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BIRATIONAL SELF-MAPS AND PIECEWISE ALGEBRAIC GEOMETRY

STÉPHANE LAMY AND JULIEN SEBAG

ABSTRACT. Let X be a smooth projective complex variety, of dimension 3, whose Hodge numbers $h^{3,0}(X), h^{1,0}(X)$ both vanish. Let $f \colon X \dashrightarrow X$ be a birational map that induces an isomorphism $U \cong V$ on (dense) open subvarieties U, V of X. Then we show that the complex varieties $(X \setminus U)_{\mathrm{red}}, (X \setminus V)_{\mathrm{red}}$ are piecewise isomorphic.

1. INTRODUCTION

In [7], MISHA GROMOV asks the following question:

Question 1. *Let X, Y be algebraic varieties that admit embeddings to a third one, say $X \hookrightarrow Z$ and $Y \hookrightarrow Z$, such that the complements $Z \setminus X$ and $Z \setminus Y$ are biregular isomorphic. How far are X and Y from being birationally equivalent?*

Let k be an arbitrary field. Let $\mathbb{Z}[\mathrm{Var}_k]$ be the free abelian group generated by the isomorphism classes of k-schemes of finite type. One defines the *Grothendieck ring of varieties over k*, denoted by $K_0(\mathrm{Var}_k)$, as the quotient of $\mathbb{Z}[\mathrm{Var}_k]$ by the following "scissor" relations: whenever X is a k-scheme of finite type, and F is a closed subscheme of X, we impose that

$$[X] = [F] + [X \setminus F].$$

The multiplication in $K_0(\mathrm{Var}_k)$ is defined by

$$[X] \cdot [X'] := [X \times_k X'].$$

for any pair of k-schemes of finite type X, X'.

Independently of [7][1], MICHAEL LARSEN and VALERY LUNTS ask, in [11, Question 1.2], the following question in order to understand the "geometric nature" of the Grothendieck ring of varieties:

Question 2. *Let X, Y be reduced separated schemes of finite type over $k = \mathbb{C}$ such that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Is it true that X, Y are piecewise isomorphic?*

Recall that two k-schemes of finite type X, Y are called *piecewise isomorphic*, if one can find a finite set I, a partition $(V_i)_{i \in I}$ of X in (locally closed) subschemes of X, and a partition $(W_i)_{i \in I}$ of Y in (locally closed) subschemes of Y, such that, for any i, $i \in I$, there exists an isomorphism of k-schemes $f_i \colon (V_i)_{\mathrm{red}} \to (W_i)_{\mathrm{red}}$. Note that this property only depends on the reduced scheme structures of X and Y. Up to now, this question is open in general (see [13], [16] for positive elements

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[1]The existence of a link between Question 1 and the structure of $K_0(\mathrm{Var}_k)$ is already mentioned in [7, 3G′′′].

of answer in this direction, or [9, Theorem 13.1] that uses a different point of view).

The fact that, without any condition on k, a positive answer to Question 2 implies a precise answer to Question 1 is easy to see. We adopt the notations of Question 1. Because of the relations lying in $K_0(\mathrm{Var}_k)$, we deduce that $[Z \setminus X] = [Z \setminus Y]$, and that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. If Question 2 has a positive answer, we conclude that X, Y are piecewise isomorphic. Because of the definition of a piecewise isomorphism, this remark proves in particular the existence of a bijection σ between the set of the irreducible components of maximal dimension of respectively X and Y, such that $(X_i)_{\mathrm{red}}$ and $(\sigma(X_i))_{\mathrm{red}}$ are birationally equivalent, for any irreducible component X_i of X.

Besides, note that Question 2 holds true in particular for any pair X, Y of separated k-schemes of finite type of dimension at most 1, if k is supposed to be algebraically closed of characteristic 0 (see [13, Proposition 6]).

In the present article, we investigate the following conjectural reformulation of Question 1 based on Question 2 (see also [13, Remark 13], or [16]).

Conjecture 1. *Let k be an algebraically closed field of characteristic zero. Let X be a separated scheme of finite type over k. Let $f\colon X \dashrightarrow X$ be a birational map that induces an isomorphism of k-schemes $U \cong V$ on dense open subsets U, V of X. Then the k-schemes $(X \setminus U)_{\mathrm{red}}, (X \setminus V)_{\mathrm{red}}$ are piecewise isomorphic.*

Note that, by a classical argument of birational geometry, if X is a smooth projective scheme over k, of arbitrary dimension d, having no rational curves, then such a map f is, in fact, an isomorphism of k-schemes. Thus Conjecture 1 holds true in that case, but remains open, and particularly relevant, for separated k-schemes of finite type having an infinite number of rational curves. (The case where the separated k-scheme of finite type X has only finitely many rational curves can be solved positively for example by [13, Theorem 5].)

Precisely, we prove the following theorem, which is the main result of this article (see Theorem 24, and Remark 25 for a slightly more general statement).

Main Theorem. *Let X be a smooth irreducible projective $\mathbb{C}$-variety, of dimension 3, whose Hodge numbers $h^{3,0}(X), h^{1,0}(X)$ both vanish. Let $f\colon X \dashrightarrow X$ be a birational map that induces a $\mathbb{C}$-isomorphism $U \cong V$ on dense open subvarieties U, V of X. Then the $\mathbb{C}$-varieties $(X \setminus U)_{\mathrm{red}}, (X \setminus V)_{\mathrm{red}}$ are piecewise isomorphic.*

The proof of such a statement is based on the following ingredients.

a) The use of the Weak Factorization Theorem for birational maps (see [1]) and its combinatoric;
b) The use of intermediate Jacobians to control the weak factorizations of f (see [3]);
c) The positive answer to Larsen–Lunts' question in the case of curves (see [13, Proposition 6]).

This article is mostly self-contained, for the convenience of the reader. Sections 3 and 4 give an overview of some tools of birational, piecewise algebraic and complex geometry that we use in the proof of Theorem 24. Besides, we establish in these

sections some technical lemmas, sometimes not explicitly described in the litera-
ture, that we need, in a crucial way, in the proof of Theorem 24 (e.g., see Lemma
10, Corollary 12, Propositions 15, 21, Corollary 23). Then Section 5 is devoted to
our main theorem and its proof (see Theorem 24). In Sections 6 and 7 we give
respectively some explicit examples and some further comments.

The authors would thank Qing Liu and François Loeser for their comments on
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2. Conventions

Let k be a field. A *curve* (resp. *surface*) is a separated k-scheme of finite type
purely of dimension 1 (resp. 2). More generally, a k-*variety* is a separated k-scheme
of finite type. If x is a point in a k-variety, we define its dimension as the transcen-
dence degree over k of the residue field $\kappa(x)$ of X at x.

Two reduced k-varieties X, Y are *birationally equivalent* if there exist a dense
open subscheme of X and a dense open subscheme of Y that are isomorphic as
k-schemes. We say that two reduced k-varieties X, Y are *stably birational* if there
exist integers $m, n \in \mathbf{N}$ such that $X \times_k \mathbb{P}_k^m, Y \times_k \mathbb{P}_k^n$ are birationally equivalent.
If k is algebraically closed, we say that an integral k-variety X is *rational*, if it
is birationally equivalent to $\mathbb{P}_k^{\dim(X)}$. More generally, if k is algebraically closed,
a reduced k-variety is said rational if its irreducible components are all rational.
An integral k-variety X, of dimension d, is said *uniruled* if there exists a k-variety
Y of dimension $d - 1$, and a dominant rational map $Y \times_k \mathbb{P}_k^1 \dashrightarrow X$. If k is an
algebraically closed field of characteristic zero, and if $d = 2$, a smooth projective
k-variety X is uniruled if and only if it is ruled.

If X, Y are two piecewise isomorphic k-varieties, the datum of the family of the
k-isomorphisms $(f_i)_{i \in I}$ (given, by definition, between the reduced locally closed
strata of X, Y) is called a *piecewise isomorphism*.

If k is an algebraically closed field, we say that a k-variety is an abelian variety
if X is a smooth irreducible projective abelian group scheme over k.

If X is a $\mathbb{C}$-variety, we denote by $X^{\mathrm{an}} := X(\mathbb{C})$ the *analytic space associated to X*.

In $K_0(\mathrm{Var}_k)$, we denote by $\mathbb{L}$ the class of the affine line, and by $[X]$ the class of
the k-scheme of finite type X. Let A be a ring. A ring morphism $K_0(\mathrm{Var}_k) \to A$ is
called a *motivic measure*.

3. Birational and piecewise algebraic geometry

In this section, we review some tools of birational and piecewise algebraic ge-
ometry that we use in the proof of Theorem 24. We also establish some useful
preliminary lemmas.

