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WEAK SOLUTIONS TO A THIN FILM MODEL WITH CAPILLARY EFFECTS AND INSOLUBLE SURFACTANT

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Abstract. The paper focuses on a model describing the spreading of an insoluble surfactant on a thin viscous film with capillary effects taken into account. The governing equation for the film height is degenerate parabolic of fourth order and coupled to a second order parabolic equation for the surfactant concentration. It is shown that nonnegative weak solutions exist under natural assumptions on the surface tension coefficient.

1. Introduction

The modeling of the spreading of an insoluble surfactant on a thin viscous film leads to a coupled system of degenerate parabolic equations describing the space and time evolution of the height $h \geq 0$ of the film and the surface concentration $\Gamma \geq 0$ of surfactant. Assuming the film thickness to be small enough so that lubrication theory is applicable, taking into account capillary effects but neglecting gravitational and intermolecular (van der Waals) forces, the following system is obtained [4, 7, 8]

$$\partial_t h + \partial_x \left(\frac{1}{3}\, h^3\, \partial_x^3 h + \frac{1}{2}\, h^2\, \partial_x \sigma(\Gamma) \right) = 0\,, \quad (t,x) \in (0,\infty) \times (0,1)\,, \tag{1}$$

$$\partial_t \Gamma + \partial_x \left(\frac{1}{2}\, h^2\, \Gamma\, \partial_x^3 h + h\, \Gamma\, \partial_x \sigma(\Gamma) \right) = D\, \partial_x^2 \Gamma\,, \quad (t,x) \in (0,\infty) \times (0,1)\,, \tag{2}$$

with homogeneous Neumann boundary conditions

$$\partial_x h(t,x) = \partial_x^3 h(t,x) = \partial_x \Gamma(t,x) = 0\,, \quad (t,x) \in (0,\infty) \times \{0,1\}\,, \tag{3}$$

and initial conditions

$$(h,\Gamma)(0) = (h_0, \Gamma_0)\,, \quad x \in (0,1)\,. \tag{4}$$

Here, $\sigma(\Gamma)$ denotes the surface tension which depends on the local concentration of surfactant, and $D > 0$ stands for the surface diffusivity of the surfactant. Since σ is defined up to a constant, we may assume without loss of generality that $\sigma(1) = 0$ as a normalization condition. As the presence of surfactant reduces surface tension, σ is a non-increasing function of Γ; for instance,

$$\sigma_\beta(s) := (\beta+1)\ \left[1 - s + \left(\frac{\beta+1}{\beta} \right)^{1/3}\ s \right]^{-3} - \beta\,, \quad s \geq 0\,, \tag{5}$$

with $\beta \in (0,\infty)$ and its limit as $\beta \to \infty$ (which is often assumed in applications)

$$\sigma_\infty(s) := 1 - s\,, \quad s \geq 0\,. \tag{6}$$

The system (1)-(4) is a fully coupled nonlinear system of parabolic equations featuring a degeneracy where h vanishes, a fact which cannot be excluded *a priori*. Thus, classical solutions are unlikely to exist for all times in general and only local existence of smooth solutions to (1)–(2) in the absence of capillarity have been shown in [10, 11]. The alternative is to study the Cauchy problem in a framework of weak solutions (see Section 2 for a precise definition) and this

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approach has been successfully employed to establish the existence of weak solutions to systems similar to (1)–(2), for instance when the capillarity effect are neglected but the gravitational ones are accounted for [6] or when Γ is replaced by $\lambda(\Gamma) = \max\{0, 1-\Gamma\} + 1$ in (2) [1, 2], a finite element numerical scheme being also developed in these two papers.

To our knowledge, existence of weak solutions has only been tackled in [7] where this is proved under further technical assumptions on the surface tension σ. Namely, given a surface tension $\sigma \in \mathcal{C}^{2,1}_{loc}(\mathbb{R})$, defining the free energy g_σ by

$$g_\sigma(1) = g_\sigma'(1) = 0, \qquad g_\sigma''(s) = -\frac{\sigma'(s)}{s} \qquad \text{for } s \in \mathbb{R}, \tag{7}$$

the authors assume the following:

(A4) The function g_σ lies in $\mathcal{C}^{2,1}_{loc}(\mathbb{R})$.
(A5) There exists $c_g > 0$ such that $g_\sigma''(s) \geq c_g$ for all $s \in \mathbb{R}$.
(A6) There exist C_g and some $r \in (0,2)$ for which $g_\sigma''(s) \leq C_g(|s|^r + 1)$ for all $s \in \mathbb{R}$.

The need in [7] to define σ and g_σ in $\mathbb{R}$ instead of the physically relevant range $[0,\infty)$ for surfactant concentration stems from the fact that the weak solution (h,Γ) to (1)-(2) constructed in [7] might not satisfy $\Gamma \geq 0$. The extension of σ and g_σ to negative values then induces several limiting conditions. Namely, the convexity (A5) of g_σ in $\mathbb{R}$ requires not only that $\sigma'(s) < 0$ for $s > 0$ as expected but also $\sigma'(s) > 0$ for $s < 0$, which implies $\sigma'(0) = 0$ and thus excludes surface tensions σ like σ_β in (5) and σ_∞ in (6). In addition, assumption (A5) yields $\sigma'(s) \leq -c_g\, s$ for $s \in \mathbb{R}$ so that σ necessarily has a quadratic decay at infinity. This again excludes the previous examples σ_β in (5) and σ_∞ in (6).

The aim of the present paper is to construct a weak solution to (1)–(2) under weaker assumptions on the surface tension σ (satisfied in particular by σ_∞) and such that both h and Γ are nonnegative throughout time evolution. More precisely, we assume that the surface tension σ satisfies:

(H1) $\sigma \in \mathcal{C}^1((0,\infty)) \cap \mathcal{C}([0,\infty))$ with $\sigma(1) = 0$.
(H2) There exist $\sigma_0, \sigma_1 \in (0,\infty)$ and $\theta \in [0,1)$ for which:

$$-\sigma_0 < \sigma'(s) \leq -\frac{\sigma_1}{1+s^\theta} \quad \text{for } s \geq 1, \qquad -\sigma_0 < \sigma'(s) < 0 \quad \text{for } s \in (0,1). \tag{8}$$

The assumptions (H1)–(H2) include physically relevant surface tensions σ, which may slowly decrease at infinity. In particular, σ_∞ is included (but not σ_β for $\beta \in (0,\infty)$).

In the next section, we introduce the definition of weak solutions and give precise statements for our existence result. To prove this result, we first construct *nonnegative* solutions in the framework of [7] under assumptions (A4)–(A6). This improves the results of [7] in that the surfactant concentration Γ stays nonnegative through time evolution. This is the content of **Section 3**. Then we extend the construction to a surface tension σ merely satisfying (H1)–(H2) by approximating σ with surface tensions σ_k satisfying (A4)–(A6) and studying compactness properties of their associated weak solutions. The construction of σ_k and the compactness argument are presented in **Section 4**.

2. Weak solutions and main results

To introduce the definition of weak solutions, we first derive energy estimates satisfied by smooth nonnegative solutions to (1)–(2). So, let us consider a non-increasing and smooth surface tension σ and a smooth solution (h,Γ) to (1)–(3) in $(0,T) \times (0,1)$ for some $T > 0$, both functions being uniformly bounded from below by a positive constant. First, we note that by (1)-(3) there holds

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\int_0^1 h\ \mathrm{d}x\right] = 0, \qquad \frac{\mathrm{d}}{\mathrm{dt}}\left[\int_0^1 \Gamma\ \mathrm{d}x\right] = 0. \tag{9}$$

Setting

$$g_\sigma(1) = g_\sigma'(1) = 0, \qquad g_\sigma''(s) = -\frac{\sigma'(s)}{s} \qquad \text{for } s \in (0,\infty), \tag{10}$$

it follows from (1)–(3) that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\int_0^1 \left(\frac{|\partial_x h|^2}{2} + g_\sigma(\Gamma)\right)\,\mathrm{d}x\right] = \int_0^1 \left[\partial_x^2 h\,\partial_x\left(\frac{h^3}{3}\,\partial_x^3 h + \frac{h^2}{2}\partial_x\sigma(\Gamma)\right)\right]\,\mathrm{d}x$$
$$+\int_0^1 \left[g_\sigma''(\Gamma)\,\partial_x\Gamma\,\left(\frac{h^2}{2}\,\Gamma\,\partial_x^3 h + h\,\Gamma\,\partial_x\sigma(\Gamma) - D\,\partial_x\Gamma\right)\right]\,\mathrm{d}x$$
$$= -\int_0^1 \left[\frac{h^3}{3}\,|\partial_x^3 h|^2 + h^2\,\partial_x\sigma(\Gamma)\,\partial_x^3 h + h|\partial_x\sigma(\Gamma)|^2 - D\frac{\sigma'(\Gamma)}{\Gamma}|\partial_x\Gamma|^2\right]\,\mathrm{d}x.$$

We abbreviate product terms by introducing J_s^2 and J_f^2, where

$$J_f = J_f[h,\Gamma] := \frac{h^{3/2}}{3}\,\partial_x^3 h + \frac{h^{1/2}}{2}\,\partial_x\sigma(\Gamma), \tag{11}$$

$$J_s = J_s[h,\Gamma] := \frac{h^{3/2}}{2}\,\partial_x^3 h + h^{1/2}\,\partial_x\sigma(\Gamma). \tag{12}$$

This yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\int_0^1 \left(\frac{|\partial_x h|^2}{2} + g_\sigma(\Gamma)\right)\,\mathrm{d}x\right]$$
$$= -\int_0^1 \left[\frac{3}{2}|J_f[h,\Gamma]|^2 + \frac{1}{2}|J_s[h,\Gamma]|^2 + \frac{1}{24}h^3|\partial_x^3 h|^2 + \frac{1}{8}h|\partial_x\sigma(\Gamma)|^2\right]\,\mathrm{d}x \tag{13}$$
$$+ D\int_0^1 \frac{\sigma'(\Gamma)}{\Gamma}|\partial_x\Gamma|^2\,\mathrm{d}x\,.$$

