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Abstract

One question of interest in an industrial conception of air planes motors is the study of the deviation of the acoustic resonance frequencies of a cavity which is linked to another one through a narrow hole. These frequencies have a direct impact on the stability of the combustion in one of these two cavities. In this work, we aim is analyzing the eigenvalue problem for the Laplace operator with Dirichlet boundary conditions. Using the Matched Asymptotic Expansions technique, we derive the asymptotic expansion of this eigenmodes. Then, these results are validated through error estimates. Finally, we show how we can design a numerical method to compute the eigenvalues of this problem. The results are compared with direct computations.

Mathematics Subject Classification. 34E05, 35J05, 65M60, 78M30, 78M35.

Keywords. Helmholtz Equation, Matched Asymptotic Expansions, Eigenvalue problem, High Order Finite Elements.

Contents

Chapter 1

Introduction and Motivation

1.1 The scientific context

In a turbo engine, the temperature of the combustion chamber can reach 2000 Kelvin. In order to protect the structure, small holes are perforated throw the wall linking the combustion chamber to the casing and fresh air is injected.

These small holes perturb the acoustic resonance frequencies and modes of the combustion chamber. This have often a negative impact on the combustion but a positive impact on the noise generated by the engine. The new environmental standard imposes a precise study of the effects of these small holes.

Unfortunately, a direct numerical approach is nowadays technically not feasible due to two main reasons.

- A fine mesh (in space and time due to the CFL condition) is compulsorily due to the small characteristic length of the holes.
- The mesh generation of a perforated structure is a hard job. This is mostly the case when the holes are numerous.

This report is a part of the ANR APam which aims in providing an efficient numerical method to take into account these small holes. The desired method should fulfill the following conditions

- mesh refinement is not required in the neighborhood of the slot.
- it must only involve quantities that can be easily computed.

Two natural approaches can be envisaged. The first one consists in replacing the effect of the wall by an equivalent transmission condition based on a surface homogenization technique, see for example [25] or [7]. The second approach consists in replacing each hole by equivalent source which intensity is derived by a multiscale analysis.

The experiments of physicist (see for example [14] and [22]) does not give a clear answer to which approach has to be considered. We have decided to approach this question with the equivalent sources point of view.

Moreover, the physical problem is really to complicated to be considered at this point. In the context of a 2-D toy model, we show that the so called technique of Matching of Asymptotic Expansions (see for example [30], [15] and [11]) permits to derive such an efficient method which can be interpreted as an equivalent point source model.

To end this bibliography, we point out that the results of this report are very close from the results of [13], where the asymptotic expansions of scattering poles are obtained for a similar problem. Moreover, the problem of a wall perforated by a small iris has been widely studied in the literature both from the theoretical and numerical point of view, see [23, 26, 29] for example.

We also mention that this problem presents a lot of similarities with the Dumbbell problem also called Helmholtz resonator (the eigenvalue problem of two cavities linked by a thin slot of length $O(1)$), see [1, 9, 4, 2, 8, 16].

To our Knowlege this report constitutes the first attempt to derive a numerical method for computing the derivation by a small hole of the eigenvalues of the Dirichlet-Laplacian.

1.2 The toy model: A 2D eigenvalue problem

Let Ω_{int} and Ω_{ext} be two open subsets of $\mathbb{R}^2$ with

$$\Omega_{int} \cap \Omega_{ext} = \emptyset \quad \text{and} \quad \exists a > 0 : \Big(\{0\} \times]-a; a[\Big) \in \partial\Omega_{int} \cap \partial\Omega_{ext}. \tag{1.1}$$

For $\delta < a$, we consider the domain Ω^δ consisting of Ω_{ext} and Ω_{int} linked by a slit of width δ

$$\Omega^\delta := \Omega_{int} \cup \Omega_{ext} \cup \Big(\{0\} \times]-\frac{\delta}{2}; \frac{\delta}{2}[\Big) \subset \mathbb{R}^2 \tag{1.2}$$

which tends when $\delta \to 0$ to

$$\Omega := \Omega_{int} \cup \Omega_{ext} \subset \mathbb{R}^2. \tag{1.3}$$

In these domains we consider the eigenvalue problems

$$\begin{cases} \text{Find } u^\delta \in \Omega^\delta \to \mathbb{R} \text{ and } \lambda^\delta \in \mathbb{R} \text{ satisfying} \\ -\Delta u^\delta(x, y) = \lambda^\delta u^\delta(x, y) \text{ in } \Omega^\delta, \\ u^\delta(x, y) = 0 \text{ on } \partial\Omega^\delta, \end{cases} \tag{1.4}$$

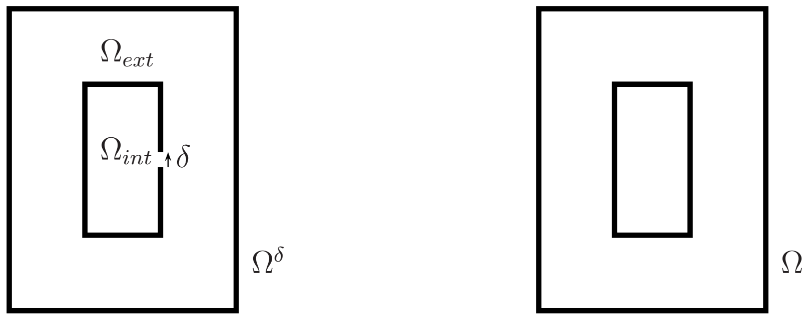

Figure 1.1: Geometry of the domain of propagation.

$$
\begin{cases}
\text{Find } u \in \Omega \to \mathbb{R} \text{ and } \lambda \in \mathbb{R} \text{ satisfying} \\
-\Delta u(x,y) = \lambda u(x,y) \text{ in } \Omega, \\
u(x,y) = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{1.5}
$$

that defines discreet sets of eigenmodes

- $(u_n^\delta, \lambda_n^\delta)_{n\geq 0}$ which can be chosen to be a bi-orthogonal basis of $L^2(\Omega^\delta)$ and $H^1(\Omega^\delta)$ and to satisfy

$$
\lambda_0^\delta \leq \lambda_1^\delta \leq \ldots \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n^\delta = +\infty. \tag{1.6}
$$

- $(u_n, \lambda_n)_{n\geq 0}$ which can be chosen to be a bi-orthogonal basis of $L^2(\Omega)$ and of $H^1(\Omega)$ and to satisfy

$$
\lambda_0 \leq \lambda_1 \leq \ldots \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n = +\infty. \tag{1.7}
$$

Some natural questions arise:

- Does the eigenvalue λ_n^δ converge to λ_n?
- Is it possible to obtain an asymptotic expansion of λ_n^δ?
- With this asymptotic expansion, is it possible to derive a numerical method to compute an approximation of λ_n^δ, with small computation cost?

For all this report and for the simplicity of a theoretical analysis, we assume that

$$
\text{The eigenvalues } (\lambda_n)_{n\geq 0}, \text{ defined by (1.5), are simple } (\lambda_n = \lambda_p \implies p = n) \tag{1.8}
$$

In the continuation we aim in proving the following Theorem which give a clear answer to these questions.

Theorem 1 *Let n be a strictly positive integer. Under the hypothesis* (1.8)*, the eigenvalue λ_n^δ can be expanded as follows*

$$\begin{aligned} &\text{if } u_n = 0 \text{ in } \Omega_{ext} \quad \text{then } \lambda_n^\delta = \lambda_n - \frac{\pi}{16}\frac{|\partial_x u_n(0^-,0)|^2}{\|u_n\|^2_{L^2(\Omega)}}\delta^2 + \underset{\delta\to 0}{o}(\delta^2), \\ &\text{if } u_n = 0 \text{ in } \Omega_{int} \quad \text{then } \lambda_n^\delta = \lambda_n - \frac{\pi}{16}\frac{|\partial_x u_n(0^+,0)|^2}{\|u_n\|^2_{L^2(\Omega)}}\delta^2 + \underset{\delta\to 0}{o}(\delta^2). \end{aligned} \tag{1.9}$$

Remark 1 *The condition* (1.8) *is to our opinion not central and is mostly considered for convenience to avoid resonance phenomena between two close eigenvalues of the Dirichlet-Laplacian in Ω^δ.*

Remark 2 *The condition* (1.8) *implies that all the eigenvectors of the Dirichlet-Laplacian of Ω are eigenvectors of the Dirichlet-Laplacian of either Ω_{int} or of Ω_{ext}. Consequently every eigenvector u_n satisfies*

$$u_n = 0 \text{ in } \Omega_{int} \text{ or in } \Omega_{int}. \tag{1.10}$$

Remark 3 *When δ is small, the formula* (1.9) *provides a way to compute an approximation of the eigenvalue λ_n^δ involving only the computation of the eigenmodes of the Dirichlet-Laplacian in Ω. This implies that, for small $\delta > 0$, no mesh refinement is required to obtain a good approximation of the eigenvalues of Ω^δ.*

1.3 Matching of asymptotic expansions

The second order asymptotic expansion of λ_n^δ

$$\lambda_n^\delta \;=\; \lambda_n^0 + \delta\lambda_n^1 + \delta^2\lambda_n^2 + o(\delta^2) \tag{1.11}$$

is sought in parallel to the second order asymptotic expansion of the eigenvector u_n^δ.

The toy model involving multiple scales (the length of the cavity $O(1)$ and the width of the slot $O(\delta)$), we look for two asymptotic expansions of u_n^δ. The first one is expressed with the unscaled variable $\mathbf{x}$ and called the far-field expansion. The second one is the near-field expansion and is written with the scaling $\mathbf{X} = \mathbf{x}/\delta$

$$\left\{ \begin{array}{lclcl} u_n^\delta(\mathbf{x}) & = & u_n^0(\mathbf{x}) + \delta u_n^1(\mathbf{x}) + \delta^2 u_n^2(\mathbf{x}) + o(\delta^2), & & \\ u_n^\delta(\delta\mathbf{X}) & = & \Pi^\delta(\mathbf{X}) & = & \Pi_n^0(\mathbf{X}) + \delta\Pi_n^1(\mathbf{X}) + \delta^2\Pi_n^2(\mathbf{X}) + o(\delta^2). \end{array} \right. \tag{1.12}$$

The far-field expansion approximates u_n^δ in a domain excluding a small neighborhood of the hole. The near-field expansion, can be used to approximate u_n^δ in a small neighborhood of the hole.

Both being two approximations of the same function u_n^δ, they have to match in some intermediate zone. More precisely, the terms of the asymptotic expansions u_n^i and Π_n^i match through common spatial behaviors.

This approach, often called Matching of Asymptotic Expansions (MAE), have been widely studied and is now rather well understood, see [30, 15] and references therein (it is impossible to give a complete bibliography). This technique is very often considered as formal but can become rigorous if one can obtain error estimates validating these expansions, see [17, 18, 27, 28].

1.4 Content

This report is organized as follows.

The Chapter 2 is devoted to the derivation of the second order asymptotic expansion of the eigenvalue and eigenvector. After having derived problems solved by u_n^i, Π_n^i, λ_n^i for $0 \leq i \leq 2$ with formal computations, we show that these problems are well-posed.

In Chapter 3, we validate this formal asymptotic expansion by obtaining an error estimate, see Theorem 3 (one can note that Theorem 1 is one of its corollary). The proof is based on a quasi-mode technique and on the classical min-max theorem and require the third order asymptotic expansion, see Appendix A.

The Chapter 4 is devoted to numerical simulations. The λ_n^δ, computed with a high order finite elements method, are compare with $\lambda_n + \delta^2 \lambda_n^2$ We observe a good agreement with theory.

Chapter 2

The second order asymptotic expansion: Leading equations

In this chapter we explain how one can get the second order asymptotic expansion of an eigenvalue λ_n^δ defined by problem (1.4)

$$\lambda_n^\delta = \lambda_n^0 + \delta\lambda_n^1 + \delta^2\lambda_n^2 + \underset{\delta\to 0}{o}(\delta^2). \tag{2.1}$$

This derivation is mostly formal and will be carried out in parallel to the derivation of the second order asymptotic expansion of the eigenfunction u_n^δ.

The toy model (1.4) involving two characteristic lengths of different magnitude (the length of the cavity L and the size of the hole $\delta << L$) it is necessary to use multiple scalings to obtain an asymptotic approximation of the eigenvector u_n^δ uniformly valid. The first scaling corresponds to the $\mathbf{x}$-variable and takes care of the cavity phenomenon. The second scaling $\mathbf{x}/\delta$ permits to describe the boundary layer phenomenons which happen in the neighborhood of the slot.

This is the reason why we will look for the expansions of the two functions $\delta \mapsto u_n^\delta(\mathbf{x})$ and $\delta \mapsto \Pi_n^\delta(\mathbf{X}) := u_n^\delta(\delta\mathbf{X})$

$$u_n^\delta(\mathbf{x}) = u_n^0(\mathbf{x}) + \delta u_n^1(\mathbf{x}) + \delta^2 u_n^2(\mathbf{x}) + \underset{\delta\to 0}{o}(\delta^2), \tag{2.2}$$

$$\Pi_n^\delta(\mathbf{X}) := u_n^\delta(\delta\mathbf{X}) = \Pi_n^0(\mathbf{X}) + \delta\Pi_n^1(\mathbf{X}) + \delta^2\Pi_n^2(\mathbf{X}) + \underset{\delta\to 0}{o}(\delta^2). \tag{2.3}$$

The derivation of the leading equations defining the terms of the asymptotic expansions $(\lambda_n^i, u_n^i, \Pi_n^i)$ is mostly formal, based on the Matching of Asymptotic Expansions technique. However, one can note that the terms of the asymptotic expansions are at the end of the day defined by well-posed problems.

Remark 4 *One can find without detail the third order asymptotic expansion in Appendix A.*

2.1 The far-field expansion

In this section, we are looking for a second order asymptotic expansion of u_n^δ in the non-scaled coordinate $\mathbf{x}$. We seek this asymptotic expansion with the form (2.2). The terms of the asymptotic expansions u_n^i $(0 \leq i \leq 2)$ will be

- defined in the far-field domain Ω which is the limit of Ω^δ when $\delta \to 0$, (see Fig.2.1).

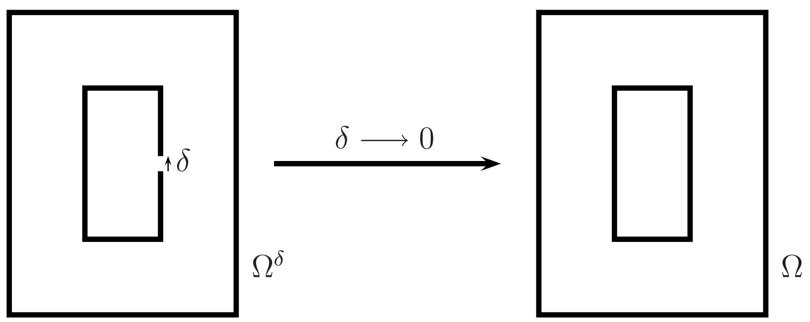

Figure 2.1: The far-field domain

- independent of δ.

They are solutions of the following problems

$$\left\{\begin{array}{ll} \text{Find } u_n^0 : \Omega \to \mathbb{R} \text{ and } \lambda_n^0 \in \mathbb{R} \text{ such that} & \\ \Delta u_n^0 + \lambda^0 u_n^0 = 0, & \text{in } \Omega, \\ u_n^0 = 0, & \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array}\right. \tag{2.4}$$

$$\left\{\begin{array}{ll} \text{Find } u_n^1 : \Omega \to \mathbb{R} \text{ and } \lambda_n^1 \in \mathbb{R} \text{ such that} & \\ \Delta u_n^1 + \lambda_n^0 u_n^1 = -\lambda_n^1 u_n^0, & \text{in } \Omega, \\ u_n^1 = 0, & \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array}\right. \tag{2.5}$$

$$\left\{\begin{array}{ll} \text{Find } u_n^2 : \Omega \to \mathbb{R} \text{ and } \lambda_n^2 \in \mathbb{R} \text{ such that} & \\ \Delta u_n^2 + \lambda_n^0 u_n^2 = -\lambda_n^2 u_n^0 - \lambda_n^1 u_n^1, & \text{in } \Omega, \\ u_n^2 = 0, & \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array}\right. \tag{2.6}$$

Obtention of the equations (2.4), (2.5) and (2.6): We use the classical route to obtain these equations. Inserting the Ansatz (2.1) and (2.2) in the equations (1.4) satisfied by u^δ and λ^δ leads to

$$\Big(\Delta + (\lambda_n^0 + \delta\lambda_n^1 + \delta^2\lambda_n^2 + \underset{\delta\to 0}{o}(\delta^2)\Big)(u_n^0 + \delta u_n^1 + \delta^2 u_n^2 + \underset{\delta\to 0}{o}(\delta^2)) = 0, \quad \text{in } \Omega, \tag{2.7}$$

or equivalently to

$$\Big(\Delta u_n^0 + \lambda_n^0 u_n^0\Big) + \delta\Big(\Delta u_n^1 + \lambda_n^0 u_n^1 + \lambda_n^1 u_n^0\Big)$$
$$+ \delta^2\Big(\Delta u_n^2 + \lambda_n^0 u_n^2 + \lambda_n^1 u_n^1 + \lambda_n^2 u_n^0\Big) + \underset{\delta\to 0}{o}(\delta^2) = 0, \quad \text{in } \Omega. \tag{2.8}$$

Now, we consider $\mathbf{x} \in \partial\Omega \setminus \{\mathbf{0}\}$. For δ small enough, $\mathbf{x} \in \partial\Omega^\delta$ and so we have

$$u_n^\delta(\mathbf{x}) = 0. \tag{2.9}$$

Inserting the Ansatz, we obtain

$$\Big(u_n^0 + \delta u_n^1 + \delta^2 u_n^2 + o(\delta^2)\Big)(\mathbf{x}) = 0. \tag{2.10}$$

The identification order by order leads to

$$u_n^0 = 0, \quad u_n^1 = 0, \quad u_n^2 = 0, \quad \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \tag{2.11}$$

This clearly leads to the result.

2.2 The near-field expansion

We introduce the scaling $X = \dfrac{x}{\delta}$, and $Y = \dfrac{y}{\delta}$, see Fig.2.2, and consider the function Π_n^δ defined by

$$\Pi_n^\delta(X,Y) = u_n^\delta(\delta X, \delta Y). \tag{2.12}$$

We are seeking a second order asymptotic expansion of Π_n^δ with the form (2.3)

$$\Pi_n^\delta(X,Y) = \Pi_n^0(X,Y) + \delta\, \Pi_n^1(X,Y) + \delta^2\, \Pi_n^2(X,Y) + o(\delta^2). \tag{2.13}$$

These functions will

- be defined on the near-field domain $\widehat{\Omega}$, see Fig. 2.2,

$$\widehat{\Omega} := \mathbb{R}^2 \setminus \left(\{0\} \times \Big(\Big]-\infty, -\frac{1}{2}\Big[\cup\Big], \frac{1}{2}, +\infty\Big[\Big)\right), \tag{2.14}$$

  which is the limit, when δ tends to zero, of

$$\Omega^\delta/\delta = \Big\{(X,Y) \in \mathbb{R}^{\mathbf{2}} \ : \ (\delta X, \delta Y) \in \Omega^\delta\Big\}, \tag{2.15}$$

- independent of δ.

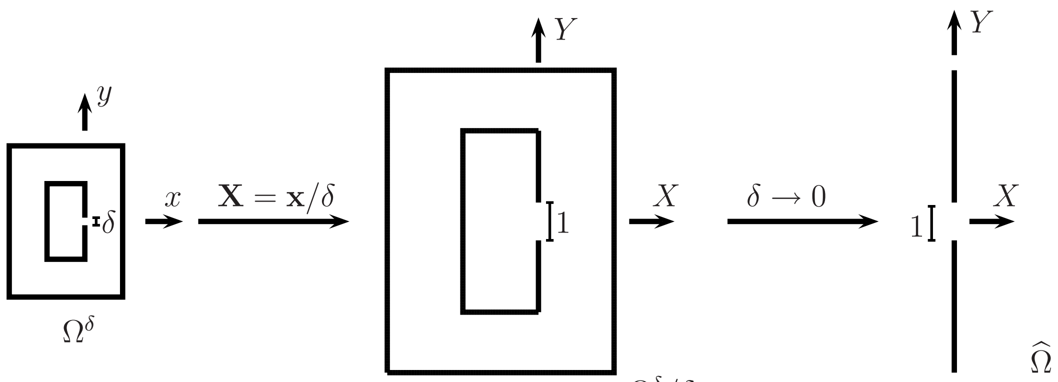

Figure 2.2: The near-field domain.

They are solutions of the following problems

$$\begin{cases} \text{Find } \Pi_n^0 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ -\Delta \Pi_n^0 = 0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^0 = 0, \qquad \text{on } \partial\widehat{\Omega}. \end{cases} \tag{2.16}$$

$$\begin{cases} \text{Find } \Pi_n^1 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ -\Delta \Pi_n^1 = 0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^1 = 0, \qquad \text{on } \partial\widehat{\Omega}. \end{cases} \tag{2.17}$$

$$\begin{cases} \text{Find } \Pi_n^2 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ -\Delta \Pi_n^2 = \lambda_n^0 \Pi_n^0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^2 = 0, \qquad \text{on } \partial\widehat{\Omega}. \end{cases} \tag{2.18}$$

Obtention of the equations (2.16), (2.17), (2.18): Scaling equation (1.4) with the change of variable $\mathbf{X} = \mathbf{x}/\delta$ ($X = x/\delta$ and $Y = y/\delta$) we have

$$\begin{cases} \Big(- \dfrac{1}{\delta^2} \Delta_{\mathbf{X}} + \lambda_n^\delta \Big) \Pi_n^\delta = 0, & \text{in } \Omega^\delta/\delta, \\ \Pi_n^\delta = 0, & \text{on } \partial\Omega^\delta/\delta. \end{cases} \tag{2.19}$$

We consider $\mathbf{X} \in \partial\widehat{\Omega}$. For δ small enough, $\mathbf{X} \in \partial\Omega^\delta/\delta$. Inserting the Ansatz (2.1) and (2.3) in equation (2.19) leads to

$$\Big(\frac{1}{\delta^2}\Delta_{\mathbf{X}} + (\lambda_n^0 + \delta\ \lambda_n^1 + \delta^2\ \lambda_n^2 + o(\delta^2))\Big)(\Pi_n^0 + \delta\ \Pi_n^1 + \delta^2\ \Pi_n^2 + \underset{\delta\to 0}{o}(\delta^2))(\mathbf{X}) = 0, \tag{2.20}$$

or equivalently to

$$\frac{1}{\delta^2}\Big(\Delta\Pi_n^0(\mathbf{X})\Big) + \frac{1}{\delta}\Big(\Delta\Pi_n^1(\mathbf{X})\Big) + \Big(\Delta\Pi_n^2(\mathbf{X}) + \lambda_n^0\Pi_n^0(\mathbf{X})\Big) + \underset{\delta\to 0}{o}(1) = 0. \tag{2.21}$$

Identifying order by order we obtain

$$-\Delta\Pi_n^0 = 0, \quad -\Delta\Pi_n^1 = 0, \quad -\Delta\Pi_n^2 = \lambda_n^0\,\Pi_n^0, \quad \text{in } \widehat{\Omega} \tag{2.22}$$

We consider $\mathbf{X} \in \partial\widehat{\Omega}$. For δ small enough, $\mathbf{X} \in \partial\Omega^\delta/\delta$ and we have

$$\Pi_n^\delta(\mathbf{X}) = 0. \tag{2.23}$$

Inserting the Ansatz (2.1), we obtain

$$\Big(\Pi_n^0 + \delta\,\Pi_n^1 + \delta^2\,\Pi_n^2 + \underset{\delta\to 0}{o}(\delta^2)\Big)(\mathbf{X}) = 0. \tag{2.24}$$

By identification order by order, it clearly leads to the following result

$$\Pi_n^0 = 0, \quad \Pi_n^1 = 0, \quad \Pi_n^2 = 0, \quad \text{on } \partial\widehat{\Omega}. \tag{2.25}$$

2.3 The Matched Asymptotic Expansions: The matching procedure

In this section, we describe an algorithm to close the problems defining the far-fields and near-fields. For $m \leq 2$, we use the following procedure to obtain these extra conditions and references therein [30, 21]:

1. We consider the far-field approximation of order m written with $\mathbf{x} = \delta\mathbf{X}$

$$\sum_{i=0}^{m} \delta^i u_n^i(\delta\mathbf{X}). \tag{2.26}$$

2. Then this sum is expanded up to $\underset{\delta\to 0}{o}(\delta^m)$. This defines the U_m^i in the $\mathbf{X}$ coordinates

$$\sum_{i=0}^{m} \delta^i u_n^i(\delta\mathbf{X}) = \sum_{i=-\infty}^{m} \delta^i (U_n^i)_m(\mathbf{X}) + \underset{\delta\to 0}{o}(\delta^m). \tag{2.27}$$

3. The matching conditions are the following

$$\begin{cases} (U_n^i)_m(\mathbf{X}) = 0, & \forall i \leq 0, \\ \Pi_n^i(\mathbf{X}) - (U_n^i)_m(\mathbf{X}) = \underset{R\to+\infty}{o}(\frac{1}{R^{m-i}}), & \forall i \in [\![0, m]\!]. \end{cases} \tag{2.28}$$

Here we will not try to explain the reason of this coupling. Note however that this coupling involves the behavior of the u_n^i in the neighborhood of zero and of Π_n^i at infinity.

Remark 5 *For $m \geq 3$, one has to consider poly-logarithmic gauge functions. Therefore, the previous algorithm has to be slightly modified to take care of this difficulty, see [21].*

2.4 The limit field

To be the limit of the eigenfunction $u_n^\delta(\mathbf{x})$, the far-field u_n^0 and the near-field Π_n^0 has to solve equations (2.4) and (2.16). Assuming regularity for $u_n^0 \in H^1(\Omega)$, we obtain that u_n^0 has to solve

$$\begin{cases} \text{Find } u_n^0 \in H^1(\Omega) \text{ such that} \\ \Delta u_n^0 + \lambda_n^0 u_n^0 = 0, \quad \text{in } \Omega, \\ u_n^0 = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.29}$$

which means that u_n^0 is an eigenvalue of the Dirichlet-Laplacian in Ω and λ_n^0 is the associated eigenvalue:

$$\exists m > 0 : \quad \lambda_n^0 = \lambda_m \text{ and } u_n^0 = u_m, \tag{2.30}$$

with (u_m, λ_m) the m^{th}-eigenpair of the Dirichlet-Laplacian in Ω, see (1.5).