3.1. **Birational maps and the Weak Factorization Theorem.** In this para-
graph, we review some well-known results on birational maps such as the Weak
Factorization Theorem established in [1, Theorem 0.1.1] and we assume that k is

an algebraically closed field of characteristic zero. Let X, Y be two k-varieties. A rational map $f\colon X \dashrightarrow Y$ that induces an isomorphism of k-schemes $U \cong V$ from a dense open subvariety of X onto a dense open subvariety of Y is called a *birational map*. In particular, there exists a unique maximal open subvariety $\mathrm{Dom}(f)$ of X, such that $f_{|\mathrm{Dom}(f)}\colon \mathrm{Dom}(f) \to Y$ is a morphism of k-schemes. If $x \in X$, we say that a birational map $f\colon X \dashrightarrow Y$ is a *local isomorphism* (for the Zariski topology) at x if there exists an open subvariety U of $\mathrm{Dom}(f)$ that contains x such that the restriction of $f_{|\mathrm{Dom}(f)}$ to U is an open immersion of k-schemes. Note that, if $f\colon X \dashrightarrow Y$ is a local isomorphism at x, then $x \in \mathrm{Dom}(f)$. Remark also that the set of points of X, at which a birational map $f\colon X \dashrightarrow Y$ is *not* a local isomorphism, is a closed subset of X, that we call *exceptional set* of f. We denote it by $\mathrm{Exc}(f)$.

The following lemmas are recalled for convenience for the reader. Lemma 4 should be kept in mind for the proof of Theorem 24.

Lemma 2. *Let X, Y be two integral k-varieties. Let $f\colon X \dashrightarrow Y$ be a birational map. Let $x \in \mathrm{Dom}(f)$ be a point. Let us set $y := f(x)$. The following assertions are equivalent.*

a) *The map f is a local isomorphism at x;*
b) *The morphism $f_{|\mathrm{Dom}(f)}$ induces an isomorphism of k-algebras $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$;*
c) *The morphism $f_{|\mathrm{Dom}(f)}$ is étale at x.*
d) *The morphism $f_{|\mathrm{Dom}(f)}$ induces an isomorphism of k-algebras $\widehat{\mathcal{O}_{Y,y}} \to \widehat{\mathcal{O}_{X,x}}$.*

Proof. Since $f_{|\mathrm{Dom}(f)}$ is dominant, the induced morphism of local k-algebras $f_x\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective.

Assume that $f_{|\mathrm{Dom}(f)}$ is étale at x. Thus, the local morphism f_x is flat, hence faithfully flat. Since f is birational, we deduce that f_x is an isomorphism. It shows that point c) implies point b). Besides, $f_{|\mathrm{Dom}(f)}$ then induces an isomorphism of k-algebras $\kappa(y) \to \kappa(x)$. So point c) is equivalent to point d). $\qquad\square$

Remark 3. Let $n \in \mathbb{N}$, $n \geq 1$. To emphasize Lemma 2, consider the birational $\mathbb{C}$-morphism $f\colon \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}$ defined by the datum of $(f_i)_{1 \leq i \leq n} \in (\mathbb{C}[x_1, \ldots, x_n])^n$. Then we deduce from the Formal Inverse Function Theorem that the set of the $\mathbb{C}$-points of $\mathbb{A}^n_{\mathbb{C}}$ at which f is not a local isomorphism is defined as the $\mathbb{C}$-points of the hypersurface of $\mathbb{A}^n_{\mathbb{C}}$ defined by the determinant of the Jacobian matrix of f, i.e., defined by the equation

$$\det((\partial f_i / \partial x_j)_{1 \leq i,j \leq n}) = 0.$$

Lemma 4. *Let $f\colon X \dashrightarrow Y$ be a birational map between two integral k-varieties. Assume that Y is normal. Let $U_0 \subset \mathrm{Dom}(f)$ be the open set of points of X at which f is a local isomorphism. Let us set $V_0 := f(U_0)$. Then*

a) *The morphism of k-schemes $f_{|U_0}\colon U_0 \to Y$ is an open immersion. So the map f induces an isomorphism of k-schemes $U_0 \to V_0$.*
b) *Assume that X, Y are proper and smooth over k. Each irreducible component of codimension 1 in $(X \setminus U_0)_{\mathrm{red}}$ is ruled.*

Proof. a) From the definition of U_0 and from Lemma 2, we deduce that the restriction of the morphism $f_{|\mathrm{Dom}(f)}$ to U_0 is separated, birational and locally quasi-finite. From Grothendieck's form of Zariski's Main Theorem applied to the morphism

$f_{|U_0}$, we conclude that $f_{|U_0}$ is an open immersion. So it induces an isomorphism of k-schemes from U_0 to V_0.

b) By eliminating the indeterminacies of f (e.g., see [1, Lemma 1.3.1]), we find a composite of blow-ups $\pi\colon Z \to X$, with smooth centers disjoint from U_0, and a birational morphism $g\colon Z \to Y$ such that $g = f \circ \pi$. Let η be the generic point of an irreducible component X_η of maximal dimension in $(X \setminus U_0)_{\mathrm{red}}$. Note that $\eta \in \mathrm{Dom}(f)$. The strict transform by π of X_η (considered with its reduced scheme structure) is a divisor Z_η of Z, birationally equivalent to X_η. But, because of the choice of U_0 and the dimension of η, the subvariety Z_η necessarily is an irreducible component of the exceptional locus of the morphism g. By Abhyankar's Theorem on exceptional sets of birational morphisms, we deduce that Z_η is ruled, i.e., birationally equivalent to $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} S$ where S is an integral k-variety of dimension $\dim(X) - 2$. So X_η is ruled. $\qquad\square$

Remark 5. Note that Lemma 4 in particular shows that the set of points of X at which f is a local isomorphism is the largest open subset U of X such that $f_{|U}$ is an isomorphism of k-schemes.

The Weak Factorization Theorem was a great step in the understanding of the geometry of birational maps. We recall below the statement of this theorem.

Let X, Y be smooth irreducible proper (resp. projective) k-varieties. Let $f\colon X \dashrightarrow Y$ be a birational map that induces an isomorphism of k-schemes $U \to V$ on a dense open subvariety U of X onto a dense open subvariety V of Y. We say that f admits a *weak factorization* if there exist birational morphisms $b_i\colon Z_i \to X_i$, $c_i\colon Z_i \to X_{i+1}$, for i, $1 \le i \le n$ between smooth irreducible proper (resp. projective) k-varieties such that:
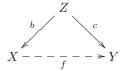

a) $X := X_1$, $Y := X_{n+1}$;
b) For any i, $1 \le i \le n$, the b_i and the c_i are composites of blow-ups, with smooth irreducible centers, disjoint from U (note that, in our notation and for convenience in the following computations, $c_n\colon Z_n \to Y$ or $b_1\colon Z_1 \to X$ could be the blow-up with empty center, i.e., $c_n = \mathrm{id}_Y$ and $b_1 = \mathrm{id}_X$).

Theorem 6 ([1, Theorem 0.1.1]). *Let X, Y be two smooth irreducible proper k-varieties. Let $f\colon X \dashrightarrow Y$ be a birational map. Then f admits a weak factorization.*

Example 7. In dimension 2, a stronger version of Theorem 6 goes back to Oscar Zariski. Let X, Y be two smooth irreducible projective k-surfaces, and let $f\colon X \dashrightarrow Y$ be a birational map. The elimination of the indeterminacies of f produces a weak factorization of f. Indeed, in that case, one knows that there exist composites of blow-ups of points $b\colon Z \to X, c\colon Z \to Y$ making the following

diagram commute (see [12, Theorem **9**.2.7]):

$$
\begin{array}{ccc}
 & Z & \\
{}^{b}\swarrow & & \searrow^{c} \\
X \;-\,-\,-\,\underset{f}{-}\,-\,-\!\!\!\rightarrow & & Y
\end{array}
$$

The existence of such a *strong factorization* in any dimension is still a conjecture.

The following remark should be kept in mind for the proof of Theorem 24.

Remark 8. Let b_{ij} be a blow-up that appears in the expression of b_i. Then two cases occur.

a) First, the strict transform of its exceptional divisor might be equal for some j' to the exceptional divisor of a blow-up $c_{i'j'}$ appearing in the expression of $c_{i'}$. Note that, in that case, the strict transform is then contracted by $c_{i'j'}$.

b) On the contrary, it can happen that no blow-up appearing in any c_i contracts the strict transform of its exceptional divisor. In that case, we can consider that strict transform in Y. Then it is equal to a divisor of Y contained in $Y \setminus V$. Note that if V is chosen as the maximal open subset on which f^{-1} is well-defined and injective, then all the irreducible components, of dimension $\dim(Y) - 1$, in $Y \setminus V$ are realized this way, and in particular are ruled. (This last assertion follows from Lemma 4, or from the Weak Factorization Theorem).

3.2. **Cohomology theories and examples of motivic measures.** In this paragraph, we assume that k is a subfield of the field of complex numbers, except when stated otherwise. Let X be a smooth proper k-variety. We denote by $H^*_{\mathrm{dR}}(X)$ the *algebraic de Rham cohomology* of X. Then

$$
\dim_k H^n_{\mathrm{dR}}(X) = \sum_{i+j=n} \dim_k H^i(X, \Omega^j_{X/k})
$$

(Hodge's decomposition theorem) for any integer n. Let us denote by $H^{p,q}(X)$ the $\mathbb{C}$-vector space $H^q(X, \Omega^p_{X/k}) \otimes_k \mathbb{C}$, for any pair p, q, $0 \leq p, q \leq \dim(X)$. The integer

$$
h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)
$$

is called the (p, q)-*Hodge number* of X. We denote by $H^*(X) := H^*(X^{\mathrm{an}}, \mathbb{Z})$ the *integral singular cohomology* of X^{an}. By Grothendieck's results ([8], Theorem 1′), we know that the canonical map $H^*_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \simeq H^*(X) \otimes_{\mathbb{Z}} \mathbb{C}$ is an isomorphism. For i, $1 \leq i \leq 2\dim(X)$, we denote by

$$
b_i(X) := \dim_{\mathbb{Q}} H^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \dim_{\mathbb{C}} H^i(X) \otimes_{\mathbb{Z}} \mathbb{C}
$$

the i-th *Betti number* of X.