Consequently, we infer that, regardless the qualitative properties of σ, (h,Γ) should satisfy

$$h \in L_\infty(0,T\,;\,H^1(0,1)), \qquad \Gamma \in L_\infty(0,T;L_1(0,1)) \tag{14}$$

together with

$$h^{3/2}\partial_x^3 h \in L_2((0,T)\times(0,1)), \qquad h^{1/2}\partial_x\sigma(\Gamma) \in L_2((0,T)\times(0,1)). \tag{15}$$

Since $D > 0$, the energy estimate (13) provides an additional estimate which depends strongly on the properties of σ, namely $\sqrt{-\sigma'(\Gamma)/\Gamma}\,\partial_x\Gamma \in L_2((0,T)\times(0,1))$. We will actually prove that, under assumption (8) on σ, this additional estimate guarantees that the solutions we construct satisfy the further regularity

$$\Gamma \in L_2((0,T)\times(0,1)) \quad \text{and} \quad \sigma(\Gamma) \in L_{4/3}(0,T;W^1_{4/3}(0,1)), \tag{16}$$

see **Lemma 9**. Let us point out here that (8) implies that σ' does not decay too fast towards $-\infty$ at infinity. Hence, assuming (14) and (15) to hold true, we realize that $J_f[h,\Gamma], J_s[h,\Gamma] \in L_2((0,T)\times(0,1))$. Since an alternative formulation of (1)–(2) reads,

$$\partial_t h + \partial_x\left(h^{3/2}\,J_f[h,\Gamma]\right) = 0 \quad \text{in} \quad (0,\infty)\times(0,1)\,, \tag{17}$$

$$\partial_t\Gamma + \partial_x\left(h^{1/2}\,\Gamma\,J_s[h,\Gamma]\right) = D\,\partial_x^2\Gamma \quad \text{in} \quad (0,\infty)\times(0,1)\,, \tag{18}$$

we infer from (14)–(16) and the embedding of $H^1(0,1)$ in $L_\infty(0,1)$ that $h^{3/2}\,J_f[h,\Gamma]$ and $h^{1/2}\,\Gamma\,J_s[h,\Gamma]$ both belong to $L_1((0,T)\times(0,1))$ and we can give a meaning to (1)–(2) at least in the following weak sense:

Definition 1. *Let $T > 0$ and σ be a surface tension such that either*

- $\sigma \in \mathcal{C}^1(0,\infty)\cap\mathcal{C}([0,\infty))$, $\sigma(1) = 0$, *and* (8) *holds true,*

or

- *$\sigma \in \mathcal{C}^{1,1}(\mathbb{R})$ is such that $\sigma(1)=0$ and g_σ, defined in (7), satisfies* (A4)–(A6).

Then, given an initial condition $(h_0,\Gamma_0)\in H^1(0,1)\times L_2(0,1)$ with $h_0\geq 0$ and $\Gamma_0\geq 0$ we say that (h,Γ) is a weak solution *in $(0,T)$ to (1)–(4) with surface tension σ and initial condition (h_0,Γ_0), if*

- *$h\geq 0$ and $\Gamma \geq 0$ satisfy*

$$\begin{aligned} &h\in L_\infty(0,T;H^1(0,1))\cap\mathcal{C}([0,T]\times[0,1]), && \Gamma\in L_\infty(0,T;L_1(0,1))\cap L_2((0,T)\times(0,1)),\\ &\partial_x^3 h\in L_2(\mathcal{P}_h(\delta)) \quad \text{for all} \quad \delta>0, && \sigma(\Gamma)\in L_1(0,T;W_1^1(0,1)),\\ &h^{3/2}\partial_x^3 h\in L_2(\mathcal{P}_h), && h^{1/2}\partial_x\sigma(\Gamma)\in L_2((0,T)\times(0,1)),\end{aligned} \tag{19}$$

where $\mathcal{P}_h(\delta):=\{(t,x)\in(0,T)\times(0,1)\ :\ h(t,x)>\delta\}$ for $\delta>0$ and $\mathcal{P}_h:=\{(t,x)\in(0,T)\times(0,1)\ :\ h(t,x)>0\}$.

- *for any $\zeta\in\mathcal{C}^\infty([0,T]\times[0,1])$ such that $\zeta(T,x)=0$ for all $x\in[0,1]$ and $\partial_x\zeta(t,x)=0$ for all $(t,x)\in[0,T]\times\{0,1\}$, there holds:*

$$\int_0^T\int_0^1\left(h\ \partial_t\zeta+\left[\frac{1}{3}\ h^3\,\mathbf{1}_{(0,\infty)}(h)\,\partial_x^3 h+\frac{1}{2}\ h^2\ \partial_x\sigma(\Gamma)\right]\ \partial_x\zeta\right)dx\,ds=-\int_0^1 h_0(x)\zeta(0,x)\,dx\,, \tag{20}$$

and

$$\int_0^T\int_0^1\left(\Gamma\ \partial_t\zeta+\left[\frac{1}{2}\ h^2\,\mathbf{1}_{(0,\infty)}(h)\,\Gamma\ \partial_x^3 h+h\ \Gamma\ \partial_x\sigma(\Gamma)\right]\ \partial_x\zeta+D\Gamma\partial_x^2\zeta\right)dx\,ds=-\int_0^1\Gamma_0(x)\zeta(0,x)\,dx\,. \tag{21}$$

With these conventions, our main result reads:

Theorem 2. *Let the surface tension $\sigma\in\mathcal{C}^1(0,\infty)\cap\mathcal{C}([0,\infty))$ satisfy $\sigma(1)=0$ and (8). Then, given an initial condition $(h_0,\Gamma_0)\in H^1(0,1)\times L_2(0,1)$ with $h_0\geq 0$, $\Gamma_0\geq 0$ and any $T>0$, there exists at least one weak solution (h,Γ) in $(0,T)$ to (1)–(4) with surface tension σ and initial condition (h_0,Γ_0) in the sense of Definition 1.*

As mentioned in the Introduction, we split the proof of **Theorem 2** into two parts. First, we focus on the nonnegativity issue of solutions to (1)–(2). In this respect, we go back to the framework considered in [7] and we prove:

Theorem 3. *Let the surface tension $\sigma\in\mathcal{C}^2(\mathbb{R})$ be such that $\sigma(1)=0$ and the free energy g_σ defined by (7) satisfies* (A4)–(A6). *Then, given an initial condition $(h_0,\Gamma_0)\in H^1((0,1))\times L_2((0,1))$ with $h_0\geq 0$, $\Gamma_0\geq 0$, and any $T>0$, there exists at least one weak solution (h,Γ) in $(0,T)$ to (1)–(2) with surface tension σ and initial condition (h_0,Γ_0) in the sense of Definition 1.*
Moreover, the solution satisfies the further regularity

$$\Gamma\in L_\infty(0,T;L_2(0,1))\cap L_2(0,T;H^1(0,1)) \tag{22}$$

and the energy estimate

$$\sup_{t\in[0,T]}\left\{\int_0^1\left[\frac{|\partial_x h(t,x)|^2}{2}+g_\sigma(\Gamma(t,x))\right]dx\right\}+\mathcal{D}[h,\Gamma]\leq\int_0^1\left(\frac{|\partial_x h_0(x)|^2}{2}+g_\sigma(\Gamma_0(x))\right)dx\,, \tag{23}$$

where

$$\begin{aligned}\mathcal{D}[h,\Gamma]\ \ :=\ \ &\int_0^T\int_0^1\left(\frac{(h^3\,\mathbf{1}_{(0,\infty)}(h(\tau,x)))}{2}1|\partial_x^3 h(\tau,x)|^2+\frac{h(\tau,x)}{8}|\partial_x\sigma(\Gamma(\tau,x))|^2\right)dx\,d\tau\\ &-D\int_0^T\int_0^1\frac{\sigma'(\Gamma(\tau,x))}{\Gamma(\tau,x)}|\partial_x\Gamma(\tau,x)|^2\,dx\,d\tau.\end{aligned}$$

With the regularity (22), any weak solution constructed in [7] is a weak solution in our sense (see the proof of **Theorem 3** for further details). The major novelty in this result is that we obtain nonnegativity of the surfactant concentration Γ. For the proof of **Theorem 2**, we consider a surface tension $\sigma \in \mathcal{C}([0,\infty)) \cap \mathcal{C}^1(0,\infty)$ and introduce a family of approximate surface tensions $(\sigma_k)_{k\in\mathbb{N}}$ satisfying the assumptions of **Theorem 3**. We achieve our result by studying the compactness properties of the family of associated weak solutions. A fundamental argument will be that, owing to assumption (8) and equation (13), the dissipation of energy is measured by

$$\int_0^T \int_0^1 \frac{|\partial_x \Gamma|^2}{\Gamma(1+\Gamma)^\theta}\, \mathrm{d}x\mathrm{d}\tau\,. \tag{24}$$

When $\theta \in [0,1)$, this quantity enables us to control $\sqrt{\Gamma}$ in some Hölder space (see **Lemma 9**). This, in turn, yields compactness on the concentration for any bounded family of solutions in $L_2((0,T)\times(0,1))$.