Since the eigenvalues of the Dirichlet-Laplacian in Ω are supposed to be simple, see (1.8), λ_n^0 is either an eigenvalue of the Dirichlet-Laplacian in Ω_{int} or in Ω_{ext}. In other words, (2.29) can be decomposed into two problems

$$\begin{cases} \text{Find } u_n^0 \in H^1(\Omega) \text{ such that} \\ \Delta u_n^0 + \lambda_n^0 u_n^0 = 0 \quad \text{in } \Omega_{int} \text{ and } u_n^0 = 0 \text{ in } \Omega_{ext}, \\ u_n^0 = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.31}$$

or

$$\begin{cases} \text{Find } u_n^0 \in H^1(\Omega) \text{ such that} \\ \Delta u_n^0 + \lambda_n^0 u_n^0 = 0 \quad \text{in } \Omega_{ext} \text{ and } u_n^0 = 0 \text{ in } \Omega_{int}, \\ u_n^0 = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2.32}$$

To get Π_n^0, we use the the matching principle, (see section 2.3)

$$\begin{cases} \text{Find } \Pi_n^0 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ -\Delta \Pi_n^0 = 0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^0 = 0, \quad \text{on } \partial\widehat{\Omega}, \\ \Pi_n^0(R,\theta) = \underset{R\to+\infty}{o}(1). \end{cases} \tag{2.33}$$

We then remark that this problem admits as solution

$$\Pi_n^0 = 0, \quad \text{in } \widehat{\Omega}. \tag{2.34}$$

Remark 6 *It is possible to prove the uniqueness of the solution of problem* (2.33)*. The details will not be given here (the existence relies on tools introduced in the following pages).*

Remark 7 *In the sequel, we will only detail the case* $u_n^0 \neq 0$ *in* Ω_{int} *and* $u_n^0 = 0$ *in* Ω_{ext}*. The case* $u_n^0 \neq 0$ *in* Ω_{int} *and* $u_n^0 = 0$ *in* Ω_{ext} *can be deduced by symmetry (replacing ext by int in the formulas).*

Property of the limit. By elliptic regularity the functions u_n^0 in Ω_{int} and u_n^0 in Ω_{ext} are infinitely differentiable on Ω in the neighborhood of $\mathbf{0}$. Consequently, the expansion of u_n^0 is given by its Taylor expansion. Written at third order, this reads

$$\left\{\begin{array}{ll} u_n^0(x,y) = x\partial_x u_n^0\big|_{\Omega_{int}}(\mathbf{0}) + xy\partial_{xy}^2 u_n^0\big|_{\Omega_{int}}(\mathbf{0}) & \\ \qquad\qquad -\partial_x^3 u_n^0\big|_{\Omega_{int}}(\mathbf{0})\dfrac{r^3}{3!}\ \sin(3\theta) + \underset{r\to 0}{O}(r^4), & \text{in } \Omega_{int}, \\ u_n^0(x,y) = 0, & \text{in } \Omega_{ext}, \end{array}\right. \tag{2.35}$$

with r and θ the polar coordinates (see Fig. 2.3)

$$x = r\sin\theta,\ y = -r\cos\theta,\ \text{with } r \geq 0,\ \text{and } 0 \leq \theta < 2\pi. \tag{2.36}$$

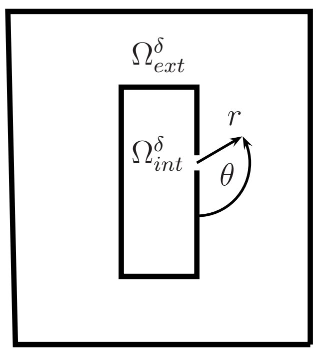

Figure 2.3: Polar coordinates.

2.5 The first order asymptotic expansion

The first order expansion is given by

$$u_n^{1,\delta} = u_n^0 + \delta u_n^1, \quad \Pi_n^{1,\delta} = \Pi_n^0 + \delta\Pi_n^1 \text{ and } \lambda_n^{1,\delta} = \lambda_n^0 + \delta\lambda_n^1$$

where all the terms of order 1 remains to be determined.

2.5.1 Derivation of the first order

Here we use the matching principle of section 2.3 to find the problems satisfied by Π_n^1 and u_n^1.

1. We consider the far-field approximation of order one written with $\mathbf{x} = \delta\mathbf{X}$

$$u_n^0(\delta\mathbf{X}) + \delta\ u_n^1(\delta\mathbf{X}) \tag{2.37}$$

2. Then, we expand this sum up to $o(\delta)$. To do so we need the spatial expansion of u_n^0 and u_n^1. The expression of u_n^0 is given by (2.35).
   The term u_n^1 is solution of (2.5). It can be locally ($r \leq r_0$) decomposed into

$$u_n^1 = u_n^{1,P}\ +\ u_n^{1,H} \tag{2.38}$$

   with

   - $u_n^{1,P}$ a particular solution of (2.5)

$$\begin{cases} \Delta u_n^{1,P} + \lambda_n^0 u_n^{1,P} = -\lambda^1 u_n^0, & \text{in } \Omega \cap \{r \leq r_0\}, \\ u_n^{1,P} = 0, & \text{on } \big(\partial\Omega \setminus \{\mathbf{0}\}\big) \cap \{r \leq r_0\}. \end{cases} \tag{2.39}$$

     Since u_n^0 is regular, $u_n^{1,P}$ can be chosen to be regular and can be expanded via its Taylor expansion

$$\begin{cases} u_n^{1,P}(x,y) \ = \ \underbrace{u_n^{1,P}(\mathbf{0})}_{0} \ + \ \underset{r\to 0}{o}(1) \ = \ \underset{r\to 0}{o}(1), & \text{in } \Omega_{int}, \\ u_n^{1,P}(x,y) \ = \ 0, & \text{in } \Omega_{ext}, \end{cases} \tag{2.40}$$

   - $u_n^{1,H}$ a homogeneous solution of the Helmholtz equation

$$\begin{cases} \Delta u_n^{1,H} + \lambda_n^0 u_n^{1,H} = 0, & \text{in } \Omega \cap \{r \leq r_0\}, \\ u_n^{1,H} = 0, & \text{on } \big(\partial\Omega \setminus \{\mathbf{0}\}\big) \cap \{r \leq r_0\}. \end{cases} \tag{2.41}$$

     By separation of variables, see Appendix C.1, $u_n^{1,H}$ in Ω_{int} (respectively $u_n^{1,H}$ in Ω_{ext}) is given by

$$u_n^{1,H}(r,\theta) \ = \ \sum_{p=1}^{+\infty} \Big((a_{int}^1)_p\ \sin(p\theta)\ J_p(\sqrt{\lambda_n^0} r) + (b_{int}^1)_p\ \sin(p\theta)\ Y_p(\sqrt{\lambda_n^0} r) \Big) \tag{2.42}$$

     and respectively

$$u_n^{1,H}(r,\theta) \ = \ \sum_{p=1}^{+\infty} \Big((a_{ext}^1)_p\ \sin(p\theta)\ J_p(\sqrt{\lambda_n^0} r) + (b_{ext}^1)_p\ \sin(p\theta)\ Y_p(\sqrt{\lambda_n^0} r) \Big). \tag{2.43}$$

Since

$$J_p(\sqrt{\lambda_n^0} r) = \underset{r\to 0}{o}(1) \text{ and } Y_p(\sqrt{\lambda_n^0} r) = \sum_{l=-p}^{0} Y_{p,l}\, (\frac{\sqrt{\lambda_0} r}{2})^l \;+\; \underset{r\to 0}{o}(1), \tag{2.44}$$

we get $u_n^{1,H}$ as follows in Ω_{int}

$$\begin{cases} u_n^{1,H}(r,\theta) &= \displaystyle\sum_{p=1}^{+\infty}\sum_{l=-p}^{0} \Big((b_{int}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} r}{2})^l \Big), \\ &= \displaystyle\sum_{l=-\infty}^{0}\;\sum_{p=max(1,-l)}^{+\infty} \Big((b_{int}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} r}{2})^l \Big) \end{cases} \tag{2.45}$$

and respectively in Ω_{ext}

$$\begin{cases} u_n^{1,H}(r,\theta) &= \displaystyle\sum_{p=1}^{+\infty}\sum_{l=-p}^{0} \Big((b_{ext}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} r}{2})^l \Big), \\ &= \displaystyle\sum_{l=-\infty}^{0}\;\sum_{p=max(1,-l)}^{+\infty} \Big((b_{ext}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} r}{2})^l \Big). \end{cases} \tag{2.46}$$

So (2.37) can be written in Ω_{int} as

$$\delta X \partial_x u_n^0(\mathbf{0}) \,+\, \delta \sum_{l=-\infty}^{0} \delta^{-l} \sum_{p=max(1,-l)}^{+\infty} \Big((b_{int}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} R}{2})^l \Big) + \underset{\delta\to 0}{o}(\delta), \tag{2.47}$$

and in Ω_{ext} as follows

$$\delta \sum_{l=-\infty}^{0} \delta^{-l} \sum_{p=max(1,-l)}^{+\infty} \Big((b_{ext}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} R}{2})^l \Big) + \underset{\delta\to 0}{o}(\delta). \tag{2.48}$$

We have now to identify (2.47) and (2.48) with $(U_n^0)_1 + \delta\ (U_n^1)_1$. Firstly, for the negative order we get

$$\sum_{p=max(1,-l)}^{+\infty} \Big((b_{int,ext}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} R}{2})^l \Big) = 0 \quad \text{for } l < -1. \tag{2.49}$$

which leads to

$$(b_{int,ext}^1)_p \; \sin(p\theta)\; Y_{p,l}\, (\frac{\sqrt{\lambda_n^0} R}{2})^l = 0 \quad \text{for } p \geq\, -l > 1. \tag{2.50}$$

Taking $l = -p$ we get $(b^1_{int,ext})_p = 0$ for all $p > 1$.
Moreover, the Bessel functions Y_1 can be expended for $z \to 0$ (see [19])

$$Y_1(z) := -\frac{2}{\pi z} + \underset{z\to 0}{o}(1). \tag{2.51}$$

So we get for u^1_n

$$\begin{cases} u^1_n(r,\theta) &= (b^1_{int})_1 \ \sin(\theta)\, Y_1\, (\sqrt{\lambda^0_n} r) \ + \underset{r\to 0}{o}(1) \quad \text{in } \Omega_{int}, \\ u^1_n(r,\theta) &= (b^1_{ext})_1 \ \sin(\theta)\, Y_1\, (\sqrt{\lambda^0_n} r) \ + \underset{r\to 0}{o}(1) \quad \text{in } \Omega_{ext}. \end{cases} \tag{2.52}$$

The equation (2.47) takes on Ω_{int} the form

$$\delta X \partial_x u^0_n|_{\Omega_{int}}(\mathbf{0}) \ - \ \frac{1}{\pi}\Big((b^1_{int})_1 \ \sin(\theta)\, \frac{2}{\sqrt{\lambda^0_n} R}\Big) + \underset{\delta\to 0}{o}(\delta) \tag{2.53}$$

and on Ω_{ext} the form

$$-\frac{1}{\pi}\Big((b^1_{ext})_1 \ \sin(\theta)\, \frac{2}{\sqrt{\lambda^0_n} R}\Big) + \underset{\delta\to 0}{o}(\delta). \tag{2.54}$$

Therefore, we get $(U^0_n)_1$, $(U^1_n)_1$ in Ω_{int} and Ω_{ext}

$$\begin{cases} (U^0_n)_1(\mathbf{X}) &= -\frac{1}{\pi}\Big((b^1_{int})_1 \ \sin(\theta)\, \frac{2}{\sqrt{\lambda^0_n} R}\Big), \quad \text{in } \Omega_{int}, \\ (U^0_n)_1(\mathbf{X}) &= -\frac{1}{\pi}\Big((b^1_{ext})_1 \ \sin(\theta)\, \frac{2}{\sqrt{\lambda^0_n} R}\Big), \quad \text{in } \Omega_{int}, \\ (U^1_n)_1(\mathbf{X}) &= X\ \partial_x u^0_n|_{\Omega_{int}}(\mathbf{0}), \quad \text{in } \Omega_{ext}, \\ (U^1_n)_1(\mathbf{X}) &= 0, \quad \text{in } \Omega_{ext}. \end{cases} \tag{2.55}$$

3. Finally, we use the matching conditions

$$\begin{cases} \Pi^0_n(\mathbf{X}) - (U^0_n)_1(\mathbf{X}) = \underset{R\to+\infty}{o}(\frac{1}{R}), \ \text{in } \Omega_{int} \text{ and } \Omega_{ext}, \\ \Pi^1_n(\mathbf{X}) - (U^1_n)_1(\mathbf{X}) = \underset{R\to+\infty}{o}(1), \ \text{in } \Omega_{int} \text{ and } \Omega_{ext}. \end{cases} \tag{2.56}$$

Since $\Pi^0_n(\mathbf{X}) = 0$ we get

$$\begin{cases} (b^1_{int})_1 = 0 \qquad \text{and} \quad (b^1_{ext})_1 = 0, \\ \Pi^1_n(\mathbf{X}) = X\ \partial_x u^0_n|_{\Omega_{int}}(\mathbf{0}) \ + \underset{R\to+\infty}{o}(1), \quad \text{in } \Omega_{int}, \\ \Pi^1_n(\mathbf{X}) = \underset{R\to+\infty}{o}(1), \quad \text{in } \Omega_{ext}. \end{cases} \tag{2.57}$$

Conclusion. We have obtained the behaviors of u_n^1 and Π_n^1

$$\begin{cases} u_n^1(\mathbf{x}) \;=\; \underset{r\to 0}{o}(1) \text{ in } \Omega_{int} \text{ and } \Omega_{ext}, \\ \Pi_n^1(\mathbf{X}) = X\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0}) \ +\ \underset{R\to+\infty}{o}(1), \quad \text{ in } \Omega_{int}, \\ \Pi_n^1(\mathbf{X}) = \underset{R\to+\infty}{o}(1), \quad \text{ in } \Omega_{ext}. \end{cases} \tag{2.58}$$

2.5.2 Existence and uniqueness of Π_n^1

In this section, we give the concrete definition of Π_n^1.

In this geometrical context (with Dirichlet boundary condition), the natural functional of Laplace problem is K_0^1

$$K_0^1 := \left\{u : \nabla u \in L^2\left(\widehat{\Omega}\right) \text{ and } \frac{u}{1+R} \in L^2\left(\widehat{\Omega}\right) \text{ such that } u = 0 \text{ on } \partial\widehat{\Omega}\right\},$$

endowed with the norm $\|.\|_{K_0^1}$ defined by

$$\|u\|_{K_0^1} = \|\nabla u\|_{L^2(\hat{\Omega})} + \left\|\frac{u}{1+R}\right\|_{L^2(\hat{\Omega})}, \quad \forall\, u\ \in K_0^1. \tag{2.59}$$

The function Π_n^1 is solution of the following problem

$$\begin{cases} \text{Find } \Pi_n^1 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ -\Delta \Pi_n^1 = 0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^1 = 0, \qquad \text{on } \partial\widehat{\Omega}, \\ \Pi_n^1 \ -\ \Psi_{int}(\mathbf{X}) X\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \in K_0^1, \end{cases} \tag{2.60}$$

with $\Psi_{int}(\mathbf{X}) = \Psi_{int}(R)$ a regular cut-off function satisfying

$$\begin{cases} \Psi_{int}(\mathbf{X}) = 0 \text{ in } \widehat{\Omega}_{ext}, \\ \Psi_{int}(\mathbf{X}) = 0 \text{ in } \widehat{\Omega}_{int} \text{ for } R < 1, \\ \Psi_{int}(\mathbf{X}) = 1 \text{ in } \widehat{\Omega}_{int} \text{ for } R > 2, \end{cases} \tag{2.61}$$

By separation of variables, see Appendix C.2, it is easy to see that the last line of (2.60) prescribes the asymptotic behavior

$$\begin{cases} \Pi_n^1(\mathbf{X}) = X\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0}) \ +\ \underset{R\to+\infty}{o}(1), \text{ in } \Omega_{int}, \\ \Pi_n^1(\mathbf{X}) = \underset{R\to+\infty}{o}(1), \text{ in } \Omega_{ext}. \end{cases} \tag{2.62}$$

The space K_0^1 equipped with $\|.\|_{K_0^1}$ satisfies the Hardy (Poincaré type) inequality, see [10] for example,

$$\exists\, \gamma > 0 :\ \gamma \|u\|_{K_0^1} \leq \|\nabla u\|_{L^2(\hat{\Omega})}, \quad \forall\, u \in K_0^1. \tag{2.63}$$

Lemma 2.5.1 *If the linear form F belongs to the functional space $(K_0^1)^*$, then the following problem*

$$\begin{cases} \text{Find } u \in K_0^1 \text{ such that} \\ \displaystyle\int \nabla u . \nabla v = F(v), \text{ for all } v \in K_0^1 \end{cases} \tag{2.64}$$

admits a unique solution.

Proof. This is a simple consequence of the Lax-Milgram theorem. Indeed, the bilinear form a is continuous,

$$|\mathsf{a}\,(u,v)\,| \leq \|\nabla u\|_{L^2(\hat{\Omega})} \times \|\nabla v\|_{L^2(\hat{\Omega})} \leq \|u\|_{K_0^1} \times \|v\|_{K_0^1}. \tag{2.65}$$

Due to (2.63), the bilinear form a is coercive ($\gamma^2 > 0$)

$$|\mathsf{a}\,(u,u)\,| = \|\nabla u\|^2_{L^2(\hat{\Omega})} \geq \gamma^2 \|u\|^2_{K_0^1}. \tag{2.66}$$

■

Corollary 2.5.1 *If $(1+R)\,F \in L^2\left(\widehat{\Omega}\right)$ then the following problem*

$$\begin{cases} \Delta u = F, & \text{in } \widehat{\Omega}, \\ u = 0, & \text{on } \partial\widehat{\Omega}, \end{cases} \tag{2.67}$$

has a unique solution.

Proof. If $\big(1+R\big)F$ belongs to the space $L^2\left(\widehat{\Omega}\right)$, then there exist a constant C such that

$$\int_{\widehat{\Omega}} Fv \leq \big\|(1+R)F\big\|_0 \times \left\|\frac{v}{1+R}\right\|_0 \;\leq\; C\,\|v\|_{K_0^1}. \tag{2.68}$$

■

Theorem 2 *The following problem*

$$\begin{cases} \text{Find } \Pi_n^1 : \widehat{\Omega} \longrightarrow \mathbb{R} \text{ such that} \\ \Pi_n^1 - \Psi_{int}(\mathbf{X})\partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})X \in K_0^1, \\ \Delta \Pi_n^1 = 0, \text{ in } \widehat{\Omega}, \\ \Pi_n^1 = 0 \text{ on } \partial\widehat{\Omega} \end{cases} \tag{2.69}$$

admits a unique solution.

Proof. We consider the function ω^1 defined by $\omega^1 = \Pi_n^1 - \Psi_{int}(\mathbf{X}) X \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})$. Using (2.69), it is easy to check that ω^1 satisfies

$$\left\{\begin{array}{l} \omega^1 \in K_0^1 \\ \Delta \omega^1 = F, \quad \text{ in } \widehat{\Omega}, \\ \omega^1 = 0, \quad \text{ on } \partial\widehat{\Omega}. \end{array}\right. \tag{2.70}$$

with

$$F(\mathbf{X}) = -\Delta \Psi_{int}(\mathbf{X}) \left(X \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0}) \right) - 2 \nabla \Psi_{int}(\mathbf{X}) \cdot \nabla \left(X \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0}) \right). \tag{2.71}$$

Since the function F, defined in the two last lines of (2.70), is compactly supported, $\big(1+R\big)F$ belongs to $L^2(\widehat{\Omega})$. Consequently, applying corollaray 2.5.1, the problem (2.69) admits a unique solution. ■

Remark 8 *By linearity, we remark that* Π_n^1 *is given by*

$$\Pi_n^1 = \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \widetilde{\Pi}^1 \text{ in } \widehat{\Omega}, \tag{2.72}$$

with the function $\widetilde{\Pi}^1$*, dependant only from the geometry, defined by*

$$\left\{\begin{array}{l} \text{Find } \widetilde{\Pi}^1 : \widehat{\Omega} \longrightarrow \mathbb{R} \text{ such that} \\ \widetilde{\Pi}^1 - \Psi_{int}(\mathbf{X}) X \in K_0^1, \\ \Delta \widetilde{\Pi}^1 = 0, \quad \text{in } \widehat{\Omega}, \\ \widetilde{\Pi}^1 = 0, \quad \text{on } \partial\widehat{\Omega}. \end{array}\right. \tag{2.73}$$

By separation of variables, see Appendix C.2, we obtain the expansion of $\widetilde{\Pi}^1$ *near infinity*

$$\left\{\begin{array}{ll} \widetilde{\Pi}^1(\mathbf{X}) = X + \alpha_{int} \dfrac{\sin\theta}{R} + \beta_{int} \dfrac{\sin(2\theta)}{R^2} + \underset{R\to+\infty}{O}\left(\dfrac{1}{R^3}\right), & \text{in } \widehat{\Omega}_{int}, \\ \widetilde{\Pi}^1(\mathbf{X}) = \alpha_{ext} \dfrac{\sin\theta}{R} + \beta_{ext} \dfrac{\sin(2\theta)}{R^2} + \underset{R\to+\infty}{O}\left(\dfrac{1}{R^3}\right), & \text{in } \widehat{\Omega}_{ext}. \end{array}\right. \tag{2.74}$$

with α_{int}*,* α_{ext}*,* β_{int}*,* β_{ext} *are reals. Consequently, the expansion of* Π_n^1 *is given by*

$$\left\{\begin{array}{ll} \Pi_n^1(\mathbf{X}) = \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0}) \Big(X + \alpha_{int} \dfrac{\sin\theta}{R} + \beta_{int} \dfrac{\sin(2\theta)}{R^2} \Big) + \underset{R\to+\infty}{O}\left(\dfrac{1}{R^3}\right), & \text{in } \widehat{\Omega}_{int}, \\ \Pi_n^1(\mathbf{X}) = \partial_x u_n^0|_{\Omega_{ext}}(\mathbf{0}) \left(\alpha_{ext} \dfrac{\sin\theta}{R} + \beta_{int} \dfrac{\sin(2\theta)}{R^2} \right) + \underset{R\to+\infty}{O}\left(\dfrac{1}{R^3}\right), & \text{in } \widehat{\Omega}_{ext}. \end{array}\right. \tag{2.75}$$

Remark 9 *The coefficients* α_{ext} *and* α_{int} *are linked by the following relationship*

$$\alpha_{ext} + \alpha_{int} = 0. \tag{2.76}$$

Indeed this can be proved in the following way. Firstly, we observe that

$$\Delta\widetilde{\Pi}^1(\mathbf{X}) \times (R\sin\theta) - \widetilde{\Pi}^1(\mathbf{X})\,\Delta(R\sin\theta) = 0 \text{ in } \widehat{\Omega}. \tag{2.77}$$

Then, integrating this expression over B_R *the ball of radius* R *and of center* $\mathbf{0}$ *we get*

$$0 = \int_{\widehat{\Omega}\cap B_R} \Delta\widetilde{\Pi}^1(\mathbf{X}) \times (R\sin\theta) - \widetilde{\Pi}^1(\mathbf{X})\,\Delta(R\sin\theta)\, d\mathbf{X}. \tag{2.78}$$

The Green formula leads to

$$0 = \int_{\partial(\widehat{\Omega}\cap B_R)} \partial_R\widetilde{\Pi}^1(\mathbf{X}) \times (R\sin\theta) - \widetilde{\Pi}^1(\mathbf{X})\partial_R(R\sin\theta)\, Rd\theta. \tag{2.79}$$

Using (2.74)*, we get after some calculation*

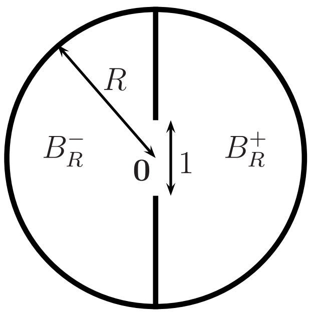

Figure 2.4: The integration domain $\widehat{\Omega}\cap B_R$

$$\Big(\int_0^{\pi} \sin^2(\theta)d\theta\Big)\alpha_{ext} + \Big(\int_{\pi}^{2\pi} \sin^2(\theta)d\theta\Big)\alpha_{int} = 0, \tag{2.80}$$

which leads to the conclusion.

