, we then conclude that (4.22) implies (4.23). Finally adding (4.22) and (4.23) gives

$$\|f_n - Q\|_{L^1} \to 0 \text{ as } \ n \to +\infty.$$

Furthermore, (4.2) and the strong convergence $\nabla\phi_{f_n} \to \nabla\phi_Q$ in L^2 imply:

$$\int_{\mathbb{R}^6} |v|^2 f_n \to \int_{\mathbb{R}^6} |v|^2 Q \quad \text{as } n \to +\infty,$$

Together with the a.e. convergence of f_n, this yields the strong L^1 convergence of $|v|^2 f_n$ to $|v|^2 Q$. Note that the uniqueness of the limit now implies the convergence of all the sequence f_n which completes the proof of (4.3).

This concludes the proof of Proposition 4.1.

□

5. Non linear stability of Q

We now turn to the proof of the nonlinear stability result stated in Theorem 1.2, which is a direct consequence of Proposition 4.1 and the known regularity of weak solutions to the Vlasov-Poisson system.

Proof of Theorem 1.2.

Step 1. Continuity claim for weak solutions

Let $f_0 \in \mathcal{E}$ and let $f(t) \in \mathcal{E}$ be a corresponding weak solution to (1.1). By the properties of weak solutions of the Vlasov-Poisson system [7, 8], we have

$$\forall t \geq 0, \qquad f(t) \in \mathrm{Eq}(f_0), \qquad \mathcal{H}(f(t)) \leq \mathcal{H}(f_0). \tag{5.1}$$

We claim:

$$\phi_f \in \mathcal{C}([0,+\infty), L^\infty(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)). \tag{5.2}$$

Note that this implies from Proposition 3.1 that

$$t \to z_{\phi_f(t)} \text{ is continuous.} \tag{5.3}$$

To prove (5.2), recall that $f \in \mathcal{C}([0,+\infty), L^1)$ (see [7, 8]) and hence (5.2) follows from: $\forall f, g \in \mathcal{E}$,

$$\|\nabla \phi_f - \nabla \phi_g\|_{L^2} + \|\phi_f - \phi_g\|_{L^\infty} \leq C_{f,g} \, \|f-g\|_{L^1}^{1/6}, \tag{5.4}$$

where $C_{f,g}$ only depends on $\|f\|_{\mathcal{E}}$ and $\|g\|_{\mathcal{E}}$. Let us prove (5.4). First, from Hölder:

$$\forall x \in \mathbb{R}^3, \quad |\phi_f - \phi_g|(x) = \left| \int_{\mathbb{R}^3} \frac{\rho_f(y) - \rho_g(y)}{4\pi |x-y|} dy \right| \lesssim \|\rho_f - \rho_g\|_{L^{5/3}}^{5/6} \|\rho_f - \rho_g\|_{L^1}^{1/6},$$

and from Hardy-Littlewood-Sobolev:

$$\|\nabla \phi_f - \nabla \phi_g\|_{L^2} \lesssim \|\rho_f - \rho_g\|_{L^{6/5}} \lesssim \|\rho_f - \rho_g\|_{L^1}^{7/12} \|\rho_f - \rho_g\|_{L^{5/3}}^{5/12}.$$

Second, by interpolation,

$$\|\rho_f - \rho_g\|_{L^{5/3}} \lesssim \|f-g\|_{L^\infty}^{2/5} \||v|^2 (f-g)\|_{L^1}^{3/5} \leq C_{f,g}.$$

Since $\|\rho_f - \rho_g\|_{L^1} \leq \|f-g\|_{L^1}$, this yields (5.4) and the continuity (5.2) of ϕ_f follows.

Step 2: Conclusion.

An equivalent reformulation of Proposition 4.1 is the following: for all $\varepsilon > 0$ small enough, there exists $\eta > 0$ such that if $f \in \mathcal{E}$ with

$$\|f^* - Q^*\|_{L^1} \leq \eta, \quad \|f\|_{L^\infty} \leq \|Q\|_{L^\infty} + M, \quad \mathcal{H}(f) \leq \mathcal{H}(Q) + \eta \tag{5.5}$$

and

$$\inf_{z \in \mathbb{R}^3} (\|\phi_f - \phi_Q(\cdot - z)\|_{L^\infty} + \|\nabla \phi_f - \nabla \phi_Q(\cdot - z)\|_{L^2}) < \delta_0, \tag{5.6}$$

then

$$\|(1+|v|^2)(f - Q(\cdot - z_{\phi_f}))\|_{L^1} \leq \varepsilon. \tag{5.7}$$

Let $\varepsilon > 0$ and let $\eta > 0$ be the associated constant. We consider an initial data $f_0 \in \mathcal{E}$ with