By Deligne's mixed Hodge theory, the assignment to a $\mathbb{C}$-variety X of its *Hodge polynomial* $X \mapsto H_X(u, v)$ defines a unique ring morphism

$$
K_0(Var_{\mathbb{C}}) \to \mathbb{Z}[u, v]
$$

(see [17, Theorem 1.1] or [5, §6.1]). Recall that, if X is a smooth proper $\mathbb{C}$-variety, its Hodge polynomial is defined as

$$
HD_X(u, v) := \sum_{p,q \geq 0} h^{p,q}(X) u^p v^q.
$$

We call *Poincaré polynomial* of a $\mathbb{C}$-variety X the polynomial $P_X \in \mathbb{Z}[u]$, defined by $P_X(u) := HD_X(u, u)$. This assignment defines a unique ring morphism

$$K_0(Var_\mathbb{C}) \to \mathbb{Z}[u].$$

Recall that $P_X(u) = \sum_{n=0}^{2 \dim X} \dim_\mathbb{C} H_{\mathrm{dR}}^n(X) u^n = \sum_{n=0}^{2 \dim X} b_n(X) u^n$ if X is a smooth proper $\mathbb{C}$-variety, by Hodge Decomposition Theorem.

One defines the *Euler characteristic* with compact support to be the ring morphism

$$\chi_c \colon K_0(Var_\mathbb{C}) \to \mathbb{Z}$$

defined by $\chi_c([X]) = P_X(-1)$, for any $\mathbb{C}$-variety X. If X is a smooth proper $\mathbb{C}$-variety, then

$$\chi_c([X]) := \sum_{n=0}^{2 \dim X} (-1)^n b_n(X).$$

This definition corresponds to the usual topological Euler characteristic associated to X^{an}.

Example 9 (see [10, Proposition 2.3(a)]). Let X, Y be two connected k-curves. Then X, Y are piecewise isomorphic if and only if $X_{\mathrm{red}}, Y_{\mathrm{red}}$ are birationally equivalent with $\chi_c(X) = \chi_c(Y)$.

These constructions have the following useful consequence.

Lemma 10. *Let k be an arbitrary field. Let us fix an algebraic closure $\bar{k}$ of k. Let X, Y be k-varieties such that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Then the dimension of X and Y are equal. Besides, the number of irreducible components of maximal dimension of $X_{\bar{k}}$ is equal to that of $Y_{\bar{k}}$.*

Proof. See, for example, [13, Proposition 4, Corollary 5]. Below, we give a proof when k equals $\mathbb{C}$, that is the framework of Theorem 24.

By the scissor relations in $K_0(\mathrm{Var}_k)$ and by the assumption, we have $[X_{\mathrm{red}}] = [X] = [Y] = [Y_{\mathrm{red}}]$. Since the statement only depends on the reduced scheme structures of X and Y, we can assume that X, Y are reduced $\mathbb{C}$-varieties. Let $X_1, \ldots, X_m$ (resp. $Y_1, \ldots, Y_n$) be the irreducible components of X (resp. Y). By Nagata's Compactification Theorem and Hironaka's Theorem on resolution of singularities, we can find smooth irreducible proper k-varieties $\bar{X}_1, \ldots, \bar{X}_n$ (resp. $\bar{Y}_1, \ldots, \bar{Y}_n$) such that, for any i, $1 \leq i \leq m$ (resp. j, $1 \leq j \leq n$), $\bar{X}_i$ is birationally equivalent to X_i (resp. $\bar{Y}_j$ is birationally equivalent to Y_j), and smooth irreducible proper k-varieties $C_1, \ldots, C_s, D_1, \ldots, D_t$ that transform the equality of classes $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$ in the following one:

$$\sum_{i=1}^{m} [\bar{X}_i] + \sum_{\alpha=1}^{s} \varepsilon_\alpha [C_\alpha] = \sum_{j=1}^{n} [\bar{Y}_j] + \sum_{\beta=1}^{t} \varepsilon_\beta [D_\beta],$$

with $\varepsilon_\alpha, \varepsilon_\beta \in \{1, -1\}$. By computing the degrees and the leading coefficients of the Poincaré polynomials associated to the left and right-hand terms, we obtain the conclusion of the statement. Indeed, by Serre Duality, one knows that:

$$h^{\dim(Z), \dim(Z)}(Z) = b_{2\dim(Z)}(Z) = 1,$$

for any smooth irreducible proper k-variety Z. $\qquad\square$

We can also prove the following lemma.

Lemma 11. *Let k be an algebraically closed field of characteristic zero. Let X, Y be piecewise isomorphic smooth irreducible proper k-varieties. Let $f\colon X \dashrightarrow Y$ be a birational map. Let*

be a weak factorization of f. If $1 \leq i \leq n$, let us denote by C_{ij}, for $1 \leq j \leq s_i$, the (non-empty) smooth irreducible center (of codimension α_j) of the blow-ups defining b_i, and by D_{ij}, for $1 \leq j \leq t_i$, the (non-empty) smooth irreducible centers (of codimension β_j) of the blow-ups defining c_i. Then

$$(1) \qquad \mathbb{L} \sum_{i=1}^{n} \sum_{j=1}^{s_i} [C_{ij}][\mathbb{P}_k^{\alpha_j - 2}] = \mathbb{L} \sum_{i=1}^{n} \sum_{j=1}^{t_i} [D_{ij}][\mathbb{P}_k^{\beta_j - 2}].$$

Proof. If $b_1 = \mathrm{id}_X$, or $c_n = \mathrm{id}_Y$, eliminate $X = X_1$ or $Y = X_{n+1}$ from the diagram. Let V be a smooth irreducible proper k-variety. Let W be the blow-up of V, with smooth irreducible center C of codimension α in V. Then, by the blow-up formula in $K_0(\mathrm{Var}_k)$, we deduce that:

$$[W] - [C][\mathbb{P}_k^{\alpha-1}] = [V] - [C],$$

or equivalently, using the identity $\mathbb{P}_k^{\gamma} = \sum_{i=0}^{\gamma} \mathbb{L}^i$,

$$[W] - \mathbb{L}[C][\mathbb{P}_k^{\alpha-2}] = [V]$$

(see, for example, [13, Example 1]). Applying this formula to each blow-up appearing in the b_i, and to each blow-up appearing in the c_j, we deduce the result by adding all the obtained relations.

$\square$

Corollary 12. *Let X, Y be piecewise isomorphic smooth irreducible proper $\mathbb{C}$-varieties. Let $f\colon X \dashrightarrow Y$ be a birational map. Let*

be a weak factorization of f. Then the number of all the (non-empty) blow-ups appearing in the expression of the b_i, $1 \leq i \leq n$, is equal to the number of all (non-empty) the blow-ups appearing in the expression of the c_i, $1 \leq i \leq n$.

Proof. The proof follows from computing the leading coefficients of Poincaré polynomials in the formula of Lemma 11. $\square$

Remark 13. *a)* With the notations of Lemma 11, assume that the dimension of X (and so that of Y by Lemma 10) is equal to 3. By looking at the proof of Lemma 11, we see that (1) can be refined in:

$$(2) \qquad \mathbb{L}\left(\sum_{i=1}^{n} \left(m_i(\mathbb{L}+1) + \sum_{\ell} [C_{ij_\ell}] \right) \right) = \mathbb{L}\left(\sum_{i=1}^{n} \left(n_i(\mathbb{L}+1) + \sum_{\ell'} [D_{ij_{\ell'}}] \right) \right)$$

where m_i (resp. n_i) is the number of blow-ups appearing in the expression of b_i (resp. c_i), with centers a rational curve (i.e., $\mathbb{P}_k^1$) or a point, and where the $C_{ij_\ell}, D_{ij_{\ell'}}$ are smooth irreducible projective k-curves, whose genus g is at least 1. If we compute Poincaré polynomials of the left and right-hand terms, then we deduce, by comparing the coefficients of the monomial of degree 3, that:

$$\sum_{i=1}^{n} \sum_{\ell} g(C_{ij_\ell}) = \sum_{i=1}^{n} \sum_{\ell'} g(D_{ij_{\ell'}}),$$

where $g(*)$ is the genus of the k-curve $*$. In Proposition 21, we will see how to improve this result by introducing intermediate Jacobians.

b) One can prove Corollary 12 directly and independently from Lemma 11 by computing Picard numbers, which increase by one under blow-up of an irreducible center.

c) Corollary 12 can be formulated when k is any algebraically closed field of characteristic zero.