Remark 10 ***Determination of*** α ***and*** β***.*** *We aim in this paragraph to determine the explicit expression of* $\alpha_{int,ext}$ *and* $\beta_{int,ext}$*, see* (2.74)*. We consider the following conformal mapping*

$$\hat{z} \mapsto z(\hat{z}) = \frac{\cosh\hat{z}}{2i} \tag{2.81}$$

which maps the complex band

$$\mathsf{B} = \Big\{ \widehat{z} = \widehat{x} + i\widehat{y} \in \mathbb{Z} : (\widehat{x}, \widehat{y}) \in \mathbb{R} \times \big[0, \pi\big] \Big\} \tag{2.82}$$

into

$$\Big\{ z = x + iy \in Z : (x, y) \in \widehat{\Omega} \Big\}. \tag{2.83}$$

This allows to determine the expression of $\widetilde{\Pi}^1$, *see* (2.73) *for its definition, which takes the form*

$$\widetilde{\Pi}^1(\mathbf{X}) = \frac{1}{2}\Im(\exp(-\widehat{z}(z))) \text{ with } z = X + iY. \tag{2.84}$$

Expanding this expression with respect to $R \to +\infty$ *leads to*

$$\begin{cases} \widetilde{\Pi}^1(\mathbf{X}) = X + \dfrac{1}{16}\dfrac{\sin\theta}{R} + \dfrac{1}{256}\dfrac{\sin(3\theta)}{R^3} + \underset{R\to+\infty}{O}\big(\dfrac{1}{R^5}\big) & \text{in } \widehat{\Omega}_{int}, \\ \widetilde{\Pi}^1(\mathbf{X}) = \quad -\dfrac{1}{16}\dfrac{\sin\theta}{R} - \dfrac{1}{256}\dfrac{\sin(3\theta)}{R^3} + \underset{R\to+\infty}{O}\big(\dfrac{1}{R^5}\big) & \text{in } \widehat{\Omega}_{ext}. \end{cases} \tag{2.85}$$

Identifying these expansions with the expansion (2.74) *we obtain*

$$\alpha_{int} = \frac{1}{16}, \quad \alpha_{ext} = -\frac{1}{16}, \quad \beta_{int} = \beta_{ext} = 0. \tag{2.86}$$

2.5.3 Obtention of u_n^1 and λ_n^1

In this section, we determine u_n^1 and λ_n^1. They are solutions of the following problem

$$\begin{cases} \text{Find } u_n^1 \in H^1(\Omega) \text{ and } \lambda_n^1 \in \mathbb{R} \text{ such that} \\ \Delta u_n^1 + \lambda_n^0 u_n^1 = -\lambda_n^1 u_n^0, \quad \text{in } \Omega, \\ u_n^1 = 0, \quad \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{cases} \tag{2.87}$$

By separation of variables, see appendix C.1, one can see that every solution u_n^1 of problem (2.87) has the behavior

$$u_n^1(\mathbf{x}) = \underset{r\to 0}{o}(1). \tag{2.88}$$

Lemma 2.5.2 *Every solution of problem* (2.87) *takes the form*

$$\lambda_n^1 = 0, \quad u_n^1 = \gamma\, u_n^0 \text{ in } \Omega_{int} \text{ and } u_n^1 = 0 \text{ in } \Omega_{ext} \text{ with } \gamma \in \mathbb{R}. \tag{2.89}$$

Proof. The function u_n^1 belongs to $H_0^1(\Omega)$. Moreover the problem (2.87) can be rewritten with its variational form

$$\left\{\begin{array}{ll} \text{Find } u_n^1 \in H_0^1(\Omega) = H_0^1(\Omega_{ext}) \times H_0^1(\Omega_{int}) \text{ and } \lambda_n^1 \in \mathbb{R} : & \\ \quad \mathsf{a}_{int}(u_n^1, v) - \lambda_n^0 (u_n^1, v)_{0,\Omega_{int}} = \lambda_n^1 \ell_{int}^1(v), & \forall v \in H_0^1(\Omega_{int}), \\ \quad \mathsf{a}_{ext}(u_n^1, v) - \lambda_n^0 (u_n^1, v)_{0,\Omega_{ext}} = 0, & \forall v \in H_0^1(\Omega_{ext}). \end{array}\right. \tag{2.90}$$

with $\mathsf{a}(u,v)$ and ℓ^1 defined for all u, v in $H_0^1(\Omega)$ by

$$\left\{\begin{array}{ll} \mathsf{a}_{int}(u,v) \;=\; \displaystyle\int_{\Omega_{int}} \Big(\nabla u \cdot \nabla v\Big) dxdy, & (u,v)_{0,\Omega_{int}} = \displaystyle\int_{\Omega_{int}} \big(uv\big) dxdy, \\ \mathsf{a}_{ext}(u,v) \;=\; \displaystyle\int_{\Omega_{ext}} \Big(\nabla u \cdot \nabla v\Big) dxdy, & (u,v)_{0,\Omega_{ext}} = \displaystyle\int_{\Omega_{ext}} \big(uv\big) dxdy, \\ \ell_{int}^1(v) = \displaystyle\int_{\Omega_{int}} \big(u^0 v\big) dxdy. & \end{array}\right. \tag{2.91}$$

Since λ_n^0 is an eigenvalue associated to the simple eigenvalue u_n^0 of the interior cavity, the Fredholm alternative allows us to say that u_n^1 exists if and only if

$$\lambda_n^1 \ell_{int}^1(u_n^0) = 0, \quad \Longleftrightarrow \quad \lambda_n^1 \int_{\Omega_{int}} \big(u_n^0\big)^2 dxdy = 0. \tag{2.92}$$

That is to say

$$\lambda_n^1 = 0. \tag{2.93}$$

Hence, the function u_n^1 solves the following problem

$$\left\{\begin{array}{ll} \text{Find } u_n^1 \in H_0^1(\Omega) \times H_0^1(\Omega) : & \\ \quad \mathsf{a}_{int}(u_n^1, v) - \lambda_n^0 (u_n^1, v)_{0,\Omega_{int}} = 0, & \forall v \in H_0^1(\Omega_{int}), \\ \quad \mathsf{a}_{ext}(u_n^1, v) - \lambda_n^0 (u_n^1, v)_{0,\Omega_{ext}} = 0, & \forall v \in H_0^1(\Omega_{ext}). \end{array}\right. \tag{2.94}$$

Since λ_n^0 is an eigenvalue of the interior cavity but not of the exterior one, we obtain that

$$u_n^1 = \gamma\, u_n^0 \text{ in } \Omega_{int} \text{ and } u_n^1 = 0 \text{ in } \Omega_{ext}, \quad \text{with } \gamma \in \mathbb{R}. \tag{2.95}$$

■

Remark 11 *The term u_n^1 of the asymptotic is defined up to a component proportional to u_n^0. In order to ensure its uniqueness, we add the additional property*

$$\int_\Omega u_n^1\, u_n^0 = 0. \tag{2.96}$$

This leads to $\gamma = 0$ and finally to

$$u_n^1 \equiv 0. \tag{2.97}$$

This is our choice for the rest of the report.

2.6 Conclusion: summary

The first order asymptotic expansion takes the form (n is an integer)

$$\lambda_n^\delta \simeq \lambda_n^0 + \delta\lambda_n^1 \text{ with } \quad \lambda_n^0 = \lambda_m \text{ and } \lambda_n^1 = 0, \tag{2.98}$$

$$u_n^\delta \simeq u_n^0 + \delta u_n^1 \text{ with } \quad u_n^0 = u_m \text{ and } u_n^1 = 0, \tag{2.99}$$

$$\Pi_n^\delta \simeq \Pi_n^0 + \delta\Pi_n^1 \text{ with } \quad \Pi_n^0 = 0 \text{ and } \Pi_n^1 = \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \widetilde{\Pi}^1, \tag{2.100}$$

see (2.73) and (2.85) for its spatial asymptotic expansion.

2.7 The second order asymptotic expansion

For the second order, we adopt the following notations

$$\begin{aligned} u_n^{2,\delta} &= u_n^0 + \delta\ u_n^1 + \delta^2\ u_n^2, \\ \lambda_n^{2,\delta} &= \lambda_n^0 + \delta\ \lambda_n^1 + \delta^2\ \lambda_n^2, \\ \Pi_n^{2,\delta} &= \Pi_n^0 + \delta\ \Pi_n^1 + \delta^2\ \Pi_n^2, \end{aligned} \tag{2.101}$$

where all the terms of second order remains to be determined.

2.7.1 Derivation of the second order

In this subsection, we will use the principle of section 2.3 to derive the problems satisfied by u_n^2, Π_n^2 and λ_n^2.

- We consider the far-field approximation of second order written in $\mathbf{x} = \delta\mathbf{X}$

$$u_n^0(\delta\mathbf{X}) + \delta^2 u_n^2(\delta\mathbf{X}) \quad (u_n^1 = 0). \tag{2.102}$$

- Then, we expand this sum up to $o(\delta^2)$. The spatial expansion of u_n^0 and u_n^2 is required. As u_n^0 is a regular function, we use its series Taylor expansion ($u_n^0(\mathbf{0}) = 0$, $\partial_y u_n^0(\mathbf{0}) = 0$, $\partial_y^2 u_n^0(\mathbf{0}) = 0$, $\partial_x^2 u_n^0(\mathbf{0}) = 0$)

$$u_n^0(x,y) \ = x\ \partial_x u_n^0(\mathbf{0}) \ + \ xy\ \partial_{xy}^2 u_n^0(\mathbf{0}) \ + \ \underset{r\to 0}{o}(r^2). \tag{2.103}$$

The term u_n^2 is solution of (2.6). It can be locally ($r \leq r_0$) decomposed into

$$u_n^2 = u_n^{2,P} \ + \ u_n^{2,H} \tag{2.104}$$

with

– $u_n^{2,P}$ a particular solution of (2.6)

$$\begin{cases} \Delta u_n^{2,P} + \lambda_n^0 u_n^{2,P} = -\lambda_n^2 u_n^0, & \text{in } \Omega \cap \{r \leq r_0\}, \\ u_n^{2,P} = 0, & \text{on } \big(\partial\Omega \setminus \{\mathbf{0}\}\big) \cap \{r \leq r_0\}. \end{cases} \tag{2.105}$$

Since u_n^0 is regular, $u_n^{2,P}$ can be chosen to be regular and can be expanded via its Taylor expansion

$$\begin{cases} u_n^{2,P}(x,y) \;=\; \underbrace{u_n^{2,P}(\mathbf{0})}_{0} \;+\; \underset{r\to 0}{o}(1) \;=\; \underset{r\to 0}{o}(1), & \text{in } \Omega_{int}, \\ u_n^{2,P}(x,y) \;=\; \underset{r\to 0}{o}(1), & \text{in } \Omega_{ext}. \end{cases} \tag{2.106}$$

– $u_n^{2,H}$ a homogeneous solution of the Helmholtz equation

$$\begin{cases} \Delta u_n^{2,H} + \lambda_n^0 u^{2,H} = 0, & \text{in } \Omega \cap \{r \leq r_0\}, \\ u_n^{2,H} = 0, & \text{on } \big(\partial\Omega \setminus \{\mathbf{0}\}\big) \cap \{r \leq r_0\}. \end{cases} \tag{2.107}$$

By separation of variables, see Appendix C.1, the term $u_n^{2,H}$ is given in Ω_{int} by

$$u_n^{2,H}(r,\theta) \;=\; \sum_{p=1}^{+\infty} \Big((a_{int}^2)_p \, \sin(p\theta)\, J_p(\sqrt{\lambda_n^0} r) + (b_{int}^2)_p \, \sin(p\theta)\, Y_p(\sqrt{\lambda_n^0} r) \Big), \tag{2.108}$$

and respectively in Ω_{ext} by

$$u_n^{2,H}(r,\theta) \;=\; \sum_{p=1}^{+\infty} \Big((a_{ext}^2)_p \, \sin(p\theta)\, J_p(\sqrt{\lambda_n^0} r) + (b_{ext}^2)_p \, \sin(p\theta)\, Y_p(\sqrt{\lambda_n^0} r) \Big). \tag{2.109}$$

In order to simplify the computations, we will suppose that (This can be proved using the technique of the first order by writing that $(U_n^i)_2 = 0$ for $i < 0$, see section 2.5.1)

$$(a_{int}^2)_p = 0 \text{ and } (a_{ext}^2)_p = 0 \text{ for } p > 2. \tag{2.110}$$

This leads to the following behavior

$$\begin{aligned} u_n^{2,H}(r,\theta) \;=\; & (a_{int}^2)_1 \, Y_1(\sqrt{\lambda_n^0} r) \sin(\theta) \\ & + \; (a_{int}^2)_2 \, Y_2(\sqrt{\lambda_n^0} r) \sin(2\theta) + \underset{r\to 0}{o}(1) \text{ in } \Omega_{int}, \end{aligned} \tag{2.111}$$

since

$$\sum_{p=1}^{+\infty} \Big((a_{ext}^2)_p \, \sin(p\theta)\, J_p(\sqrt{\lambda_n^0} r) \Big) = \underset{r\to 0}{o}(1); \tag{2.112}$$

respectively

$$
\begin{aligned}
u_n^{2,H}(r,\theta) \; &= \; (a_{ext}^2)_1 \; Y_1(\sqrt{\lambda_n^0} r) \sin(\theta) \\
&\quad + \; (a_{ext}^2)_2 \; Y_2(\sqrt{\lambda_n^0} r) \sin(2\theta) + \underset{r\to 0}{o}(1) \text{ in } \Omega_{ext}. \quad (2.113)
\end{aligned}
$$

The Bessel functions Y_1, and Y_2 can be expanded for $z \to 0$ (see [19])

$$
\begin{cases}
Y_1(z) := -\dfrac{2}{\pi z} + \underset{z\to 0}{o}(1), \\
Y_2(z) := -\dfrac{1}{\pi}\left(\dfrac{2}{z}\right)^2 - \dfrac{1}{\pi} + \underset{z\to 0}{o}(1).
\end{cases} \quad (2.114)
$$

Then, (2.111) and (2.113) can be rewritten respectively as follows

$$
\begin{aligned}
u_n^2(r,\theta) &= (a_{int}^2)_1 \Big(- \frac{2}{\pi\sqrt{\lambda_n^0}\; r} \sin(\theta) \Big) \\
&\quad + (a_{int}^2)_2 \left(-\frac{1}{\pi} \left(\frac{2}{\sqrt{\lambda_n^0}\; r} \right)^2 - \frac{1}{\pi} \right) \sin(2\theta) + \underset{r\to 0}{o}(1), \text{ in } \Omega_{int}, \quad (2.115)
\end{aligned}
$$

$$
\begin{aligned}
u_n^2(r,\theta) &= (a_{ext}^2)_1 \Big(- \frac{2}{\pi\sqrt{\lambda_n^0}\; r} \sin(\theta) \Big) \\
&\quad + (a_{ext}^2)_2 \left(-\frac{1}{\pi} \left(\frac{2}{\sqrt{\lambda_n^0}\; r} \right)^2 - \frac{1}{\pi} \right) \sin(2\theta) + \underset{r\to 0}{o}(1), \text{ in } \Omega_{ext}. \quad (2.116)
\end{aligned}
$$

With $r = R\,\delta$, $x = X\delta$, $y = Y\delta$, (2.102) can be written in Ω_{int} as

$$
\begin{aligned}
&\delta \; X \; \partial_x u_n^{\mathbf{0}}|_{\Omega_{int}}(\mathbf{0}) \; + \; \delta^2 \; XY \; \partial_{xy}^2 u_n^0|_{\Omega_{int}}(\mathbf{0}) \\
&\quad + \; \delta^2 \Big[(a_{int}^2)_1 \left(-\frac{2}{\pi\sqrt{\lambda_n^0}\; R\delta} \right) \sin(\theta) + \\
&\quad (a_{int}^2)_2 \left(-\frac{1}{\pi} \left(\frac{2}{\sqrt{\lambda_n^0}\; R\delta} \right)^2 - \frac{1}{\pi} \right) \sin(2\theta) \Big] + \underset{\delta\to 0}{o}(\delta^2) \quad (2.117)
\end{aligned}
$$

and in Ω_{ext} as

$$\delta^2 \Big[(a^2_{ext})_1 \left(-\frac{2}{\pi \sqrt{\lambda^0_n}\ R\delta} \right) \sin(\theta) +$$

$$(a^2_{ext})_2 \left(-\frac{1}{\pi} \left(\frac{2}{\sqrt{\lambda^0_n}\ R\delta} \right)^2 - \frac{1}{\pi} \right) \sin(2\theta) \Big] + \underset{\delta \to 0}{o}(\delta^2). \quad (2.118)$$

Then we order (2.117) (resp. (2.118)) with respect to the order of δ

$$\Big(-\frac{1}{\pi} \frac{4\ (a^2_{int})_2}{\lambda^0_n\ R^2} \sin(2\theta) \Big) + \delta \left(X\ \partial_x u^{\mathbf{0}}_n |_{\Omega_{int}}(\mathbf{0})\ -\ \frac{2\ (a^2_{int})_1\ \sin(\theta)}{\pi \sqrt{\lambda^0_n}\ R} \right)$$

$$+\ \delta^2 \left(XY\ \partial^2_{xy} u^0_n |_{\Omega_{int}}(\mathbf{0})\ +\ (a^2_{int})_2 \left(-\frac{1}{\pi} \right) \right)\ +\ \underset{\delta \to 0}{o}(\delta^2), \quad (2.119)$$

respectively

$$\left(-\frac{1}{\pi} \frac{4\ (a^2_{ext})_2}{\lambda^0_n\ R^2} \sin(2\theta) \right) + \delta \left(-\frac{2\ (a^2_{ext})_1\ \sin(\theta)}{\pi \sqrt{\lambda^0_n}\ R} \right) + \delta^2\ (a^2_{ext})_2 \left(-\frac{1}{\pi} \right) + \underset{\delta \to 0}{o}(\delta^2). \quad (2.120)$$

Therefore, $(U^0_n)_2$, $(U^1_n)_2$, $(U^2_n)_2$ in Ω_{int} and in Ω_{ext} are given by

$$(U^0_n)_2 = -\frac{4\ (a^2_{int})_2\ \sin(2\theta)}{\pi \lambda^0_n\ R^2},\ \text{in } \Omega_{int} \text{ and } (U^0_n)_2 = -\frac{4\ (a^2_{ext})_2\ \sin(2\theta)}{\pi \lambda^0_n\ R^2},\ \text{in } \Omega_{ext}, \quad (2.121)$$

$$(U^1_n)_2 = X\ \partial_x u^{\mathbf{0}}_n |_{\Omega_{int}}(\mathbf{0})\ -\ \frac{2\ (a^2_{int})_1\ \sin(\theta)}{\pi \sqrt{\lambda^0_n} R},\ \text{in } \Omega_{int}$$

$$\text{and } (U^1_n)_2 = -\frac{2\ (a^2_{ext})_1\ \sin(\theta)}{\pi \sqrt{\lambda^0_n} R},\ \text{in } \Omega_{ext}, \quad (2.122)$$

$$(U^2_n)_2 = XY\ \partial^2_{xy} u^0_n |_{\Omega_{int}}(\mathbf{0})\ +\ (a^2_{int})_2 \left(-\frac{1}{\pi} \right),\ \text{in } \Omega_{int}$$

$$\text{and } (U^2_n)_2 = (a^2_{ext})_2 \left(-\frac{1}{\pi} \right),\ \text{in } \Omega_{ext}. \quad (2.123)$$

- In the continuation we shall match $(U_n^i)_2$ with the asymptotic expansions at infinity of the near-field Π_n^i. We recall that the behaviors of Π_n^0 and Π_n^1 defined by (2.34) and (2.60) are (the behavior of Π_n^2 will be exactly $(U_n^2)_2$)

$$\Pi_n^0(\mathbf{X}) \;=\; 0, \tag{2.124}$$

$$\begin{cases} \Pi_n^1(\mathbf{X}) = \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\Big(X + \alpha_{int}\dfrac{\sin\theta}{R}\Big) + \underset{R\to+\infty}{o}\left(\dfrac{1}{R}\right), & \text{in } \widehat{\Omega}_{int}, \\ \Pi_n^1(\mathbf{X}) = \partial_x u_n^0|_{\Omega_{ext}}(\mathbf{0})\;\alpha_{ext}\dfrac{\sin\theta}{R} + \underset{R\to+\infty}{o}\left(\dfrac{1}{R}\right), & \text{in } \widehat{\Omega}_{ext}. \end{cases} \tag{2.125}$$

- Finally, we use the matching conditions of second order

$$\begin{cases} \Pi_n^0(\mathbf{X}) - (U_n^0)_2(\mathbf{X}) = \underset{R\to+\infty}{o}(\dfrac{1}{R^2}), & \text{in } \Omega_{int} \text{ and } \Omega_{ext}, \quad (a) \\ \Pi_n^1(\mathbf{X}) - (U_n^1)_2(\mathbf{X}) = \underset{R\to+\infty}{o}(\dfrac{1}{R}), & \text{in } \Omega_{int} \text{ and } \Omega_{ext}, \quad (b) \\ \Pi_n^2(\mathbf{X}) - (U_n^2)_2(\mathbf{X}) = \underset{R\to+\infty}{o}(1), & \text{in } \Omega_{int} \text{ and } \Omega_{ext}. \quad (c) \end{cases} \tag{2.126}$$

Due to equations (2.121), (2.124), (2.126,a), we obtain $(a^2_{int,ext})_2 = 0$, and using (2.122), (2.125), (2.126,b), we deduce $(a^2_{int,ext})_1 \;=\; -\dfrac{\pi\sqrt{\lambda_n^0}}{2}\,\partial_x u_n^0(\mathbf{0})\;\alpha_{int,ext}$. Finally, we use (2.123), and (2.126,c) to obtain the behavior of u_n^2 and Π_n^2.

- **Conclusion.** The following behaviors of u_n^2 are required in order that the matching occurs:

$$\begin{cases} u_n^2(\mathbf{x}) = -\dfrac{\pi\sqrt{\lambda_n^0}}{2}\,\partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\;\alpha_{int}\;Y_1(\sqrt{\lambda_n^0}r)\;\sin(\theta) \;+\; \underset{r\to 0}{o}(1), & \text{in } \Omega_{int}, \\ u_n^2(\mathbf{x}) = -\dfrac{\pi\sqrt{\lambda_n^0}}{2}\,\partial_x u_n^0|_{\Omega_{ext}}(\mathbf{0})\;\alpha_{ext}\;Y_1(\sqrt{\lambda_n^0}r)\;\sin(\theta) \;+\; \underset{r\to 0}{o}(1), & \text{in } \Omega_{ext}, \end{cases} \tag{2.127}$$

or equivalently

$$\begin{cases} u_n^2(\mathbf{x}) = \;\partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\;\alpha_{int}\;\dfrac{\sin(\theta)}{r} \;+\; \underset{r\to 0}{o}(1), & \text{in } \Omega_{int}, \\ u_n^2(\mathbf{x}) = \;\partial_x u_n^0|_{\Omega_{ext}}(\mathbf{0})\;\alpha_{ext}\;\dfrac{\sin(\theta)}{r} \;+\; \underset{r\to 0}{o}(1), & \text{in } \Omega_{ext}, \end{cases} \tag{2.128}$$

$$\begin{cases} \Pi_n^2(\mathbf{X}) \;=\; XY\;\partial^2_{xy} u_n^0|_{\Omega_{int}}(\mathbf{0}) + \underset{R\to+\infty}{o}(1), & \text{in } \widehat{\Omega}_{int}, \\ \Pi_n^2(\mathbf{X}) \;=\; \underset{R\to+\infty}{o}(1), & \text{in } \widehat{\Omega}_{ext}, \end{cases} \tag{2.129}$$

with α_{int} and α_{ext} defined by the spatial expansion of $\widetilde{\Pi}^1$ given by (2.73) and (2.74).

Remark 12 *In the continuation the definition of the U's will be required. We give here their forms*

$$(U_n^0)_2 = 0 \text{ in } \Omega_{int} \text{ and } \Omega_{ext}, \tag{2.130}$$

$$(U_n^1)_2 = \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\Big(X+\alpha_{int}\frac{\sin\theta}{R}\Big) \text{ in } \Omega_{int} \text{ and } (U_n^1)_2 = \partial_x u_n^0|_{\Omega_{ext}}(\mathbf{0})\ \alpha_{ext}\frac{\sin\theta}{R} \text{ in } \Omega_{ext}, \tag{2.131}$$

$$(U_n^2)_2 = XY\ \partial_{xy}^2 u_n^0|_{\Omega_{int}}(\mathbf{0}) \text{ in } \Omega_{int} \text{ and } (U_n^2)_2 = 0 \text{ in } \Omega_{ext}. \tag{2.132}$$

2.7.2 Existence and uniqueness of Π_n^2

In this section we give the concrete definition of Π_n^2.

The function Π^2 is solution of the following problem

$$\begin{cases} \text{Find } \Pi_n^2 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ -\Delta \Pi_n^2 = 0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^2 = 0, \qquad \text{on } \partial\widehat{\Omega}, \\ \Pi_n^2(\mathbf{X}) \ - \ \Psi_{int}(\mathbf{X})\ XY\ \partial_{xy}^2 u_n^0|_{\Omega_{int}}(\mathbf{0}) \ \in K_0^1, \end{cases} \tag{2.133}$$

with Ψ_{int} a regular cut-off function defined by (2.61). By separation of variables, see Appendix C.2, it is easy to see that the last line of (2.133) prescribes the asymptotic behavior

$$\begin{cases} \Pi_n^2(\mathbf{X}) = XY\ \partial_{xy}^2 u_n^0(\mathbf{0}) \ + \underset{R\to+\infty}{o}(1), & \text{in } \Omega_{int}, \\ \Pi_n^2(\mathbf{X}) = \underset{R\to+\infty}{o}(1), & \text{in } \Omega_{ext}. \end{cases} \tag{2.134}$$

Introducing the auxiliary function $\omega_n^2(\mathbf{X}) = \Pi_n^2(\mathbf{X}) \ - \ \Psi_{int}(\mathbf{X})\ XY\ \partial_{xy}^2 u_n^0|_{\Omega_{int}}(\mathbf{0})$ and applying corollary 2.5.1, it is easy to prove that the problem (2.133) is well-posed.

2.7.3 Existence and uniqueness of u_n^2 and λ_n^2

Here we give the concrete definition of u_n^2 and λ_n^2. In the last chapters, we have seen that they are solutions of the following problem

$$\left\{\begin{array}{l} \text{Find } u_n^2 : \Omega \to \mathbb{R} \text{ and } \lambda_n^2 \in \mathbb{R} \text{ such that} \\ \quad \Delta u_n^2 + \lambda_n^0 u_n^2 = -\lambda_n^2 u_n^0, \quad \text{in } \Omega, \\ \quad u_n^2 = 0, \qquad\qquad\qquad\qquad \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \\ u_n^2(\mathbf{x}) - \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \dfrac{\sin(\theta)}{r} \in H^1(\Omega_{int}), \\ u_n^2(\mathbf{x}) - \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{ext}\ \dfrac{\sin(\theta)}{r} \in H^1(\Omega_{ext}), \end{array}\right. \tag{2.135}$$

By separation of variables, see Appendix C.1, one can note that the two last lines of (2.135) prescribes the asymptotic behavior (2.128).

