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Abstract This paper is devoted to the analysis of non-negative solutions for a degenerate parabolic-elliptic Patlak-Keller-Segel system with critical nonlinear diffusion in a bounded domain with homogeneous Neumann boundary conditions. Our aim is to prove the existence of a global weak solution under a smallness condition on the mass of the initial data, there by completing previous results on finite blow-up for large masses. Under some higher regularity condition on solutions, the uniqueness of solutions is proved by using a classical duality technique.

Keywords: Chemotaxis; Keller-Segel model; Parabolic equation; Elliptic equation; Global existence; Uniqueness.

1 Introduction

Chemotaxis is the movement of biological organisms oriented towards the gradient of some substance, called the chemoattractant. The Patlak-Keller-Segel (PKS) model (see [13], [12] and [17]) has been introduced in order to explain chemotaxis cell aggregation by means of a coupled system of two equations: a drift-diffusion type equation for the cell density u, and a reaction diffusion equation for the chemoattractant concentration φ. It reads

$$(PKS)\begin{cases} \partial_t u = \operatorname{div}(\nabla u^m - u\cdot\nabla\varphi) & x\in\Omega, t>0,\\ -\Delta\varphi = u - <u> & x\in\Omega, t>0,\\ <\varphi(t)> = 0 & t>0,\\ \partial_\nu u = \partial_\nu\varphi = 0 & x\in\partial\Omega, t>0,\\ u(0,x) = u_0(x) & x\in\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain, ν the outward unit normal vector to the boundary $\partial\Omega$ and $m\geq 1$. An important parameter in this model is the total mass M of cells, which is formally conserved through the evolution:

$$M = <u> = \frac{1}{|\Omega|}\int_\Omega u(t,x)\,dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx. \tag{2}$$

Several studies have revealed that the dynamics of (1) depend sensitively on the parameters N, m and M. More precisely, if $N=2$ and $m=1$, it is well-known that the solutions of (1) may blow up in finite time if M is sufficiently large (see [17, 16]) while solutions are global in time for M sufficiently small [17], see also the survey articles [4, 10].

The situation is very different when $m=1$ and $N\neq 2$. In fact, if $N=1$, there is global existence of solutions of (1) whatever the value of the mass of initial data M, see [8] and the references therein. If $N\geq 3$, for all $M>0$, there are initial data u_0 with mass M for which the corresponding solutions of (1) explode in finite time (see [16]). Thus, in dimension $N \geq 3$ and $m = 1$, the threshold phenomenon does not take place as in dimension 2, but we expect the same phenomenon when $N \geq 3$ and m is equal to the *critical* value $m = m_c = \frac{2(N-1)}{N}$. More precisely, we consider a more general version of (1) where the first equation of (1) is replaced by

$$\partial_t u = \operatorname{div}(\phi(u)\ \nabla u - u\ \nabla\varphi), \quad t > 0, \quad x \in \Omega,$$

and the diffusivity ϕ is a positive function in $C^1([0,\infty[)$ which does not grow to fast at infinity. In [8], the authors proved that there is a critical exponent such that, if the diffusion has a faster growth than the one given by this exponent, solutions to (1) (with $\phi(u)$ instead of mu^{m-1}) exist globally and are uniformly bounded, see also [6, 14] for $N = 2$. More precisely, the main results in [8] read as follows:

- If $\phi(u) \geq c(1+u)^p$ for all $u \geq 0$ and some $c > 0$ and $p > 1 - \frac{2}{N}$ then all solutions of (1) are global and bounded.
- If $\phi(u) \leq c(1+u)^p$ for all $u \geq 0$ and some $c > 0$ and $p < 1 - \frac{2}{N}$ then there exist initial data u_0 such that

$$\lim_{t\to T} ||u(.,t)||_\infty = \infty, \text{ for some finite T} > 0.$$

Except for $N = 2$, the critical case $m = \frac{2(N-1)}{N}$ is not covered by the analysis of [8]. Recently, Cieślak and Laurençot in [7] show that if $\phi(u) \leq c(1+u)^{1-\frac{2}{N}}$ and $N \geq 3$, there are solutions of (1) blowing up in finite time when M exceeds an explicit threshold. In order to prove that, when $N \geq 3$ and $m = \frac{2\ (N-1)}{N}$, we have a threshold phenomenon similar to dimension $N = 2$ with $m = 1$, it remains to show that solutions of (1) are global when M is small enough. The goal of this paper is to show that this is indeed true, see Theorem 2.2 below.