3.3. **Larsen–Lunts' motivic measure and some consequences.** In this paragraph we assume that k is an algebraically closed field of characteristic zero. In [11], M. LARSEN and V. LUNTS have constructed a motivic measure that is a very important piece in the understanding of the Grothendieck ring of varieties and its structure. Precisely we have:

Theorem 14 (Larsen–Lunts)**.** *Let $\mathbb{Z}[\mathfrak{sb}]$ be the ring generated by the classes of smooth irreducible proper k-varieties for the stably birational equivalence relation. There exists a unique surjective ring morphism*

$$\mathrm{SB} \colon K_0(\mathrm{Var}_k) \to \mathbb{Z}[\mathfrak{sb}],$$

that assigns to the class in $K_0(\mathrm{Var}_k)$ of a smooth irreducible proper k-variety X its stably birational class in $\mathbb{Z}[\mathfrak{sb}]$. Besides, its kernel is the ideal of $K_0(\mathrm{Var}_k)$ generated by $\mathbb{L}$.

It is important to note that the description of the image of the morphism SB in terms of stably birational classes of k-varieties only holds for classes of smooth irreducible proper k-varieties, which form a set of generators of the ring $K_0(\mathrm{Var}_k)$. The image of a k-variety which is not irreducible proper and smooth is defined as a linear combination of stably birational equivalence classes of smooth irreducible proper k-varieties. In particular, the image by SB of such a variety can be "non-effective". To illustrate this point, consider the following example: It is easy to show that, if Y is a rational k-variety of dimension 1, then [13, Proposition 7] implies the following formula

$$\mathrm{SB}([Y]) = \chi_c([Y]) - c_Y,$$

where c_Y is the number of irreducible components of dimension 1 of Y. See also [10, Remark 1.6]. (Note that the canonical homomorphism $\mathbb{Z} \to K_0(\mathrm{Var}_k)$ is injective.)

The realization morphism has the following useful consequence.

Proposition 15 ([13, Proposition 6])**.** *Let X, Y be k-varieties such that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Assume that the dimension of X is at most 1. Then X, Y are piecewise isomorphic.*

Proof. The proof below follows the arguments of the proof of [13, Proposition 6]. Since the statement only depends on the reduced scheme structures of X and Y, we can suppose that X and Y are reduced. Assume first that the dimension of X is one. By Lemma 10 the dimension of Y is equal to 1 too. Let $\bar{X}_1, \ldots, \bar{X}_n$ (resp. $\bar{Y}_1, \ldots, \bar{Y}_n$) be smooth irreducible projective models of the irreducible components $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_n$) of X (resp. of Y). So, the relation $[X] = [Y]$ can be transformed, in $K_0(\mathrm{Var}_k)$, in the following one:

$$\sum_{i=1}^{n} [\bar{X}_i] = \sum_{i=1}^{n} [\bar{Y}_i] + m,$$

with $m \in \mathbb{Z}$. By applying the morphism SB, we conclude that $m = 0$, and, up to renumbering, that $\mathrm{SB}(\bar{X}_i) = \mathrm{SB}(\bar{Y}_i)$. Thus $\bar{X}_i, \bar{Y}_i$ are stably birational, for any i, $1 \leq i \leq n$. Let us fix i, $1 \leq i \leq n$. From a birational map $\bar{X}_i \times_k \mathbb{P}_k^s \dashrightarrow \bar{Y}_i \times_k \mathbb{P}_k^s$, we deduce, if $\bar{X}_i$ is rational (i.e., k-isomorphic to $\mathbb{P}_k^1$, because of the assumption on k), the existence of a dominant morphism of k-schemes $\mathbb{P}_k^1 \to \bar{Y}_i$. Thus, by Lüroth Problem, we conclude that $\bar{Y}_i$ is k-isomorphic to $\mathbb{P}_k^1$, and that X_i, Y_i are birationally equivalent. If X_i is not rational, the symmetry of the argument proves that Y_i is not rational. By [13, Theorem 2], there exists an isomorphism of k-schemes $\bar{X}_i \to \bar{Y}_i$. So, X_i, Y_i are again birationally equivalent. These remarks imply the existence of dense open subvarieties U of X and V of Y, and of a k-isomorphism $f_1 \colon U \to V$. The complements of U in X and of V in Y are disjoint sums of points. A (non canonical) k-isomorphism $f_0 \colon (X \setminus U)_{\mathrm{red}} \to (Y \setminus V)_{\mathrm{red}}$ exists if and only if the cardinals of the underlying sets of $(X \setminus U)_{\mathrm{red}}$ and $(Y \setminus V)_{\mathrm{red}}$ are equal. But this is the case thanks to the equality $\mathrm{Card}((X \setminus U)_{\mathrm{red}}) = \chi_c([(X \setminus U)_{\mathrm{red}}]) = \chi_c([(Y \setminus V)_{\mathrm{red}}]) = \mathrm{Card}((X \setminus U)_{\mathrm{red}})$.

If the dimension of X is zero, then X, Y are disjoint sums of points. We conclude in that case directly by computing Euler characteristics. $\qquad\square$

Remark 16. Assume that $\mathbb{L}$ is not a zero divisor in $K_0(\mathrm{Var}_k)$. Then it follows from the formula of Lemma 11 that

$$\sum_{i=1}^{n} \sum_{j=1}^{s_i} [C_{ij}][\mathbb{P}_k^{\alpha_j - 2}] = \sum_{i=1}^{n} \sum_{j=1}^{t_i} [D_{ij}][\mathbb{P}_k^{\beta_j - 2}].$$

If all the C_{ij}, D_{ij} are smooth irreducible projective k-curves or points, it follows from Proposition 15 that there exists a bijection σ from the set of the C_{ij} to the set of the D_{ij} such that, for any C_{ij}, the curves $C_{ij}, \sigma(C_{ij})$ are k-isomorphic. See §7 for a discussion around cancellation problems.

4. Jacobians

4.1. **Principally polarized abelian varieties.** Recall that a *polarization* on a complex torus V/Λ is a positive definite hermitian form $\mathcal{H}$ on V such that the imaginary part $\mathrm{Im}\,\mathcal{H}$ takes integral values on $\Lambda \times \Lambda$. Equivalently a polarization is given by an ample line bundle L on X; the first class Chern $c_1(L)$ then corresponds to $\mathrm{Im}\,\mathcal{H}$. A polarization is called *principal* if the Hermitian form is unimodular, or equivalently if $H^0(X, L) = \mathbb{C}$. The pair $(X, \mathcal{H})$, or equivalently (X, L), is called a *principally polarized abelian variety*. In this situation a divisor associated with a section of L is called a theta divisor and is denoted Θ. The datum of Θ is another way to specify a polarization on X. Note that such a divisor Θ is always reduced.

A morphism $f\colon (X_1, \mathcal{H}_1) \to (X_2, \mathcal{H}_2)$ between principally polarized abelian varieties is defined as a morphism of $\mathbb{C}$-varieties such that $\mathcal{H}_1 = f^*\mathcal{H}_2$. In particular such a morphism is always injective.

For any line bundle L on a principally polarized abelian variety (X, Θ) we can define a morphism $X \to Pic^0(X)$, $x \mapsto \tau_x^* L \otimes L^{-1}$, where τ_x is the translation by x. The kernel of this morphism is a disjoint union of subtori of X; we denote by $K(L)^0$ the irreducible component of the kernel that contains the identity, and by $\pi\colon X \to X/K(L)^0$ the quotient map. Then if $H^0(X, L) \neq 0$, there exists an ample line bundle Θ on $X/K(L)^0$ such that $\pi^*\Theta = L$ (see [4, Theorem VI.5.1]).

There is a natural notion of product for principally polarized abelian varieties (see [4, VI.6.1]): The product $(X_1, \Theta_1) \times (X_2, \Theta_2)$ is defined as $(X_1 \times X_2, \Theta)$ where $\Theta = p_1^*\Theta_1 + p_2^*\Theta_2$, and p_1, p_2 are the first and second projections on $X_1 \times X_2$. We say that a principally polarized abelian variety (X, Θ) is *indecomposable* if it cannot be written as such a (nontrivial) product. Then any principally polarized abelian variety (X, Θ) admits an essentially unique factorization into indecomposable factors, which is directly related to the irreducible components of Θ (see [4, VI.9] or [3, Lemma 3.20]):

Theorem 17. *Let (X, Θ) be a principally polarized abelian variety, and let $D_1, \cdots, D_s$ be the irreducible components of Θ. For each $k = 1, \ldots, s$, denote $X_k = X/K(D_k)^0$, $\pi_k\colon X \to X_k$ the projection and Θ_k a divisor on X_k such that $\pi_k^*\Theta_k = D_k$. Then*

$$(X, \Theta) = (X_1, \Theta_1) \times \cdots \times (X_s, \Theta_s)$$

is a factorization into indecomposable principally polarized abelian varieties. Furthermore, if $(X, \Theta) = \prod_{k=1,\ldots,s'}(Y_k, \Theta_k')$ is another such factorization, then $s = s'$ and up to a change of index we have (X_k, Θ_k) isomorphic to (Y_k, Θ_k') for all k.

4.2. **Intermediate Jacobians.** In this paragraph, we quickly overview the theory of intermediate Jacobians, following [3] and [18].