The following Lemma ensures the existence and uniqueness of u_n^2 and λ_n^2 up to knowledge of the u_n^0-component of u_n^2. This component will be arbitrary chosen.

Lemma 2.7.1 *The solution of problem* (2.135) *exists. Moreover if* (u_n^2, λ_n^2) *and* $(u_{n,*}^2, \lambda_{n,*}^2)$ *are solutions, one has* $\lambda_n^2 = \lambda_{n,*}^2$ *and*

$$\lambda_n^2 = -\alpha_{int}\ \pi \frac{|\partial_x u_n^0|_{\partial\Omega_{int}}(\mathbf{0})|^2}{\|u_n^0\|_0^2}, \tag{2.136}$$

$$\exists \gamma \in \mathbb{R} : u_{n,*}^2 - u_n^2 = \gamma u_n^0. \tag{2.137}$$

Proof. In order to prove the existence of u_n^2, we introduce the auxiliary function ω_n^2

$$\left\{\begin{array}{ll} \omega_n^2 = u_n^2 - \chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \dfrac{\sin(\theta)}{r}, & \text{in } \Omega_{int} \quad (\in H^1\big(\Omega_{int}\big)), \\ \omega_n^2 = u_n^2 - \chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{ext}\ \dfrac{\sin(\theta)}{r}, & \text{in } \Omega_{ext} \quad (\in H^1\big(\Omega_{ext}\big)), \end{array}\right. \tag{2.138}$$

with χ the regular cut-off function satisfying

$$\left\{\begin{array}{l} \chi(z) = 0, \text{ if } z \leq 1, \\ \chi(z) = 1, \text{ if } z \geq 2. \end{array}\right. \tag{2.139}$$

Using (2.135), ω^2 is solution of the problem

$$\left\{\begin{array}{l} \text{Find } \omega_n^2 \in H_0^1(\Omega) \text{ such that} \\ \Delta\omega_n^2 + \lambda_n^0 \omega_n^2 = F_n^2 \text{ in } \Omega_{int} \quad \text{and} \quad \omega_n^2 = 0 \text{ in } \partial\Omega_{int}, \\ \Delta\omega_n^2 + \lambda_n^0 \omega_n^2 = F_n^2 \text{ in } \Omega_{ext} \quad \text{and} \quad \omega_n^2 = 0 \text{ in } \partial\Omega_{ext}, \end{array}\right. \tag{2.140}$$

with $F_n^2 : \Omega \to \mathbb{C}$ defined by

$$\left\{ \begin{array}{lll} F_n^2 &=& -\lambda_n^2 u_n^0 - \big(\Delta + \lambda_n^0\big)\big(\chi(r)\, \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \alpha_{int}\, \dfrac{\sin(\theta)}{r}\big), \qquad \text{in } \Omega_{int}, \\ F_n^2 &=& -\big(\Delta + \lambda_n^0\big)\big(\chi(r)\, \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \alpha_{ext}\, \dfrac{\sin(\theta)}{r}\big), \qquad \text{in } \Omega_{ext}. \end{array} \right. \tag{2.141}$$

Since λ_n^0 is an eigenvalue associated to the eigenvector u_n^0 (which is simple) of the laplacian in Ω_{int}, the problem (2.140) defining ω_n^2 has solutions if and only if

$$\int_\Omega F_n^2\, u_n^0 = 0. \tag{2.142}$$

Moreover this solution is determined up to its u_n^0–component, ie. if ω^2 and ω_*^2 are two solutions of (2.140) then

$$\exists \gamma \in \mathbb{R} : \omega_n^2 = \omega_*^2 + \gamma u_n^0. \tag{2.143}$$

- Since $u_n^0 \equiv 0$ in Ω_{int}, the condition (2.142) takes the form

$$\lambda_n^2 \int_{\Omega_{int}} \big(u_n^0\big)^2 + \int_{\Omega_{int}} \big(\Delta + \lambda_n^0\big)\big(\chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\big) u_n^0 \ = \ 0. \tag{2.144}$$

Since $\Delta u_n^0 + \lambda_n^0 u_n^0 = 0$, this leads to

$$\begin{aligned} \lambda_n^2\, \|u_n^0\|_0^2 = &- \int_{\Omega_{int}} \Delta\big(\chi(r)\, \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \alpha_{int}\, \frac{\sin(\theta)}{r}\big) u_n^0 \\ &+ \int_{\Omega_{int}} \chi(r)\, \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \alpha_{int}\, \frac{\sin(\theta)}{r}\big(\Delta u_n^0\big), \text{ in } \Omega_{int}. \end{aligned} \tag{2.145}$$

Introducing the ball B_η of center $\mathbf{0}$ and radius η (see Figure (2.5)). Since the domain $\Omega_{int} \setminus B_\eta$ tends to Ω_{int} when $\eta \to 0$, we have (Lebesgues Theorem)

$$\begin{aligned} \lambda_n^2\, \|u_n^0\|_0^2 = \lim_{\eta \to 0} \Big[&- \int_{\Omega_{int} \setminus B_\eta} \Delta\big(\chi(r)\, \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \alpha_{int}\, \frac{\sin(\theta)}{r}\big) u_n^0 \\ &+ \int_{\Omega_{int} \setminus B_\eta} \chi(r)\, \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\, \alpha_{int}\, \frac{\sin(\theta)}{r}\big(\Delta u_n^0\big)\Big], \text{ in } \Omega_{int}. \end{aligned} \tag{2.146}$$

Two Green formulas lead to

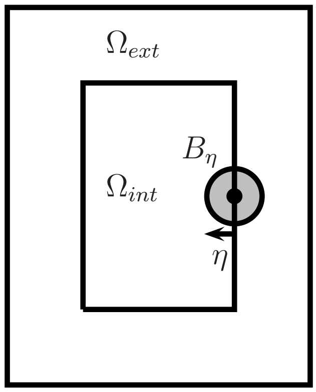

Figure 2.5: The ball B_η.

$$\begin{aligned}
&\int_{\Omega_{int}\setminus B_\eta} \Delta\big(\chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\big)u_n^0 \\
&\quad = -\Big[\int_0^\pi \partial_r\big(\chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\big)u_n^0\ r\ d\theta\Big](r=\eta) \\
&\qquad - \int_{\Omega_{int}\setminus B_\eta} \nabla\big(\chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\big)\nabla u_n^0\ r\ dr\ d\theta, \quad (2.147)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_{int}\setminus B_\eta} \chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\big(\Delta u_n^0\big) \\
&\quad = -\Big[\int_0^\pi \chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\ \partial_r u_n^0|_{\Omega_{int}}(r)\ r\ d\theta\Big](r=\eta) \\
&\qquad - \int_{\Omega_{int}\setminus B_\eta} \nabla\big(\chi(r)\ \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\big)\nabla u_{int}^0\ r\ dr\ d\theta. \quad (2.148)
\end{aligned}$$

Inserting (2.147), and (2.148) in (2.146), we obtain ($\chi(\eta) = 1$ and $\partial_r\chi(\eta) = 0$, since η is small)

$$\begin{aligned}
\lambda_n^2\|u_n^0\|_0^2 \ =\ &\lim_{\eta\to 0}\Big\{\int_0^\pi \partial_r\big(\partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\big)u_n^0\ r\ d\theta \\
&- \int_0^\pi \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \frac{\sin(\theta)}{r}\ \partial_r u_n^0|_{\Omega_{int}}\ r\ dr\ d\theta\Big\}(r=\eta),\ \text{in } \Omega_{int}. \quad (2.149)
\end{aligned}$$

Using the first order Taylor expansion of u_n^0 and of $\partial_r u_n^0$ in Ω_{int} (see (2.35)), we obtain

$$\left\{\begin{array}{lcl} u_n^0(r,\theta) & = & x\,\partial_x u_n^0(\mathbf{0}) + O(r^2) = r\ \sin\theta\ \partial_x u_n^0(\mathbf{0}) + \underset{r\to 0}{O}(r^2), \\ \partial_r u_n^0(r,\theta) & = & \sin\theta\ \partial_x u_n^0(\mathbf{0}) + \underset{r\to 0}{O}(r). \end{array}\right. \tag{2.150}$$

Inserting (2.150) in (2.149), we have

$$\lambda_n^2 \|u_n^0\|_0^2 \quad = \quad \lim_{\eta\to 0}\Big\{ -2\int_0^{\pi} \big(\partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\big)^2\ \alpha_{int}\ \ \sin^2(\theta) d\theta + \underset{\eta\to 0}{O}(\eta)\Big\}. \tag{2.151}$$

Taking the limit we get (2.136).

- Under condition (2.136), we have seen that ω_n^2 is defined up to its u_n^0–component, see (2.143), we get the last result. Taking into account (2.138), we finally obtain (2.137).

■

2.7.4 Spatial expansion of the second order fields

In the continuation, a precise expression of the behavior of u_n^2 and Π_n^2 will be required. By separation of variables, see Appendix B, one can prove that they have the following expressions

$$\Pi_n^2(\mathbf{X}) = \partial^2_{xy} u_n^0|_{\Omega_{int}}(\mathbf{0})\ \Big(XY + \mu_{int}\ \frac{\sin(\theta)}{R}\Big) + \underset{R\to+\infty}{O}(\frac{1}{R^2}) \text{ in } \widehat{\Omega}_{int}, \tag{2.152}$$

$$\Pi_n^2(\mathbf{X}) = \partial^2_{xy} u_n^0|_{\Omega_{ext}}(\mathbf{0})\mu_{ext}\ \frac{\sin(\theta)}{R} + \underset{R\to+\infty}{O}(\frac{1}{R^2}) \text{ in } \widehat{\Omega}_{ext}, \tag{2.153}$$

$$u_n^2(\mathbf{x}) = \partial_x u_n^0|_{\Omega_{int}}(\mathbf{0})\ \alpha_{int}\ \Big(\frac{1}{r} - \frac{\lambda_n^0 r}{2}\Big(\ln\frac{\sqrt{\lambda_n^0} r}{2}\Big) + \gamma_{int}\ r\Big)\ \sin(\theta)\ +\ \underset{r\to 0}{O}(r^2) \text{ in } \Omega_{int}, \tag{2.154}$$

$$u_n^2(\mathbf{x}) = \partial_x u_n^0|_{\Omega_{ext}}(\mathbf{0})\ \alpha_{ext}\ \Big(\frac{1}{r} - \frac{\lambda_n^0 r}{2}\Big(\ln\frac{\sqrt{\lambda_n^0} r}{2}\Big) + \gamma_{ext}\ r\Big)\ \sin(\theta)\ +\ \underset{r\to 0}{O}(r^2) \text{ in } \Omega_{ext}, \tag{2.155}$$

with γ_{int} depending on the u_n^0-component of u_n^2 (it can be chosen to be zero).

Remark 13 *For Π_n^2, we can be much more precise. Using the conformal mapping of remark 10, we obtain the expression of Π_n^2*

$$\Pi_n^2(\mathbf{X}) = -2\ \partial^2_{xy} u_n^0|_{\Omega_{int}}(\mathbf{0})\ \Im(\exp(-2\widehat{z}(z))) \tag{2.156}$$

with the conformal mapping

$$z(\widehat{z}) = \frac{1}{2i}\cosh(\widehat{z}), \quad \widehat{z} = \widehat{X} + i\widehat{Y} \quad \textit{and} \quad z = X + iY. \tag{2.157}$$

Expanding this expression for $R \longrightarrow +\infty$

$$\left\{\begin{array}{lcl} \Pi_n^2(\mathbf{X}) & = & \partial_{xy}^2 u_n^0|_{\Omega_{int}}(\mathbf{0})\left(XY + \frac{1}{256}\frac{\sin(2\theta)}{R^2}\right) + O(\frac{1}{R^4}) \ \textit{in} \ \widehat{\Omega}_{int}, \\ \Pi_n^2(\mathbf{X}) & = & \partial_{xy}^2 u_n^0|_{\Omega_{int}}(\mathbf{0})\left(\qquad - \frac{1}{256}\frac{\sin(2\theta)}{R^2}\right) + O(\frac{1}{R^4}) \ \textit{in} \ \widehat{\Omega}_{ext}. \end{array}\right. \tag{2.158}$$

Chapter 3

Theoritical result: Error estimates

For $n \in \mathbb{N}$ we defined in the last chapter the eigenvalue terms λ_n^0, λ_n^1, λ_n^2 the far-field terms u_n^0, u_n^1, u_n^2 and the near-fields terms Π_n^0, Π_n^1, Π_n^2 by well-posed problems, see Chapter 2 or Appendix A.

Since the derivation of these problems was a consequence of the formal (not based only on rigorous consideration) technique of Matching of Asymptotic Expansions, there is no evidence that $\lambda_n^0 + \delta\lambda_n^1 + \delta^2\lambda_n^2$ is an asymptotic expansion of an eigenvalue λ_n^δ of the Dirichlet-Laplacian in Ω^δ.

The following Theorem, that we aim to prove in this chapter, give a concreet answer to this question.

Theorem 3 *Let λ_n^δ be the n^{th} eigenvalue of the Dirichlet-Laplacian in Ω^δ. Let λ_n and u_n be the n^{th} eigenvalue and eigenvector of the Dirichlet-Laplacian in Ω. Under hypothesis* (1.8)*, we have*

$$\forall n \in \mathbb{N} \quad \exists C > 0 \text{ and } \delta_0 > 0 : \quad \forall \delta \in [0, \delta_0] \quad |\lambda_n^\delta - \lambda_n - \delta^2\lambda_n^2| \leq C\,\delta^3\,\Big|\ln\delta\Big| \tag{3.1}$$

with

$$\begin{cases} \lambda_n^2 = -\dfrac{\pi}{16}\dfrac{|\partial_x u_n\big|_{\Omega_{int}}(\mathbf{0})|^2}{\|u_n\|^2_{L^2(\Omega_{int})}}, & \text{if } u_n = 0 \text{ in } \Omega_{ext}, \\ \lambda_n^2 = -\dfrac{\pi}{16}\dfrac{|\partial_x u_n\big|_{\Omega_{ext}}(\mathbf{0})|^2}{\|u_n\|^2_{L^2(\Omega_{ext})}}, & \text{if } u_n = 0 \text{ in } \Omega_{int}. \end{cases} \tag{3.2}$$

Remark 14 *One can note that the last Theorem reveals the first order Taylor expansion of $\delta \mapsto \lambda_n^\delta$*

$$\lambda_n^\delta = \lambda_n + \delta^2\lambda_n^2 + \underset{\delta\to 0}{o}(\delta^2). \tag{3.3}$$

The proof of Theorem 3 will be decomposed into two steps.

- Section 3.1 is devoted to the first step. For every n, we will prove the existence of an eigenvalue of the Dirichlet-Laplacian of Ω^δ in a $\delta^3\ \ln\delta$ neighborhood of $\lambda_n + \delta^2\ \lambda_n^2$. The key argument will be Theorem 7 of Appendix B.
- This result being demonstrated, it is still possible to do not have a one by one correspondence between the eigenvalues λ_n^δ and λ_n. In Section 3.2 we prove this one by one mapping using the min-max principle, see Theorem 6 of Appendix B.

3.1 First step

This section is devoted to the proof of the following theorem.

Theorem 4 *Let λ_n and u_n be the n^{th} eigenvalue and eigenvector of the Dirichlet-Laplacian in Ω. Under hypothesis* (1.8)*, we have:*

There exists $\delta_0 > 0$ such that for all $\delta \in]0, \delta_0[$ there exists an eigenvalue λ^δ of the Dirichlet-Laplacian of Ω^δ, see (1.4), satisfying

$$\left|\lambda^\delta - (\lambda_n + \delta^2\lambda_n^2)\right| \leq C\ \delta^3\ |\ln(\delta)|. \tag{3.4}$$

with λ_n^2 given by

$$\begin{cases} \lambda_n^2 = -\dfrac{\pi}{16}\dfrac{|\partial_x u_n\big|_{\Omega_{int}}(\mathbf{0})|^2}{\|u_n\|^2_{L^2(\Omega_{int})}}, & \text{if } u_n = 0 \text{ in } \Omega_{ext}, \\ \lambda_n^2 = -\dfrac{\pi}{16}\dfrac{|\partial_x u_n\big|_{\Omega_{ext}}(\mathbf{0})|^2}{\|u_n\|^2_{L^2(\Omega_{ext})}}, & \text{if } u_n = 0 \text{ in } \Omega_{int}. \end{cases} \tag{3.5}$$

Proof. Since Ω is not connected, λ_n^0 is either an eigenvalue of Ω_{int} or of Ω_{ext} ie. $u_n^0 = 0$ in Ω_{int} or in Ω_{ext}. Due to symmetry, one has only to consider the case where λ^0 is an eigenvalue of Ω_{int}.

The first step of the proof consists in constructing a quasi-mode: A uniformly valid approximation or global approximation of the eigenvector u^δ (The definition of u_n^i, Π_n^i and U_n^i can be found in Appendix A and the notation "ˆ" refers to the change of variable $\widehat{\Pi}(\mathbf{x}) = \Pi(\mathbf{x}/\delta)$)

$$\widetilde{w}_n^\delta = \chi^\delta\left(u_n^0 + \delta\ u_n^1 + \delta^2\ u_n^2 + \delta^3\ u_n^3\right) + \Psi\left(\widehat{\Pi}_n^0 + \delta\ \widehat{\Pi}_n^1 + \delta^2\ \widehat{\Pi}_n^2 + \delta^3\ \widehat{\Pi}_n^{3,0} + \delta^3 \ln\delta\ \widehat{\Pi}_n^{3,1}\right)$$

$$- \chi^\delta\Psi\left((\widehat{U}_n^0)_3 + \delta\ (\widehat{U}_n^1)_3 + \delta^2\ (\widehat{U}_n^2)_3 + \delta^3\ (\widehat{U}_n^{3,0})_3 + \delta^3\ln\delta\ (\widehat{U}_n^{3,1})_3\right), \tag{3.6}$$

i.e

$$\widetilde{w}_n^\delta = \chi^\delta \left(u_n^0 + +\delta^2\ u_n^2\right) + \Psi\Big(\delta\ \widehat{\Pi}_n^1 + \delta^2\ \widehat{\Pi}_n^2 + \delta^3\ \widehat{\Pi}_n^{3,0} + \delta^3 \ln\delta\ \widehat{\Pi}_n^{3,1}\Big)$$

$$- \chi^\delta \Psi\Big((\widehat{U}_n^0)_3 + \delta\ (\widehat{U}_n^1)_3 + \delta^2\ (\widehat{U}_n^2)_3 + \delta^3\ (\widehat{U}_n^{3,0})_3 + \delta^3 \ln\delta\ (\widehat{U}_n^{3,1})_3\Big), \quad (3.7)$$

and in proving the following non trivial estimate

$$\Big|\mathsf{a}(\widetilde{w}_n^\delta, v) - \left(\lambda_n^0 + \delta^2\ \lambda_n^2\right)\ (\widetilde{w}_n^\delta, v)_0\Big| \leq C\ \delta^3\ |\ln\delta|\ \|\widetilde{w}_n^\delta\|_0 \|v\|_0, \quad \forall v \in H_0^1(\Omega^\delta). \quad (3.8)$$

Obtention of estimate (3.8)**.** Firstly, we use the Green formula

$$\Big|\mathsf{a}\left(\widetilde{w}_n^\delta, v\right) - \left(\lambda_n^0 + \delta^2\lambda_n^2\right)\left(\widetilde{w}_n^\delta, v\right)\Big| = \Big|\int_{\Omega^\delta} \left(\Delta\widetilde{w}_n^\delta + \left(\lambda_n^0 + \delta^2\lambda_n^2\right)\ \widetilde{w}_n^\delta\right)\ v\Big|. \quad (3.9)$$

The right hand side of (3.9) be decomposed into two terms: The first one over Ω_{int} and the second one over Ω_{ext}.

$$\Big|\mathsf{a}\left(\widetilde{w}_n^\delta, v\right) - \left(\lambda_n^0 + \delta^2\lambda_n^2\right)\left(\widetilde{w}_n^\delta, v\right)\Big| = \Big|\int_{\Omega_{int}} \left(\Delta\widetilde{w}_n^\delta + \left(\lambda_n^0 + \delta^2\lambda_n^2\right)\ \widetilde{w}_n^\delta\right) v\Big|$$

$$+ \Big|\int_{\Omega_{ext}} \left(\Delta\widetilde{w}_n^\delta + \left(\lambda_n^0 + \delta^2\lambda_n^2\right)\ \widetilde{w}_n^\delta\right) v\Big|, \quad \forall v \in H_0^1(\Omega^\delta). \quad (3.10)$$

Here we only estimate the Ω_{int}-part. The Ω_{ext}-part can be estimated in the same way.

Then, we explicit the expression of $\Delta\widetilde{w}_n^\delta + \left(\lambda_n^0 + \delta^2\lambda_n^2\right)\ \widetilde{w}_n^\delta$ as follows (χ^δ and Ψ

does not commute with the laplacian)

$$
\begin{aligned}
\Delta \widetilde{w}_n^\delta + (\lambda_n^0 + \delta^2 \lambda_n^2)\ \widetilde{w}_n^\delta = \quad & \chi^\delta \Big(\underbrace{(\Delta + \lambda_n^0) u_n^0}_{=0,\ \text{see (2.4)}} + \delta^2 \big(\underbrace{(\Delta + \lambda_n^0) u_n^2 + \lambda_n^2 u_n^0}_{=0,\ \text{see (2.135)}} \big) \\
& \qquad + \delta^4\ \lambda_n^2\ u_n^2 \Big) \\
& \Psi \Big(\underbrace{\delta \Delta \widehat{\Pi}_n^1}_{=0,\ \text{see (2.60)}} + \underbrace{\delta^2 \Delta \widehat{\Pi}_n^2}_{=0,\ \text{see (2.133)}} + \underbrace{\delta^3 \Delta \widehat{\Pi}_n^{3,0}}_{=-\lambda_n^0\ \delta\ \widehat{\Pi}^1,\ \text{see (A.14)}} + \\
& \qquad \underbrace{\delta^3 \ln \delta \Delta \widehat{\Pi}_n^{3,1}}_{=0,\ \text{see (A.15)}} \\
& \quad + \big(\lambda_n^0 + \delta^2\ \lambda_n^2\big) \big(\delta \widehat{\Pi}_n^1 + \delta^2\ \widehat{\Pi}_n^2 + \delta^3\ \widehat{\Pi}_n^{3,0} \\
& \qquad + \delta^3 \ln \delta\ \widehat{\Pi}_n^{3,1} \big) \Big) \\
& -\chi^\delta \Psi \Big(\underbrace{\delta \Delta\ (\widehat{U}_n^1)_3}_{=0,\ \text{see (2.131)}} + \underbrace{\delta^2\ \Delta (\widehat{U}_n^2)_3}_{=0,\ \text{see (2.132)}} + \underbrace{\delta^3 \Delta\ (\widehat{U}_n^{3,0})_3}_{=-\delta\ \lambda_n^0 (\widehat{U}_n^1)_3} \\
& \qquad + \underbrace{\delta^3 \ln \delta\ \Delta\ (\widehat{U}_n^{3,1})_3}_{=0,\ \text{see (2.131)}} \\
& \quad + \big(\lambda_n^0 + \delta^2\ \lambda_n^2\big) \big(\delta (\widehat{U}_n^1)_3 + \delta^2 (\widehat{U}_n^2)_3 + \delta^3 (\widehat{U}_n^{3,0})_3 \\
& \quad + \delta^3 \ln \delta\ (\widehat{U}_n^{3,1})_3 \big) \Big) \\
& + 2\ \nabla \chi^\delta . \nabla\ \big(u_n^0\ +\ \delta^2\ u_n^2\ -\ \delta\ (\widehat{U}_n^1)_3 - \delta^2\ (\widehat{U}_n^2)_3 \\
& \qquad\qquad - \delta^3\ (\widehat{U}_n^{3,0})_3 - \delta^3 \ln \delta\ (\widehat{U}_n^{3,1})_3 \big) \\
& + \Delta \chi^\delta\ \big(u_n^0\ +\ \delta^2\ u_n^2\ -\ \delta\ (\widehat{U}_n^1)_3 - \delta^2\ (\widehat{U}_n^2)_3 \\
& \qquad\qquad - \delta^3\ (\widehat{U}_n^{3,0})_3 - \delta^3 \ln \delta\ (\widehat{U}_n^{3,1})_3 \big) \\
& + 2\ \nabla \Psi . \nabla \big(\delta \widehat{\Pi}_n^1 + \delta^2\ \widehat{\Pi}_n^2 + \delta^3\ \widehat{\Pi}_n^{3,0} + \delta^3 \ln \delta\ \widehat{\Pi}_n^{3,1} \\
& \qquad - \delta (\widehat{U}_n^1)_3 - \delta^2 (\widehat{U}_n^2)_3 - \delta^3\ (\widehat{U}_n^{3,0})_3 - \delta^3 \ln \delta\ (\widehat{U}_n^{3,1})_3 \big) \\
& + \Delta \Psi\ \big(\delta \widehat{\Pi}_n^1 + \delta^2\ \widehat{\Pi}_n^2 + \delta^3\ \widehat{\Pi}_n^{3,0} + \delta^3 \ln \delta\ \widehat{\Pi}_n^{3,1} \\
& \qquad - \delta (\widehat{U}_n^1)_3 - \delta^2 (\widehat{U}_n^2)_3 - \delta^3\ (\widehat{U}_n^{3,0})_3 - \delta^3 \ln \delta\ (\widehat{U}_n^{3,1})_3 \big)
\end{aligned}
\tag{3.11}
$$

since

$$\nabla(\chi^\delta \Psi) = \nabla \chi^\delta + \nabla \Psi. \tag{3.12}$$

This simplifies in

$$
\begin{aligned}
\Delta\widetilde{w}_n^\delta + (\lambda_n^0 + \delta^2\lambda_n^2)\, \widetilde{w}_n^\delta \;=\; & \chi^\delta\Big(\delta^4\, \lambda_n^2\, u_n^2\Big) \\
& +\Psi\Big(\delta^3\, \lambda_n^2\, \big(\widehat{\Pi}_n^1 - \chi^\delta \widehat{U}_n^1\big) \\
& \quad +\big(\lambda_n^0 + \delta^2\, \lambda_n^2\big)\, \big(\delta^2\, \big(\widehat{\Pi}_n^2 - \chi^\delta \widehat{U}_n^2\big) + \delta^3\, \big(\widehat{\Pi}_n^{3,0} - \chi^\delta \widehat{U}_n^{3,0}\big) \\
& \qquad +\delta^3 \ln\delta\, \big(\widehat{\Pi}_n^{3,1} - \chi^\delta \widehat{U}_n^{3,1}\big)\big)\Big) \\
& +2\, \nabla\chi^\delta.\nabla\, \big(u_n^0 \;+\; \delta^2\, u_n^2 \;-\; \delta\, (\widehat{U}_n^1)_3 - \delta^2\, (\widehat{U}_n^2)_3 \\
& \qquad -\delta^3\, (\widehat{U}_n^{3,0})_3 - \delta^3 \ln\delta\, (\widehat{U}_n^{3,1})_3\big) \\
& +\Delta\chi^\delta\, \big(u_n^0 \;+\; \delta^2\, u_n^2 \;-\; \delta\, (\widehat{U}_n^1)_3 - \delta^2\, (\widehat{U}_n^2)_3 \\
& \qquad -\delta^3\, (\widehat{U}_n^{3,0})_3 - \delta^3 \ln\delta\, (\widehat{U}_n^{3,1})_3\big) \\
& +2\, \nabla\Psi.\nabla\big(\delta\widehat{\Pi}_n^1 + \delta^2\, \widehat{\Pi}_n^2 + \delta^3\, \widehat{\Pi}_n^{3,0} + \delta^3 \ln\delta\, \widehat{\Pi}_n^{3,1} \\
& \qquad -\delta(\widehat{U}_n^1)_3 - \delta^2(\widehat{U}_n^2)_3 - \delta^3\, (\widehat{U}_n^{3,0})_3 - \delta^3 \ln\delta\, (\widehat{U}_n^{3,1})_3\big) \\
& +\Delta\Psi\, \big(\delta\widehat{\Pi}_n^1 + \delta^2\, \widehat{\Pi}_n^2 + \delta^3\, \widehat{\Pi}_n^{3,0} + \delta^3 \ln\delta\, \widehat{\Pi}_n^{3,1} \\
& \qquad -\delta(\widehat{U}_n^1)_3 - \delta^2(\widehat{U}_n^2)_3 - \delta^3\, (\widehat{U}_n^{3,0})_3 - \delta^3 \ln\delta\, (\widehat{U}_n^{3,1})_3\big).
\end{aligned}
\tag{3.13}
$$

We shall now estimate each of these terms.