By combining Theorem 2.2 with the blow-up result obtained in [7], we conclude that, for $N \geq 3$ and $m = \frac{2(N-1)}{N}$, there exists $0 < M_1 \leq M_2 < \infty$ such that the solutions of (1) are global if the mass M of the initial data u_0 is in $[0, M_1)$, and may explode in finite time if $M > M_2$. An important open question is whether $M_1 = M_2$ when Ω is a ball in $\mathbb{R}^N$ and u_0 is a radially symmetric function. Notice that, in the radial case, this result is true when $N = 2$ and $m = 1$, and the threshold value of the mass for blow-up is $M_1 = M_2 = 8\pi$, see [6, 16, 15, 18]. Again, for $N = 2$ and $m = 1$, but for regular, connected and bounded domain, it has been shown that $M_1 = 4\pi = \frac{M_2}{2}$ (see [15, 16] and the references therein). Such a result does not seem to be known for $N \geq 3$ and $m = \frac{2(N-1)}{N}$.

Still, in the whole space $\Omega = \mathbb{R}^N$ when the equation for φ in (1) is replaced by the Poisson equation $\varphi = E_N * u$, with E_N being the Poisson kernel, it has been shown in [9, 5, 2, 20, 21, 3] that:

- When $N \geq 3$ and $1 \leq m < 2 - \frac{2}{N}$, this modified version of (1) has a global weak solution if $M = \|u_0\|_1$ is sufficiently small, while finite time blow-up occurs for some initial data with sufficiently large mass.
- When $N \geq 2$ and $m > 2 - \frac{2}{N}$, this modified version of (1) has a global weak solution whatever the value of M.

- When $N \geq 2$ and $m = 2 - \frac{2}{N}$, there is a threshold mass $M_c > 0$ such that solutions to this modified version of (1) exist globally if $M = \|u_0\|_1 \leq M_c$, and might blow up in finite time if $M > M_c$.

From now on, we assume that

$$N \geq 3 \quad \text{and} \quad m = \frac{2(N-1)}{N}.$$

2 Main Theorem

Throughout this paper , we deal with weak solutions of (1). Our definition of weak solutions now reads:

Definition 2.1. *Let $T \in (0;\infty]$. A pair (u, φ) of functions $u : \Omega \times [0,T) \longrightarrow [0,\infty)$, $\varphi : \Omega \times [0,T) \longrightarrow \mathbb{R}$ is called a weak solution of (1) in $\Omega \times [0,T)$ if*

- $u \in L^\infty((0,T); L^\infty(\Omega))$; $u^m \in L^2((0,T); H^1(\Omega))$ *and* $< u >= M$.
- $\varphi \in L^2((0,T); H^1(\Omega))$ *and* $< \varphi >= 0$.
- (u, φ) *satisfies the equation in the sense of distributions ; i.e,*

$$-\int_0^T \int_\Omega (\nabla u^m \cdot \nabla\psi - u\nabla\varphi \cdot \nabla\psi - u\ \partial_t\psi)\ \mathrm{d}x\mathrm{d}t = \int_\Omega u_0(x)\ \psi(0,x)\ \mathrm{d}x,$$

$$\int_0^T \int_\Omega \nabla\varphi \cdot \nabla\psi\ dxdt = \int_0^T \int_\Omega (u - M)\ \psi\ \mathrm{d}x\mathrm{d}t,$$

for any continuously differentiable function $\psi \in C^1([0,T] \times \overline{\Omega})$ with $\psi(T) = 0$ and $T > 0$.

For $\varphi \in H^1(\Omega)$ satisfying $< \varphi >= 0$, we denote by C_s the Sobolev constant where

$$||\nabla\varphi||_2 \geq C_s ||\varphi||_{2^*}, \quad \text{where } 2^* = \frac{2N}{N-2}. \tag{3}$$

The main theorem gives the existence and uniqueness of a time global weak solution to (1) which corresponds to a degenerate version of the "Nagai model" for the semi-linear Keller-Segel system, when $u_0 \in L^\infty(\Omega)$ and the initial data is assumed to be small.

Theorem 2.2. *Define*

$$M_* := \left(\frac{2\ C_s^2}{(m-1)\ |\Omega|^{\frac{2}{N}}} \right)^{\frac{N}{2}}, \tag{4}$$

where C_s is the Sobolev constant in (3).
Assume that u_0 is nonnegative function in $L^\infty(\Omega)$, which satisfies

$$||u_0||_1 < M_*. \tag{5}$$

Then the equation (1) has a global weak solution (u, φ) in the sense of Definition 2.1. Moreover, if we assume that

$$\varphi \in L^\infty((0,T); W^{2,\infty}(\Omega)) \tag{6}$$

for all $T > 0$ then this solution is unique.