Let X be a smooth irreducible projective $\mathbb{C}$-variety. For $n \in \mathbb{N}$ such that $2n-1 \leq \dim X$, consider the $\mathbb{C}$-vector space

$$V_n = H^{n-1,n}(X) \times H^{n-2,n+1}(X) \times \cdots \times H^{0,2n-1}(X).$$

Then the *nth intermediate Jacobian* of X is defined as the complex torus:

$$J_n(X) = V_n/H^{2n-1}(X, \mathbb{Z}),$$

where $H^{2n-1}(X, \mathbb{Z})$ is identified to its image via the group morphism $\iota\colon H^{2n-1}(X, \mathbb{Z}) \to V_n$, which is obtained by composing the canonical morphism $H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(X, \mathbb{C})$ and the projection $H^{2n-1}(X, \mathbb{C}) \to V_n$. In general $J_n(X)$ has no reason to define an abelian variety.

First we specialize to the case where $\dim X = 3$ and $n = 2$. Consider the hermitian form defined on V_2 by

$$\mathcal{H}(\alpha, \beta) = i \int_X \alpha \wedge \overline{\beta}.$$

Recall that if $\omega \in H^{1,1}(X)$ is the Kähler form induced by the ambient Fubini-Study metric, the Lefschetz operator $L\colon H^k(X) \to H^{k+2}$ is defined by $L(\alpha) = \omega \wedge \alpha$, and

the primitive subspace $H^{p,q}_{\mathrm{prim}}(X)$ is defined as the kernel of the restriction of L^{4-p-q} to $H^{p,q}(X)$. One can show that V_2 admits a decomposition

$$V_2 = H^{1,2}_{\mathrm{prim}}(X) \times LH^{0,1}(X) \times H^{0,3}(X),$$

and that $\mathcal{H}$ is positive definite on $H^{1,2}_{\mathrm{prim}}(X)$ and negative definite on $LH^{0,1}(X) \times H^{0,3}(X)$. If we assume

$$H^{0,1}(X) = H^{0,3}(X) = 0,$$

we obtain a principal polarization on $J(X) = J_2(X)$ (see [18, Theorem 6.32] or [3, Lemma 3.4]). Remark that threefolds which are complete intersections gives examples of varieties satisfying $H^{0,1}(X) = H^{0,3}(X) = 0$.

If instead we specialize to the case where $\dim X = 1$ and $n = 1$, we recover the usual Jacobian $J(X) = J_1(X)$ of a curve X. In this case a canonical principal polarization comes from the hermitian metric defined on $H^{0,1}(X)$ by

$$\mathcal{H}(\alpha, \beta) = -i \int_X \alpha \wedge \overline{\beta}.$$

Recall the classical result of Torelli which says that we can recover a curve from its principally polarized Jacobian (see [6, p. 359]):

Theorem 18. *Let X, X' be two smooth irreducible projective $\mathbb{C}$-curves. If their Jacobians $(J(X), \mathcal{H})$ and $(J(X'), \mathcal{H}')$ are isomorphic as principally polarized varieties, then X is isomorphic to X'.*

Note also that for a curve X, a theta divisor on $J(X)$ is irreducible, because it is the image (up to translation) of the $(g-1)$-symmetric product of X under a natural morphism $X^{(g-1)} \to J(X)$ (see [6, p. 338]). In particular $J(X)$ is indecomposable by Theorem 17.

4.3. **Blow-ups.** For our purposes it is important to understand how the intermediate Jacobian of a threefold is affected under the blow-up of a smooth center. First, at the level of the cohomology, we have the following fact (this is a particular case of [18, Theorem 7.31]):

Proposition 19. *Let X be a smooth irreducible projective $\mathbb{C}$-variety of dimension 3, and let $Z \to X$ be the blow-up of a smooth irreducible center C.*

(1) If C is a point, then we have canonical isomorphisms of $\mathbb{C}$-vector spaces

$$H^{1,1}(Z) \cong H^{1,1}(X) \oplus \mathbb{C};$$
$$H^{p,q}(Z) \cong H^{p,q}(X) \text{ for } (p,q) \neq (1,1).$$

(2) If C is a curve, then we have canonical isomorphisms of $\mathbb{C}$-vector spaces

$$H^{1,1}(Z) \cong H^{1,1}(X) \oplus \mathbb{C};$$
$$H^{2,1}(Z) \cong H^{2,1}(X) \oplus H^{1,0}(C);$$
$$H^{1,2}(Z) \cong H^{1,2}(X) \oplus H^{0,1}(C);$$
$$H^{p,q}(Z) \cong H^{p,q}(X) \text{ for other } (p,q).$$

Note in particular that the condition $H^{0,3}(X) = H^{0,1}(X) = 0$ is preserved under such blow-ups. Now at the level of the Jacobians we have (see [3, Lemma 3.11]):

Proposition 20. *Let X be a smooth irreducible projective $\mathbb{C}$-variety of dimension 3, with $H^{0,3}(X) = H^{0,1}(X) = 0$, and let $Z \to X$ be the blow-up of a smooth irreducible center C.*

(1) If C is a point or a rational curve, then $J(Z) \cong J(X)$.

(2) If C is a non rational curve, then $J(Z) \cong J(X) \times J(C)$.

These are isomorphisms of principally polarized abelian varieties, where $J(X)$, $J(Z)$ and $J(C)$ are equipped with their canonical polarizations defined above.

4.4. **Consequences.** For our main result, Theorem 24, we are mainly interested in varieties of dimension 3 over $\mathbb{C}$. In this context, we have the following result.

Proposition 21. *Let X be a smooth irreducible projective $\mathbb{C}$-variety of dimension 3 whose Hodge numbers $h^{3,0}(X), h^{1,0}(X)$ both vanish. Let $f\colon X \dashrightarrow X$ be a birational map. Let*
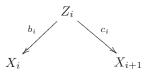

be a weak factorization of f. For each $1 \leq i \leq n$, let $\mathcal{C}_{b_i}$ (resp. $\mathcal{C}_{c_i}$) be the (finite) set of all the blow-ups π_C, with center a smooth irreducible non rational $\mathbb{C}$-curve C, appearing in the expression of the birational morphisms b_i (resp. c_i). Then there exists an isomorphism of principally polarized abelian varieties over $\mathbb{C}$

$$J(X) \times \left(\prod_{\pi_C \in \cup_{i=1}^n \mathcal{C}_{b_i}} J(C) \right) \simeq J(X) \times \left(\prod_{\pi_{C'} \in \cup_{i=1}^n \mathcal{C}_{c_i}} J(C') \right).$$

In particular, we have the following properties:

a) *The sets $\mathcal{C}_b := \cup_{i=1}^n \mathcal{C}_{b_i}$ and $\mathcal{C}_c := \cup_{i=1}^n \mathcal{C}_{c_i}$ have same cardinality;*

b) *There exists a bijection $\sigma\colon \mathcal{C}_b \to \mathcal{C}_c$ such that, for any π_C, $\pi_C \in \mathcal{C}_b$, the center C of π_C is isomorphic over k to the center of $\sigma(\pi_C)$.*

Proof. Let us fix i, $1 \leq i \leq n$, and consider the diagram of blow-ups with irreducible smooth centers

coming from the weak factorization of f. Recall that the morphisms b_i and c_i are finite sequences of blow-ups along irreducible smooth centers. By Proposition 20, we know that only blow-ups along non rational curves affect the intermediate Jacobians, and precisely we obtain a canonical isomorphism of principally polarized abelian varieties over $\mathbb{C}$, for each i, $1 \leq i \leq n$:

$$J(X_i) \times \left(\prod_{\pi_C \in \mathcal{C}_{b_i}} J(C) \right) \cong J(Z_i) \cong J(X_{i+1}) \times \left(\prod_{\pi_{C'} \in \mathcal{C}_{c_i}} J(C') \right).$$

By taking the product of respectively the left and right-hand sides of these n isomorphisms, and canceling the factors $J(X_2), \ldots, J(X_n)$ which appear once on both side

(this makes use of Theorem 17), we obtain the expected isomorphism of principally polarized abelian varieties over $\mathbb{C}$

$$(3) \qquad J(X) \times \left(\prod_{\pi_C \in \mathcal{C}_b} J(C) \right) \simeq J(X) \times \left(\prod_{\pi_{C'} \in \mathcal{C}_c} J(C') \right).$$

Then assertion a) follows from (3) and Theorem 17; and assertion b) follows from a) and Torelli's Theorem 18. $\qquad \square$

Corollary 22. *Let X be a smooth irreducible projective $\mathbb{C}$-variety of dimension 3 whose Hodge numbers $h^{3,0}(X), h^{1,0}(X)$ both vanish. Let $f \colon X \dashrightarrow X$ be a birational map. Let*

be a weak factorization of f. Let $\tilde{\mathcal{C}}_b$ (resp. $\tilde{\mathcal{C}}_c$) be the (finite) set of all of the blow-ups with center a point or a rational curve, appearing in the expression of any of the birational morphism b_i (resp. c_i), $1 \leq i \leq n$. Then there exists a bijection $\tilde{\mathcal{C}}_b \to \tilde{\mathcal{C}}_c$.