Estimate of $\ell^1(v) = \displaystyle\int_{\Omega_{int}} \chi^\delta\Big(\delta^4\, \lambda_n^2\, u_n^2\Big)v$. Since the support of χ^δ is $\Omega \cap \{r \geq \delta\}$ we can reduce the domain of integration to

$$
|\ell^1(v)| \leq \int_{\Omega_{int}\cap\{r\geq\delta\}} \Big|\chi^\delta\Big(\delta^4\, \lambda_n^2\, u_n^2\Big)v\Big|. \tag{3.14}
$$

Then, equation (2.154) leads to

$$
\Big|u_n^2(\mathbf{x})\Big| \;\leqslant \frac{C}{\delta}, \quad \forall r \geq \delta,\ \text{in } \Omega_{int}. \tag{3.15}
$$

It follows

$$
|\ell^1(v)| \leq C\delta^3 \|v\|_{L^1(\Omega_{int}\cap\{r\geq\delta\})}. \tag{3.16}
$$

Finally due to a Cauchy-Schwartz inequality, we have

$$
|\ell^1(v)| \leq C\, \delta^3\, \|v\|_{L^2(\Omega_{int})} \leq C\, \delta^3\, |\ln\delta|\, \|v\|_{L^2(\Omega_{int})}. \tag{3.17}
$$

Estimate of

$$
\begin{aligned}
\ell^2(v) = \int_{\Omega_{int}} \Psi\Big[&\delta^3\, \lambda_n^2\Big(\widehat{\Pi}_n^1 - \chi^\delta(\widehat{U}_n^1)_3\Big) + (\lambda_n^0 + \delta^2\, \lambda_n^2)\Big(\delta^2\big(\widehat{\Pi}_n^2 - \chi^\delta(\widehat{U}_n^2)_3\big) \\
&+ \delta^3\big(\widehat{\Pi}_n^{3,0} - \chi^\delta(\widehat{U}_n^{3,0})_3\big) + \delta^3\, \ln\delta\big(\widehat{\Pi}_n^{3,1} - \chi^\delta(\widehat{U}_n^{3,1})_3\big)\Big)\Big]v.
\end{aligned}
\tag{3.18}
$$

We split the integral into four parts

$$\ell^2(v) = \lambda_n^2\, \delta^3 \int_{\Omega_{int}} \Psi\big(\widehat{\Pi}_n^1 - \chi^\delta(\widehat{U}_n^1)_3\big) v$$
$$+ \big(\lambda_n^0 + \delta^2\, \lambda_n^2\big)\delta^2 \int_{\Omega_{int}} \Psi\big(\widehat{\Pi}_n^2 - \chi^\delta(\widehat{U}_n^2)_3\big) v$$
$$+ \big(\lambda_n^0 + \delta^2\, \lambda_n^2\big)\delta^3 \int_{\Omega_{int}} \Psi\big(\widehat{\Pi}_n^{3,0} - \chi^\delta(\widehat{U}_n^{3,0})_3\big) v$$
$$+ \big(\lambda_n^0 + \delta^2\, \lambda_n^2\big)\delta^3\, \ln\delta \int_{\Omega_{int}} \Psi\big(\widehat{\Pi}_n^{3,1} - \chi^\delta(\widehat{U}_n^{3,1})_3\big) v. \quad (3.19)$$

Since $\|\Psi\|_{L^\infty(\Omega_{int})} \leq 1$, we use the Cauchy-Schwartz inequality and obtain

$$\big|\ell^2(v)\big| \leq C\, \Big[\delta^3\, \big\|\widehat{\Pi}_n^1 - \chi^\delta(\widehat{U}_n^1)_3\big\|_{L^2(\Omega_{int})} + \delta^2\big\|\widehat{\Pi}_n^2 - \chi^\delta(\widehat{U}_n^2)_3\big\|_{L^2(\Omega_{int})}\Big]\|v\|_{L^2(\Omega_{int})}$$
$$+ \delta^3\Big[\big\|\widehat{\Pi}_n^{3,0} - \chi^\delta(\widehat{U}_n^{3,0})_3\big\|_{L^\infty(\Omega_{int})}$$
$$+ \big|\ln\delta\big|\, \big\|\widehat{\Pi}_n^{3,1} - \chi^\delta(\widehat{U}_n^{3,1})_3\big\|_{L^\infty(\Omega_{int})}\Big]\, \|v\|_{L^1(\Omega_{int})}. \quad (3.20)$$

Going back to the near-field coordinate

$$\big|\ell^2(v)\big| \leq C\, \Big[\delta^4\, \big\|\Pi_n^1 - \chi(U_n^1)_3\big\|_{L^2(\widehat{\Omega}_{int})} + \delta^3\big\|\Pi_n^2 - \chi(U_n^2)_3\big\|_{L^2(\widehat{\Omega}_{int})}\Big]\, \|v\|_{L^2(\Omega_{int})}$$
$$+ \delta^3\big\|\Pi_n^{3,0} - \chi(U_n^{3,0})_3\big\|_{L^\infty(\widehat{\Omega}_{int})}$$
$$+ \delta^3\, \big|\ln\delta\big|\, \big\|\Pi_n^{3,1} - \chi(U_n^{3,1})_3\big\|_{L^\infty(\widehat{\Omega}_{int})}\Big]\, \|v\|_{L^1(\Omega_{int})}. \quad (3.21)$$

Due to equations (A.19) one gets

$$\begin{cases} \Pi_n^1(\mathbf{X}) - \chi(R)\ (U_n^1)_3(\mathbf{X}) = \underset{R\to+\infty}{O}(\frac{1}{R^3}), \\ \Pi_n^2(\mathbf{X}) - \chi(R)\ (U_n^2)_3(\mathbf{X}) = \underset{R\to+\infty}{O}(\frac{1}{R^2}), \\ \Pi_n^{3,0}(\mathbf{X}) - \chi(R)\ (U_n^{3,0})_3(\mathbf{X}) = \underset{R\to+\infty}{O}(\frac{1}{R}), \\ \Pi_n^{3,1}(\mathbf{X}) - \chi(R)\ (U_n^{3,1})_3(\mathbf{X}) = \underset{R\to+\infty}{O}(\frac{1}{R}), \end{cases} \quad (3.22)$$

and consequently

$$\begin{cases} \big\|\Pi_n^1 - \chi(U_n^1)_3\big\|_{L^2(\widehat{\Omega}_{int})} \leq C, \\ \big\|\Pi_n^2 - \chi(U_n^2)_3\big\|_{L^2(\widehat{\Omega}_{int})} \leq C, \\ \big\|\Pi_n^{3,0} - \chi(U_n^{3,0})_3\big\|_{L^\infty(\widehat{\Omega}_{int})} \leq C, \\ \big\|\Pi_n^{3,1} - \chi(U_n^{3,1})_3\big\|_{L^\infty(\widehat{\Omega}_{int})} \leq C. \end{cases} \quad (3.23)$$

Finally, we get the bound

$$\left|\ell^2(v)\right| \leq C\ \delta^3\ \left\|v\right\|_{L^2(\Omega_{int})} + C\ \delta^3\ \left|\ln\delta\right|\ \left\|v\right\|_{L^1(\Omega_{int})} \leq C\ \delta^3\ \left|\ln\delta\right|\ \left\|v\right\|_{L^2(\Omega)}. \quad (3.24)$$

Estimate of

$$\ell^3(v) = \int_{\Omega_n} 2\ \nabla\chi^\delta\cdot\nabla\ \Big(u_n^0 + \delta^2\ u_n^2 - \delta\ (\widehat{U}_n^1)_3 - \delta^2\ (\widehat{U}_n^2)_3 - \delta^3\ (\widehat{U}_n^{3,0})_3 - \delta^3\ \ln\delta\ (\widehat{U}_n^{3,1})_3\Big) v.$$

The vector field $\mathbf{x} \mapsto \nabla\chi^\delta(\mathbf{x})$ has its support in C^δ, see Figure 3.1

$$\begin{cases} C^\delta = C^\delta_{int} \cup C^\delta_{ext}, \\ C^\delta_{int} := \left\{(r,\theta) : \delta \leq r \leq 2\delta, \text{and } \pi \leq \theta \leq 2\pi\right\}, \\ C^\delta_{ext} := \left\{(r,\theta) : \delta \leq r \leq 2\delta, \text{and } 0 \leq \theta \leq \pi\right\}. \end{cases} \quad (3.25)$$

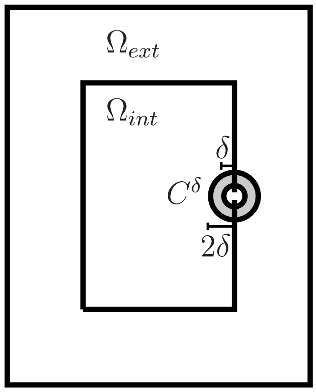

Figure 3.1: Presentation of C^δ.

So one can reduce the domain of integration to C^δ

$$\ell^3(v) = \int_{C^\delta_{int}} 2\ \nabla\chi^\delta \cdot \nabla\ \Big(u_n^0\ +\ \delta^2\ u_n^2\ -\ \delta\ (\widehat{U}_n^1)_3$$

$$\delta^2\ (\widehat{U}_n^2)_3 - \delta^3\ (\widehat{U}_n^{3,0})_3 - \delta^3\ \ln\delta\ (\widehat{U}_n^{3,1})_3\Big) v. \quad (3.26)$$

A Young inequality leads to

$$\left|\ell^3(v)\right| \leq \left\|\nabla\chi^\delta\right\|_{L^\infty(C^\delta_{int})}\Big\|\nabla\,\big(u^0_n \;+\; \delta^2\;u^2_n$$

$$-\;\delta\;(\widehat{U}^1_n)_3 - \delta^2\;(\widehat{U}^2_n)_3$$

$$-\,\delta^3\;(\widehat{U}^{3,0}_n)_3 - \delta^3\;\ln\delta\;(\widehat{U}^{3,1}_n)_3\big)\Big\|_{L^\infty(C^\delta_{int})}\Big\|v\Big\|_{L^1(C^\delta_{int})}. \quad (3.27)$$

Bounding the two external terms,

$$\begin{cases} \left\|\nabla\chi^\delta\right\|_{L^\infty(C^\delta_{int})} \;\leq\; \dfrac{C}{\delta}, \\ \left\|v\right\|_{L^1(C^\delta_{int})} \leq C\;\delta\;\left\|v\right\|_{L^2(C^\delta_{int})} \;\leq\; C\;\delta\;\left\|v\right\|_{L^2(\Omega)}, \quad \text{(Cauchy–Schwartz ineq.)} \end{cases} \quad (3.28)$$

we obtain

$$\left|\ell^3(v)\right| \leq C\Big\|\nabla\,\big(u^0_n \;+\; \delta^2\;u^2_n \;-\; \delta\;(\widehat{U}^1_n)_3 - \delta^2\;(\widehat{U}^2_n)_3$$

$$-\,\delta^3\;(\widehat{U}^{3,0}_n)_3 - \delta^3\;\ln\delta\;(\widehat{U}^{3,1}_n)_3\big)\Big\|_{L^\infty(C^\delta_{int})}\Big\|v\Big\|_{L^2(\Omega)}. \quad (3.29)$$

Due to (A.18) and taking into account (A.16) and (A.21), we have

$$\nabla\,\big(u^0_n \;+\; \delta^2\;u^2_n\big)(\mathbf{x}) \;=$$

$$\nabla\,\big(\;\delta\;(\widehat{U}^1_n)_3 + \delta^2\;(\widehat{U}^2_n)_3 + \delta^3\;(\widehat{U}^{3,0}_n)_3 + \delta^3\;\ln\delta\;(\widehat{U}^{3,1}_n)_3\big)(\mathbf{x})$$

$$+\underset{r\to 0}{O}(r^3) + \delta^2\underset{r\to 0}{O}(r\ln r). \quad (3.30)$$

This allows to obtain in C^δ_{int} $(\delta \leq r \leq 2\delta)$

$$\Big\|\nabla\,\big(u^0_n \;+\; \delta^2\;u^2_n \;-\; \delta\;(\widehat{U}^1_n)_3 - \delta^2\;(\widehat{U}^2_n)_3$$

$$-\,\delta^3\;(\widehat{U}^{3,0}_n)_3 - \delta^3\;\ln\delta\;(\widehat{U}^{3,1}_n)_3\big)\Big\|_{L^\infty(C^\delta_{int})} \leq\; C\;\Big(\delta^3 + \delta^3\big|\ln\delta\big|\Big) \quad (3.31)$$

Hence, we get the result

$$\left|\ell^3(v)\right| \;\leq\; C\;\delta^3\;|\ln(\delta)|\;\left\|v\right\|_{L^2(\Omega)}. \quad (3.32)$$

Estimate of

$$\ell^4(v) = \int_{\Omega_{int}} \Delta\chi^\delta \Big(u_n^0 + \delta^2 \, u_n^2 - \delta \, (\widehat{U}_n^1)_3 - \delta^2 \, (\widehat{U}_n^2)_3 - \delta^3 \, (\widehat{U}_n^{3,0})_3 - \delta^3 \, \ln\delta \, (\widehat{U}_n^{3,1})_3\Big) v.$$

Since the support of $\Delta\chi^\delta$ is C^δ, we can reduce the domain of integration

$$\begin{aligned} \big|\ell^4(v)\big| \le \frac{C}{\delta^2} \, \big\|u_n^0 + \delta^2 \, u_n^2 - \delta \, (\widehat{U}_n^1)_3 - \delta^2 \, (\widehat{U}_n^2)_3 \\ - \delta^3 \, (\widehat{U}_n^2)_3 - \delta^3 \, \ln\delta \, (\widehat{U}_n^{3,1})_3\big\|_{L^\infty(C^\delta_{int})} \big\|v\big\|_{L^1(C^\delta_{int})}. \end{aligned} \quad (3.33)$$

Using the triangular inequality we obtain

$$\begin{aligned} &\big|u_n^0(\mathbf{x}) + \delta^2 \, u_n^2(\mathbf{x}) - \delta(\widehat{U}_n^1)_3(\mathbf{x}) - \delta^2 \, (\widehat{U}_n^2)_3(\mathbf{x}) \\ &\qquad - \delta^3 \, (\widehat{U}_n^{3,0})_3(\mathbf{x}) - \delta^3 \, \ln\delta \, (\widehat{U}_n^{3,0})_3(\mathbf{x})\big| \\ &\le \Big|u_n^0(\mathbf{x}) - x \, \partial_x u_n^0\big|_{\Omega_{int}}(\mathbf{0}) - xy \, \partial^2_{xy} u_n^0\big|_{\Omega_{int}}(\mathbf{0}) - x^3 \frac{\partial^3_x u_n^0\big|_{\Omega_{int}}(\mathbf{0})}{3!}\Big| \\ &\quad + \delta^2 \big|u_n^2(\mathbf{x}) - \partial_x u_n^0\big|_{\Omega_{int}}(\mathbf{0}) \, \alpha_{int}\Big(\frac{1}{r} - \frac{\lambda^0 r}{2}\Big(\ln\frac{\sqrt{\lambda^0} r}{2}\Big) + \gamma_{int} \, r\Big) \, \sin(\theta)\big|. \end{aligned} \quad (3.34)$$

In C^δ_{int}, we have $\delta \le r \le 2\delta$. According to (A.17), we get for $\mathbf{x} \in C^\delta_{int}$

$$\left\{ \begin{aligned} &\Big|u_n^0(\mathbf{x}) - x \, \partial_x u_n^0\big|_{\Omega_{int}}(\mathbf{0}) - xy \, \partial^2_{xy} u_n^0\big|_{\Omega_{int}}(\mathbf{0}) - x^3 \, \frac{\partial^3_x u_n^0\big|_{\Omega_{int}}(\mathbf{0})}{3!}\Big| \le C \, \delta^4, \\ &\big|u_n^2(\mathbf{x}) - \partial_x u_n^0\big|_{\Omega_{int}}(\mathbf{0}) \, \alpha_{int}\Big(\frac{1}{r} - \frac{\lambda^0 r}{2}\Big(\ln\frac{\sqrt{\lambda^0} r}{2}\Big) + \gamma_{int} \, r\Big) \, \sin(\theta)\big| \le C \, \delta^2 \, |\ln\delta|. \end{aligned} \right. \quad (3.35)$$

This leads to

$$\begin{aligned} \big\|u_n^0 + \delta^2 \, u_n^2 - \delta \, (\widehat{U}_n^1)_3 - \delta^2 \, (\widehat{U}_n^2)_3 \\ - \delta^3 \, (\widehat{U}_n^{3,0})_3 - \delta^3 \, \ln\delta \, (\widehat{U}_n^{3,1})_3\big\|_{L^\infty(C^\delta_{int})} \le C \, \big(\delta^4 + \delta^4 |\ln\delta|\big). \end{aligned} \quad (3.36)$$

Finally, inserting (3.28), and (3.36) in (3.33) we get the result

$$\big|\ell^4(v)\big| \le C \, \delta^3 \, |\ln(\delta)| \, \big\|v\big\|_{L^2(\Omega)}. \quad (3.37)$$

Estimate of

$$\begin{aligned} \ell^5(v) = \int_{\Omega_{int}} 2 \, \nabla\Psi \cdot \nabla\Big(\delta\big(\widehat{\Pi}_n^1 - (\widehat{U}_n^1)_3\big) + \delta^2\big(\widehat{\Pi}_n^2 - (\widehat{U}_n^2)_3\big) \\ + \delta^3\big(\widehat{\Pi}_n^{3,0} - (\widehat{U}_n^{3,0})_3\big) + \delta^3 \, \ln\delta \, \big(\widehat{\Pi}_n^{3,1} - (\widehat{U}_n^{3,1})_3\big)\Big) v. \end{aligned} \quad (3.38)$$

Since the support of $\nabla\Psi$ is

$$\begin{cases} \mathcal{C} = \mathcal{C}_{int} \cup \mathcal{C}_{ext}, \\ \mathcal{C}_{int} := \{(r,\theta) : 1 \le r \le 2, \text{and } \pi \le \theta \le 2\pi\}, \\ \mathcal{C}_{ext} := \{(r,\theta) : 1 \le r \le 2, \text{and } 0 \le \theta \le \pi\}, \end{cases} \tag{3.39}$$

we can reduce the domain of integration

$$\ell^5(v) = \int_{\mathcal{C}} 2\, \nabla\Psi \cdot \nabla\Big(\delta\big(\widehat{\Pi}_n^1 - (\widehat{U}_n^1)_3\big) + \delta^2\big(\widehat{\Pi}_n^2 - (\widehat{U}_n^2)_3\big)$$

$$+ \delta^3\big(\widehat{\Pi}_n^{3,0} - (\widehat{U}_n^{3,0})_3\big) + \delta^3\, \ln\delta\, \big(\widehat{\Pi}_n^{3,1} - (\widehat{U}_n^{3,1})_3\big)\Big)v. \tag{3.40}$$

The Young inequality leads to

$$\begin{aligned} \big|\ell^5(v)\big| \;\le\; 2\,\|\nabla\Psi\|_{L^\infty(\mathcal{C})}\big(\delta\,\|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^1 - (\widehat{U}_n^1)_3\big)\|_{L^\infty(\mathcal{C})} \\ +\; \delta^2\,\|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^2 - (\widehat{U}_n^2)_3\big)\|_{L^\infty(\mathcal{C})} \\ +\; \delta^3\,\|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^{3,0} - (\widehat{U}_n^{3,0})_3\big)\|_{L^\infty(\mathcal{C})} \\ +\; \delta^3\, \ln\delta\,\|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^{3,1} - (\widehat{U}_n^{3,1})_3\big)\|_{L^\infty(\mathcal{C})}\big)\|v\|_{L^1(\mathcal{C}_{int})}. \end{aligned} \tag{3.41}$$

For $\mathbf{x} \in \mathcal{C}$, we have $\frac{1}{\delta} \le \frac{r}{\delta} \le \frac{2}{\delta}$. Due to (A.20), after scaling the gradient in the near field coordinate, there exists a constant C such that for all $\mathbf{x} \in \mathcal{C}$

$$\begin{cases} \big|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^1 - (\widehat{U}_n^1)_3\big)\big| \;=\; \dfrac{1}{\delta}\,\big|\nabla_{\mathbf{X}}\big(\Pi_n^1(\dfrac{\mathbf{x}}{\delta}) - (\widehat{U}_n^1)_3(\dfrac{\mathbf{x}}{\delta})\big)\big| \le C\,\delta^3, \\ \big|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^2 - (\widehat{U}_n^2)_3\big)\big| \;=\; \dfrac{1}{\delta}\,\big|\nabla_{\mathbf{X}}\big(\Pi_n^2(\dfrac{\mathbf{x}}{\delta}) - (U_n^1)_3(\dfrac{\mathbf{x}}{\delta})\big)\big| \le C\,\delta^2, \\ \big|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^{3,0} - (\widehat{U}_n^{3,0})_3\big)\big| \;=\; \dfrac{1}{\delta}\,\big|\nabla_{\mathbf{X}}\big(\Pi_n^{3,0}(\dfrac{\mathbf{x}}{\delta}) - (U_n^{3,0})_3(\dfrac{\mathbf{x}}{\delta})\big)\big| \le C\,\delta,, \\ \big|\nabla_{\mathbf{x}}\big(\widehat{\Pi}_n^{3,1} - (\widehat{U}_n^{3,1})_3\big)\big| \;=\; \dfrac{1}{\delta}\,\big|\nabla_{\mathbf{X}}\big(\Pi_n^{3,1}(\dfrac{\mathbf{x}}{\delta}) - (U_n^{3,1})_3(\dfrac{\mathbf{x}}{\delta})\big)\big| \le C\,\delta. \end{cases} \tag{3.42}$$

This allows to obtain the result

$$\big|\ell^5(v)\big| \;\le\; C\,\delta^3\,\big|\ln\delta\big|\,\big\|v\big\|_{L^2(\Omega)}. \tag{3.43}$$

Estimate of

$$\ell^6(v) = \int_{\Omega_{int}} \Delta\Psi\,\big(\delta\big(\widehat{\Pi}_n^1 - (\widehat{U}_n^1)_3\big) + \delta^2\big(\widehat{\Pi}_n^2 - (\widehat{U}_n^2)_3\big)$$

$$+ \delta^3\big(\widehat{\Pi}_n^{3,0} - (\widehat{U}_n^{3,0})_3\big) + \delta^3\, \ln\delta\,\big(\widehat{\Pi}_n^{3,1} - (\widehat{U}_n^{3,1})_3\big)\big)v. \tag{3.44}$$

Since the support of $\Delta\Psi$ is $\mathcal{C}$, we can reduce the domain of integration

$$\ell^6(v) = \int_{\mathcal{C}_{int}} \Delta\Psi \ \big(\delta\big(\widehat{\Pi}^1_n - (\widehat{U}^1_n)_3\big) + \delta^2\big(\widehat{\Pi}^2_n - (\widehat{U}^2_n)_3\big) + \delta^3\big(\widehat{\Pi}^{3,0}_n - (\widehat{U}^{3,0}_n)_3\big) + \delta^3 \ \ln\delta \ \big(\widehat{\Pi}^{3,1}_n - (\widehat{U}^{3,1}_n)_3\big)\big)v. \quad (3.45)$$