In order to prove the previous theorem, we introduce the following approximated equations

$$
(KS)_\delta \left\{ \begin{array}{rcll} \partial_t u_\delta & = & \operatorname{div}\left(\nabla(u_\delta+\delta)^m - u_\delta \nabla\varphi_\delta\right) & x\in\Omega, t>0, \\ -\Delta\varphi_\delta & = & u_\delta - < u_\delta > & x\in\Omega, t>0, \\ \partial_\nu u_\delta = \partial_\nu\varphi_\delta & = & 0 & x\in\partial\Omega, t>0, \\ u_\delta(0,x) & = & u_0(x) & x\in\Omega, \end{array} \right.
$$

where $\delta \in (0,1)$, and we show that under a smallness condition on the mass of initial data, the Liapunov function

$$
L_\delta(u,\varphi) = \int_\Omega (b_\delta(u) + \frac{1}{2}|\nabla\varphi_\delta|^2 - u_\delta\ \varphi_\delta)\ \mathrm{d}x,
$$

yields the L^m bound of $u_\delta(t)$ independent of δ. Then using Gagliardo-Nirenberg and Poincaré inequalities, we obtain for $p > m$, the L^p bound for $u_\delta(t)$ independent of δ. As a consequence of Sobolev embedding theorem, we improve the regularity of φ_δ. And thus, under the same assumptions on the initial data, Moser's iteration technique yields the uniform bound of u_δ. Then, thanks to the local well-posedness result [8, Theorem 3.1] we obtain the existence of a global solution of $(KS)_\delta$. The existence of solutions stated in Theorem 2.2 is then proved using a compactness method; for that purpose we show an additional estimate on $\partial_t u_\delta^m$ which, together with the already derived estimates, guarantees the compactness in space and time of the family $(u_\delta)_{\delta\in(0,1)}$. Finally, in the presence of nonlinear diffusion and under some additional regularity assumption on φ_δ, we prove the uniqueness using a classical duality technique.

3 Approximated Equations

The first equation of (1) is a quasilinear parabolic equation of degenerate type. Therefore, we cannot expect the system (1) to have a classical solution at the point where u vanishes. In order to prove Theorem 2.2, we use a compactness method and introduce the following approximated equations of (KS):

$$
(KS)_\delta \left\{ \begin{array}{rcll} \partial_t u_\delta & = & \operatorname{div}\left(\nabla(u_\delta+\delta)^m - u_\delta \nabla\varphi_\delta\right) & x\in\Omega, t>0, \\ -\Delta\varphi_\delta & = & u_\delta - < u_\delta > & x\in\Omega, t>0, \\ \partial_\nu u_\delta = \partial_\nu\varphi_\delta & = & 0 & x\in\partial\Omega, t>0, \\ u_\delta(0,x) & = & u_0(x) & x\in\Omega, \end{array} \right. \tag{7}
$$

where $\delta \in (0,1)$.
The main purpose of this section is to construct the time global strong solution of (7).

3.1 Existence of global strong solution of $(KS)_\delta$

Theorem 3.1. *For $\delta \in (0,1)$ and $T > 0$, we consider an initial condition $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and such that $||u_0||_1 < M_*$ where M_* is defined in (4). Then $(KS)_\delta$ has a global strong solution $(u_\delta, \varphi_\delta)$ which is bounded in $L^\infty((0,T)\times\Omega)$ for all $T > 0$ uniformly with respect to $\delta \in (0,1)$.*

The starting point of the proof of Theorem 3.1 is the following local well-posedness result [8, Theorem 1.3]:

Lemma 3.2. *Let the same assumptions as that in Theorem 3.1 hold. There exists a maximal existence time $T^\delta_{max} \in (0,\infty]$ and a unique solution $(u_\delta,\varphi_\delta)$ of $(KS)_\delta$ in $[0,T^\delta_{max})\times\Omega$. Moreover,*

$$\text{if } T^\delta_{max} < \infty \text{ then } \lim_{t\to T^\delta_{max}} ||u_\delta(t,.)||_\infty = \infty.$$

In addition $< u_\delta(t) >=< u_0 >= M$ for all $t \in [0, T^\delta_{\max})$.

To prove Theorem 3.1 we need to prove some lemmas which control L^m norm, L^p norm and L^∞ norm of the solution u_δ of (7).

3.2 L^p -estimates, $1 \le p \le \infty$.

Our goal is to show that if $||u_0||_1$ is small enough then all solutions are global in time and uniformly bounded.

Let us first prove the L^m bound for u_δ.