$$\|f_0 - Q\|_{L^1} < \eta, \quad \|f_0\|_{L^\infty} \leq \|Q\|_{L^\infty} + M \quad \text{and} \quad \mathcal{H}(f_0) \leq \mathcal{H}(Q) + \eta$$

and a corresponding weak solution $f(t)$ of (1.1). Observe that, by the contractivity of the symmetric rearrangement in L^1 (see [30]), we have

$$\|f_0^* - Q^*\|_{L^1} = \|f_0 - Q\|_{L^1} \leq \eta. \tag{5.8}$$

Moreover, (5.4) implies that, for η small enough,

$$\|\nabla\phi_{f_0} - \nabla\phi_Q(\cdot - z_{\phi_{f_0}})\|_{L^2} + \|\phi_f(0) - \phi_Q(\cdot - z_{\phi_{f_0}})\|_{L^\infty} \leq \frac{\delta_0}{2}.$$

From (5.1), we first deduce that the corresponding solution $f(t)$ of (1.1) satisfies (5.5) for all $t \geq 0$. Hence, if we prove that

$$\forall t \geq 0, \qquad \|\nabla\phi_f(t) - \nabla\phi_Q(\cdot - z_{\phi_f(t)})\|_{L^2} + \|\phi_f(t) - \phi_Q(\cdot - z_{\phi_f(t)})\|_{L^\infty} < \delta_0, \tag{5.9}$$

then (5.7) holds true for all $t \geq 0$, which is nothing but (1.17). Now (5.9) follows for $\eta > 0$ small enough from a straightforward bootstrap argument using the continuity (5.2), (5.3) and the bound (5.4). The proof of Theorem 1.2 is complete. ☐

Appendix A. Proof of Lemma 3.2

Proof. The proof is similar to the one in [29] and we briefly sketch the argument for the sake of completeness. Recall that the set $\mathcal{X}$ is convex, thus $\phi + \lambda h = (1-\lambda)\phi(x) + \lambda\widetilde{\phi}(x)$ belongs to $\mathcal{X}$ for all $\lambda \in [0,1]$ and $a_{\phi+\lambda h}$ is well-defined.

Step 1. Proof of (i).

Let $e_1 < 0$ be fixed. For all $e \leq e_1$, we consider the domain

$$D_{\phi+\lambda h}(e) = \left\{x \in \mathbb{R}^3 : \quad (\phi + \lambda h)(x) < e\right\}.$$

From (2.4), we have

$$a_{\phi+\lambda h}(e) = \frac{8\pi\sqrt{2}}{3}\int_{D_{\phi+\lambda h}(e)} (e - \phi(x) - \lambda h(x))_+^{3/2}\, dx.$$

We clearly have

$$D_{\phi+\lambda h}(e) \subset D_\phi(e_1) \cup D_{\widetilde{\phi}}(e_1).$$

Since $\phi(x)$ and $\widetilde{\phi}(x)$ go to zero at the infinity, $D_\phi(e_0)$ and $D_{\widetilde{\phi}}(e_0)$ are bounded. Hence for all $e \leq e_1$, $D_{\phi+\lambda h}(e)$ is contained in a fixed compact domain of $\mathbb{R}^3$. As in addition the functions ϕ and $\widetilde{\phi}$ are continuous, the Lebesgue dominated convergence theorem may thus be applied to obtain the continuity and the differentiability of $a_{\phi+\lambda h}(e)$ with respect to λ and e. The expression (3.4) follows.

Step 2. Continuity of the function $\lambda \mapsto a^{-1}_{\phi+\lambda h}(s)$.

Let $s \in \mathbb{R}_+^*$. In this step, we prove that the function $\lambda \mapsto a^{-1}_{\phi+\lambda h}(s)$ is continuous. To this aim, we consider a sequence $\lambda_n \in [0,1]$ converging to λ_0 as $n \to +\infty$ and prove that $a^{-1}_{\phi+\lambda_n h}(s)$ converges to $a^{-1}_{\phi+\lambda_0 h}(s)$. We set

$$e_n = a^{-1}_{\phi+\lambda_n h}(s) \in (\min(\phi + \lambda_n h), 0) \subset (\min\phi + \min\widetilde{\phi}, 0).$$

Hence, up to a subsequence, e_n converges to some $e \leq 0$ as $n \to +\infty$.