Proof. This follows from Proposition 21, which says that $\mathcal{C}_b$ and $\mathcal{C}_c$ have the same cardinality, and from Corollary 12, which says that $\mathcal{C}_b \cup \tilde{\mathcal{C}}_b$ and $\mathcal{C}_c \cup \tilde{\mathcal{C}}_c$ have the same cardinality. $\qquad \square$

Corollary 23. *Let X be a smooth irreducible projective $\mathbb{C}$-variety of dimension 3 whose Hodge numbers $h^{3,0}(X), h^{1,0}(X)$ both vanish. Let $f \colon X \dashrightarrow X$ be a birational map. Let*

be a weak factorization of f. Let C be a smooth irreducible projective curve C of genus $g(C) \geq 1$. Then the number of blow-ups, with center isomorphic to C, appearing in the b_i, $1 \leq i \leq n$, is equal to the number of blow-ups, with center isomorphic to C, appearing in the c_i, $1 \leq i \leq n$.

Proof. The result follows from Proposition 21, from the unicity of the factorization into indecomposable factors (Theorem 17) and from Torelli's Theorem 18. $\qquad \square$

5. The main theorem

5.1. **Preliminary results and remarks.** We assume that k is algebraically closed of characteristic zero. Let X be a k-variety, of dimension $d \geq 0$, and let $f \colon X \dashrightarrow X$ be a map that induces an isomorphism $U \cong V$ on dense open subvarieties U, V of X.

5.1.1. *Components of maximal dimension.* By Lemma 10, we see that the k-varieties $(X \setminus U)_{\mathrm{red}}$ and $(X \setminus V)_{\mathrm{red}}$ have automatically the same dimension, and the same number of irreducible components of maximal dimension.

If U is not the complement of the exceptional set of f, the variety $(X \setminus U)_{\mathrm{red}}$ could contain non-uniruled irreducible components of maximal dimension (see Lemma 4). But then, the set of non-uniruled irreducible components of maximal dimension of $(X \setminus U)_{\mathrm{red}}$ is automatically in bijection with that of of $(Y \setminus U)_{\mathrm{red}}$, by [13, Theorem 2]. In particular, if U_0 is the complement of the exceptional set of f, and $V_0 := f(U_0)$, then the set of all the irreducible components of maximal dimension in $U_0 \setminus U$ is in bijection with that of $V_0 \setminus V$.

5.1.2. *State of the art.* We gather here what was known about Conjecture 1 before this work.

a) Let X be a k-variety of dimension $d \geq 0$. Assume that X contains only finitely many rational curves. It follows from [13, Theorem 5] that Conjecture 1 holds true. (See also [9, §13].)

b) Assume that the dimension of $(X \setminus U)_{\mathrm{red}}$ is at most 1, then Conjecture 1 holds true for X in that case by Proposition 15 (see also [13, Proposition 6]).
    This remark implies in particular that Conjecture 1 holds true for any k-variety X of dimension 2.

c) Assume that X is projective, of dimension at least 3, that the projective k-varieties $(X \setminus U)_{\mathrm{red}}, (X \setminus V)_{\mathrm{red}}$ are smooth surfaces over k. Then Conjecture 1 holds true for X in that case by [13, Theorem 4].

d) Assume that the k-varieties $(X \setminus U)_{\mathrm{red}}, (X \setminus V)_{\mathrm{red}}$ have dimension 2, and that their irreducible components of maximal dimension are rational, then Conjecture 1 holds true for X in that case by [16]. One can prove directly this statement by Lemma 10 and Proposition 15.

e) Let X be a smooth irreducible proper k-variety with a birational map $f : X \dashrightarrow X$ which admits a weak factorization obtained by blowing-up smooth irreducible projective rational k-surfaces, smooth irreducible projective rational k-curves or points. Then, if the dimension of $(X \setminus U)_{\mathrm{red}}$ is at most 2, the k-varieties $(X \setminus U)_{\mathrm{red}}$ and $(X \setminus V)_{\mathrm{red}}$ are piecewise isomorphic. Indeed, the assumptions implies that each irreducible component of dimension 2 of $(X \setminus U)_{\mathrm{red}}$ or $(X \setminus V)_{\mathrm{red}}$ is rational. We conclude by Lemma 10 and Proposition 15.

5.2. **Proof of Main Theorem.** In this paragraph, we prove the following theorem which is the main result of this article.

Theorem 24. *Let X be a smooth irreducible projective $\mathbb{C}$-variety of dimension 3, whose Hodge numbers $h^{1,0}(X), h^{3,0}(X)$ both vanish. Let $f : X \dashrightarrow X$ be a birational map that induces an isomorphism $U \cong V$ on dense open subvarieties U, V of X. Then the $\mathbb{C}$-varieties $(X \setminus U)_{\mathrm{red}}, (X \setminus V)_{\mathrm{red}}$ of X are piecewise isomorphic.*

Proof. First we observe that it is sufficient to prove that the irreducible components of dimension 2 of $(X \setminus U)_{\mathrm{red}}$ and $(X \setminus V)_{\mathrm{red}}$ are mutually birationally equivalent. Indeed, if there exist an open dense subscheme U_1 of $(X \setminus U)_{\mathrm{red}}$, an open dense subscheme V_1 of $(X \setminus V)_{\mathrm{red}}$, and an isomorphism $f_2 : U_1 \to V_1$ of $\mathbb{C}$-varieties, then Proposition 15 produces a piecewise isomorphism (f_0, f_1) from $((X \setminus U) \setminus U_1)_{\mathrm{red}}$ to $((X \setminus V) \setminus V_1)_{\mathrm{red}}$. The datum of (f_0, f_1, f_2) is a piecewise isomorphism between

$(X \setminus U)_{\mathrm{red}}$ and $(X \setminus V)_{\mathrm{red}}$. From now, we will only prove that there exists a bijection σ from the set of all the irreducible components of dimension 2 of $(X \setminus U)_{\mathrm{red}}$ to those of $(X \setminus V)_{\mathrm{red}}$ such that, for any such a component F of $(X \setminus U)_{\mathrm{red}}$, the $\mathbb{C}$-varieties F and $\sigma(F)$ are birationally equivalent.

Now the proof proceeds from a precise analysis of the weak factorization of f, via Proposition 21. First, by §5.1.1 we can assume without loss in generality that U is the largest open subset of X on which the restriction of f is an isomorphism. In particular all components of dimension 2 in $(X \setminus U)_{\mathrm{red}}$ and $(X \setminus V)_{\mathrm{red}}$ are ruled.
Let

$$
\begin{array}{ccccccc}
 & Z_1 & & & & Z_n & \\
b_1 \swarrow & & \searrow c_1 & & b_n \swarrow & & \searrow c_n \\
X & & X_2 & \cdots & X_n & & X
\end{array}
$$

be a weak factorization of f. Let η be the generic point of an irreducible component of dimension 2 of $(X \setminus U)_{\mathrm{red}}$. Thanks to our hypothesis, we know that there exists a smooth irreducible projective curve C_η such that the residue field $\kappa(\eta)$ of X at η is isomorphic to $\kappa(C_\eta)(T)$, the quotient field of the ring of polynomials in T with coefficients in the functions field of C_η. We also know that such a point must be contracted by a morphism c_i, because of our assumption on U and the dimension of η. If we replace c_i by b_i, the generic points η' of the irreducible components of dimension 2 of $(X \setminus V)_{\mathrm{red}}$ verify the same properties.

So, it is sufficient to prove that for any smooth irreducible projective curve C of genus $g(C) \geq 1$, the number of generic points η in $(X \setminus U)_{\mathrm{red}}$ with residue field isomorphic to $\kappa(C)(T)$ is equal to the number of generic points η' in $(X \setminus V)_{\mathrm{red}}$ with $\kappa(\eta') \simeq \kappa(C)(T)$. Indeed, by Lemma 10, we conclude that the number of rational irreducible components in $(X \setminus U)_{\mathrm{red}}$ will coincide with that in $(X \setminus V)_{\mathrm{red}}$.

Let C be a smooth irreducible projective curve C of genus $g(C) \geq 1$. By Corollary 23, we know that in the weak factorization of f, the number of blow-ups with center C appearing in the b_i is equal to those appearing in the c_i.

We define now a numbering on the set of the blow-ups, with center C, that appear in the b_i as follows. In b_1, consider the first blow-up of this kind. If the strict transform of its exceptional divisor is the exceptional divisor of a blow-up appearing in a c_j, we denote it by $b(1)$, and by $c(1)$ the corresponding blow-up in c_j. If not, consider the second blow-up in b_1 and so on. When we have tested all the blow-ups of this kind in b_1, we apply the same procedure to b_2 and so on. Then, if it exists, we have found the first numbered blow-up $b(1)$, with center C. It determines ipso facto $c(1)$ that is also a blow-up with center C. At this step, we search for the second numbered blow-up $b(2)$ such that the strict transform of its exceptional divisor is k-isomorphic to the exceptional divisor of a blow-up appearing in a $c_{j'}$, distinct from the blow-up $c(1)$ by construction, and so on.