Lemma 3.3. *Let the same assumptions as that in Theorem 3.1 hold and $(u_\delta,\varphi_\delta)$ be the nonnegative maximal solution of $(KS)_\delta$. Then, u_δ satisfies the following estimate*

$$||u_\delta(t)||_m \le C_0, \text{ for all } t\in[0,T^\delta_{\max})$$

and $||u_\delta(t)||_1 = ||u_0||_1$ where C_0 is a constant independent of T^δ_{max} and δ.

Proof. In this proof, the solution to equation (7) should be denoted by $(u_\delta,\varphi_\delta)$ but for simplicity we drop the index.
Let us define the functional L_δ by

$$L_\delta(u,\varphi) = \int_\Omega (b_\delta(u) + \frac{1}{2}|\nabla\varphi|^2 - u\ \varphi)\ \mathrm{d}x,$$

where

$$b_\delta(u) := \int_1^u \int_1^z \frac{m(\sigma+\delta)^{m-1}}{\sigma}\ d\sigma\ dz,$$

such that $b_\delta(1) = b'_\delta(1) = 0$ and $b(u) \ge 0$. According to [11] it is a Liapunov functional for $(KS)_\delta$. Indeed,

$$\begin{aligned}
\frac{d}{dt}L_\delta(u(t),\varphi(t)) &= \int_\Omega b'_\delta(u)\ \partial_t u\ dx - \int_\Omega \Delta\varphi\ \partial_t\varphi\ dx - \int_\Omega \partial_t u\ \varphi\ dx - \int_\Omega u\ \partial_t\varphi\ dx \\
&= \int_\Omega \partial_t u\ (b'_\delta(u) - \varphi)\ dx - \int_\Omega (\Delta\varphi + u)\ \partial_t\varphi\ dx \\
&= \int_\Omega \operatorname{div}\big(m\ (u+\delta)^{m-1}\ \nabla u - u\ \nabla\varphi\big)\ (b'_\delta(u) - \varphi)\ dx - \int_\Omega < u(t) >\ \partial_t\varphi\ dx \\
&= -\int_\Omega (m\ (u+\delta)^{m-1}\ \nabla u - u\ \nabla\varphi)\ (b''_\delta(u)\ \nabla u - \nabla\varphi)\ dx - M\ \frac{d}{dt}\int_\Omega \varphi\ dx \\
&= -\int_\Omega u\ (b''_\delta(u)\ \nabla u - \nabla\varphi)^2\ dx \\
&\le 0.
\end{aligned}$$

Then, we can conclude that for all $t \in [0, T^\delta_{max})$ we have $L_\delta(u(t),\varphi(t)) \le L_\delta(u_0,\varphi_0)$. Using Sobolev inequality (3), Hölder inequality, and Young inequality we obtain

$$\int_\Omega u\ \varphi\ dx \le ||\varphi||_{2^*}\ ||u||_{\frac{2N}{N+2}} \le C_s^{-1}||\nabla\varphi||_2\ ||u||_{\frac{2N}{N+2}} \le \frac{1}{2}||\nabla\varphi||_2^2 + \frac{C_s^{-2}}{2}||u||^2_{\frac{2N}{N+2}}.$$

Since $\frac{2\,N}{N+2} < m$, and using interpolation inequality we get,

$$||u||_{\frac{2N}{N+2}} \leq ||u||_1^{\frac{1}{N}} \ ||u||_m^{\frac{N-1}{N}} \leq M^{\frac{1}{N}} \ |\Omega|^{\frac{1}{N}} \ ||u||_m^{\frac{m}{2}}.$$

Then,

$$\int_\Omega u \ \varphi \ \mathrm{d}x \leq \frac{1}{2}||\nabla\varphi||_2^2 + \frac{C_s^{-2}}{2} \ M^{\frac{2}{N}} \ |\Omega|^{\frac{2}{N}} \ ||u||_m^m.$$

Substituting this into the Liapunov functional, we find:

$$\begin{aligned} L_\delta(u,\varphi) &\geq \int_\Omega (b_\delta(u) + \frac{1}{2}|\nabla\varphi|^2) \ \mathrm{d}x - \frac{1}{2}||\nabla\varphi||_2^2 - \frac{C_s^{-2}}{2} \ M^{\frac{2}{N}} \ |\Omega|^{\frac{2}{N}} \ ||u||_m^m \\ &\geq \int_\Omega b_\delta(u) \ \mathrm{d}x - \frac{C_s^{-2}}{2} \ M^{\frac{2}{N}} \ |\Omega|^{\frac{2}{N}} \ ||u||_m^m. \end{aligned}$$

We next observe that:

$$\begin{aligned} b_\delta(u) &= m\int_1^u \int_1^z \frac{(\delta+s)^{m-1}}{s} \ dsdz \geq m\int_1^u \int_1^z s^{m-2} \ dsdz \\ &\geq \frac{u^m}{m-1} - \frac{m}{m-1}u + 1 \geq \frac{u^m}{m-1} - \frac{m}{m-1}u. \end{aligned}$$