Let us prove that $e<0$ by contradiction. Assume that $e=0$. For n large enough such that $\frac{\lambda_0}{2}\le\lambda_n\le\frac{1+\lambda_0}{2}$, we have

$$\begin{aligned} s=a_{\phi+\lambda_n h}(e_n) &= \frac{8\pi\sqrt{2}}{3}\int_{\mathbb{R}^3}\left(e_n-(1-\lambda_n)\phi(x)-\lambda_n\widetilde{\phi}(x)\right)_+^{3/2}dx\\ &\ge \frac{8\pi\sqrt{2}}{3}\int_{\mathbb{R}^3}\left(e_n-\frac{1-\lambda_0}{2}\phi(x)-\frac{\lambda_0}{2}\widetilde{\phi}(x)\right)_+^{3/2}dx=a_\psi(e_n), \end{aligned}$$

where $\psi(x)=\frac{1-\lambda_0}{2}\phi(x)+\frac{\lambda_0}{2}\widetilde{\phi}(x)$. From Lemma 2.2, we have $\lim_{t\to 0-}a_\psi(t)=+\infty$, which implies that $\lim_{n\to+\infty}a_\psi(e_n)=+\infty$, a contradiction.

Therefore, we have $e_n\to e<0$. The continuity of $(\lambda,e)\mapsto a_{\phi+\lambda h}(e)$ proved in Step 1 gives that

$$s=a_{\phi+\lambda_n h}(e_n)\to a_{\phi+\lambda_0 h}(e)\text{ as }n\to+\infty.$$

Thus $e=a^{-1}_{\phi+\lambda_0 h}(s)$. This ends the proof of (ii).

Step 3. Differentiability of $\lambda\mapsto a^{-1}_{\phi+\lambda h}(s)$.

Denoting $\phi_0=\phi+\lambda_0 h$ and $\phi_\lambda=\phi+\lambda h$, we write

$$\begin{aligned} \frac{a^{-1}_{\phi_\lambda}(s)-a^{-1}_{\phi_0}(s)}{\lambda} &= \frac{a^{-1}_{\phi_\lambda}(s)-a^{-1}_{\phi_0}(s)}{a_{\phi_0}(a^{-1}_{\phi_\lambda}(s))-a_{\phi_0}(a^{-1}_{\phi_0}(s))}\,\frac{a_{\phi_0}(a^{-1}_{\phi_\lambda}(s))-a_{\phi_0}(a^{-1}_{\phi_0}(s))}{\lambda}\\ &= A_1(\lambda)\,A_2(\lambda), \end{aligned} \tag{A.1}$$

where we have set

$$A_1(\lambda)=\frac{a^{-1}_{\phi_\lambda}(s)-a^{-1}_{\phi_0}(s)}{a_{\phi_0}(a^{-1}_{\phi_\lambda}(s))-a_{\phi_0}(a^{-1}_{\phi_0}(s))},\quad A_2(\lambda)=\frac{a_{\phi_0}(a^{-1}_{\phi_\lambda}(s))-a_{\phi_\lambda}(a^{-1}_{\phi_\lambda}(s))}{\lambda},$$

and where we simply used that $a_{\phi_0}(a^{-1}_{\phi_0}(s))=s=a_{\phi_\lambda}(a^{-1}_{\phi_\lambda}(s))$. Let us examinate separately the convergence of the two factors A_1 and A_2 in (A.1). From Step 2, we have

$$\lim_{\lambda\to 0}a^{-1}_{\phi_\lambda}(s)=a^{-1}_{\phi_0}(s),$$

hence

$$\lim_{\lambda\to 0}A_1(\lambda)=\frac{1}{a'_{\phi_0}(a^{-1}_{\phi_0}(s))}=\frac{1}{4\pi\sqrt{2}\displaystyle\int_{\mathbb{R}^3}(a^{-1}_{\phi_0}(s)-\phi_0(x))_+^{1/2}dx}. \tag{A.2}$$

Now, (3.4) and Step 2 imply:

$$\lim_{\lambda\to 0}A_2(\lambda)=4\pi\sqrt{2}\int_{\mathbb{R}^3}(a^{-1}_{\phi_0}(s)-\phi_0(x))_+^{1/2}h(x)dx. \tag{A.3}$$

Therefore, (A.1), (A.2) and (A.3) give (3.5). This concludes the proof of Lemma 3.2. □

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CNRS and IRMAR, Université de Rennes 1, France
E-mail address: `mohammed.lemou@univ-rennes1.fr`

IRMAR, Université de Rennes 1, France
E-mail address: `florian.mehats@univ-rennes1.fr`

IMT, Université Paul Sabatier, Toulouse, France
E-mail address: `pierre.raphael@math.univ-toulouse.fr`