3. Existence of nonnegative solutions for decaying surface tensions

In this section, we assume $\sigma \in \mathcal{C}^2(\mathbb{R})$ is such that $\sigma(1) = 0$ and the free energy g_σ, as defined in (7), satisfies (A4)–(A6) and we construct nonnegative weak solutions to (1)–(2). In [7, Sect.3.4], the authors remark that, for proving nonnegativity of the surfactant concentration of weak solutions to (1)–(4), a difficulty arises when multiplying equation (2) by $\Gamma_- = -\min\{0,\Gamma\}$. Indeed, under assumptions (A4)–(A6), the very low regularity of Γ implies only that $\partial_t\Gamma \in L_{3/2}(0,T;(W_3^1(0,1))^*)$ and $\Gamma_- \in L_2(0,T;H^1(0,1))$. This regularity does not allow to define the duality bracket $\langle \partial_t\Gamma, \Gamma_-\rangle$. To construct weak solutions with nonnegative surfactant concentrations, we go back to the strategy applied in [7]: construction of solutions to a regularized problem via a Galerkin method, followed by a compactness argument when the regularization parameter goes to 0. We introduce a supplementary truncation operator in the regularized problem in order to guarantee that the solutions to the regularized problems have nonnegative surfactant concentrations.

Throughout this section, we fix a nonnegative initial condition $(h_0,\Gamma_0) \in H^1(0,1)\times L_2(0,1)$. We also introduce a Lipschitz continuous truncation function $\mathcal{T}$ such that

$$\mathcal{T}(s) = \begin{cases} s & \text{if } s \in (0,1), \\ 2-s & \text{if } s\in[1,2], \\ 0 & \text{if } s \geq 2, \end{cases} \qquad \mathcal{T}(-s) = -\mathcal{T}(s) \text{ if } s<0, \tag{25}$$

and put $\mathcal{T}_k := k\mathcal{T}(\cdot/k)$ for $k \geq 1$. Then, we set

$$\sigma_k(s) := \int_1^s \mathcal{T}_k(\sigma'(r))\mathrm{d}r \quad \text{ for } s\in\mathbb{R}. \tag{26}$$

We emphasize that this construction ensures that $\sigma_k \in \mathcal{C}^{1,1}(\mathbb{R})$ has bounded first and second derivatives. Associated to this truncation of σ, we introduce a truncation of the identity

$$\tau_k(s) := s\,\frac{\sigma_k'(s)}{\sigma'(s)} \quad \text{ for } s\in\mathbb{R}. \tag{27}$$

We note that the construction above is well-defined because

$$0 \geq \sigma_k'(s) \geq \sigma'(s) \quad \text{ for all } \quad s\in\mathbb{R}. \tag{28}$$

With these conventions, our regularized problem reads

$$\partial_t h + \partial_x\left([a_3(h)+1/k]\ \partial_x^3 h + a_2(h)\ \partial_x\sigma_k(\Gamma)\right) = 0\,, \qquad (t,x)\in(0,\infty)\times(0,1)\,, \tag{29}$$

$$\partial_t \Gamma + \partial_x\left(a_2(h)\ \tau_k(\Gamma)\ \partial_x^3 h + a_1(h)\ \Gamma\ \partial_x\sigma_k(\Gamma)\right) = D\ \partial_x^2\Gamma\,, \quad (t,x)\in(0,\infty)\times(0,1)\,, \tag{30}$$

subject to (3)–(4), where k is a positive integer. The notation $a_i(h)$ stands for $(\max\{0,h\})^i/i$ for $i=1,2,3$. This is the same convention as in [7] so that conditions (A1)–(A8) therein are satisfied.

3.1. **Existence for** (29)-(30)**.** To begin with, we fix $k \geq 1$ and prove:

Lemma 4. *Consider an initial condition* $(h_0, \Gamma_0) \in H^1((0,1)) \times L_2((0,1))$ *with* $h_0 \geq 0$ *and* $\Gamma_0 \geq 0$. *For any* $k \geq 1$ *and* $T > 0$, *there exists at least a couple of functions* (h, Γ) *having the regularity*

$$h \in L_\infty(0,T;H^1(0,1)) \cap L_2(0,T;H^3(0,1)) \ , \quad \Gamma \in L_\infty(0,T;L_2(0,1)) \cap L_2(0,T;H^1(0,1)), \tag{31}$$

$$\partial_t h \in L_2(0,T;(H^1(0,1))^*) \ , \quad \partial_t \Gamma \in L_{3/2}(0,T;(W_3^1(0,1))^*), \tag{32}$$

and satisfying,

$$\int_0^T \langle \partial_t h, \zeta \rangle \, ds - \int_0^T \int_0^1 \left(a_2(h) \partial_x \sigma_k(\Gamma) + [a_3(h) + 1/k] \ \partial_x^3 h \right) \partial_x \zeta \, dx ds = 0, \tag{33}$$

for all $\zeta \in L_2(0,T;H^1(0,1))$, *together with*

$$\int_0^T \langle \partial_t \Gamma, \zeta \rangle \, ds - \int_0^T \int_0^1 \left(a_1(h) \ \Gamma \ \partial_x \sigma_k(\Gamma) + a_2(h) \ \tau_k(\Gamma) \ \partial_x^3 h - D \, \partial_x \Gamma \right) \partial_x \zeta \, dx ds = 0, \tag{34}$$

for all $\zeta \in L_3(0,T;W_3^1(0,1))$ *and*

$$(h(0,\cdot), \Gamma(0,\cdot)) = (h_0, \Gamma_0) \, , \tag{35}$$

the latter being meaningful as $h \in \mathcal{C}([0,T];(H^1(0,1))^*)$ *and* $\Gamma \in \mathcal{C}([0,T];(W_3^1(0,1))^*)$ *by* (31) *and* (32). *Moreover, there holds the energy inequality*

$$\sup_{t \in [0,T]} \left\{ \int_0^1 \left[\frac{|\partial_x h(t,x)|^2}{2} + g_\sigma(\Gamma(t,x)) \right] dx \right\} + \mathcal{D}_k[\tilde{h}, \Gamma] \leq \int_0^1 \left[\frac{|\partial_x h_0(x)|^2}{2} + g_\sigma(\Gamma_0(x)) \right] dx, \tag{36}$$

where

$$\tilde{\mathcal{D}}_k[h,\Gamma] := \int_0^T \int_0^1 \left\{ \left[\frac{1}{k} + \frac{a_3(h)}{7} \right] |\partial_x^3 h|^2 - D \frac{\sigma'(\Gamma)}{\Gamma} |\partial_x \Gamma|^2 + \frac{a_1(h)}{8} |\partial_x \sigma_k(\Gamma)|^2 \right\} dx ds.$$

Remark 5. *Note that, in* (36), σ_k *only appears in the last term of* $\tilde{\mathcal{D}}_k[h,\Gamma]$.

Proof. We follow here the Galerkin method from [7, Section 3]. The system (29)–(30) is actually almost identical to the regularized system used in [7, Section 3] except that the truncation function τ_k is replaced by the identity there. Since τ_k is a bounded and Lipschitz continuous function, the analysis performed in [7, Section 3] carries over to (29)–(30) with only slight changes, the main one arising in the derivation of the energy inequality. We will thus only give a sketch of the proof and refer to [7, Section 3] for details.

The first step is an alternative formulation of (29)–(30) in therms of h and the new unknown function $v := g_\sigma'(\Gamma)$, the latter being well-defined thanks to the convexity (A5) of g_σ. Denoting the inverse function of g_σ' by W, we have

$$\partial_t h + \partial_x \left([a_3(h) + 1/k] \ \partial_x^3 h - a_2(h) \tau_k(W(v)) \partial_x v \right) = 0 \, , \tag{37}$$

$$\partial_t W(v) + \partial_x \left(a_2(h) \ \tau_k(W(v)) \ \partial_x^3 h - a_1(h) \ W(v) \tau_k(W(v)) \partial_x v \right) = D \ \partial_x^2 W(v) \, , \tag{38}$$

in $(0,\infty) \times (0,1)$. As already mentioned, (37)–(38) is the same as the system studied in [7, Section 3] except for the terms involving the bounded and Lipschitz continuous function τ_k. Not surprisingly, considering the same Galerkin approximation to (37)–(38) as in [7, Section 3], one can prove the local existence of solutions to the Galerkin approximations exactly in the same way as in [7, Section 3.1]. To obtain the global existence, we argue as in [7, Section 3.2] by deriving an energy estimate for the Galerkin approximations. Since there is a slight modification necessary, let us sketch the proof for (37)–(38), the argument being the same at the level of the Galerkin approximations. We multiply (37) by $-\partial_x^2 h$, (38) by $v = g_\sigma'(\Gamma)$, integrate over $(0,1)$, and add the resulting identities to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left[\frac{|\partial_x h|^2}{2} + g_\sigma(\Gamma) \right] \mathrm{d}x + \int_0^1 \left\{ \left[\frac{1}{k} + a_3(h) \right] |\partial_x^3 h|^2 - D \, \frac{\sigma'(\Gamma)}{\Gamma} |\partial_x \Gamma|^2 + a_1(h) (\sigma'(\Gamma) \sigma_k'(\Gamma)) |\partial_x \Gamma|^2 \right\} \mathrm{d}x = I,$$

where (see (7) and (27))

$$I := -\int_0^1 a_2(h)\ \partial_x\Gamma\ \partial_x^3 h \left(\sigma_k'(\Gamma) + \sigma'(\Gamma)\frac{\tau_k(\Gamma)}{\Gamma}\right) \mathrm{d}x = -\int_0^1 2a_2(h)\ \partial_x\sigma_k(\Gamma)\ \partial_x^3 h\, \mathrm{d}x.$$