Then:

$$\begin{aligned} L_\delta(u,\varphi) &\geq \frac{1}{m-1} \ ||u||_m^m - \frac{C_s^{-2}}{2} \ |\Omega|^{\frac{2}{N}} \ M^{\frac{2}{N}} \ ||u||_m^m - \frac{m}{m-1} \ M \ |\Omega| \\ &= \left(\frac{1}{m-1} - \frac{C_s^{-2}}{2} \ M^{\frac{2}{N}} \ |\Omega|^{\frac{2}{N}}\right) \ ||u||_m^m - \frac{m}{m-1} \ M \ |\Omega|. \end{aligned}$$

Let us define ω_M by

$$\omega_M := \frac{1}{m-1} - \frac{C_s^{-2}}{2} \ M^{\frac{2}{N}} \ |\Omega|^{\frac{2}{N}} = \frac{|\Omega|^{\frac{2}{N}}}{2\ C_s^2} \ (M_*^{\frac{2}{N}} - M^{\frac{2}{N}}).$$

Since $M = ||u_0||_1 < M_*$, then ω_M is positive. Finally we get,

$$L_\delta(u_0,\varphi_0) + \frac{m}{m-1} \ M \ |\Omega| \geq L_\delta(u(t),\varphi(t)) + \frac{m}{m-1} \ M \ |\Omega| \geq \omega_M \ ||u(t)||_m^m \ \text{ for } t \in [0, T_{max}^\delta).$$

In addition, we can see that $L_\delta(u_0,\varphi_0) \leq C$ where C is independent of $\delta \in (0,1)$. In fact,

$$L_\delta(u_0,\varphi_0) = \int_\Omega (b_\delta(u_0) + \frac{1}{2}|\nabla\varphi_0|^2 - u_0 \ \varphi_0) \ \mathrm{d}x,$$

and, since $(\delta+s)^{m-1} \leq \delta^{m-1} + s^{m-1} \leq 1 + s^{m-1}$ we obtain

$$\begin{aligned} b_\delta(u_0) = m\int_1^{u_0} \int_1^z \frac{(\delta+s)^{m-1}}{s} \ dsdz &\leq m\int_1^{u_0} \int_1^z \frac{1+s^{m-1}}{s} \ dsdz \\ &\leq m(u_0 \ln u_0 - u_0 + 1) + \frac{m}{m-1}\left(\frac{u_0^m}{m} - u_0 + 1\right). \end{aligned}$$

Using Young inequality we get

$$L_\delta(u_0,\varphi_0) \leq m \ ||u_0||_2^2 + m \ |\Omega| + \frac{||u_0||_m^m}{m-1} + \frac{m \ |\Omega|}{m-1} + \frac{1}{2}||\nabla\varphi_0||_2^2 + \frac{1}{2}||u_0||_2^2 + \frac{1}{2}||\varphi_0||_2^2.$$

since $u_0 \in L^\infty(\Omega)$ and $\varphi_0 \in H^1(\Omega)$ we get $L_\delta(u_0,\varphi_0) \leq C$ where C is independent of δ and the proof of the lemma is complete. □

Thanks to Lemma 3.3, let us now show that for all $p > m$ the L^p bound for u_δ.

Lemma 3.4. *Let the same assumptions as that in Theorem 3.1 hold. Then for all $T > 0$ and all $p \in (1, \infty)$ there exists $C(p, T)$ independent on δ such that, for all $t \in [0, T^\delta_{\max}) \cap [0, T]$, the solution $(u_\delta, \varphi_\delta)$ to $(KS)_\delta$ satisfies*

$$||u_\delta(t)||_p \leq C(p, T), \tag{8}$$

and

$$\int_0^t \int_\Omega (\delta + u_\delta)^{m-1}\ u_\delta^{p-2}\ |\nabla u_\delta|^2\ dxds \leq C(p, T). \tag{9}$$

To prove the previous lemma we need the following preliminary result [20].