The Young inequality leads to

$$\big|\ell^6(v)\big| \le \big\|\Delta\Psi\big\|_{L^\infty(\mathcal{C}_{int})}\Big(\delta\big\|\widehat{\Pi}^1_n - (\widehat{U}^1_n)_3\big\|_{L^\infty(\mathcal{C}_{int})} + \delta^2 \ \big\|\widehat{\Pi}^2_n - (\widehat{U}^2_n)_3\big\|_{L^\infty(\mathcal{C}_{int})} \delta^3 \ \big\|\widehat{\Pi}^{3,0}_n - (\widehat{U}^{3,0}_n)_3\big\|_{L^\infty(\mathcal{C}_{int})} + \delta^3 \ \ln\delta \ \big\|\widehat{\Pi}^{3,0}_n - (\widehat{U}^{3,0}_n)_3\big\|_{L^\infty(\mathcal{C}_{int})}\Big)\|v\|_{L^1(\mathcal{C}_{int})}. \quad (3.46)$$

Since

$$\begin{cases} \big\|\Delta\Psi\big\|_{L^\infty(\mathcal{C}_{int})} & \le \ \big\|\Delta\Psi\big\|_{L^\infty(\Omega)} \ \le \ C, \\ \|v\|_{L^1(\mathcal{C}_{int})} & \le \ \|v\|_{L^1(\Omega)} \ \le \ C \ \|v\|_{L^2(\Omega)}. \end{cases} \quad (3.47)$$

one has

$$\big|\ell^6(v)\big| \le C \ \Big(\delta\big\|\widehat{\Pi}^1_n - (\widehat{U}^1_n)_3\big\|_{L^\infty(\mathcal{C}_{int})} + \delta^2 \ \big\|\widehat{\Pi}^2_n - (\widehat{U}^2_n)_3\big\|_{L^\infty(\mathcal{C}_{int})} \delta^3 \ \big\|\widehat{\Pi}^{3,0}_n - (\widehat{U}^{3,0}_n)_3\big\|_{L^\infty(\mathcal{C}_{int})} + \delta^3 \ \ln\delta \ \big\|\widehat{\Pi}^{3,0}_n - (\widehat{U}^{3,0}_n)_3\big\|_{L^\infty(\mathcal{C}_{int})}\Big)\|v\|_{L^2(\Omega)}. \quad (3.48)$$

Then for $\mathbf{x} \in \mathcal{C}$ we have $1/\delta < r/\delta < 2/\delta$. According to (A.19), we get the control

$$\begin{cases} \big|\widehat{\Pi}^1_n(\mathbf{x}) - (\widehat{U}^1_n)_3(\mathbf{x})\big| = \big|\Pi^1_n(\frac{\mathbf{x}}{\delta}) - (U^1_n)_3(\frac{\mathbf{x}}{\delta})\big| \le C \ \delta^3, \quad \forall \mathbf{x} \in \mathcal{C}, \\ \big|\widehat{\Pi}^2_n(\mathbf{x}) - (\widehat{U}^2_n)_3(\mathbf{x})\big| = \big|\Pi^2_n(\frac{\mathbf{x}}{\delta}) - (U^2_n)_3(\frac{\mathbf{x}}{\delta})\big| \le C \ \delta^2, \quad \forall \mathbf{x} \in \mathcal{C}, \\ \big|\widehat{\Pi}^{3,0}_n(\mathbf{x}) - (\widehat{U}^{3,0}_n)_3(\mathbf{x})\big| = \big|\Pi^{3,0}_n(\frac{\mathbf{x}}{\delta}) - (U^{3,0}_n)_3(\frac{\mathbf{x}}{\delta})\big| \le C \ \delta, \quad \forall \mathbf{x} \in \mathcal{C}, \\ \big|\widehat{\Pi}^{3,1}_n(\mathbf{x}) - (\widehat{U}^{3,1}_n)_3(\mathbf{x})\big| = \big|\Pi^{3,1}_n(\frac{\mathbf{x}}{\delta}) - (U^{3,1}_n)_3(\frac{\mathbf{x}}{\delta})\big| \le C \ \delta, \quad \forall \mathbf{x} \in \mathcal{C}. \end{cases} \quad (3.49)$$

Finally, inserting (3.49) in (3.48) , we get the estimate

$$\left|\ell^6(v)\right| \;\leq\; C\,\delta^3\,\left|\ln\delta\right|\,\|v\|_{L^2(\Omega)}. \tag{3.50}$$

Conclusion It follows from (3.10), (3.13) (3.17), (3.24), (3.32), (3.37), (3.43), (3.50),

$$\left|\mathsf{a}(\widetilde{w}^\delta, v) - \left(\lambda_n^0 + \delta^2\,\lambda_n^2\right)(\widetilde{w}^\delta, v)_0\right| \leq C\,\delta^3\,|\ln\delta|\,\|v\|_0, \quad \forall v \in H_0^1(\Omega^\delta). \tag{3.51}$$

To obtain (3.8), we remark that

$$\|\widetilde{w}^\delta\|_{L^2(\Omega)} \geq \|\widetilde{w}^\delta\|_{L^2(\Omega\cap\{r\geq 2\})} = \|u_n^0 + \delta^2 u_n^2\|_{L^2(\Omega\cap\{r\geq 2\})}. \tag{3.52}$$

Since u^0 is not vanishing in Ω_{int}

$$\|\widetilde{w}^\delta\|_{L^2(\Omega)} \geq C > 0 \tag{3.53}$$

with C independent of δ and apply Theorem 7 of Appendix B. ■

3.2 Second step

In the last section, we have proved, see Theorem 3, the existence of a λ_p^δ in a small neighborhood of λ_n. In this section we show that $p = n$ and consequently demonstrate Theorem 3.

Lemma 3.2.1 *For all $n > 0$, we have*

$$\lim_{\delta\to 0} \lambda_n^\delta = \beta_n \leq \lambda_n, \tag{3.54}$$

with $\beta_n \in \mathbb{R}$.

Proof. Let $n \in \mathbb{N}$ be fixed. First, we remark that for $\delta' < \delta$, $H_0^1\big(\Omega^{\delta'}\big) \subset H_0^1\big(\Omega^\delta\big)$. Due to the min-max principle (see Theorem 6 of Appendix B), we have

$$\lambda_n^{\delta'} = \min_{\substack{V\subset H_0^1(\Omega^{\delta'})\\ dim(V)=n}} \;\max_{\substack{u_n^{\delta'}\in V\\ u_n^{\delta'}\neq 0}} R(u_n^{\delta'}) \geq \min_{\substack{V\subset H_0^1(\Omega^{\delta})\\ dim(V)=n}} \;\max_{\substack{u_n^{\delta}\in V\\ u_n^{\delta}\neq 0}} R(u_n^{\delta}) = \lambda_n^\delta. \tag{3.55}$$

This prove that, $\delta \to \lambda_n^\delta$ is decreasing. On the other hand, we remark that $H_0^1\big(\Omega\big) \subset H_0^1\big(\Omega^\delta\big)$. The same argument leads to $\lambda_n^\delta \leq \lambda_n$. This leads to the existence of $\beta_n \in \mathbb{R}$ such that $\lim_{\delta\to 0} \lambda_n^\delta = \beta_n \leq \lambda_n$. ■

In the continuation, we associate to λ_n^δ a normalized eigenvector u_n^δ in $H_0^1\big(\Omega^\delta\big)$ satisfying $\|u_n^\delta\|_{H^1(\Omega^\delta)} = 1$, i.e we have

$$\big(\nabla u_n^\delta, \nabla v\big)_{L^2(\Omega^\delta)} - \lambda_n^\delta\big(u_n^\delta, v\big)_{L^2(\Omega^\delta)} = 0 \quad \forall v \in H_0^1\big(\Omega^\delta\big). \tag{3.56}$$

Lemma 3.2.2 *The mapping* $\delta \to u_n^\delta \in H_0^1(\Omega)$ *admits an adherence value* $\tilde{u}_n$ *for the weak topology of* $H_0^1\big(\Omega\big)$ *at* $\delta = 0$ *satisfying*

(i) $\tilde{u}_n \in H_0^1\big(\Omega\big)$.

(ii) $\big(\nabla\tilde{u}_n, \nabla v\big)_{L^2(\Omega)} - \beta_n\big(\tilde{u}_n, v\big)_{L^2(\Omega)} = 0 \quad \forall v \in H_0^1\big(\Omega^\delta\big)$.

(iii) $\tilde{u}_n \neq 0$.

Proof. Since u_n^δ is bounded in $H^1\big(\Omega\big)$, there exists a sequence $\big(\delta_p\big)_{p\in\mathbb{N}^*}$ and $\tilde{u}_n \in H^1\big(\Omega\big)$ satisfying

$$\delta_p \to 0 \text{ and } \tilde{u}_n^{\delta_p} \to \tilde{u}_n \text{ in } H^1\big(\Omega\big) \text{ for } p \to +\infty. \tag{3.57}$$

Let $V \subset \mathbb{R}$ be a neighborhood of zero. For p large enough, one has

$$\tilde{u}_n^{\delta_p} \;=\; 0 \text{ in } L^2\big(\partial\Omega\backslash V\big). \tag{3.58}$$

Since the trace operator is compact from $H^1\big(\Omega\big)$ to $L^2\big(\partial\Omega\backslash V\big)$, then $\tilde{u}_n \;=\; 0$ in $L^2\big(\partial\Omega\backslash V\big)$. Thus, we obtain

$$\tilde{u}_n \;=\; 0 \text{ in } L^2\big(\partial\Omega\big), \tag{3.59}$$

which implies $\tilde{u}_n \in H_0^1\big(\Omega\big)$.
To get (ii), we remark that $H_0^1\big(\Omega\big) \subset H_0^1\big(\Omega^\delta\big)$. Consequently, due to (3.56), one has

$$\big(\nabla u_n^{\delta_p}, \nabla v\big)_{L^2(\Omega)} - \lambda_n^{\delta_p}\big(u_n^{\delta_p}, v\big)_{L^2(\Omega)} = 0 \quad \forall v \in H_0^1\big(\Omega\big). \tag{3.60}$$

For p tending to infinity, we obtain (see lemma 3.2.1)

$$\big(\nabla\tilde{u}_n, \nabla v\big)_{L^2(\Omega)} - \beta_n\big(\tilde{u}_n, v\big)_{L^2(\Omega)} = 0 \quad \forall v \in H_0^1\big(\Omega\big). \tag{3.61}$$

To obtain (iii), we act by contradiction. Let us suppose that $\tilde{u}_n \;=\; 0$. As the space $H_0^1\big(\Omega\big)$ is compact in the space $L^2\big(\Omega\big)$, we have

$$u_n^{\delta_p} \to 0 \text{ in } L^2\big(\Omega\big) \;\Leftrightarrow\; \big(u_n^{\delta_p}, u_n^{\delta_p}\big)_{L^2\left(\Omega\right)} \underset{p\to+\infty}{\to} 0. \tag{3.62}$$

Consequently, we get

$$\big(\nabla u_n^{\delta_p}, \nabla u_n^{\delta_p}\big)_{L^2(\Omega)} \;=\; \lambda_n^{\delta_p}\,\big(u_n^{\delta_p}, u_n^{\delta_p}\big)_{L^2(\Omega)} \underset{\delta_p\to 0}{\to} \beta_n.0 \;=\; 0, \tag{3.63}$$

which is impossible because $\|u_n^{\delta_p}\|_{H^1(\Omega^\delta)} \;=\; 1$. ■

Theorem 5 *If* (1.8) *is satisfied, we have the following result*

$$\lambda_n^\delta \underset{\delta\to 0}{\to} \lambda_n \textit{ for all } n > 0. \tag{3.64}$$

Proof. According to Lemma 3.2.2, β_n is an eigenvalue of the Dirichlet-Laplacian. To achieve the proof of theorem 5, it suffices now to verify that there exists a unique p such that $\lambda_p^\delta \to \lambda_n$, when δ tends to 0 (see Lemma 3.2.1). We act by contradiction. Suppose that there exist p and m such that

$$\lambda_p^\delta \underset{\delta\to 0}{\longrightarrow} \lambda_n, \text{ and } \lambda_m^\delta \underset{\delta\to 0}{\longrightarrow} \lambda_n. \tag{3.65}$$

Since u_p^δ and u_m^δ are associated to two different eigenvalues of the Dirichlet-Laplacian of Ω^δ, we have

$$\left(u_p^\delta, u_m^\delta\right)_{L^2(\Omega^\delta)} \;=\; 0. \tag{3.66}$$

Since $H_0^1\big(\Omega\big)$ is compact in $L^2\big(\Omega\big)$, we obtain for the adherence values at $\delta = 0$ of u_p^δ, and u_m^δ introduced in Lemma 3.2.2,

$$\left(\tilde{u}_p, \tilde{u}_m\right)_{L^2(\Omega)} \;=\; 0. \tag{3.67}$$

According to Lemma 3.2.2, we deduce that the eigenvalue λ_n is not simple, which is impossible. ∎

Chapter 4

Numerical simulation

4.1 Introduction and presentation of simulation

For two geometries, we will numerically compare (a very precise direct numerical approximation) of the exact eigenvalue λ_n^δ to its second order approximation

$$\lambda_n^{2,\delta} = \lambda_n^0 + \delta^2 \lambda_n^2. \qquad (4.1)$$

- The first geometry is chosen in order to obtain explicit fomula for $\lambda_n^{2,\delta}$.
- For the second geometry, a numerical simulation is required in order to compute an approximation of $\lambda_n^{2,\delta}$.

The numerical experiments are performed using GETFEM a high order finite elements library (see http://home.gna.org/getfem/) on triangular meshes. We aim in this chapter in checking the feasibility of the method and the agreement between the theory and these simulations.

4.2 A semi-analytical test

For different δ, we consider domains Ω^δ (see (1.2) and Fig. 4.1) corresponding to

$$\Omega_{int} =]-2, 0[\times]-2.5, 1.5[\quad \text{and} \quad \Omega_{ext} =]0, 2.5[\times]-2.5, 1[, \text{ see Fig. 1.1.} \qquad (4.2)$$

A very precise approximation of λ_n^δ is computed via a P^3-continuous finite element on a very refined triangular mesh (h=0.03125, see FIG. 4.2). Taking into account that the eigenvalues and eigenvectors of the Dirichlet-Laplacian of $\big[0, a\big] \times \big[0, b\big]$ are given by formula

$$\lambda_{n,m} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \qquad u_{n,m}\big(x, y\big) = \sin\big(\frac{n\pi}{a} x\big) \sin\big(\frac{m\pi}{b} y\big). \qquad (4.3)$$

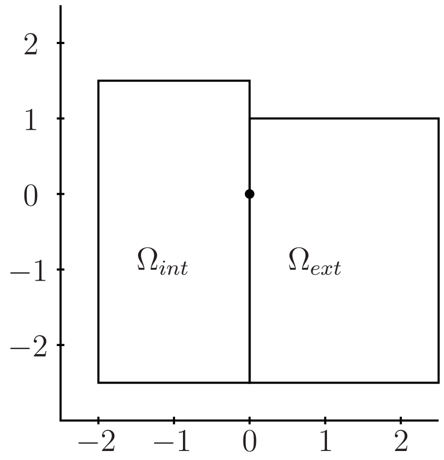

Figure 4.1: The domain Ω

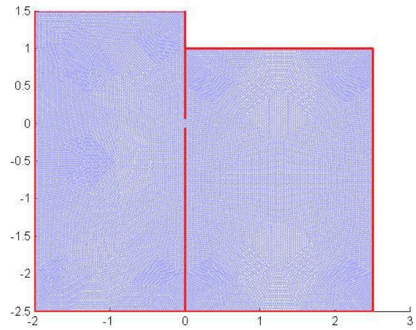

Figure 4.2: A first computational mesh ($h = 0.03125$, $\delta = 0.125$).

One can analytically compute the λ_n and λ_n^2

n	λ_n	λ_n^2
0	2.38	−0.087
1	3.08	−0.207
2	4.80	−0.135
3	4.93	−0.121
4	7.12	−0.347
5	8.02	−0.036

(4.4)

The reader can find the results of the numerical experiments in Fig. 4.3 and 4.4. In order to perform the computation we were practically limited to $\delta > 0.0625$. Anyway the results are to our opinion really convincing and in very good agreement with the theory, see Theorem 3. For smaller δ, we are convinced that this method should give better results (due to memory limitation it was not possible to obtain a precise value of λ_n^δ).

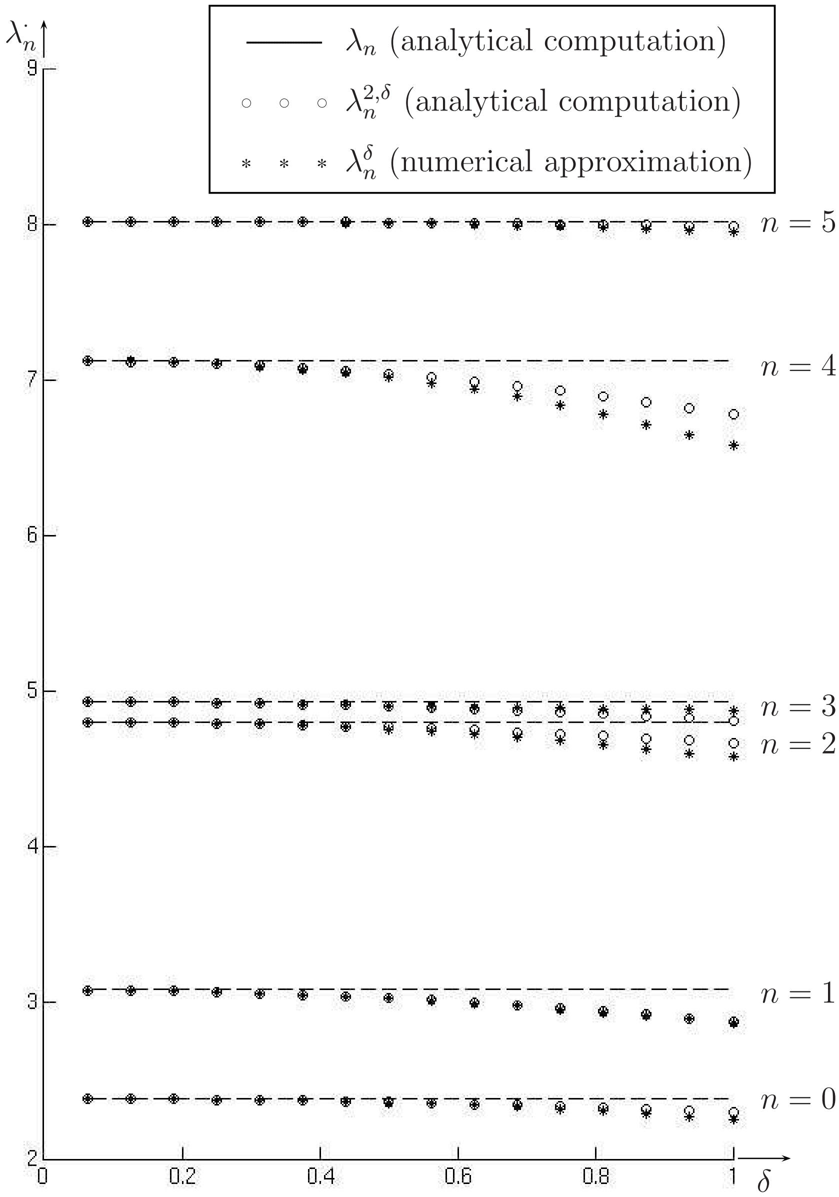

Figure 4.3: Comparison of the numerical value of λ_n^{δ} with its second order asymptotic expansion (analytical value) $\lambda_n^{2,\delta} = \lambda_n + \delta^2 \lambda_n^2$.

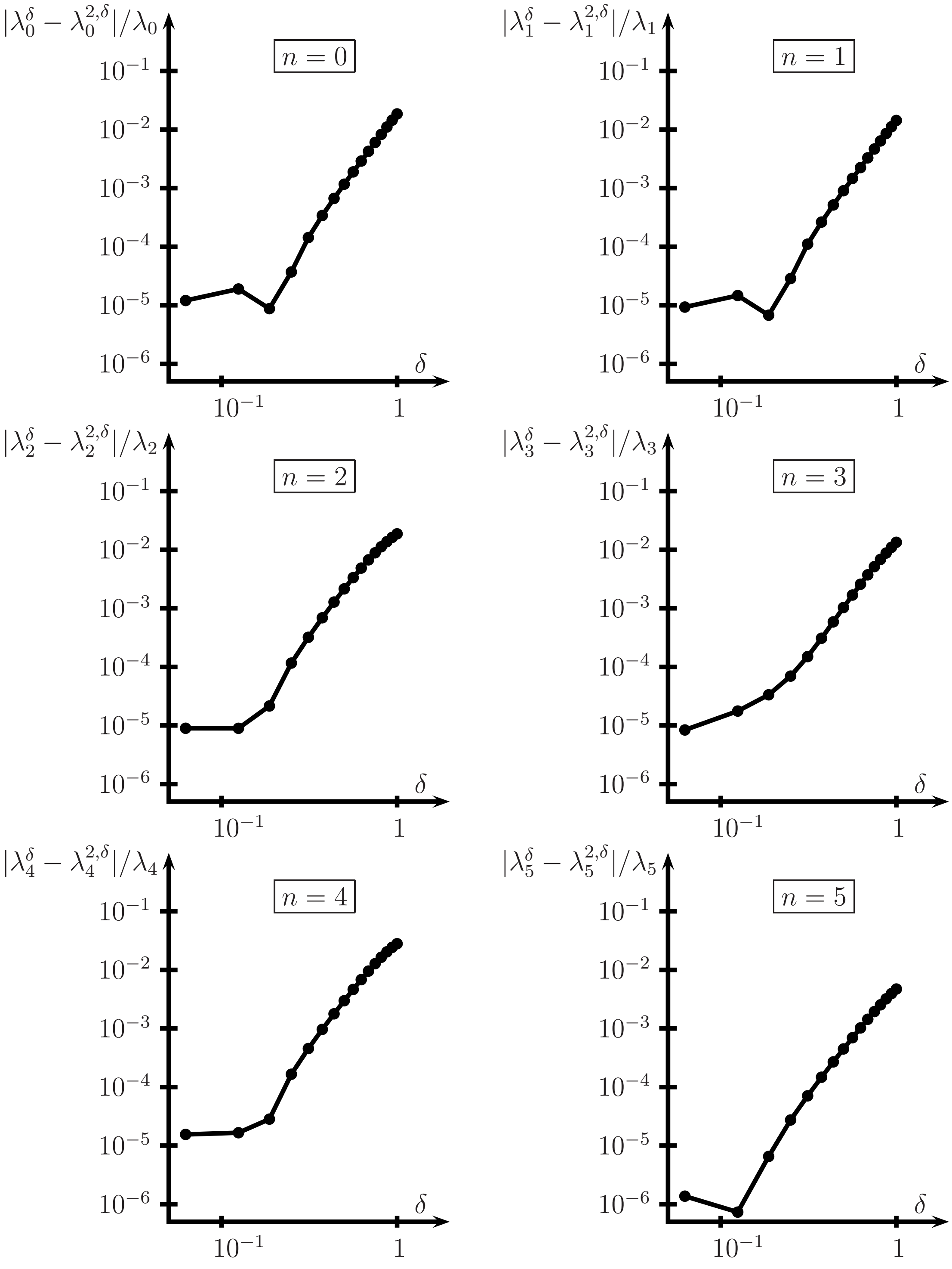

Figure 4.4: Relative error $|\lambda_n^\delta - \lambda_n^{2,\delta}|/\lambda_n$ with respect to δ for $n \in [0, 5]$ in log-log scale.

4.3 Numerical simulation

In contrary to the previous section, the eigenvalues of the Dirichlet Laplacian of Ω, see Fig. 4.5, can not be explicitly computed but just numerically. Like in the last chapter a very precise approximation of λ_n^δ is computed via a P^3-continuous finite element on a very refined triangular mesh (h=0.03125, see Fig. 4.7). The λ_n and λ_n^2 are computed using the same finite element but on a non refined mesh (we do not have to take into account the hole, see Fig. 4.6)

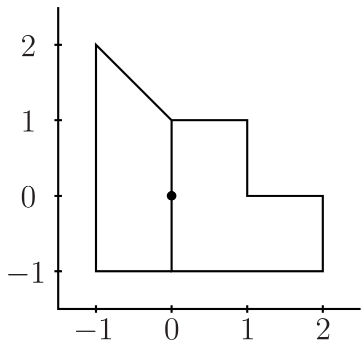

Figure 4.5: The domain Ω.

n	λ_n	λ_n^2
0	9.6	-1.2714
1	11.6	-2.8253
2	15.2	-1.8023
3	16.9	-0.6439
4	19.7	-0.0025
5	25.3	-1.4571

(4.5)

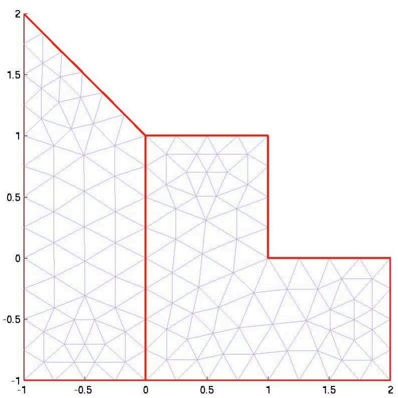

Figure 4.6: The computational mesh for λ_n and λ_n^2

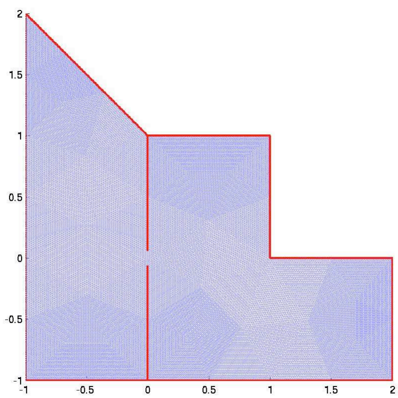

Figure 4.7: The computational mesh for λ_n^δ

The result of this numerical experiment is shown in Fig. 4.8. This confirms the feasibility of the method (one does not need to use a mesh refinement to compute an approximation of the eigenvalues).