Since $\sigma_k' \le 0$, it follows from (28) that $(\sigma_k')^2 \le \sigma'\sigma_k'$ while Young's inequality ensures that

$$|2a_2(h)\ \partial_x\sigma_k(\Gamma)\ \partial_x^3 h| \le \frac{7a_1(h)}{8}\ |\partial_x\sigma_k(\Gamma)|^2 + \frac{6a_3(h)}{7}\ |\partial_x^3 h|^2\,,$$

so that we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left[\frac{|\partial_x h|^2}{2} + g_\sigma(\Gamma)\right] \mathrm{d}x + \int_0^1 \left\{\left[\frac{1}{k} + a_3(h)\right] |\partial_x^3 h|^2 - D\frac{\sigma'(\Gamma)}{\Gamma}|\partial_x\Gamma|^2 + a_1(h)|\partial_x\sigma_k(\Gamma)|^2\right\} \mathrm{d}x$$
$$\le \int_0^1 \left[\frac{7a_1(h)}{8}\ |\partial_x\sigma_k(\Gamma)|^2 + \frac{6a_3(h)}{7}\ |\partial_x^3 h|^2\right] \mathrm{d}x\,.$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left[\frac{|\partial_x h|^2}{2} + g_\sigma(\Gamma)\right] \mathrm{d}x + \int_0^1 \left\{\left[\frac{1}{k} + \frac{a_3(h)}{7}\right] |\partial_x^3 h|^2 - D\frac{\sigma'(\Gamma)}{\Gamma}|\partial_x\Gamma|^2 + \frac{a_1(h)}{8}\ |\partial_x\sigma_k(\Gamma)|^2\right\} \mathrm{d}x \le 0\,,$$

whence (36) after time integration.
The convergence of the Galerkin approximations to a solution to (29)–(30) satisfying the properties listed in **Lemma 4** is then carried out as in [7, Section 3.3] to which we refer. □

At this point, we show that the idea to introduce truncation functions τ_k and σ_k yields the nonnegativity of Γ. This relies on a gain of regularity for $\partial_t\Gamma$.

Lemma 6. *Consider an initial condition $(h_0,\Gamma_0) \in H^1((0,1)) \times L_2((0,1))$ with $h_0 \ge 0$ and $\Gamma_0 \ge 0$. Given $k \ge 1$ and $T > 0$, any solution (h,Γ) to (31)-(36) and (3)–(4) in the sense of* **Lemma 4** *satisfies $\partial_t\Gamma \in L_2(0,T;(H^1(0,1))^*)$ and $\Gamma \ge 0$ a.e. in $(0,T)\times(0,1)$.*

Proof. Owing to (31), the embedding of $H^1(0,1)$ in $L_\infty(0,1)$, and the compactness of the supports of σ_k' and τ_k (which follows from (A5) and the properties of $\mathcal{T}$), there holds

$$a_1(h)\ \Gamma\ \partial_x\sigma_k(\Gamma) + a_2(h)\ \tau_k(\Gamma)\ \partial_x^3 h = a_1(h)\Gamma\sigma_k'(\Gamma)\partial_x\Gamma + a_2(h)\ \tau_k(\Gamma)\ \partial_x^3 h \in L_2((0,T)\times(0,1)),$$

and $D\partial_x\Gamma \in L_2((0,T)\times(0,1))$. As a consequence (34) also holds true for all $\zeta \in L_2(0,T;H^1(0,1))$ and $\partial_t\Gamma \in L_2(0,T;(H^1(0,1))^*)$. Then, if $\beta \in \mathcal{C}^2(\mathbb{R})$ is such that β' is Lipschitz continuous, we have $\beta'(\Gamma) \in L_2(0,T;H^1(0,1))$ and

$$\frac{d}{dt}\int_0^1 \beta(\Gamma)\,\mathrm{d}x = \langle\partial_t\Gamma, \beta'(\Gamma)\rangle\,.$$

Assuming furthermore that β is convex, i.e. $\beta'' \ge 0$, there holds, for any $t \in (0,T)$,

$$\int_0^1 \beta(\Gamma(t))\,\mathrm{d}x \le \int_0^1 \beta(\Gamma_0)\,\mathrm{d}x + \int_0^T\int_0^1 \left|\left(a_1(h)\ \Gamma\ \partial_x\sigma_k(\Gamma) + a_2(h)\ \tau_k(\Gamma)\ \partial_x^3 h\right)\beta''(\Gamma)\partial_x\Gamma\right|\,\mathrm{d}x\mathrm{d}s.$$

To finish off the proof, we apply this inequality to a family of functions approximating the negative part of Γ. Namely, we fix a nonnegative $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi \not\equiv 0$ has support in $(-1,0)$ and define β_1 by

$$\beta_1(0) = 0, \quad \beta_1'(s) := -\frac{\int_s^\infty \chi(\alpha)\mathrm{d}\alpha}{\int_{-\infty}^\infty \chi(\alpha)\mathrm{d}\alpha} \quad \text{for} \quad s \in \mathbb{R}\,.$$

We then set $\beta_\varepsilon(s) := \varepsilon\beta_1(s/\varepsilon)$ for $s\in\mathbb{R}$ and $\varepsilon>0$. Taking $\beta=\beta_\varepsilon$ for $\varepsilon>0$ in the above inequality, there holds, for each $t\in(0,T)$,

$$\int_0^1 \beta_\varepsilon(\Gamma(t))\,\mathrm{d}x \le \int_0^T\int_0^1 \left|\left(a_1(h)\ \Gamma\ \partial_x\sigma_k(\Gamma)+a_2(h)\ \tau_k(\Gamma)\ \partial_x^3 h\right)\beta_\varepsilon''(\Gamma)\partial_x\Gamma\right|\,\mathrm{d}x\mathrm{d}s\,,$$

since $\beta_\varepsilon(\Gamma_0)=0$ due to $\Gamma_0\ge 0$. Observing that $|\tau_k(s)\beta_\varepsilon''(s)|\le|s\beta_\varepsilon''(s)|\le C(\chi)$ for $s\in\mathbb{R}$, we have

$$\begin{aligned}\int_0^T\int_0^1 &\left|\left(a_1(h)\ \Gamma\ \partial_x\sigma_k(\Gamma)+a_2(h)\ \tau_k(\Gamma)\ \partial_x^3 h\right)\beta_\varepsilon''(\Gamma)\partial_x\Gamma\right|\,\mathrm{d}x\mathrm{d}s\\ &\le C(\chi)\int_{\{|\Gamma|<\varepsilon\}}\left[|a_1(h)\ \partial_x\sigma_k(\Gamma)\partial_x\Gamma|+\left|a_2(h)\ \partial_x^3 h\ \partial_x\Gamma\right|\right]\,\mathrm{d}x\mathrm{d}s\\ &\le C(\chi)\left[\|a_1(h)\sigma_k'(\Gamma)\|_{L_\infty(0,T;L_\infty(0,1))}+\|a_2(h)\|_{L_\infty(0,T;L_\infty(0,1))}\right]\int_{\{|\Gamma|<\varepsilon\}}\left[|\partial_x\Gamma|^2+\left|\partial_x^3 h\ \partial_x\Gamma\right|\right]\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

As $\partial_x\Gamma$ and $\partial_x^3 h$ both belong to $L_2((0,T)\times(0,1))$ and $\partial_x\Gamma=0$ a.e. in $\{\Gamma=0\}$ by [9, Lemma A.4], we obtain in the limit $\varepsilon\to 0$

$$\int_0^1 \max\{-\Gamma(t,x),0\}\,\mathrm{d}x\le 0,\qquad\text{for}\qquad t\in(0,T).$$

This completes the proof. □

3.2. **Proof of Theorem 3.** Let σ be as in the statement of **Theorem 3** and consider an initial condition $(h_0,\Gamma_0)\in H^1((0,1))\times L_2((0,1))$ with $h_0\ge 0$, $\Gamma_0\ge 0$ and $T>0$. First, applying **Lemma 4** and **Lemma 6**, we obtain a sequence $(h_k,\Gamma_k)_{k\ge1}$ of solutions to (29)–(30), (3)–(4) for which $\partial_t\Gamma_k\in L_2(0,T;(H^1(0,1))^*)$ and $\Gamma_k\ge 0$ a.e. in $(0,T)\times(0,1)$. In particular, for each $k\ge 1$, the time regularity of h_k and Γ_k, together with the initial conditions (35) $(h_k(0,\cdot),\Gamma_k(0,\cdot))=(h_0,\Gamma_0)$, yield the integration by parts formula:

$$\int_0^T\langle\partial_t h_k,\zeta\rangle\,\mathrm{d}t=-\int_0^1 h_0(x)\zeta(0,x)\mathrm{d}x-\int_0^T\int_0^1 h_k(s,x)\partial_t\zeta(s,x)\mathrm{d}x\mathrm{d}t\,,$$

$$\int_0^T\langle\partial_t \Gamma_k,\zeta\rangle\,\mathrm{d}t=-\int_0^1 \Gamma_0(x)\zeta(0,x)\mathrm{d}x-\int_0^T\int_0^1 \Gamma_k(s,x)\partial_t\zeta(s,x)\mathrm{d}x\mathrm{d}t$$

for any test function $\zeta\in\mathcal{C}^\infty([0,T]\times[0,1])$ such that $\zeta(T,x)=0$ for all $x\in[0,1]$ and $\partial_x\zeta(t,x)=0$ for all $(t,x)\in[0,T]\times\{0,1\}$. Hence, taking such a test function ζ in (33)-(34) we obtain :

$$\int_0^T\int_0^1\left(h_k\ \partial_t\zeta+\left[\left(a_3(h_k)+\frac{1}{k}\right)\ \partial_x^3 h_k+a_2(h_k)\ \partial_x\sigma(\Gamma_k)\right]\ \partial_x\zeta\right)\,\mathrm{d}x\mathrm{d}t=-\int_0^1 h_0(x)\zeta(0,x)\mathrm{d}x\,,\tag{39}$$

$$\int_0^T\int_0^1\left(\Gamma_k\ \partial_t\zeta+\left[a_2(h_k)\ \tau_k(\Gamma_k)\ \partial_x^3 h_k+a_1(h_k)\ \Gamma_k\ \partial_x\sigma_k(\Gamma_k)\right]\partial_x\zeta+D\Gamma_k\partial_x^2\zeta\right)\,\mathrm{d}x\mathrm{d}t=-\int_0^1\Gamma_0(x)\zeta(0,x)\mathrm{d}x\,.\tag{40}$$

So, the proof reduces to find a weak cluster point (h,Γ) of the sequence $((h_k,\Gamma_k))_{k\ge1}$ that has the regularity (19) and for which we can pass to the limit in the two previous equations.