Lemma 3.5. *Consider $0 < q_1 < q_2 \leq 2^*$. There is C_1 depending only on N such that*

$$||u||_{q_2} \leq C_1^\theta\ ||u||^\theta_{H^1(\Omega)}\ ||u||^{1-\theta}_{q_1},\ \text{ for } u \in H^1(\Omega), \tag{10}$$

with

$$\theta = \frac{2N\ (q_2 - q_1)}{q_2[(N+2)q_1 + 2N(1-q_1)]} \in [0, 1].$$

Proof. For $u \in H^1(\Omega)$ we have by Sobolev inequality

$$||u||_{2^*} \leq C_N ||u||_{H^1}. \tag{11}$$

By interpolation inequality we have for $0 < q_1 < q_2 \leq 2^*$

$$||u||_{q_2} \leq ||u||^\theta_{2^*}\ ||u||^{1-\theta}_{q_1}, \tag{12}$$

where $\frac{1}{q_2} = \frac{\theta(N-2)}{2N} + \frac{1-\theta}{q_1}$. Hence, substitute (11) into (12) and the lemma is proved. □

Now, we recall the following generalized Poincaré inequality.

Lemma 3.6. *For $u \in H^1(\Omega)$ we have for $0 < q_1 \leq 1$ the following inequality*

$$||u||^2_{H^1} \leq C_2(q_1)\ (||\nabla u||^2_2 + ||u||^2_{q_1}),$$

where C_2 depends only on Ω and q_1.

Now using the last two lemmas, let us prove Lemma 3.4.

Proof. In this proof, the solution to equation (7) should be denoted by $(u_\delta, \varphi_\delta)$ but for simplicity we drop the index.
We choose $p > 1$, $K \geq 0$ and we multiply the first equation in (7) by $(u - K)_+^{p-1}$ and

integrate by parts using the boundary conditions for u and φ to see that

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{dt}}||(u-K)_+||_p^p &= -m(p-1)\int_\Omega (\delta+u)^{m-1}\ (u-K)_+^{p-2}\ |\nabla u|^2\ \mathrm{d}x \\
&+ (p-1)\int_\Omega u\ \nabla\varphi\ (u-K)_+^{p-2}\cdot\nabla u\ \mathrm{d}x \\
&= -m(p-1)\int_\Omega (\delta+u-K+K)^{m-1}\ (u-K)_+^{p-2}\ |\nabla u|^2\ \mathrm{d}x \\
&+ (p-1)\int_\Omega (u-K+K)\ \nabla\varphi\cdot(u-K)_+^{p-2}\ \nabla u\ \mathrm{d}x \\
&\le -m(p-1)\int_\Omega (u+\delta-K)^{m-1}\ (u-K)_+^{p-2}\ |\nabla u|^2\ \mathrm{d}x \\
&+ (p-1)\int_\Omega (u-K)_+^{p-1}\ \nabla\varphi\cdot\nabla u\ \mathrm{d}x + (p-1)K\int_\Omega \nabla\varphi\ (u-K)_+^{p-2}\cdot\nabla u\ \mathrm{d}x \\
&\le -m(p-1)\int_\Omega (u+\delta-K)^{m-1}\ (u-K)_+^{p-2}\ |\nabla u|^2\ \mathrm{d}x \\
&- \frac{p-1}{p}\int_\Omega (u-K)_+^{p}\ \Delta\varphi\ \mathrm{d}x - K\int_\Omega (u-K)_+^{p-1}\ \Delta\varphi\ \mathrm{d}x \\
&\le -m(p-1)\int_\Omega (u+\delta-K)^{m-1}\ (u-K)_+^{p-2}\ |\nabla u|^2\ \mathrm{d}x + (I),
\end{aligned}$$

where, thanks to the second equation in (7),

$$\begin{aligned}
(I) &= \frac{p-1}{p}\int_\Omega (u-K)_+^p\ (u-M)\ \mathrm{d}x + K\int_\Omega (u-K)_+^{p-1}\ (u-M)\ \mathrm{d}x \\
&= \frac{p-1}{p}||(u-K)_+||_{p+1}^{p+1} + \frac{p-1}{p}(K-M)||(u-K)_+||_p^p \\
&+ K||(u-K)_+||_p^p + K(K-M)||(u-K)_+||_{p-1}^{p-1} \\
&\le K^2\ ||(u-K)_+||_{p-1}^{p-1} + 2K\ ||(u-K)_+||_p^p + ||(u-K)_+||_{p+1}^{p+1}.
\end{aligned}$$