At the end, we have numbered all the blow-ups with center C appearing in the b_i, by $b(\ell)$, $1 \leq \ell \leq s$, except a finite number of them which are those whose strict transform are the irreducible components of $(X \setminus V)_{\mathrm{red}}$ birationally equivalent to

$C \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$. Then we can split the formula in Proposition 21 as follows

$$(4) \quad J(X) \times \left(\prod_{\tilde{C} \neq C} J(\tilde{C}) \right) \times \left(\prod_{1 \leq \ell \leq s} J(C) \right) \times \left(\prod_{\ell \in \Gamma} J(C) \right)$$

$$\simeq J(X) \times \left(\prod_{\tilde{C}' \neq C} J(\tilde{C}') \right) \times \left(\prod_{1 \leq \ell \leq s} J(C) \right) \times \left(\prod_{\ell' \in \Gamma'} J(C) \right)$$

where Γ is a set in bijection with the (finite) set of irreducible component in $(X \setminus V)_{\mathrm{red}}$, birationally equivalent to $C \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$, and where, by symmetry, Γ' is a (finite) set in bijection with the set of irreducible component in $(X \setminus U)_{\mathrm{red}}$, birationally equivalent to $C \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$. By unicity of the factorization of a principally polarized abelian variety (Theorem 17), we deduce from (4) that Γ is in bijection with Γ'. In other words the numbers of irreducible components birationally equivalent to $C \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ in $(X \setminus U)_{\mathrm{red}}$ and $(X \setminus V)_{\mathrm{red}}$ are equal. $\qquad\square$

Remark 25. The arguments of the proof of Theorem 24 also imply the following statement. Let X, Y be smooth projective $\mathbb{C}$-varieties of dimension 3 (possibly reducible), with $h^{3,0}(X) = h^{1,0}(X) = h^{3,0}(Y) = h^{1,0}(Y) = 0$, and such that there exists an isomorphism $J(X) \cong J(Y)$ of principally polarized abelian varieties over $\mathbb{C}$. If there exists a birational map $f \colon X \dashrightarrow Y$ that induces an isomorphism of $\mathbb{C}$-schemes $U \to V$ between dense open subvarieties U of X and V of Y, then the $\mathbb{C}$-varieties $(X \setminus U)_{\mathrm{red}}$ and $(Y \setminus V)_{\mathrm{red}}$ are piecewise isomorphic.

Corollary 26. *Let X be a smooth irreducible projective $\mathbb{C}$-variety, of dimension 3, whose Hodge numbers $h^{3,0}(X), h^{1,0}(X)$ both vanish. Let $f \colon X \dashrightarrow X$ be a birational map that induces an isomorphism of $\mathbb{C}$-schemes $U \cong V$ on dense open subvarieties U, V of X. Let A be a ring, and $\mu \colon K_0(\mathrm{Var}_k) \to A$ be a motivic measure. Then*

$$\mu([(X \setminus U)_{\mathrm{red}}]) = \mu([(X \setminus V)_{\mathrm{red}}]).$$

In particular, the projective $\mathbb{C}$-varieties $(X \setminus U)_{\mathrm{red}}, (X \setminus V)_{\mathrm{red}}$ have the same Hodge–Deligne polynomial, the same Poincaré polynomial, the same (topological) Euler characteristic, and $\mathrm{SB}([(X \setminus U)_{\mathrm{red}}]) = \mathrm{SB}([(X \setminus V)_{\mathrm{red}}])$.

Proof. By Theorem 24, $[(X \setminus U)_{\mathrm{red}}] = [(X \setminus V)_{\mathrm{red}}]$ in $K_0(Var_{\mathbb{C}})$. So, by the definition of a motivic measure, $\mu([(X \setminus U)_{\mathrm{red}}]) = \mu([(X \setminus V)_{\mathrm{red}}])$. Recall that the Hodge–Deligne realization morphism, the Poincaré realization morphism, the (topological) Euler characteristic, the morphism SB are examples of motivic measure (see §3.2). $\qquad\square$

6. Examples

One particularly rich source of examples for Theorem 24 is when we consider X equal to the projective space. We give below three explicit examples in this setting. The first one is in dimension 2, and illustrates the fact that the exceptional loci of a transformation and its inverse might be quite far from being isomorphic, despite the fact that they are piecewise isomorphic.

The two last examples are in dimension 3, and as such are direct illustrations of Theorem 24.

Example 27 (see [2, Example 2.1.14]). Consider the plane Cremona map $f\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ defined by the datum of the three following homogeneous polynomials, expressed in homogeneous coordinates x, y, z:

$$F = (x^3 - yz(x+y))(x^2 - yz)(x+y),$$
$$G = x^2(x^2 - yz)(x+y)^2,$$
$$H = x^3(x^3 - yz(x+y)).$$

The determinant of the Jacobian matrix $\begin{pmatrix} \partial_x F & \partial_y F & \partial_z F \\ \partial_x G & \partial_y G & \partial_z G \\ \partial_x H & \partial_y H & \partial_z H \end{pmatrix}$ is equal to

$$\mathrm{Jac}(f) = -6x^3 y^2 z^2 (x+y)^3 (x+z)^3 (y-z)^2.$$

We deduce the (reduced) equation of the contracted curves:

$$xyz(x+y)(x+z)(y-z) = 0.$$

We see that the exceptional locus is equal to the disjoint union of

- Seven points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$, $[-1:1:0]$, $[-1:1:1]$, $[0:1:1]$, $[-1:0:1]$;
- Six curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ minus three points.

The inverse of f has the form

$f^{-1}\colon [x:y:z] \dashrightarrow$
$$[-xyz(x+y)(x+z)(y-z) : -x^2 y(x+z)(y-z)^2 : z^2(y+x)^2(x^2 - yz)]$$

with Jacobian determinant

$$\mathrm{Jac}(f^{-1}) = 6x^4 y(x+y)^2(x^2 - yz)(x^3 - y^2 z - xyz)(x^2 y + 2x^3 - y^2 z - xyz).$$

So the exceptional locus of f^{-1} has six irreducible components described as follows with their reduced structures: three lines, a conic and two nodal cubics. We can write this locus as the disjoint union of

- Three points $[0:0:1]$, $[0:1:0]$, and $[-1:1:1]$;
- One curve isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ minus one point (line $y = 0$), two curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ minus two points (corresponding to the two other lines), and three other curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ minus three points (corresponding to the conic and the nodal cubics).

We see that we can realize a piecewise isomorphism between the exceptional locus of f and f^{-1} by choosing an arbitrary additional point on each one of the lines $x = 0$ and $x + y = 0$, and two arbitrary additional points on the line $y = 0$.

Example 28. We give now an example in dimension 3. Consider a general transformation of $\mathbb{P}^3_{\mathbb{C}}$ of bidegree $(2, 4)$, meaning that f has degree 2 and f^{-1} has degree 4: see [15], where this transformation is denoted by $\mathrm{gen}^{[4]}$. One representant (up to left and right composition by linear automorphisms) is for instance

$$f\colon [x:y:z:t] \dashrightarrow [yz : xz : xy : t(x+y+z)]$$

with inverse

$$f^{-1}\colon [x:y:z:t] \dashrightarrow [(yz+xz+xy)yz : (yz+xz+xy)xz : (yz+xz+xy)xy : xyzt].$$

We compute the Jacobian determinants of f and f^{-1}:

$$\mathrm{Jac}(f) = xyz(x + y + z);$$
$$\mathrm{Jac}(f^{-1}) = x^2 y^2 z^2 (yz + xz + xy)^3.$$

The exceptional set of f is equal to the disjoint union of:

- The point $p = [0 : 0 : 0 : 1]$;
- Six curves isomorphic to $\mathbb{P}^1_{\mathbb{C}} \setminus \{p\}$;
- Four surfaces isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ minus three lines with a common point.

On the other hand the exceptional set of f^{-1} is equal to the disjoint union of:

- The point $p = [0 : 0 : 0 : 1]$;
- Three curves isomorphic to $\mathbb{P}^1_{\mathbb{C}} \setminus \{p\}$;
- Three surfaces isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ minus two lines;
- A surface isomorphic to a quadratic cone minus three rules, which is also isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ minus three lines with a common point.

By choosing a general line passing through p in each plane $x = 0$, $y = 0$, $z = 0$, we can recover a stratification of $\mathrm{Exc}(f^{-1})$ which realizes a piecewise isomorphism with $\mathrm{Exc}(f)$.

Example 29. The following example is probably the simplest birational self-map g of $\mathbb{P}^3_{\mathbb{C}}$ such that

(a) $\mathrm{Exc}(g)$ and $\mathrm{Exc}(g^{-1})$ are not isomorphic;
(b) $\mathrm{Exc}(g)$ contains an irreducible component which is not a rational surface.

Consider the smooth plane cubic $C \subset \mathbb{P}^2_{\mathbb{C}}$ with homogeneous equation

$$q_3(x, y, z) := y^2 z - x^3 + xz^2 = 0.$$

Then

$$f : (x, y, z) \dashrightarrow \left(\frac{xq_3}{x^3}, \frac{yq_3}{x^3}, \frac{zq_3}{x^3} \right)$$

is a birational transformation of $\mathbb{P}^3_{\mathbb{C}}$ (expressed in an affine chart $\mathbb{C}^3$), with inverse

$$f^{-1} : (x, y, z) \dashrightarrow \left(\frac{x^4}{q_3}, \frac{yx^3}{q_3}, \frac{zx^3}{q_3} \right).$$

Note that we could construct a similar example starting from any irreducible plane curve C of arbitrary degree (see [14]). The associated transformation contracts a surface birational to $C \times \mathbb{P}^1_{\mathbb{C}}$: We see that transformations contracting only rational surfaces are far from being the rule.