First, we note that the conservation laws (9) are also satisfied by (h_k,Γ_k). Consequently, due to (36) and the Poincaré inequality, we have uniform bounds for

- $(h_k)_{k\ge1}$ in $L_\infty(0,T;H^1(0,1))$ and $(g_\sigma(\Gamma_k))_{k\ge1}$ in $L_\infty(0,T;L_1(0,1))$,
- $(\sqrt{a_3(h_k)}\partial_x^3 h_k)_{k\ge1}$, $(\sqrt{a_1(h_k)}\partial_x\sigma_k(\Gamma_k))_{k\ge1}$, and $(\sqrt{-\sigma'(\Gamma_k)/\Gamma_k}\partial_x\Gamma_k)_{k\ge1}$ in $L_2((0,T)\times(0,1))$.

Owing to the bound (A5) from below on σ', this yields a uniform bound on $(\Gamma_k)_{k\ge1}$ in $L_\infty(0,T;L_2(0,1))$ and $L_2(0,T;H^1(0,1))$, and the sequence of fluxes, given by

$$\begin{aligned} J_s^k &:= \frac{a_2(h_k)}{a_1(h_k)^{1/2}}\ \frac{\tau_k(\Gamma_k)}{\Gamma_k}\ \partial_x^3 h_k + a_1(h_k)^{1/2}\ \partial_x\sigma_k(\Gamma_k),\\ J_f^k &:= \left(\frac{a_3(h_k)}{3}+\frac{1}{3k}\right)^{1/2}\ \partial_x^3 h_k + a_2(h_k)\ \left(3a_3(h_k)+\frac{3}{k}\right)^{-1/2}\ \partial_x\sigma_k(\Gamma_k), \end{aligned}$$

are also bounded in $L_2((0,T)\times(0,1))$ by (36).

Repeating the arguments in [3, Section 2] and [7, Section 3.4], we may extract a subsequence (not relabeled) and find functions h and Γ such that the following convergences hold:

- $h_k \to h$ in $\mathcal{C}([0,T]\times[0,1])$ and $\Gamma_k\to\Gamma$ in $L_2(0,T;L_p(0,1))$ for all $p\in[1,\infty)$,
- $\sqrt{3a_3(h_k)+3/k}\,\partial_x^3 h_k \rightharpoonup H$ in $L_2((0,T)\times(0,1))$ with $H=\sqrt{3a_3(h)}\,\partial_x^3 h$ a.e. in $\{h\neq 0\}$,
- $\partial_x\Gamma_k\rightharpoonup\partial_x\Gamma$ in $L_2((0,T)\times(0,1))$.

Arguing as in the proof of [7, Equation (3.28)], the previous convergences imply that

$$(3a_3(h_k)+3/k)\,\partial_x^3 h_k \rightharpoonup h^{3/2}\mathbf{1}_{(0,\infty)}(h)\,\partial_x^3 h \quad\text{in}\quad L_2((0,T)\times(0,1))\ .$$

Next, interpolating the bounds on $(\Gamma_k)_{k\ge1}$ with the help of [5, Proposition I.3.2], we deduce that $(\Gamma_k)_{k\ge1}$ is bounded in $L_6((0,T)\times(0,1))$ and that the convergence of $(\Gamma_k)_{k\ge1}$ to Γ takes actually place in $L_p((0,T)\times(0,1))$ for all $p\in[2,6)$. Now, since

$$0\le\tau_k(s)\le s\quad\text{for all}\quad s\ge0\quad\text{and}\quad \tau_k(s)=s\quad\text{for}\quad 0\le s\le s_k:=\left[(k/C_g)^{r/(r+1)}-1\right]^{1/r}$$

(see assumption (A6) and (27) for the definitions of r and τ_k, respectively), we have, for $p\ge1$,

$$\begin{aligned} \int_0^T\int_0^1\left|\frac{\tau_k(\Gamma_k)}{\Gamma_k}-1\right|^p\,\mathrm{d}x\mathrm{d}t &= \int_{\{\Gamma_k>s_k\}}\left|\frac{\tau_k(\Gamma_k)}{\Gamma_k}-1\right|^p\,\mathrm{d}x\mathrm{d}t\le 2^p\int_{\{\Gamma_k>s_k\}}\mathrm{d}x\mathrm{d}t\\ &\le \frac{2^p}{s_k^6}\int_{\{\Gamma_k>s_k\}}\Gamma_k^6\,\mathrm{d}x\mathrm{d}t\le\frac{C(p,T)}{s_k^6}\,. \end{aligned}$$

Since $s_k\to\infty$ as $k\to\infty$, we conclude that $\tau_k(\Gamma_k)/\Gamma_k\to1$ in $L_p((0,T)\times(0,1))$ for any $p\ge1$. Similarly, since $\sigma_k'(s)=\sigma'(s)$ for $s\in[0,s_k]$ and $\sigma'\in\mathcal{C}^1(\mathbb{R})$, it follows from (A6) and (28) that, given $p_0\in[1,6/(r+1))$, $R\ge1$, and $k\ge1$ such that $s_k\ge R$, we have

$$\begin{aligned} &\int_0^T\int_0^1|\sigma_k'(\Gamma_k)-\sigma'(\Gamma)|^{p_0}\,\mathrm{d}x\mathrm{d}t\\ \le\ &\int_{\{\max\{\Gamma_k,\Gamma\}\le R\}}|\sigma'(\Gamma_k)-\sigma'(\Gamma)|^{p_0}\,\mathrm{d}x\mathrm{d}t+\int_{\{\Gamma_k>R\}\cup\{\Gamma>R\}}|\sigma_k'(\Gamma_k)-\sigma'(\Gamma)|^{p_0}\,\mathrm{d}x\mathrm{d}t\\ \le\ &\|\sigma''\|_{L_\infty(0,R)}\int_{\{\max\{\Gamma_k,\Gamma\}\le s_k\}}|\Gamma_k-\Gamma|^{p_0}\,\mathrm{d}x\mathrm{d}t\\ +\ &C(p_0,C_g,r)\int_{\{\Gamma_k>R\}\cup\{\Gamma>R\}}\left(\Gamma_k^{(r+1)p_0}+\Gamma^{(r+1)p_0}\right)\mathrm{d}x\mathrm{d}t\\ \le\ &\|\sigma''\|_{L_\infty(0,R)}\int_0^T\int_0^1|\Gamma_k-\Gamma|^{p_0}\,\mathrm{d}x\mathrm{d}t+\frac{C(p_0,C_g,r)}{R^{6-(r+1)p_0}}\int_{\{\Gamma_k>R\}\cup\{\Gamma>R\}}\left(\Gamma_k^6+\Gamma^6\right)\mathrm{d}x\mathrm{d}t\\ \le\ &\|\sigma''\|_{L_\infty(0,R)}\int_0^T\int_0^1|\Gamma_k-\Gamma|^{p_0}\,\mathrm{d}x\mathrm{d}t+\frac{C(p_0,C_g,r,T)}{R^{6-(r+1)p_0}}\,. \end{aligned}$$

Letting first $k \to \infty$ and then $R \to \infty$ yield that $\sigma_k'(\Gamma_k) \to \sigma'(\Gamma)$ in $L_{p_0}((0,T)\times(0,1))$ for any $p_0 \in [1, 6/(r+1))$. As $r < 2$ we note that we may choose $p_0 > 2$ in the previous convergence which, combined with the weak convergence of $(\partial_x \Gamma_k)_{k\ge1}$ in $L_2((0,T)\times(0,1))$ implies that $\partial_x \sigma_k(\Gamma_k) \rightharpoonup \partial_x \sigma(\Gamma)$ in $L_{q_0}((0,T)\times(0,1))$ for some $q_0 > 1$. Consequently, $\sqrt{a_1(h_k)}\, \partial_x \sigma_k(\Gamma_k) \rightharpoonup \sqrt{a_1(h)} \partial_x \sigma(\Gamma)$ and

$$a_2(h_k)\ \left(3a_3(h_k) + \frac{3}{k}\right)^{-1/2}\ \partial_x \sigma_k(\Gamma_k) \rightharpoonup \frac{\sqrt{a_1(h)}}{2}\ \partial_x \sigma(\Gamma) \quad \text{in} \quad L_2((0,T)\times(0,1)).$$

Thus, we conclude that $J_s^k \rightharpoonup H/2 + \sqrt{a_1(h)} \partial_x \sigma(\Gamma)$ and $J_f^k \rightharpoonup H + (\sqrt{a_1(h)}/2) \partial_x \sigma(\Gamma)$ in $L_2((0,T)\times(0,1))$. Combining these convergences with the convergence of $(h_k)_{k\ge1}$ to h in $\mathcal{C}([0,T]\times[0,1])$ and that of $(\Gamma_k)_{k\ge1}$ to Γ in $L_2((0,T)\times(0,1))$ allows us to pass to the limit in (39)–(40).