Since for $a>0$ and $b>0$ we have $a^{p-1}b \le a^{p+1} + b^{\frac{p+1}{2}}$ and $a^p b \le a^{p+1} + b^{p+1}$ then,

$$(I) \le 3||(u-K)_+||_{p+1}^{p+1} + (2K)^{p+1} + K^{p+1}, \tag{13}$$

and we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}||(u-K)_+||_p^p &\le -m(p-1)\int_\Omega (u+\delta-K)^{m-1}\ (u-K)_+^{p-2}\ |\nabla u|^2\ \mathrm{d}x \\
&+ 3p||(u-K)_+||_{p+1}^{p+1} + C_p\ K^{p+1},
\end{aligned} \tag{14}$$

for all $t\in[0,T_{\max}^\delta)$.
The term $||(u-K)_+||_{p+1}^{p+1}$ can be estimated with the help of Lemma 3.5 and Lemma 3.6. Assuming now that $p>2$ we remark that $0<\frac{2}{p+m-1}\le 1$ and $1<\frac{2(p+1)}{p+m-1}=\frac{2N}{N-2}\,\frac{1+p}{1+\frac{Np}{N-2}}\le \frac{2N}{N-2}$, then thanks to Lemma 3.5 and Lemma 3.6 we obtain

$$\begin{aligned}
||(u-K)_+^{\frac{p+m-1}{2}}||_{\frac{2(p+1)}{p+m-1}}^{\frac{2(p+1)}{p+m-1}} &\le C(p)\ \left(||\nabla(u-K)_+^{\frac{p+m-1}{2}}||_2^{\frac{2(p+1)}{p+m-1}\theta}\ ||(u-K)_+^{\frac{p+m-1}{2}}||_{\frac{2}{p+m-1}}^{\frac{2(p+1)}{p+m-1}(1-\theta)}\right. \\
&+ \left. ||(u-K)_+^{\frac{p+m-1}{2}}||_{\frac{2}{p+m-1}}^{\frac{2(p+1)}{p+m-1}}\right),
\end{aligned} \tag{15}$$

where

$$\theta = \frac{p+m-1}{p+1} \in (0,1). \tag{16}$$

Since

$$||(u-K)_+^{\frac{p+m-1}{2}}||_{\frac{2(p+1)}{p+m-1}}^{\frac{2(p+1)}{p+m-1}} = \int_\Omega (u-K)_+^{p+1}\ \mathrm{d}x = ||(u-K)_+||_{p+1}^{p+1}, \tag{17}$$

$$||(u-K)_+^{\frac{p+m-1}{2}}||_{\frac{2}{p+m-1}}^{\frac{2(p+1)}{p+m-1}(1-\theta)} = \left(\int_\Omega (u-K)_+\ \mathrm{d}x\right)^{(p+1)(1-\theta)} = ||(u-K)_+||_1^{\frac{2}{N}}, \tag{18}$$

and by Lemma 3.3

$$||(u-K)_+||_1 = \int_{u\geq K} (u-K)\ \mathrm{d}x \leq \frac{1}{K^{m-1}}\int_{u\geq K} K^{m-1}\ u\ dx \leq \frac{||u||_m^m}{K^{m-1}} \leq \frac{C_0^m}{K^{m-1}}, \tag{19}$$

we substitute (17), (18) and (19) into (15) and obtain

$$||(u-K)_+||_{p+1}^{p+1} \leq C_3(p)\left\{||\nabla(u-K)_+^{\frac{m+p-1}{2}}||_2^2\ K^{\frac{-2(m-1)}{N}} + K^{-(m-1)(p+1)}\right\}. \tag{20}$$

We may choose $K=K_*$ large enough such that

$$3\ p\ C_3(p)\ K_*^{\frac{-2(m-1)}{N}} \leq \frac{2\ p\ (p-1)\ m}{(m+p-1)^2},$$

Hence

$$\frac{\mathrm{d}}{\mathrm{dt}}||(u-K_*)_+||_p^p \leq C(p)\ K_*^{p+1},$$

so that

$$||(u(t)-K_*)_+||_p^p \leq C(p)\ t + ||u_0||_p^p,\ \text{ for } t\in[0,T_{\max}^\delta).$$

As

$$\begin{aligned}\int_\Omega |u|^p\ \mathrm{d}x &\leq \int_{u<2K_*}(2\ K_*)^{p-1}\ |u|\ \mathrm{d}x + \int_{u\geq 2K_*}|u-K_*+K_*|^p\ \mathrm{d}x\\ &\leq (2K_*)^{p-1}\ M + \int_{u\geq 2K_*}(2\ |u-K_*|)^p\ \mathrm{d}x,\\ &\leq (2K_*)^{p-1}\ M + 2^p\ ||(u-K_*)_+||_p^p,\end{aligned}$$

the previous inequality warrants that

$$||u(t)||_p \leq C(p,T), \quad t\in[0,T_{\max})\cap[0,T], \tag{21}$$

where $C(p,T)$ is a constant independent of δ.

We next take $K=0$ in (14), integrate with respect to time and use (8) to obtain (9). □

Thanks to Lemma 3.4, we can improve the regularity of φ_δ.