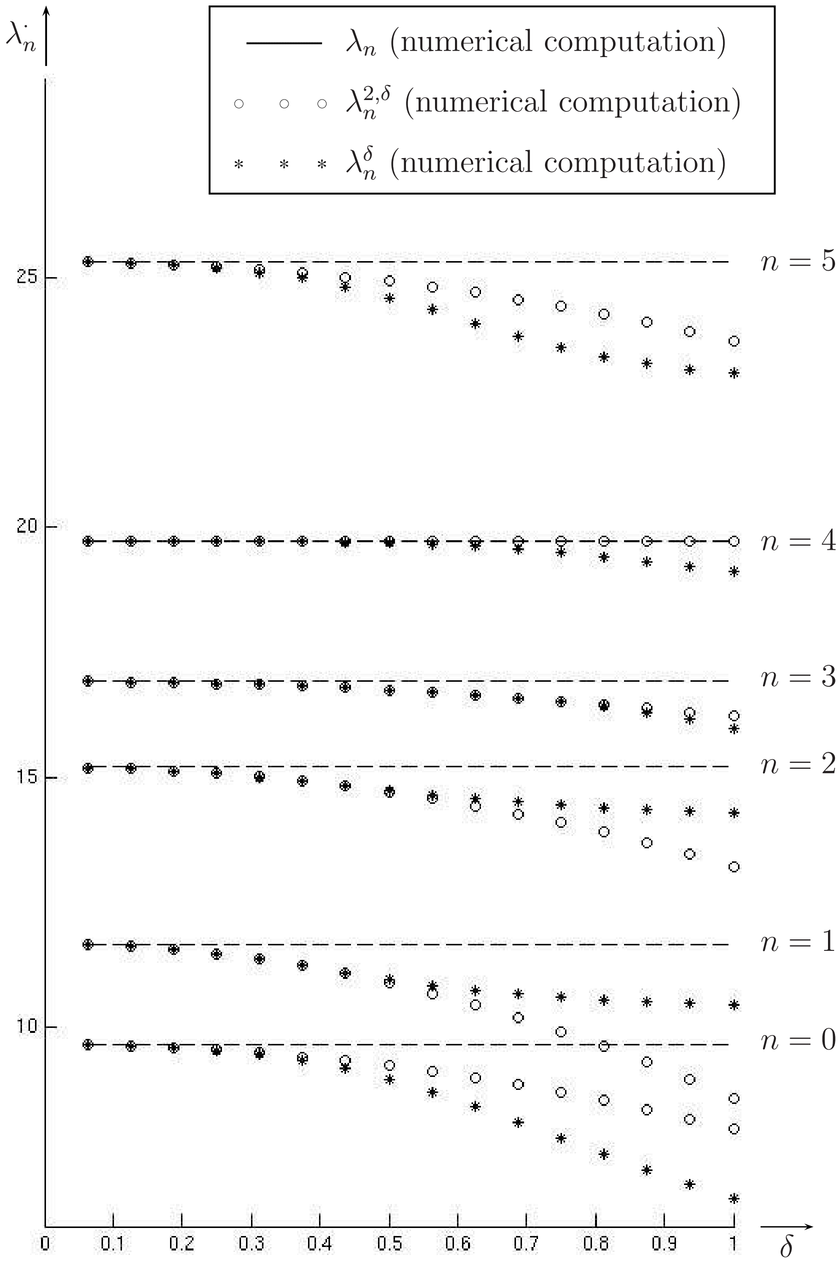

Figure 4.8: Result of the second numerical experiment.

Eigenvectors for $\delta = 0.0125$.

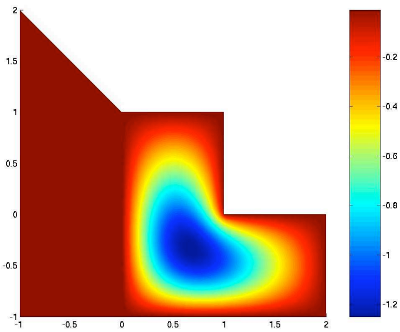

Figure 4.9: The eigenvector u_0

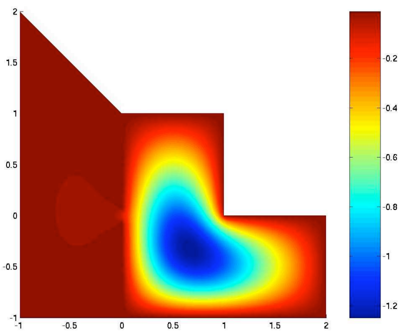

Figure 4.10: The eigenvector u_0^δ

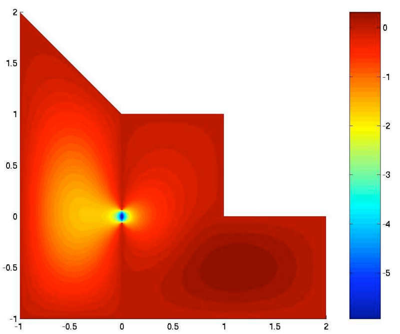

Figure 4.11: $\dfrac{u_0^\delta - u_0}{\delta^2}$

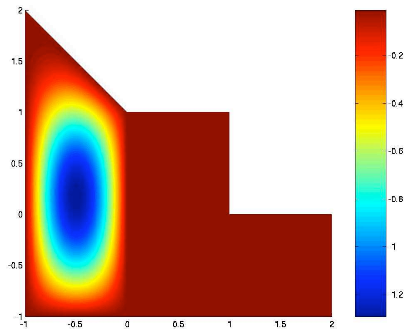

Figure 4.12: The eigenvector u_1

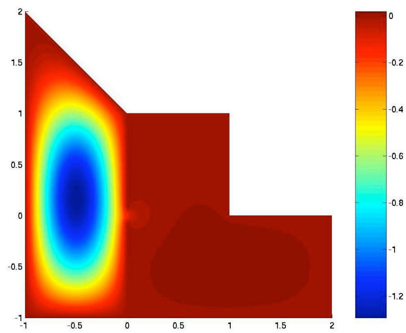

Figure 4.13: The eigenvector u_1^δ

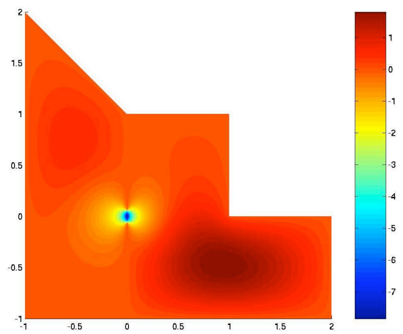

Figure 4.14: $\dfrac{u_1^\delta - u_1}{\delta^2}$

Eigenvectors for $\delta = 0.0125$.

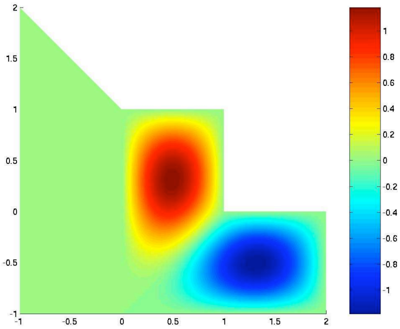

Figure 4.15: The eigenvector u_2

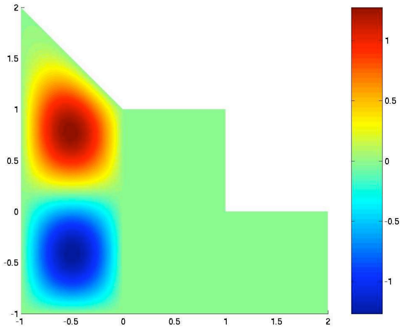

Figure 4.18: The eigenvector u_3

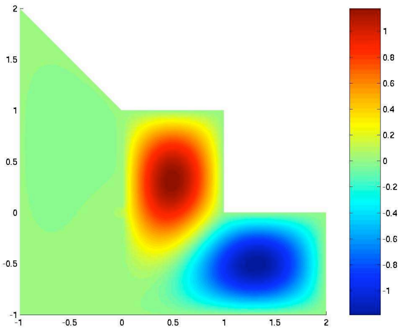

Figure 4.16: The eigenvector u_2^δ

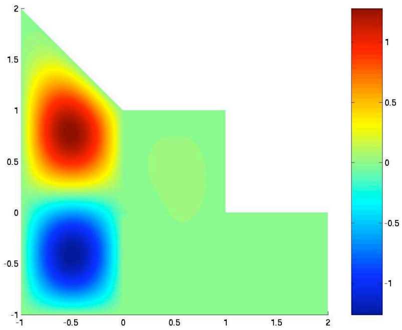

Figure 4.19: The eigenvector u_3^δ

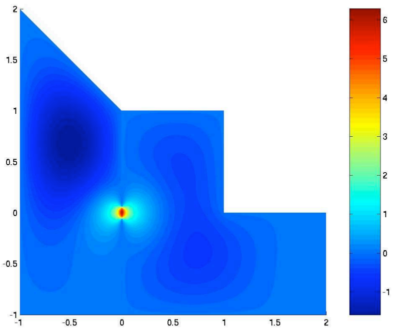

Figure 4.17: $\dfrac{u_2^\delta - u_2}{\delta^2}$

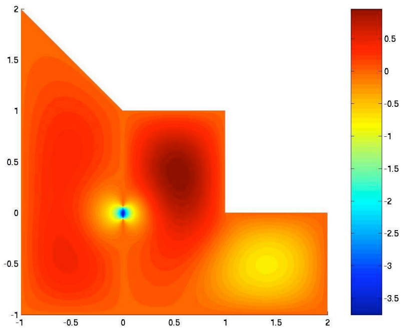

Figure 4.20: $\dfrac{u_3^\delta - u_3}{\delta^2}$

Eigenvectors for $\delta = 0.0125$.

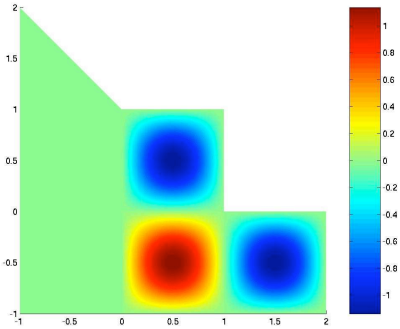

Figure 4.21: The eigenvector u_4

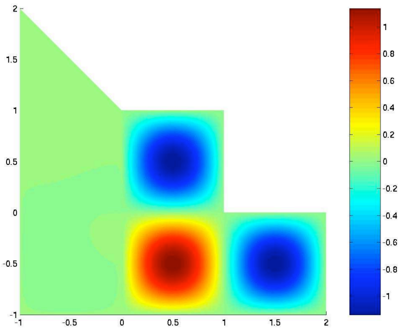

Figure 4.22: The eigenvector u_4^δ

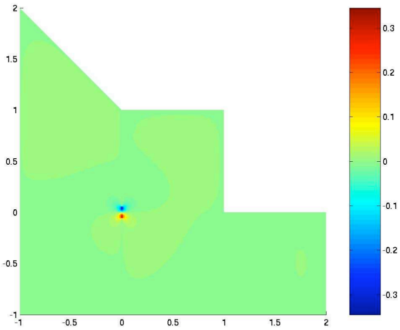

Figure 4.23: $\dfrac{u_4^\delta - u_4}{\delta^2}$

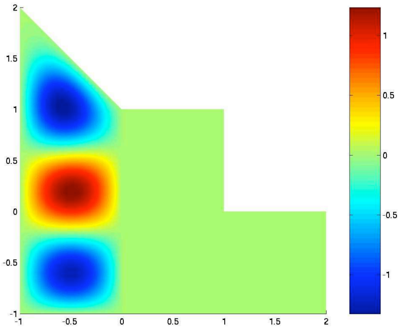

Figure 4.24: The eigenvector u_5

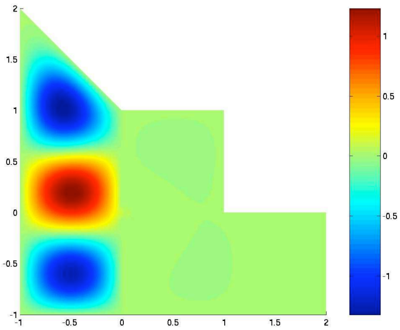

Figure 4.25: The eigenvector u_5^δ

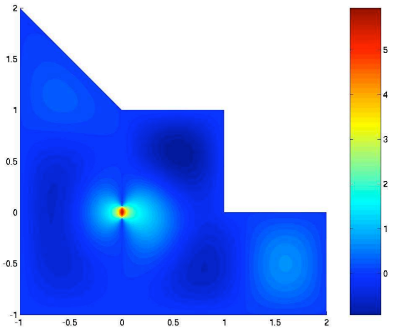

Figure 4.26: $\dfrac{u_5^\delta - u_5}{\delta^2}$

Chapter 5

Conclusion

In the framework of the 2D Dirichlet eigenvalue problem of the Laplace operator, we have obtained a second order asymptotic expansion of an eigenvalue problem on a domain consisting of two cavities linked by a small iris (see problem (1.4) and (1.5)).

$$\lambda_n^\delta = \begin{cases} \lambda_n - \dfrac{\pi}{16} \dfrac{\left|\partial_x u_n|_{\Omega_{int}}(\mathbf{0})\right|^2}{\|u_n\|^2_{L^2(\Omega_{int})}} \delta^2 + O\big(\delta^3 \ \ln(\delta)\big), & \text{if } u_n = 0 \text{ in } \Omega_{ext}, \\ \lambda_n - \dfrac{\pi}{16} \dfrac{\left|\partial_x u_n|_{\Omega_{ext}}(\mathbf{0})\right|^2}{\|u_n\|^2_{L^2(\Omega_{ext})}} \delta^2 + O\big(\delta^3 \ \ln(\delta)\big), & \text{if } u_n = 0 \text{ in } \Omega_{int}. \end{cases} \tag{5.1}$$

This provides an easy way to compute an approximation of the Dirichlet-Laplacian eigenvalues when the width of the iris is small without any mesh refinement. The theoretical results are in good agreement with numerical tests.

Acknowledgment: This work has been supported by the French National Agency (ANR) in the frame of its programm Systèmes Complexes et Modélisation: project APAM (Acoustique et Paroi Multi-perforée).

Appendix A

The third order asymptotic expansion

Let us recall that λ_n and u_n are the n^{th} eigenvalue and eigenvector of the Diriclet-Laplacian in Ω, (see (1.5)) and that λ_n and u_n are the n^{th} eigenvalue and eigenvector of the Diriclet-Laplacian in Ω^δ, (see (1.5)).

In this section, we give without detail the third order asymyptotic expansion of λ^δ and u^δ. A polynomial gauge is not sufficient to describe the asymptotic, one has to consider an extra polynomial-logarithmic gauge function ($\delta^3\ \ln(\delta)$). We introduce the following notations.

$$\lambda_n^\delta \simeq \lambda_n^0 + \delta\ \lambda_n^1 + \delta^2\ \lambda_n^2 + \delta^3\ \lambda_n^3, \tag{A.1}$$

$$u_n^\delta \simeq u_n^0 + \delta\ u_n^1 + \delta^2\ u_n^2 + \delta^3\ u_n^3, \tag{A.2}$$

$$u_n^\delta(\mathbf{X}\delta) = \Pi_n^\delta(\mathbf{X}) \simeq \Pi_n^0 + \delta\ \Pi_n^1 + \delta^2\ \Pi_n^2 + \delta_n^3\ \Pi_n^2 + \delta^3\ \Pi_n^{3,0} + \delta^3 \ln\delta\ \Pi_n^{3,1}, \tag{A.3}$$

We mention that the second order asymptotic expansion has been formally derived in Chapter 2 and the second order asymptotic expansion has been mathematically validated in Chapter 3.

A.1 The interior case ($u_n = 0$ in Ω_{ext})

A.1.1 The eigenvalue expansion

$$\lambda_n^0 = \lambda_n, \tag{A.4}$$

$$\lambda_n^1 = 0, \tag{A.5}$$

$$\lambda_n^2 = -\frac{\pi}{16}\frac{|\partial_x u_n\big|_{\Omega_{int}}(\mathbf{0})|^2}{\|u_n\|^2_{L^2(\Omega_{int})}}. \tag{A.6}$$

A.1.2 The far-field expansion

$$u_n^0 = u_n, \tag{A.7}$$

$$u_n^1 = 0, \tag{A.8}$$

$$\left\{\begin{array}{l} \text{Find } u_n^2 : \Omega \to \mathbb{R} \text{ and } \lambda_n^2 \in \mathbb{R} \text{ such that} \\ \Delta u_n^2 + \lambda_n u_n^2 = -\lambda_n^2 u_n, \quad \text{in } \Omega, \\ u_n^2 = 0, \quad \text{on } \partial\Omega \setminus \{\mathbf{0}\}, \\ u_n^2(\mathbf{x}) - \partial_x u_n|_{\Omega_{int}}(\mathbf{0}) \, \dfrac{1}{16} \, \dfrac{\sin(\theta)}{r} \in H^1(\Omega_{int}), \\ u_n^2(\mathbf{x}) + \partial_x u_n|_{\Omega_{int}}(\mathbf{0}) \, \dfrac{1}{16} \, \dfrac{\sin(\theta)}{r} \in H^1(\Omega_{ext}), \end{array}\right. \tag{A.9}$$

$$u_n^3 = 0. \tag{A.10}$$

A.1.3 The near-field expansion

$$\Pi_n^0 = 0, \tag{A.11}$$

$$\left\{\begin{array}{l} \text{Find } \Pi_n^1 : \widehat{\Omega} \longrightarrow \mathbb{R} \text{ such that} \\ \Pi_n^1 - \partial_x u_n|_{\Omega_{int}}(\mathbf{0}) \Psi_{int}(\mathbf{X}) X \in K_0^1, \\ \Delta \Pi_n^1 = 0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^1 = 0, \quad \text{on } \partial\widehat{\Omega}. \end{array}\right. \tag{A.12}$$

$$\left\{\begin{array}{l} \text{Find } \Pi_n^2 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ -\Delta \Pi_n^2 = 0, \quad \text{in } \widehat{\Omega}, \\ \Pi_n^2 = 0, \quad \text{on } \partial\widehat{\Omega}, \\ \Pi_n^2(\mathbf{X}) \; - \; \Psi_{int}(\mathbf{X}) \, XY \, \partial_{xy}^2 u_n^0|_{\Omega_{int}}(\mathbf{0}) \; \in K_0^1, \end{array}\right. \tag{A.13}$$

$$
\left\{
\begin{aligned}
&\text{Find } \Pi_n^{3,0} : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\
&\Delta \Pi_n^{3,0} = -\lambda_n \Pi_n^1, && \text{in } \widehat{\Omega}, \\
&\Pi_n^{3,0} = 0, && \text{on } \partial\widehat{\Omega}, \\
&\Pi_n^{3,0}(\mathbf{X}) - (U_n^{3,0})_3(\mathbf{X}) = \underset{R\to+\infty}{o}(1), && \text{in } \widehat{\Omega}_{int} \text{ and } \widehat{\Omega}_{ext}.
\end{aligned}
\right.
\tag{A.14}
$$

$$
\left\{
\begin{aligned}
&\text{Find } \Pi_n^{3,1} : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\
&\Delta \Pi_n^{3,1} = 0, && \text{in } \widehat{\Omega}, \\
&\Pi_n^{3,1} = 0, && \text{on } \partial\widehat{\Omega}, \\
&\Pi_n^{3,1}(\mathbf{X}) - (U_n^{3,1})_3(\mathbf{X}) = \underset{R\to+\infty}{o}(1), && \text{in } \widehat{\Omega}_{int} \text{ and } \widehat{\Omega}_{ext}.
\end{aligned}
\right.
\tag{A.15}
$$

with

$$
\left\{
\begin{aligned}
(U_n^{3,0})_3(\mathbf{X}) &= \partial_x^3 u_n|_{\Omega_{int}}(\mathbf{0})\frac{X^3}{3!} \\
&\quad + \frac{\partial_x u_n|_{\Omega_{int}}(\mathbf{0})}{16}\left(-\frac{\lambda_n}{2}\ln\left(\frac{\sqrt{\lambda_n}R}{2}\right)+\gamma_{int}\right)X, \text{ in } \widehat{\Omega}_{int}, \\
(U_n^{3,0})_3(\mathbf{X}) &= -\frac{\partial_x u_n|_{\Omega_{int}}(\mathbf{0})}{16}\left(-\frac{\lambda_n}{2}\ln\left(\frac{\sqrt{\lambda_n}R}{2}\right)+\gamma_{ext}\right)X, \text{ in } \widehat{\Omega}_{ext}, \\
(U_n^{3,1})_3(\mathbf{X}) &= -\frac{\partial_x u_n|_{\Omega_{int}}(\mathbf{0})}{32}\lambda_n X, \text{ in } \widehat{\Omega}_{int}, \\
(U_n^{3,1})_3(\mathbf{X}) &= \frac{\partial_x u_n|_{\Omega_{int}}(\mathbf{0})}{32}\lambda_n X, \text{ in } \widehat{\Omega}_{ext}.
\end{aligned}
\right.
\tag{A.16}
$$

A.1.4 Spatial asymptotic expansions of the far-field coefficients

$$\begin{cases} u_n\big(r,\theta\big) - x\ \partial_x u_n\big(\mathbf{0}\big)\ -\ xy\ \partial^2_{xy} u_n\big(\mathbf{0}\big)\ -\ x^3\ \dfrac{\partial^3_x u_n\big(\mathbf{0}\big)}{3!} = \underset{r\to 0}{O}(r^4) \text{ in } \Omega_{int}, \\ u_n\big(r,\theta\big) = 0 \text{ in } \Omega_{ext}, \\ u^2_n(\mathbf{x}) - \partial_x u_n(\mathbf{0})\ \dfrac{1}{16}\Big(\dfrac{1}{r} - \dfrac{\lambda_n r}{2}\Big(\ln \dfrac{\sqrt{\lambda_n} r}{2}\Big) + \gamma_{int}\ r\Big)\ \sin(\theta) = \underset{r\to 0}{O}(r^2 \ln r) \text{ in } \Omega_{int}, \\ u^2_n(\mathbf{x}) + \partial_x u_n(\mathbf{0})\ \dfrac{1}{16}\Big(\dfrac{1}{r} - \dfrac{\lambda_n r}{2}\Big(\ln \dfrac{\sqrt{\lambda_n} r}{2}\Big) + \gamma_{ext}\ r\Big)\ \sin(\theta) = \underset{r\to 0}{O}(r^2 \ln r) \text{ in } \Omega_{ext}. \end{cases} \tag{A.17}$$

$$\begin{cases} \nabla\Big(u_n\big(r,\theta\big) - x\ \partial_x u_n\big(\mathbf{0}\big)\ -\ xy\ \partial^2_{xy} u_n\big(\mathbf{0}\big)\ -\ x^3\ \dfrac{\partial^3_x u_n\big(\mathbf{0}\big)}{3!}\Big) = \underset{r\to 0}{O}(r^3) \text{ in } \Omega_{int}, \\ \nabla\Big(u_n\big(r,\theta\big)\Big) = 0 \text{ in } \Omega_{ext}, \\ \nabla\Big(u^2_n(\mathbf{x}) - \partial_x u_n(\mathbf{0})\ \dfrac{1}{16}\Big(\dfrac{1}{r} - \dfrac{\lambda_n r}{2}\Big(\ln \dfrac{\sqrt{\lambda_n} r}{2}\Big) + \gamma_{int}\ r\Big)\ \sin(\theta)\Big) = \underset{r\to 0}{O}(r \ln r) \text{ in } \Omega_{int}, \\ \nabla\Big(u^2_n(\mathbf{x}) + \partial_x u_n(\mathbf{0})\ \dfrac{1}{16}\Big(\dfrac{1}{r} - \dfrac{\lambda_n r}{2}\Big(\ln \dfrac{\sqrt{\lambda_n} r}{2}\Big) + \gamma_{ext}\ r\Big)\ \sin(\theta)\Big) = \underset{r\to 0}{O}(r \ln r) \text{ in } \Omega_{ext}. \end{cases} \tag{A.18}$$

A.1.5 Spatial asymptotic expansions of the near-field coefficients

$$\begin{cases} \Pi^0_n(\mathbf{X}) - (U^0_n)_3(\mathbf{X}) &= 0, \\ \Pi^1_n(\mathbf{X}) - (U^1_n)_3(\mathbf{X}) &= \underset{R\to+\infty}{O}(\frac{1}{R^3}), \\ \Pi^2_n(\mathbf{X}) - (U^2_n)_3(\mathbf{X}) &= \underset{R\to+\infty}{O}(\frac{1}{R^2}), \\ \Pi^{3,0}_n(\mathbf{X}) - (U^{3,0}_n)_3(\mathbf{X}) &= \underset{R\to+\infty}{O}(\frac{1}{R}), \\ \Pi^{3,1}_n(\mathbf{X}) - (U^{3,1}_n)_3(\mathbf{X}) &= \underset{R\to+\infty}{O}(\frac{1}{R}), \end{cases} \tag{A.19}$$

$$
\begin{cases}
\nabla_X\Big(\Pi_n^0(\mathbf{X}) - (U_n^0)_3(\mathbf{X})\Big) &= 0,\\
\nabla_X\Big(\Pi_n^1(\mathbf{X}) - (U_n^1)_3(\mathbf{X})\Big) &= \underset{R\to+\infty}{O}(\frac{1}{R^4}),\\
\nabla_X\Big(\Pi_n^2(\mathbf{X}) - (U_n^2)_3(\mathbf{X})\Big) &= \underset{R\to+\infty}{O}(\frac{1}{R^3}),\\
\nabla_X\Big(\Pi_n^{3,0}(\mathbf{X}) - (U_n^{3,0})_3(\mathbf{X})\Big) &= \underset{R\to+\infty}{O}(\frac{1}{R^2}),\\
\nabla_X\Big(\Pi_n^{3,1}(\mathbf{X}) - (U_n^{3,1})_3(\mathbf{X})\Big) &= \underset{R\to+\infty}{O}(\frac{1}{R^2}),
\end{cases}
\tag{A.20}
$$

with

$$
\begin{cases}
(U_n^0)_3(\mathbf{X}) &= 0, \text{ in } \widehat{\Omega}_{int} \text{ and } \widehat{\Omega}_{ext},\\
(U_n^1)_3(\mathbf{X}) &= \partial_x u_n|_{\Omega_{int}}(\mathbf{0})\big(X + \frac{1}{16}\frac{\sin\theta}{R}\big), \text{ in } \widehat{\Omega}_{int},\\
(U_n^1)_3(\mathbf{X}) &= -\partial_x u_n|_{\Omega_{int}}(\mathbf{0})\frac{1}{16}\frac{\sin\theta}{R}, \text{ in } \widehat{\Omega}_{ext},\\
(U_n^2)_3(\mathbf{X}) &= \partial_{xy}^2 u_n|_{\Omega_{int}}(\mathbf{0})XY, \text{ in } \widehat{\Omega}_{int},\\
(U_n^2)_3(\mathbf{X}) &= 0, \text{ in } \widehat{\Omega}_{ext}.
\end{cases}
\tag{A.21}
$$