That h is nonnegative can be obtained as in [7, Section 3.4] while the nonnegativity of Γ is preserved by the weak limit. Concerning the energy estimate (23), we recall (A5) and prove as above that

$$\sqrt{-\sigma_k'(\Gamma_k)/\Gamma_k} \to \sqrt{-\sigma'(\Gamma)/\Gamma} \text{ in } L_2((0,T)\times(0,1)).$$

Consequently, $(\sqrt{-\sigma_k'(\Gamma_k)/\Gamma_k}\partial_x \Gamma_k)_{k\ge1}$ converges weakly in $L_1((0,T)\times(0,1))$ to $\sqrt{-\sigma'(\Gamma)/\Gamma}\partial_x \Gamma$, and we can pass to the weak limit in the energy estimate. This completes the proof of **Theorem 3**.

4. Existence of nonnegative solutions for slowly decaying surface tension

From **Theorem 3** we obtain existence of weak solutions for a class of surface tension σ decreasing at least quadratically to $-\infty$. We now extend with **Theorem 2** the existence result to a class containing surface tensions which decrease slowly to $-\infty$ at infinity (but not too slowly, see (H2)) and are thus closer to applications. To this end, we fix a surface tension σ satisfying (H1)–(H2) and an initial condition $(h_0, \Gamma_0) \in H^1(0,1) \times L_2(0,1)$ satisfying $h_0 \ge 0$ and $\Gamma_0 \ge 0$. We split the proof of Theorem 2 into three steps: we first construct a sequence $(\sigma_k)_{k\ge1}$ of surface tensions approximating σ and enjoying the properties (A4)-(A6) for $k \ge 4$ (with constants depending of course on k). Owing to this construction, we may apply Theorem 3 to obtain, for each $k \ge 4$, a nonnegative weak solution (h_k, Γ_k) to (1)–(4) satisfying (23). We then show that $(h_k, \Gamma_k)_{k\ge4}$ is compact in suitable function spaces. In the last step, we identify the equations satisfied by the cluster points (h, Γ) of $(h_k, \Gamma_k)_{k\ge4}$.

4.1. **Construction of approximate surface tensions.** For $k \ge 1$, we set $\tilde\sigma_k(1) := 0$ and

$$\tilde\sigma_k'(s) := \begin{cases} \left[k\sigma'\left(\frac{1}{k}\right) - k\right] s & \text{for} \quad s < \frac{1}{k}, \\ \sigma'(s) & \text{for} \quad \frac{1}{k} \le s \le k, \\ \sigma'(k) - \frac{s}{k^{1+\theta}} & \text{for} \quad s > k. \end{cases} \tag{41}$$

Recall that θ is defined in (8). Denoting a family of even mollifiers by $(\chi_\varepsilon)_{\varepsilon>0}$, we introduce then the approximate surface tension σ_k by

$$\sigma_k' := \chi_{1/k^2} * \tilde\sigma_k', \qquad \sigma_k(1) = 0. \tag{42}$$

The following proposition verifies that we can apply **Theorem 3** to any approximate surface tension.

Proposition 7. *Given $k \ge 4$, the free energy $g_k := g_{\sigma_k}$ associated to σ_k via formula (7) satisfies (A4)–(A6).*

Proof. By construction, $\sigma_k' \in \mathcal{C}^\infty(\mathbb{R})$ and, owing to the properties of χ_{1/k^2}, straightforward computations yield that

$$\sigma_k'(s) \quad = \quad \left[k\sigma'\left(\frac{1}{k}\right) - k\right]\ s \quad \text{for all } s \le \frac{k-1}{k^2}, \tag{43}$$

$$\sigma_k'(s) \quad = \quad \sigma'(k) - \frac{s}{k^{1+\theta}} \quad \text{for all } s \ge k + \frac{1}{k^2}. \tag{44}$$

In particular, it follows from (43) that $[s \mapsto \sigma_k'(s)/s] \in \mathcal{C}^\infty(\mathbb{R})$ and g_k satisfies (A4). Next, if $s \in ((k-1)/k^2, k+(1/k^2))$, we have $s-(1/k^2) \ge (k-2)/k^2 \ge 1/(2k)$ and it follows from (41), (H1), and (H2) that

$$\begin{aligned}
\sigma_k'(s) &\le \int_{\mathbb{R}} \left[k\sigma'\left(\frac{1}{k}\right)\ r\ \mathbf{1}_{(0,1/k)}(r) + \sigma'(r)\ \mathbf{1}_{(1/k,k)}(r) + \sigma'(k)\ \mathbf{1}_{(k,\infty)}(r) \right]\ \chi_{1/k^2}(s-r)\,\mathrm{d}r \\
&\le \int_{\mathbb{R}} \left[\frac{1}{2}\ \sigma'\left(\frac{1}{k}\right)\ \mathbf{1}_{(0,1/k)}(r) + \frac{1}{2}\ \sigma'(r)\ \mathbf{1}_{(1/k,k)}(r) + \frac{1}{2}\ \sigma'(k)\ \mathbf{1}_{(k,\infty)}(r) \right]\ \chi_{1/k^2}(s-r)\,\mathrm{d}r \\
&\le \frac{1}{2} \sup_{[1/k,k]} \{\sigma'\} \int_{\mathbb{R}} \chi_{1/k^2}(s-r)\,\mathrm{d}r = \frac{1}{2} \sup_{[1/k,k]} \{\sigma'\} < 0\,.
\end{aligned} \tag{45}$$

We then infer from (43)–(45) that

$$g_k''(s) = -\frac{\sigma_k'(s)}{s} \ge \begin{cases} k & \text{if} \quad s \le \dfrac{k-1}{k^2}\,, \\ -\dfrac{1}{4k} \sup\limits_{[1/k,k]} \{\sigma'\} & \text{if} \quad \dfrac{k-1}{k^2} \le s \le k + \dfrac{1}{k^2}\,, \\ \dfrac{1}{k^{1+\theta}} & \text{if} \quad k + \dfrac{1}{k^2} < s\,, \end{cases}$$

and we obtain the existence of a constant $c_k > 0$ for which (A5) holds. Finally, it follows from (8) that, for $s \in ((k-1)/k^2, k+(1/k^2))$,

$$\begin{aligned}
\sigma_k'(s) &\ge -\int_{\mathbb{R}} \left[(1+\sigma_0)\ \mathbf{1}_{(0,1/k)}(r) + \sigma_0\ \mathbf{1}_{(1/k,k)}(r) + \left(\sigma_0 + \frac{r}{k^{1+\theta}}\right)\ \mathbf{1}_{(k,\infty)}(r) \right]\ \chi_{1/k^2}(s-r)\,\mathrm{d}r \\
&\ge -(2+\sigma_0) \int_{\mathbb{R}} \chi_{1/k^2}(s-r)\,\mathrm{d}r = -(2+\sigma_0)\,.
\end{aligned}$$

Noting that (8) and (43)–(44) guarantee this lower bound also for $s \in [0,(k-1)/k^2)$ and $s \ge k+(1/k^2)$, we conclude that

$$\sigma_k'(s) \ge -(2+\sigma_0) \quad \text{for} \quad s \ge 0\,. \tag{46}$$

In addition, it follows from (8) and (43) that $\sigma_k'(s)/s \ge -(1+\sigma_0)\ k$ for $s \in (-\infty,(k-1)/k^2]$. These two facts give

$$g_k''(s) = -\frac{\sigma_k'(s)}{s} \le \begin{cases} k(1+\sigma_0) & \text{for} \quad s < \dfrac{k-1}{k^2}, \\ \dfrac{k^2(2+\sigma_0)}{k-1} & \text{for} \quad s \ge \dfrac{k-1}{k^2}, \end{cases} \tag{47}$$

and we obtain (A6) with $r=0$ (so that it also holds true for arbitrary $r \in (0,2)$). □

The previous proposition and **Theorem 3** ensure that, for any $T>0$ and $k \ge 4$, there exists at least a nonnegative weak solution (h_k,Γ_k) to (1)–(4) with surface tension σ_k and initial condition (h_0,Γ_0). We prepare the study of compactness properties of the sequence $(h_k,\Gamma_k)_{k\ge4}$ by deriving technical properties of the approximate surface tensions $(\sigma_k)_{k\ge4}$.

Proposition 8. *If θ is the exponent given by (8), then there exist constants $C_1, C_2 \in (0,\infty)$ such that, for $k \ge 4$,*

$$0 \le g_k(s) \le C_1\left(1+s^2\right) \quad \text{for all} \quad s \ge 0\,, \tag{48}$$

$$-(2+\sigma_0) \le \sigma_k'(s) \le -C_2\, \frac{ks}{(1+s)^\theta\,(1+ks)} \quad \text{for all} \quad s \ge 0\,. \tag{49}$$

Moreover, $(\sigma_k)_{k\ge4}$ converges uniformly to σ on compact subsets of $[0,\infty)$.