The transformations f and f^{-1} have same exceptional loci, so we now compose f with the cubo-cubic involution

$$\sigma : (x, y, z) \dashrightarrow \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right)$$

to get a more interesting example. We set $g = \sigma \circ f$, so $g^{-1} = f^{-1} \circ \sigma$, and explicitly

$$g : (x, y, z) \dashrightarrow \left(\frac{x^3}{xq_3}, \frac{x^3}{yq_3}, \frac{x^3}{zq_3} \right),$$

$$g^{-1} : (x, y, z) \dashrightarrow \left(\frac{y^2 z^2}{xq_4}, \frac{xyz^2}{xq_4}, \frac{xy^2 z}{xq_4} \right),$$

where $q_4(x, y, z) := x^3 z - y^2 z^2 + x^2 y^2$ satisfies $q_3 \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) = \frac{q_4(x, y, z)}{x^3 y^2 z^2}$.

	$\mathrm{Exc}(g)$	$\mathrm{Exc}(g^{-1})$
Points	•6 points $p_0, \dots, p_5$	•4 points $p_0, \dots, p_3$
Curves	•7 components $\cong \mathbb{C}^*$: lines $(p_0, p_i), i = 1, \dots 5$, and $(p_1, p_2), (p_2, p_3)$. •1 component $\cong \mathbb{P}^1_{\mathbb{C}} \setminus 4$ points : line (p_1, p_3) minus $\{p_1, p_3, p_4, p_5\}$ •1 component $\cong C \setminus 4$ points : $q_3 = 0$ minus $\{p_2, p_3, p_4, p_5\}$	•6 components $\cong \mathbb{C}^*$: Lines $(p_0, p_i), i = 1, \dots 3$, and $(p_1, p_2), (p_2, p_3), (p_1, p_3)$. •1 component $\cong C \setminus 4$ points : $q_4 = 0$ minus $\{p_1, p_2, p_3\}$
Surfaces	•2 components $\cong \mathbb{A}^2_{\mathbb{C}}$ minus 2 lines : come from $x = 0, z = 0$. •1 component $\cong \mathbb{A}^2_{\mathbb{C}}$ minus 4 lines : comes from $y = 0$. •1 component $\cong (\mathbb{C}^*)^2 \setminus \{q_3 = 0\}$: comes from $t = 0$. •Cone over $C \setminus \{4$ points$\}$	•3 components $\cong \mathbb{A}^2_{\mathbb{C}}$ minus 2 lines : come from $x = 0, y = 0, z = 0$. •1 component $\cong (\mathbb{C}^*)^2 \setminus \{q_4 = 0\}$: comes from $t = 0$. •Cone over $C' \setminus \{3$ points$\}$

TABLE 1. Stratification of exceptional loci in Example 29.

In homogeneous coordinates we have

$$g \colon [x : y : z : t] \dashrightarrow [x^2yzt : x^3zt : x^3yt : yzq_3],$$
$$g^{-1} \colon [x : y : z : t] \dashrightarrow [y^2z^2t : xyz^2t : xy^2zt : xq_4].$$

We compute the Jacobian determinants:

$$\mathrm{Jac}(g) = -5x^7y^2z^2t^2q_3;$$
$$\mathrm{Jac}(g^{-1}) = -5x^2y^4z^4t^2q_4.$$

The exceptional locus of g is the union of the 4 coordinates planes and of the cone over the smooth elliptic curve $\{t = q_3(x, y, z) = 0\}$, isomorphic to C. Similarly, the exceptional locus of g is the union of the 4 coordinates planes and of the cone over the singular quartic curve $C' = \{t = q_4(x, y, z) = 0\}$.

We set

$$p_0 = [0 : 0 : 0 : 1], \qquad p_1 = [0 : 0 : 1 : 0],$$
$$p_2 = [0 : 1 : 0 : 0], \qquad p_3 = [1 : 0 : 0 : 0],$$
$$p_4 = [1 : 0 : 1 : 0], \qquad p_5 = [-1 : 0 : 1 : 0],$$

and we note that in the plane $t = 0$, the singular locus of the curve $xyzq_3 = 0$ is equal to the five points $\{p_1, \dots, p_5\}$, and the singular locus of $xyzq_4 = 0$ is equal to the three points $\{p_1, p_2, p_3\}$.

Observe also that there exists a birational morphism $\pi \colon C \to C'$, since C is isomorphic to the desingularisation of C', and π induces an isomorphism between $C' \setminus \{p_1, p_2, p_3\}$ and C minus four points. Indeed p_2 is an ordinary double point for C', hence $\pi^{-1}(p_2)$ is equal to two points, and p_1, p_3 are respectively a smooth and a cuspidal double point for C' hence their preimage by π both consist of one single point.

We have natural stratifications of the exceptional loci, as detailed in Table 1.

Note that by construction the quadratic involution $[x : y : z] \dashrightarrow [yz : xz : xy]$ realizes an isomorphism between $(\mathbb{C}^*)^2 \setminus \{q_3 = 0\}$ and $(\mathbb{C}^*)^2 \setminus \{q_4 = 0\}$. We can realize a piecewise isomorphism by picking two well-chosen new rules in $\{y = 0\} \subset \mathrm{Exc}(g^{-1})$, and at most four well-chosen new rules in both cones over the elliptic curve in $\mathrm{Exc}(g)$ and $\mathrm{Exc}(g^{-1})$.

7. FURTHER COMMENTS

In this section, we assume for simplicity that k is an algebraically closed field of characteristic zero. As we have shown in the introduction, Conjecture 1 is linked to a stronger problem: Question 2 (see [11, Question 1.2]).

The strategy of resolution of Question 2 in [13] is stopped by different problems of cancellation (geometric and motivic). Besides, some preexisting geometric problems of cancellation can be formulated, or interpreted via Question 2, in the Grothendieck ring of varieties.

We list here some problems related to these cancellation questions.

7.1. **Zero divisor.** Is $\mathbb{L}$ a zero divisor in $K_0(\mathrm{Var}_k)$? That question corresponds to a "piecewise" cancellation problem. Let $\mathfrak{a} \in K_0(\mathrm{Var}_k)$ such that $\mathbb{L} \cdot \mathfrak{a} = 0$. One can represent the element $\mathfrak{a}$ as a formal some of classes of the following type

$$\mathfrak{a} = \sum_{i=1}^{m}[X_i] - \sum_{i=1}^{n}[Y_i],$$

where all the X_i, Y_i are smooth irreducible projective k-varieties. Now, if $\mathbb{L}$ is not a zero divisor in $K_0(\mathrm{Var}_k)$ then $\mathfrak{a} = 0$. Thus we are studying the validity of the following formula

$$\mathbb{L} \cdot ([\sqcup_{i=1}^m X_i]) = \mathbb{L} \cdot ([\sqcup_{i=1}^n Y_i]) \;\Rightarrow\; [\sqcup_{i=1}^m X_i] = [\sqcup_{i=1}^n Y_i].$$

Note that, if $\mathbb{L}$ is a regular element of $K_0(\mathrm{Var}_k)$, we deduce formally that Question 2 admits a positive answer for any pair of k-varieties of dimension 2 (see [16]). In that case, Conjecture 1 would be true for any k-varieties of dimension 3.

7.2. **Rationality.** Is it possible to characterize rationality in $K_0(\mathrm{Var}_k)$? That question admits a positive answer in dimension at most 2, by [13, Proposition 7,Lemma 12]. In particular, is it possible that a stably rational (or unirational, i.e., dominated by $\mathbb{P}_k^{\dim(X)}$), non-rational, smooth proper k-variety X, of dimension at least 3, has the same class in $K_0(\mathrm{Var}_k)$ than a rational smooth proper k-variety (of the same dimension)?

7.3. **Cancellation.** In affine geometry, Zariski's Cancellation Problem can be formulated as follows. If X, Y is a pair of affine k-varieties such that $X \times_k \mathbb{A}_k^1, Y \times_k \mathbb{A}_k^1$ are k-isomorphic, is it true that X, Y are k-isomorphic? One knows that some further assumptions are necessary to hope for a positive answer: indeed there exist counterexamples (e.g., Danielewski's counterexample) formed by a couple X, Y of non-isomorphic affine k-varieties such that $X \times_k \mathbb{A}_k^1, Y \times_k \mathbb{A}_k^1$ are k-isomorphic.

We can formulate the question in $K_0(\mathrm{Var}_k)$. We are interested in couples of affine k-varieties X, Y such that $\mathbb{L}[X] = \mathbb{L}[Y]$. If $\mathbb{L}$ is a regular element of $K_0(\mathrm{Var}_k)$, we conclude that any such a couple verifies $[X] = [Y]$. If Question 2 admits a positive answer, then X, Y must be piecewise isomorphic. Up to our knowledge, and up to

now, all the counterexamples to Zariski's Cancellation Problem (in affine geometry) do not contradict this conclusion. Moreover, it follows from [13, Theorem 2, Proposition 6], that any counterexample which does not verify that piecewise condition must be of dimension at most 3.

So it seems to us interesting to study the following question (see [16]). Does there exist a couple of integral affine k-varieties X, Y, of dimension $d \geq 3$, such that $X \times_k \mathbb{A}_k^1, Y \times_k \mathbb{A}_k^1$ are k-isomorphic, and X, Y not birationally equivalent?

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