A.2 The exterior case ($u_n = 0$ in Ω_{int})

A.2.1 The eigenvalue expansion

$$\lambda_n^0 = \lambda_n, \tag{A.22}$$

$$\lambda_n^1 = 0, \tag{A.23}$$

$$\lambda_n^2 = -\frac{\pi}{16}\frac{|\partial_x u_n\big|_{\Omega_{ext}}(\mathbf{0})|^2}{\|u_n\|^2_{L^2(\Omega_{ext})}}. \tag{A.24}$$

A.2.2 The far-field expansion

$$u_n^0 = u_n, \tag{A.25}$$

$$u_n^1 = 0, \tag{A.26}$$

$$
\begin{cases}
\text{Find } u_n^2 : \Omega \to \mathbb{R} \text{ and } \lambda_n^2 \in \mathbb{R} \text{ such that} \\
\Delta u_n^2 + \lambda_n u_n^2 = -\lambda_n^2 u_n, \quad \text{in } \Omega, \\
u_n^2 = 0, \quad \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \\
u_n^2(\mathbf{x}) - \partial_x u_n|_{\Omega_{ext}}(\mathbf{0}) \, \dfrac{1}{16} \, \dfrac{\sin(\theta)}{r} \in H^1(\Omega_{ext}), \\
u_n^2(\mathbf{x}) + \partial_x u_n|_{\Omega_{ext}}(\mathbf{0}) \, \dfrac{1}{16} \, \dfrac{\sin(\theta)}{r} \in H^1(\Omega_{int}),
\end{cases}
\tag{A.27}
$$

$$
u_n^3 = 0. \tag{A.28}
$$

A.2.3 The near-field expansion

$$
\Pi_n^0 = 0, \tag{A.29}
$$

$$
\begin{cases}
\text{Find } \Pi_n^1 : \widehat{\Omega} \longrightarrow \mathbb{R} \text{ such that} \\
\Pi_n^1 - \partial_x u_n|_{\Omega_{ext}}(\mathbf{0}) \Psi_{ext}(\mathbf{X}) X \in K_0^1, \\
\Delta \Pi_n^1 = 0, \quad \text{in } \widehat{\Omega}, \\
\Pi_n^1 = 0, \quad \text{on } \partial\widehat{\Omega}.
\end{cases}
\tag{A.30}
$$

$$
\begin{cases}
\text{Find } \Pi_n^2 : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\
-\Delta \Pi_n^2 = 0, \quad \text{in } \widehat{\Omega}, \\
\Pi_n^2 = 0, \quad \text{on } \partial\widehat{\Omega}, \\
\Pi_n^2(\mathbf{X}) \; - \; \Psi_{ext}(\mathbf{X}) \, XY \, \partial_{xy}^2 u_n|_{\Omega_{ext}}(\mathbf{0}) \; \in K_0^1,
\end{cases}
\tag{A.31}
$$

$$
\begin{cases}
\text{Find } \Pi_n^{3,0} : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\
\Delta \Pi_n^{3,0} = -\lambda_n \Pi_n^1, \quad \text{in } \widehat{\Omega}, \\
\Pi_n^{3,0} = 0, \quad \text{on } \partial\widehat{\Omega}, \\
\Pi_n^{3,0}(\mathbf{X}) - (U_n^{3,0})_3(\mathbf{X}) = \underset{R \to +\infty}{o}(1), \quad \text{in } \widehat{\Omega}_{int} \text{ and } \widehat{\Omega}_{ext}.
\end{cases}
\tag{A.32}
$$

$$\begin{cases} \text{Find } \Pi_n^{3,1} : \widehat{\Omega} \to \mathbb{R} \text{ such that} \\ \Delta \Pi_n^{3,1} = 0, & \text{in } \widehat{\Omega}, \\ \Pi_n^{3,1} = 0, & \text{on } \partial\widehat{\Omega}, \\ \Pi_n^{3,1}(\mathbf{X}) - (U_n^{3,1})_3(\mathbf{X}) = \underset{R\to+\infty}{o}(1), & \text{in } \widehat{\Omega}_{int} \text{ and } \widehat{\Omega}_{ext}. \end{cases} \tag{A.33}$$

with

$$\begin{cases} (U_n^{3,0})_3(\mathbf{X}) &= \partial_x^3 u_n|_{\Omega_{ext}}(\mathbf{0}) \dfrac{X^3}{3!} \\ & \quad + \dfrac{\partial_x u_n|_{\Omega_{ext}}(\mathbf{0})}{16} \left(-\dfrac{\lambda_n}{2} \ln\left(\dfrac{\sqrt{\lambda_n} R}{2}\right) + \gamma'_{ext} \right) X, \text{ in } \widehat{\Omega}_{ext}, \\ (U_n^{3,0})_3(\mathbf{X}) &= -\dfrac{\partial_x u_n|_{\Omega_{ext}}(\mathbf{0})}{16} \left(-\dfrac{\lambda_n}{2} \ln\left(\dfrac{\sqrt{\lambda_n} R}{2}\right) + \gamma'_{int} \right) X, \text{ in } \widehat{\Omega}_{int}, \\ (U_n^{3,1})_3(\mathbf{X}) &= -\dfrac{\partial_x u_n|_{\Omega_{ext}}(\mathbf{0})}{32} \lambda_n X, \text{ in } \widehat{\Omega}_{ext}, \\ (U_n^{3,1})_3(\mathbf{X}) &= \dfrac{\partial_x u_n|_{\Omega_{ext}}(\mathbf{0})}{32} \lambda_n X, \text{ in } \widehat{\Omega}_{int}. \end{cases} \tag{A.34}$$

Appendix B

Prerequisite on eigenvalue problem

In this section, we recall briefly some classical results on eigenvalues of the Dirichlet-Laplacian. One can find a survey of this very old topic in [19].

B.1 The eigenvalues of the Dirichlet-Laplacian

Let Ω be a bounded open domain of $\mathbb{R}^2$ with Lipschitz boundary. We denote by $L^2(\Omega)$ the space of square integrable functions and by $H_0^1(\Omega)$ the space

$$H_0^1(\Omega) \ = \ \Big\{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega \Big\}. \tag{B.1}$$

These spaces are equipped with the $L^2(\Omega)$ and $H_0^1(\Omega)$ inner products and the associated norms

$$\begin{cases} (u,v)_{L^2(\Omega)} = \displaystyle\int_\Omega uv, \quad \|u\|_0 = (u,u)^{\frac{1}{2}}_{L^2(\Omega)}, \\ \mathsf{a}(u,v) = \displaystyle\int_\Omega \nabla u \cdot \nabla v, \quad |u|_1 = (\mathsf{a}(u,u))^{\frac{1}{2}}. \end{cases} \tag{B.2}$$

The Dirichlet-Laplacian can be defined as an unbounded operator on $L^2(\Omega)$

$$\Delta : u \longmapsto \Delta u = \partial_x^2 u + \partial_y^2 u \tag{B.3}$$

with domain

$$\mathsf{D}(\Delta) = \Big\{ u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega) \Big\}. \tag{B.4}$$

The eigenvalues of the laplacian (their opposites for positivity) are defined like the solution of the following problem

$$\begin{cases} \text{Find } \lambda \in \mathbb{R} \text{ such that } \exists u \in \mathsf{D}(\Delta),\ u \neq 0 \text{ satisfying} \\ -\Delta u = \lambda u, \end{cases} \tag{B.5}$$

or equivalently by the variational problem

$$\begin{cases} \text{Find } \lambda \in \mathbb{R} \text{ such that } \exists u \in H_0^1(\Omega),\ u \neq 0 \text{ satisfying} \\ \mathsf{a}(u,v) = \lambda(u,v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{B.6}$$

Since Ω is bounded, the spectral theory of self-adjoint compact operator ensure the existence of a countable set of eigenvalues $\{\lambda_n > 0\}_{n>0}$

$$\lambda_0 \leq \lambda_1 \leq \cdots \text{ and } \lim_{n\to+\infty} \lambda_n = +\infty \tag{B.7}$$

with associated eigenvectors $\omega_n \in H_0^1(\Omega)$ ($\omega_n \neq 0$) which can be chosen to be an orthogonal basis of $L^2(\Omega)$ and of $H_0^1(\Omega)$

$$(\omega_n, \omega_m)_{L^2(\Omega)} = 0 \text{ and } (\nabla\omega_m, \nabla\omega_n)_{L^2(\Omega)} = 0 \quad \text{for } n \neq m. \tag{B.8}$$

B.2 The min-max principle

Theorem 6 (min-max) *The n^{th}-eigenvalue of the Dirichlet-Laplacian in Ω is given by*

$$\lambda_n = \min_{\substack{V \subset H_0^1(\Omega) \\ dim(V) = n}} \max_{\substack{u \in V \\ u \neq 0}} R(u) \tag{B.9}$$

with $R(u)$ the Rayleigh quotient

$$R(u) = \frac{\mathsf{a}(u,u)}{(u,u)_{L^2(\Omega)}}. \tag{B.10}$$

Proof. One can refer to Theorem 6.2-2 of [24] for the proof. ∎

B.3 A theorem of localisation of eigenvalues

We recall now a basic tool often used to derive asymptotic expansions of eigenvalues.

Theorem 7 *If there exists $u \in H_0^1(\Omega)$, $\gamma \in \mathbb{R}$, and $\varepsilon \in \mathbb{R}_+^*$ such that*

$$\left|\mathsf{a}(u,v) - \gamma(u,v)\right| \leq \varepsilon \|u\|_0 \|v\|_0, \quad \forall v \in H_0^1(\Omega) \tag{B.11}$$

then there exists an eigenvalue λ of the operator $-\Delta$ with domain $\mathsf{D}(\Delta)$ such that

$$\gamma - \varepsilon \leq \lambda \leq \gamma + \varepsilon. \tag{B.12}$$

Proof. We act by contradiction. We suppose that there exists no eigenvalues satisfying the inequality (B.12). Let $(\lambda_i)_{i>0}$ be the set eigenvalues of the Dirichlet-laplacian $-\Delta$, and $(\omega_i)_{i>0}$ the orthogonal eigenfunctions family associated to these eigenvalues. For all u, $v \in H_0^1(\Omega)$, there exist $(u_p)_{p>0}$, and $(v_p)_{p>0}$ (the coordinates of u and v in the basis $(\omega_i)_{i>0}$) such that

$$u = \sum_{p=1}^{+\infty} u_p \omega_p \text{ and } v = \sum_{p=1}^{+\infty} v_p \omega_p. \tag{B.13}$$

Therefore, one has

$$\mathsf{a}\,(u, v) = \sum_{p=1}^{+\infty} \lambda_p u_p v_p. \tag{B.14}$$

Consequently, we get

$$\begin{aligned} \left|\mathsf{a}(u,v) - \gamma(u,v)\right| &= \Big|\sum_{p=1}^{+\infty} \lambda_p u_p v_p - \gamma \sum_{p=1}^{+\infty} u_p v_p\Big|, \\ &= \Big|\sum_{p=1}^{+\infty} (\lambda_p - \gamma)\, u_p v_p\Big|. \end{aligned} \tag{B.15}$$

By taking $v_p = \dfrac{|\lambda_p - \gamma|}{\lambda_p - \gamma}$ ($\lambda_p \neq \gamma$ by hypothesis) and using equation (B.15) we obtain

$$\Big|\sum_{p=1}^{+\infty} \lambda_p u_p v_p - \gamma \sum_{p=1}^{+\infty} u_p v_p\Big| = \sum_{p=1}^{+\infty} |\lambda_p - \gamma| u_p^2 > \varepsilon \sum_{p=1}^{+\infty} u_p^2 \geq \varepsilon \|u\|_0^2 . \|v\|_0^2, \tag{B.16}$$

which is impossible, and the existence of an eigenvalue in the ε-neighborhood of γ holds. ■

Note that this Theorem does not involve constant depending on Ω. Consequently, this Theorem will be one of the key point of the asymptotic analysis carried through this report where the domain depend on the small parameter δ. The reader can also refer to [12, 5, 3, 6] to see another use of this Theorem.

Appendix C

Some results on separation of variables.

C.1 Separation of variables for the far-field

In the continuation, we denote by r and θ the polar coordinates

$$x = r \sin\theta,\ y = -r\cos\theta,\ \text{with } r \geq 0,\ \text{and } 0 \leq \theta < 2\pi. \tag{C.1}$$

Let B_{ext} be a neighborhood of zero (ρ is a real which quantifies the size of this neighborhood)

$$B_{ext} = \Big\{ \mathbf{x} \in \mathbb{R}^2 : x > 0 \text{ and } r < \rho \Big\} \tag{C.2}$$

For $\lambda > 0$, we are interested in expanding solutions of the following equations

$$\begin{cases} u \in C^{\infty}(\overline{B_{ext}}) \setminus \{\mathbf{0}\}, \\ \Delta u + \lambda u = 0 \text{ in } B_{ext}, \\ u(0, y) = 0 \text{ for } 0 < |y| < \rho. \end{cases} \tag{C.3}$$

By separation of variables, the solutions of this equation can be written with the following form (the reader can refer for example to [27] for more details)

$$u(r, \theta) := \sum_{n=1}^{+\infty} \Big(a^n_{ext} J_n(\sqrt{\lambda} r) + b^n_{ext} Y_n(\sqrt{\lambda} r) \Big) \sin n\theta, \tag{C.4}$$

where $J_n(z)$ and $Y_n(z)$ for $n \in \mathbb{N}$ are the Bessel functions (see for example [20, 31]) defined by the following series (which converges unconditionally)

$$\begin{cases} J_n(z) = \sum_{l=-\infty}^{+\infty} J_{n,l} \left(\frac{z}{2}\right)^l, \\ Y_n(z) = \sum_{l=-\infty}^{+\infty} Y_{n,l} \left(\frac{z}{2}\right)^l + \frac{2}{\pi} \sum_{l=-\infty}^{+\infty} J_{n,l} \left(\frac{z}{2}\right) \log \frac{z}{2}, \end{cases} \tag{C.5}$$

with $J_{n,p}$ and $Y_{n,p}$ given by

$$\begin{cases} J_{n,n+l} = 0 & \text{if } l > 0 \text{ or } l \text{ odd }, \\ J_{n,n+2l} = \frac{(-1)^l}{l!\,(l+n)!} & \text{if } l \geq 0, \end{cases} \tag{C.6}$$

$$\begin{cases} Y_{n,-n+l} = 0 & \text{if } l < 0 \text{ or } l \text{ odd }, \\ J_{n,-n+2l} = -\frac{1}{\pi} \frac{(n-l-1)!}{l!} & \text{if } 0 \leq l \leq n, \\ Y_{n,n+2l} = -\frac{1}{\pi} \frac{(-1)^l}{l!\,(l+n)!} \left(\psi(l+1) + \psi(l+n+1)\right) & \text{if } 0 \leq l, \end{cases} \tag{C.7}$$

with ($\gamma = 0,5772157...$ is the Euler number)

$$\psi(1) = -\gamma, \qquad \psi(k+1) = -\gamma + \sum_{m=1}^{k} \frac{1}{m}, \forall k \in \mathbb{N}^*. \tag{C.8}$$

Remark 15 *Asymptotic expansion of $J_n(z)$ and $Y_n(z)$ in the neighborhood of zero: The following equivalent will be required in the report*

$$J_n(z) \underset{z \to 0}{\sim} \frac{(z/2)^n}{n!}, \quad Y_n(z) \underset{z \to 0}{\sim} -\frac{(n-1)!}{\pi} \left(\frac{z}{2}\right)^{-n} \text{ for } n \geq 1. \tag{C.9}$$

In the case of a regular functions ($u \in C^\infty(B_{ext})$) one can simplify (C.4) in

$$u(r, \theta) = \sum_{p=1}^{+\infty} a_{ext}^p J_p \left(\sqrt{\lambda} r\right) sin(p\,\theta). \tag{C.10}$$

Taking into account the behavior of J_p we get

$$u(r, \theta) = \sum_{p=1}^{N} a_{ext}^p J_p \left(\sqrt{\lambda} r\right) sin(p\,\theta) + \underset{r \to 0}{o}(r^N), \quad \forall N \in \mathbb{N}. \tag{C.11}$$

By symmetry, in B_{int}

$$B_{int} = \Big\{\mathbf{x} \in \mathbb{R}^2 : x < 0 \text{ and } r < \rho\Big\} \tag{C.12}$$

every solution of

$$\begin{cases} u \in C^\infty(\overline{B_{int}}) \setminus \{\mathbf{0}\}, \\ \Delta u + \lambda u = 0 \text{ in } B_{int}, \\ u(0,y) = 0 \text{ for } 0 < |y| < \rho. \end{cases} \tag{C.13}$$

can be expanded as follows

$$\begin{aligned} u_{int}(r,\theta) &= \sum_{p=1}^{+\infty} \Big(a_{int}^p J_p(\sqrt{\lambda} r) + b_{int}^p Y_p(\sqrt{\lambda} r)\Big) \sin p\theta, \quad \text{in } B_{int}, \\ u_{ext}(r,\theta) &= \sum_{p=1}^{+\infty} \Big(a_{ext}^p J_p(\sqrt{\lambda} r) + b_{int}^p Y_p(\sqrt{\lambda} r)\Big) \sin p\theta, \quad \text{in } B_{ext}. \end{aligned} \tag{C.14}$$

C.2 Separation of variables for the near field

Let $\mathcal{B}_{int}$ and $\mathcal{B}_{ext}$ be the two neighborhood of infinity

$$\begin{cases} \mathcal{B}_{int} = \Big\{\mathbf{X} \in \mathbb{R}^2 : X < 0 \text{ and } R > 1\Big\}, \\ \mathcal{B}_{ext} = \Big\{\mathbf{X} \in \mathbb{R}^2 : X > 0 \text{ and } R > 1\Big\}. \end{cases} \tag{C.15}$$

We consider $\Pi = (\Pi_{int}, \Pi_{ext})$ solution of the laplace equation with Dirichlet boundary conditions

$$\begin{gathered} \Pi = (\Pi_{int}, \Pi_{ext}) \in \big(C^\infty(\overline{\mathcal{B}_{int}}) \setminus \{\mathbf{0}\}\big) \times \big(C^\infty(\overline{\mathcal{B}_{ext}}) \setminus \{\mathbf{0}\}\big), \\ \begin{cases} \Delta\Pi_{int} = 0 \text{ in } \mathcal{B}_{int}, \\ \Pi_{int}(0,Y) = 0 \text{ for } |Y| > 1. \end{cases} \qquad \begin{cases} \Delta\Pi_{ext} = 0 \text{ in } \mathcal{B}_{ext}, \\ \Pi_{ext}(0,Y) = 0 \text{ for } |Y| > 1. \end{cases} \end{gathered} \tag{C.16}$$

This function can be expanded via its modal expansion

$$\begin{cases} \Pi_{int}(R,\theta) = \sum_{p=1}^{+\infty} \big(\alpha_{int}^p R^p + \beta_{int}^p R^{-p}\big)\, sin(p\,\theta), \quad \text{in } \mathcal{B}_{int}, \\ \Pi_{ext}(R,\theta) = \sum_{p=1}^{+\infty} \big(\alpha_{ext}^p R^p + \beta_{ext}^p R^{-p}\big)\, sin(p\,\theta), \quad \text{in } \mathcal{B}_{ext} \end{cases} \tag{C.17}$$

Bibliography

[1] C. Anné, *A note on the generalized dumbbell problem*, Proc. Amer. Math. Soc., 123 (1995), pp. 2595–2599.

[2] J. M. Arrieta, *Neumann eigenvalue problems on exterior perturbations of the domain*, J. Differential Equations, 118 (1995), pp. 54–103.

[3] A. Bamberger and A.-S. Bonnet, *Mathematical analysis of the guided modes of an optical fiber*, SIAM J. Math. Anal., 21 (1990), pp. 1487–1510.

[4] J. T. Beale, *Scattering frequencies of reasonators*, Comm. Pure Appl. Math., 26 (1973), pp. 549–563.

[5] V. Bonnaillie-Noël and M. Dauge, *Asymptotics for the low-lying eigenstates of the Schrödinger operator with magnetic field near corners*, Ann. Henri Poincaré, 7 (2006), pp. 899–931.

[6] A.-S. Bonnet-BenDhia, *Mathematical analysis of conductive and superconductive transmission lines*, in Mathematical and numerical aspects of wave propagation (Golden, CO, 1998), SIAM, Philadelphia, PA, 1998, pp. 12–21.

[7] A.-S. Bonnet-BenDhia, D. Drissi, and N. Gmati, *Simulation of muffler's transmission losses by a homogenized finite element method*, J. Comput. Acoust., 12 (2004), pp. 447–474.

[8] R. Brown, P. D. Hislop, and A. Martinez, *Eigenvalues and resonances for domains with tubes: Neumann boundary conditions*, J. Differential Equations, 115 (1995), pp. 458–476.

[9] R. M. Brown, P. D. Hislop, and A. Martinez, *Lower bounds on the interaction between cavities connected by a thin tube*, Duke Math. J., 73 (1994), pp. 163–176.

[10] G. Caloz, M. Costabel, M. Dauge, and G. Vial, *Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer*, Asymptot. Anal., 50 (2006), pp. 121–173.

[11] D. Crighton, A. Dowling, J. Ffowks Williams, M. Heckl, and F. Leppington, *Modern Methods in Analytical acoustics*, Lecture Notes, Springer-Verlag, London, 1992. An Asymptotic Analysis.

[12] M. Dauge, I. Djurdjevic, E. Faou, and A. Rössle, *Eigenmode asymptotics in thin elastic plates*, J. Math. Pures Appl. (9), 78 (1999), pp. 925–964.

[13] R. R. Gadyl'shin, *Surface potentials and the method of matching asymptotic expansions in the helmholtz resonator problem*, (Russian) Algebra i Analiz 4 (1992), no. 2, 88–115, translation in St. Petersburg Math. J., 4 (1993), pp. 273–296.

[14] J. M. Garcia, S. Mendez, O. Staffelbach, G. Vermorel, and T. Poinsot, *Growth of rounding errors and the repetitivity of large eddy simulations*, AIAA Journal, 46 (2008), pp. 1773–1781.

[15] A. M. Il′in, *Matching of asymptotic expansions of solutions of boundary value problems*, vol. 102 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1992. Translated from the Russian by V. Minachin [V. V. Minakhin].

[16] S. Jimbo and Y. Morita, *Remarks on the behavior of certain eigenvalues on a singularly perturbed domain with several thin channels*, Comm. Partial Differential Equations, 17 (1992), pp. 523–552.

[17] P. Joly and S. Tordeux, *Matching of asymptotic expansions for wave propagation in media with thin slots. I. The asymptotic expansion*, Multiscale Model. Simul., 5 (2006), pp. 304–336 (electronic).

[18] ——, *Matching of asymptotic expansions for waves propagation in media with thin slots. II. The error estimates*, M2AN Math. Model. Numer. Anal., 42 (2008), pp. 193–221.

[19] J. R. Kuttler and V. G. Sigillito, *Eigenvalues of the Laplacian in two dimensions*, SIAM Rev., 26 (1984), pp. 163–193.

[20] N. N. Lebedev, *Special functions and their applications*, Dover Publications Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.

[21] A. Makhlouf, *Justification et amlioration de modles d'antennes patch par la mthode des dveloppements asymptotiques raccords*, Phd thesis, Université de Toulouse, 2008.

[22] S. Mendez and F. Nicoud, *Large-eddy simulation of a bi-periodic turbulent flow with effusion*, J. Fluid Mech., 598 (2008), pp. 27–65.

[23] J. Rauch and M. Taylor, *Electrostatic screening*, J. Mathematical Phys., 16 (1975), pp. 284–288.

[24] P.-A. Raviart and J.-M. Thomas, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983.

[25] J. Sanchez-Hubert and E. Sánchez-Palencia, *Acoustic fluid flow through holes and permeability of perforated walls*, J. Math. Anal. Appl., 87 (1982), pp. 427–453.

[26] A. Taflov, K. Umashanker, B. Becker, F. Harfoush, and K. S. Yee, *Detailed fdtd analysis of electromagnetic fields penetrating narrow slots and lapped joints in think conducting screens*, IEEE Trans Antenna and Propagation, 36 (1988), pp. 247–257.

[27] S. Tordeux, *Méthodes asymptotiques pour la propagation des ondes dans les milieux comportant des fentes*, Phd thesis, Université de Versailles, 2004.

[28] S. Tordeux, G. Vial, and M. Dauge, *Matching and multiscale expansions for a model singular perturbation problem*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 637–642.

[29] E. O. Tuck, *Matching problems involving flow through small holes*, in Advances in applied mechanics, Vol. 15, Academic Press, New York, 1975, pp. 89–158.

[30] M. Van Dyke, *Perturbation methods in fluid mechanics*, The Parabolic Press, Stanford, Calif., annotated ed., 1975.

[31] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition.

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