Proof. The first inequality in (49) having already been proved in (46), we concentrate on the second inequality and first establish a similar estimate for $\tilde{\sigma}_k'$. As the surface tension σ satisfies (8), there holds

$$\begin{aligned} \tilde{\sigma}_k'(s) &\le -\frac{\sigma_1}{(1+s^\theta)} \le -\frac{2^{\theta-1}\sigma_1}{(1+s)^\theta} \quad \text{for} \quad s \in \left[\frac{1}{k}, k\right], \\ \tilde{\sigma}_k'(s) &\le -\frac{s}{k^{1+\theta}} \le -\frac{1}{(1+s)^\theta} \quad \text{for} \quad s \ge k, \end{aligned}$$

whence

$$\tilde{\sigma}_k'(s) \le -\frac{C\,ks}{(1+ks)(1+s)^\theta} \quad \text{for} \quad s \ge \frac{1}{k}$$

since $ks/(1+ks) \le 1$ for $s \ge 0$. Also,

$$\tilde{\sigma}_k'(s) \le -ks \le -\frac{ks}{(1+ks)(1+s)^\theta} \quad \text{for} \quad s \in \left[0, \frac{1}{k}\right].$$

Consequently, if $s \ge (k-1)/k^2$, we have $s - (1/k^2) \ge (k-2)/k^2$ and, as $k \ge 4$,

$$2s \ge s + \frac{1}{k} \ge s + \frac{1}{k^2} \ge s - \frac{1}{k^2} \ge \frac{s}{2},$$

we have

$$\sigma_k'(s) \le -\frac{C\left(k(s-\frac{1}{k^2})\right)}{(1+k(\frac{1}{k^2}+s))(1+(\frac{1}{k^2}+s))^\theta} \le -\frac{C_2\,ks}{(1+ks)(1+s)^\theta} \quad \text{for} \quad s \ge \frac{k-1}{k^2}.$$

Since $\sigma_k'(s) = \tilde{\sigma}_k'(s)$ for $s \le (k-1)/k^2$ by (43), we end up with

$$\sigma_k'(s) \le -\frac{C_2\,ks}{(1+s)^\theta(1+ks)} \quad \text{for} \quad s \ge 0,$$

and thus obtain (49). We next note that, given $R > 0$ and $s \in [0, R]$, it follows from (8) that, for $k \ge R$, we have

$$\begin{aligned} |\tilde{\sigma}_k(s) - \sigma(s)| &= \left| \int_{\min\{s,1/k\}}^{1/k} \left[\left(k\sigma'\left(\frac{1}{k}\right) - k \right) r - \sigma'(r) \right] dr \right| \\ &\le \left(\left| \sigma'\left(\frac{1}{k}\right) \right| + 1 \right) \frac{1}{2k} + \sigma\left(\frac{1}{k}\right) - \sigma\left(\min\left\{s, \frac{1}{k}\right\}\right) \\ &\le \frac{1+\sigma_0}{2k} + \sigma(0) - \sigma\left(\frac{1}{k}\right) \end{aligned}$$

and

$$|\tilde{\sigma}_k'(s)| \le \frac{1+\sigma_0}{k} \quad \text{for} \quad s \in \left[-\frac{1}{k^2}, 0\right].$$

Consequently, owing to the continuity of σ in $[0,\infty)$ and the properties of the convolution, the sequences $(\tilde{\sigma}_k)_k$ and $(\sigma_k)_k$ converge uniformly to σ on compact subsets of $[0,\infty)$.

Finally, integrating the bound (46) gives $g_k(s) \le (2+\sigma_0)\ (s\ln s - s + 1) \le (2+\sigma_0)(1+s^2)$ for $s \ge 0$, whence (48). □

4.2. **Compactness.** Let $T > 0$. The main difference here with the strategy employed in **Section 3.2** is that we no longer have an estimate on $(\Gamma_k)_k$ in $L_\infty(0,T;L_2(0,1))$ but only in $L_\infty(0,T;L_1(0,1))$, and this requires a different approach to the compactness issue for $(\Gamma_k)_k$. Let us collect the estimates available for $(h_k, \Gamma_k)_k$ which result from (1)–(4), (23), and the nonnegativity of g_k:

(1) Conservation of matter: for $t \in [0,T]$, there holds

$$\int_0^1 h_k(t,x)\mathrm{d}x = \int_0^1 h_0(x)\mathrm{d}x\,, \qquad \int_0^1 \Gamma_k(t,x)\mathrm{d}x = \int_0^1 \Gamma_0(x)\mathrm{d}x\,. \tag{50}$$

(2) Energy estimate: for $t \in [0,T]$, there holds

$$\begin{aligned}\frac{1}{2}\int_0^1 |\partial_x h_k(t,x)|^2\,\mathrm{d}x &+ \int_0^T\int_0^1 \left[\frac{h_k^3(s,x)\,\mathbf{1}_{(0,\infty)}(h_k(s,x))}{21}|\partial_x^3 h_k(s,x)|^2 + \frac{h_k(s,x)}{8}|\partial_x\sigma_k(\Gamma_k(s,x))|^2\right]\mathrm{d}x\mathrm{d}s \\ &- D\int_0^T\int_0^1 \frac{\sigma_k'(\Gamma_k(s,x))}{\Gamma_k(s,x)}|\partial_x\Gamma_k(s,x)|^2\mathrm{d}x\mathrm{d}s \le \int_0^1\left[\frac{|\partial_x h_0(x)|^2}{2} + g_k(\Gamma_0(x))\right]\mathrm{d}x.\end{aligned} \tag{51}$$

Moreover, both h_k and Γ_k are nonnegative a.e. in $(0,T)\times(0,1)$, and $\|g_k(\Gamma_0)\|_1 \le C_1\,(1+\|\Gamma_0\|_2^2)$ by (48). Consequently, (50) and (51), together with the lower bound (49) on $-\sigma_k'$ and the Poincaré inequality yield:

(B.1) $(h_k)_k$ is bounded in $L_\infty(0,T;H^1(0,1))$ and $(\Gamma_k)_k$ is bounded in $L_\infty(0,T;L_1(0,1))$.

(B.2) $(h_k^{3/2}\,\mathbf{1}_{(0,\infty)}(h_k)\,\partial_x^3 h_k)_k$, $(\partial_x\Gamma_k/(1+\Gamma_k)^{(1+\theta)/2})_k$, and $(\sqrt{h_k}\partial_x\sigma_k(\Gamma_k))_k$ are bounded in $L_2((0,T)\times(0,1))$.

We then infer from (20) (with surface tension σ_k), **(B.1)**, and the embedding of $H^1(0,1)$ in $L_\infty(0,1)$ that

$$(h_k)_k \;\text{ is bounded in }\; L_\infty((0,T)\times(0,1)) \quad\text{and}\quad (\partial_t h_k)_k \;\text{ is bounded in }\; L_2(0,T;(H^1(0,1))^*)\,. \tag{52}$$

Next, we prove the following embedding:

Lemma 9. *Let Γ be a nonnegative function in $L_1(0,1)$ such that $(1+\Gamma)^{(1-\theta)/2} \in H^1(0,1)$. Then there exists $C_\theta < \infty$ depending only on θ such that, after possibly redefining Γ on a set of measure zero, $\Gamma \in \mathcal{C}^{0,(1-\theta)/2}([0,1])$ together with*

$$\|\Gamma\|_{\mathcal{C}^{0,(1-\theta)/2}([0,1])} \le C_\theta\left[1+\int_0^1\Gamma(x)dx\right]\left[1+\int_0^1\frac{|\partial_x\Gamma(x)|^2}{(1+\Gamma(x))^{(1+\theta)}}dx\right].$$

Proof. Set

$$G := \frac{4}{(1-\theta)^2}\|\partial_x(1+\Gamma)^{(1-\theta)/2}\|_2^2 = \int_0^1\frac{|\partial_x\Gamma(x)|^2}{(1+\Gamma(x))^{1+\theta}}\mathrm{d}x < \infty.$$

We assume Γ to be smooth for simplicity and focus on the distance $\sqrt{1+\Gamma(x)} - \sqrt{1+\Gamma(y)}$ for $0 \le x \le y \le 1$. Then, by Hölder's inequality

$$\begin{aligned}\left|\sqrt{1+\Gamma(x)} - \sqrt{1+\Gamma(y)}\right| &\le \int_x^y\frac{|\partial_x\Gamma(z)|}{\sqrt{1+\Gamma(z)}}\mathrm{d}z \le \left[\int_x^y\frac{|\partial_x\Gamma(z)|^2}{(1+\Gamma(z))^{1+\theta}}\mathrm{d}z\right]^{1/2}\left[\int_x^y(1+\Gamma(z))^\theta\mathrm{d}z\right]^{1/2} \\ &\le \sqrt{G}\,\left[\int_x^y(1+\Gamma(z))\,\mathrm{d}z\right]^{\theta/2}\,|y-x|^{(1-\theta)/2} \le \sqrt{G}\,\left(1+\|\Gamma\|_1\right)^{\theta/2}\,|y-x|^{(1-\theta)/2}\,.\end{aligned}$$

Since

$$\int_0^1\sqrt{1+\Gamma(z)}\mathrm{d}z \le (1+\|\Gamma\|_1)^{1/2}\,,$$

integrating the above inequality with respect to y over $(0,1)$ ensures that $\|\sqrt{1+\Gamma}\|_\infty \le (1+\|\Gamma\|_1)^{1/2} + \sqrt{G}\,\left(1+\|\Gamma\|_1\right)^{\theta/2}$, so that there exists C_θ depending on θ only such that

$$\|\sqrt{1+\Gamma}\|_{\mathcal{C}^{0,(1-\theta)/2}([0,1])} \le C_\theta\left[1+\int_0^1\Gamma(x)\mathrm{d}x\right]^{1/2}\left[1+\int_0^1\frac{|\partial_x\Gamma(x)|^2}{(1+\Gamma(x))^{1+\theta}}\mathrm{d}x\right]^{1/2}.$$

We conclude using the classical trick $\Gamma = (\sqrt{1+\Gamma})^2 - 1$. □

Now, **(B.1)**, **(B.2)**, and **Lemma 9** yield that

$$(\Gamma_k)_k \;\text{ is bounded in }\; L_\infty(0,T;L_1(0,1)) \cap L_1(0,T;\mathcal{C}^{0,(1-\theta)/2}([0,1]))\,. \tag{53}$$

In particular, since $\|\Gamma_k\|_2^2 \le \|\Gamma_k\|_\infty\, \|\Gamma_k\|_1$, we have that

$$(\Gamma_k)_k \;\text{ is bounded in }\; L_2((0,T)\times(0,1))\,. \tag{54}$$

Owing to (49), a first consequence of (54) is that $(\sigma_k(\Gamma_k))_k$ is also bounded in $L_2((0,T)\times(0,1))$. Furthermore, it follows from (51), (49), and (54) that

$$\begin{aligned}
\int_0^T\int_0^1 |\partial_x\sigma_k(\Gamma_k)|^{4/3}\,\mathrm{d}x\mathrm{d}s &= \int_0^T\int_0^1 \left(\frac{-\sigma_k'(\Gamma_k)}{\Gamma_k}\right)^{2/3} |\partial_x\Gamma_k|^{4/3}\ \left(\Gamma_k\ |\sigma_k'(\Gamma_k)|\right)^{2/3}\,\mathrm{d}x\mathrm{d}s \\
&\le \left(\int_0^T\int_0^1 \frac{-\sigma_k'(\Gamma_k)}{\Gamma_k}\ |\partial_x\Gamma_k|^2\,\mathrm{d}x\mathrm{d}s\right)^{2/3} \left(\int_0^T\int_0^1 \left(\Gamma_k\ |\sigma_k'(\Gamma_k)|\right)^2\,\mathrm{d}x\mathrm{d}s\right)^{1/3} \\
&\le C(T)\ (2+\sigma_0)^2\ \left(\int_0^T\int_0^1 \Gamma_k^2\,\mathrm{d}x\mathrm{d}s\right)^{1/3} \le C(T)\,.
\end{aligned}$$

Consequently,

$$(\sigma_k(\Gamma_k))_k \;\text{ is bounded in }\; L_{4/3}(0,T;W^1_{4/3}(0,1))\,. \tag{55}$$

Finally, (21) (with surface tension σ_k), **(B.1)**, **(B.2)**, (52), and (53) guarantee that

$$(\partial_t\Gamma_k)_k \;\text{ is bounded in }\; L_1(0,T;(H^2_N(0,1))^*)\,, \tag{56}$$

where $H^2_N(0,1) := \{w\in H^2(0,1)\ :\ \partial_x w(0) = \partial_x w(1) = 0\}$. Hence, owing to the compactness of the embeddings of $H^1(0,1)$ and $\mathcal{C}^{0,(1-\theta)/2}([0,1])$ in $\mathcal{C}([0,1])$ and the continuity of the embedding of $\mathcal{C}([0,1])$ in either $(H^1(0,1))^*$ or $(H^2_N(0,1))^*$, we infer from **(B.1)**, (52), (53), (56), and [12, Corollary 4] that there are a subsequence of $(h_k,\Gamma_k)_k$ (not relabeled) and functions h and Γ such that

$$h_k \to h \text{ in } \mathcal{C}([0,T]\times[0,1]), \qquad \Gamma_k \to \Gamma \text{ in } L_1(0,T;\mathcal{C}([0,1]))\,. \tag{57}$$

In addition, $(\partial_x h_k)_k$ being bounded in $L_\infty(0,T;L_2(0,1))$ by **(B.1)** and $(\partial_x\sigma_k(\Gamma_k))_k$ being bounded in $L_{4/3}((0,T)\times(0,1))$ by (55), we have, up to an extraction of a subsequence and for some function Σ,

$$\partial_x h_k \rightharpoonup \partial_x h \quad \text{weakly-}\star \text{ in } L_\infty(0,T;L_2(0,1)) \text{ and } \partial_x\sigma_k(\Gamma_k) \rightharpoonup \Sigma \quad \text{in } L_{4/3}((0,T)\times(0,1))\,. \tag{58}$$

As a consequence of **(B.1)**, (54), (57), and (58), we get that the limits satisfy

$$h\in L_\infty(0,T;H^1(0,1)), \quad h\ge 0\,, \quad \Gamma\in L_\infty(0,T;L_1(0,1))\cap L_2((0,T)\times(0,1))\,, \quad \Gamma\ge 0\,. \tag{59}$$

Finally, thanks to **(B.1)**, (58), and (59), we have

$$\int_0^T\int_0^1 |\Gamma_k(t,x)-\Gamma(t,x)|^2\,\mathrm{d}x\mathrm{d}t \le \sup_{s\in[0,T]}\{\|\Gamma_k(s)\|_1 + \|\Gamma(s)\|_1\}\ \int_0^T \|\Gamma_k(t)-\Gamma(t)\|_\infty\,\mathrm{d}t \underset{k\to\infty}{\longrightarrow} 0\,,$$

so that we also have

$$\Gamma_k \to \Gamma \text{ in } L_2((0,T)\times(0,1))\,. \tag{60}$$

4.3. **Identifying the limit system.** According to the uniform bounds **(B.1)**, **(B.2)**, and (52), we first obtain that, up to an extraction of a subsequence and for some function $\overline{\jmath}_1$ and $\overline{\jmath}_2$,

$$h_k^{3/2}\ \mathbf{1}_{(0,\infty)}(h_k)\ \partial_x^3 h_k \rightharpoonup \overline{\jmath}_1, \qquad h_k^{1/2}\ \partial_x\sigma_k(\Gamma_k) \rightharpoonup \overline{\jmath}_2, \qquad \text{in } L_2((0,T)\times(0,1)). \tag{61}$$

Arguing as in [3, Section 3] and [7, Section 3.4], we first deduce from **(B.2)** and (57) that $\partial_x^3 h$ belongs to $L_2(\mathcal{P}(\delta))$ for all $\delta>0$ where $\mathcal{P}(\delta) := \{(t,x)\in(0,T)\times(0,1)\ :\ h(t,x)>\delta\}$ and $\overline{\jmath}_1 = h^{3/2}\partial_x^3 h$ in $\{(t,x)\in(0,T)\times(0,1)\ :\ h(t,x)>0\}$. Combining this result with (57) yields

$$\lim_{k\to\infty}\int_0^T\int_0^1 h_k^3\,\mathbf{1}_{(0,\infty)}(h_k)\,\partial_x^3 h_k\ \partial_x\zeta\,\mathrm{d}x\mathrm{d}s = \int_0^T\int_0^1 h^3\,\mathbf{1}_{(0,\infty)}(h)\,\partial_x^3 h\ \partial_x\zeta\,\mathrm{d}x\mathrm{d}s\,,$$

$$\lim_{k\to\infty}\int_0^T\int_0^1 h_k^2\,\mathbf{1}_{(0,\infty)}(h_k)\,\Gamma_k\,\partial_x^3 h_k\ \partial_x\zeta\,\mathrm{d}x\mathrm{d}s = \int_0^T\int_0^1 h^2\,\mathbf{1}_{(0,\infty)}(h)\,\Gamma\,\partial_x^3 h\ \partial_x\zeta\,\mathrm{d}x\mathrm{d}s\,,$$

for any $\zeta\in\mathcal{C}^\infty([0,T]\times[0,1])$ such that $\zeta(T,x)=0$ for all $x\in[0,1]$ and $\partial_x\zeta(t,x)=0$ for all $(t,x)\in[0,T]\times\{0,1\}$. We next claim the strong convergence

$$\sigma_k(\Gamma_k)\longrightarrow\sigma(\Gamma) \quad \text{in } L_2((0,T)\times(0,1)). \tag{62}$$

Indeed, on the one hand, we readily infer from (49) and (60) that

$$\int_0^T\int_0^1 |\sigma_k(\Gamma_k)-\sigma_k(\Gamma)|^2\ \mathrm{d}x\mathrm{d}t \le (2+\sigma_0)^2\ \int_0^T\int_0^1 |\Gamma_k-\Gamma|^2\,\mathrm{d}x\mathrm{d}t \underset{k\to\infty}{\longrightarrow} 0\,.$$

On the other hand, it follows from Proposition 8 and (49) that $\sigma_k(\Gamma)\to\sigma(\Gamma)$ a.e. in $(0,T)\times(0,1)$ with $|\sigma_k(\Gamma)|\le(2+\sigma_0)\ (1+\Gamma)\in L_2((0,T)\times(0,1))$, whence

$$\lim_{k\to\infty}\int_0^T\int_0^1 |\sigma_k(\Gamma)-\sigma(\Gamma)|^2\ \mathrm{d}x\mathrm{d}t = 0$$

by the Lebesgue dominated convergence theorem. Thus, (62) holds true.

Together with (55), the convergence (62) ensures that $\sigma(\Gamma)\in L_{4/3}(0,T;W^1_{4/3}(0,1))$ and $\Sigma=\partial_x\sigma(\Gamma)$ in (58). Next, collecting (57), (58), and (62) yields $\overline{\jmath}_2 = h^{1/2}\partial_x\sigma(\Gamma)$, so that $h^{1/2}\partial_x\sigma(\Gamma)\in L_2((0,T)\times(0,1))$. It is then straightforward to pass to the limit as $k\to\infty$ in the remaining terms in the weak formulation (20)-(21) for (h_k,Γ_k) and conclude that (h,Γ) is a weak solution to (1)–(4) with surface tension σ and initial data (h_0,Γ_0). This completes the proof of **Theorem 2**.

Remark 10. *We shall point out that our strategy to prove* **Theorem 2** *by approximating the surface tension* $\sigma\in\mathcal{C}([0,\infty))\cap\mathcal{C}^1(0,\infty)$ *by surface tensions* $(\sigma_k)_{k\in\mathbb{N}}$ *satisfying the assumptions of* **Theorem 3** *does not yield existence of nonnegative weak solutions for the limiting case* $\theta=1$. *More precisely, if* $\theta=1$, *then the analogue of* **Lemma 9** *for the energy dissipation merely yields a control on* $\sqrt{\Gamma}$ *in the space of continuous functions (instead of a Hölder space as in the case* $\theta\in[0,1)$*), and we thus lose compactness of the concentration of any bounded family of solutions in* $L_2((0,T)\times(0,1))$. *It seems that this threshold is of high importance. Indeed, provided* $-\sigma'(\Gamma)$ *is dominated by* $1/(1+\Gamma)$ *at infinity, a good choice of multiplier for* (1)–(2) *yields that the integral* (24) *(with* $\theta=1$*) measures the dissipation of energy for any small solution.*

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