Lemma 3.7. *Let the same assumptions as that in Theorem 3.1 hold, the solution φ_δ satisfies*

$$||\nabla\varphi_\delta(t)||_\infty \leq L(T),\ \ t\in[0,T_{\max}^\delta)\cap[0,T]$$

where $T>0$ and L is a positive constant independent of δ.

Proof. Using standard elliptic regularity estimates for φ_δ, we infer from Lemma 3.4 that given $T>0$, and $p \in (1,\infty)$, there is $C(p,T)$ such that

$$||\varphi_\delta(t)||_{W^{2,p}} \le C(p)\ ||u_\delta(t)||_p \le C(p,T), \text{ for } t \in [0,T_{\max}) \cap [0,T].$$

Lemma 3.7 then readily follows from Sobolev embedding theorem upon choosing $p > N$. ☐

Lemma 3.8. *Let $N \ge 3$, $r \ge 4$, $u \in L^{\frac{r}{4}}(\Omega)$, and $u^{\frac{r+m-1}{2}} \in H^1(\Omega)$. Then it holds that*

$$||u||_r \le C_1^{\frac{2\theta}{r+m-1}}\ ||u||_{\frac{r}{4}}^{1-\theta}\ ||u^{\frac{r+m-1}{2}}||_{H^1}^{\frac{2\theta}{r+m-1}} \tag{22}$$

with

$$\theta = \frac{3\ N(r+m-1)}{(3N+2)r+4N(m-1)}\ \in (0,1). \tag{23}$$

Proof. For $r \ge 4$, we can see that

$$||u||_r = \left(\int_\Omega (u^{\frac{r+m-1}{2}})^{\frac{2r}{r+m-1}}\ dx\right)^{\frac{1}{r}} = ||u^{\frac{r+m-1}{2}}||_{\frac{2r}{r+m-1}}^{\frac{2}{r+m-1}},$$

and

$$\frac{r}{2(r+m-1)} < 1 < \frac{2r}{r+m-1} < 2 < \frac{2N}{N-2}.$$

By Lemma 3.5,

$$||u||_r = ||u^{\frac{r+m-1}{2}}||_{\frac{2r}{r+m-1}}^{\frac{2}{r+m-1}} \le \left(C_1^\theta\ ||u^{\frac{r+m-1}{2}}||_{H^1(\Omega)}^\theta\ ||u^{\frac{r+m-1}{2}}||_{\frac{r}{2(r+m-1)}}^{1-\theta}\right)^{\frac{2}{r+m-1}}$$

and

$$\begin{aligned}\theta &= \frac{2N\ (\frac{2r}{r+m-1} - \frac{r}{2\ (r+m-1)})}{\frac{2r}{r+m-1}\ \left(2N(1-\frac{r}{2(r+m-1)}) + (N+2)\ \frac{r}{2\ (r+m-1)}\right)} \\ &= \frac{3N\ (r+m-1)}{(3N+2)\ r+4N\ (m-1)} \in (0,1).\end{aligned}$$

In addition, we have

$$||u^{\frac{r+m-1}{2}}||_{\frac{r}{2(r+m-1)}} = \left(\int_\Omega |u|^{\frac{r+m-1}{2}\ \frac{r}{2(r+m-1)}}\ dx\right)^{\frac{2(r+m-1)}{r}} = ||u||_{\frac{r}{4}}^{\frac{r+m-1}{2}},$$

and we obtain (22). ☐

We are now in a position to prove the uniform $L^\infty(\Omega)$ bound for u_δ.

Lemma 3.9. *Let the same assumptions as that in Theorem 3.1 hold, and $(u_\delta,\varphi_\delta)$ be the nonnegative maximal solution of (7). For all $T>0$, there is $C_\infty(T)$ such that*

$$||u_\delta(t)||_\infty \le C_\infty(T), \text{ for all } t \in [0,T_{\max}^\delta) \cap [0,T],$$

where $C_\infty(T)$ is a positive constant independent on δ.

Proof. In this proof we omit the index δ, and we employ Moser's iteration technique developed in [1, 21] to show the uniform norm bound for u.
We multiply the first equation in (7) by u^{r-1}, where $r \geq 4$, and integrate it over Ω. Then, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{||u||_r^r}{r} &= -\int_\Omega (\nabla(u+\delta)^m - u\,\nabla\varphi)\cdot\nabla u^{r-1}\,\mathrm{d}x \\
&= -m(r-1)\int_\Omega (u+\delta)^{m-1}\,u^{r-2}\,|\nabla u|^2\,\mathrm{d}x + (r-1)\int_\Omega u^{r-1}\,\nabla\varphi\cdot\nabla u\,\mathrm{d}x \\
&\leq -m(r-1)\int_\Omega u^{m+r-3}\,|\nabla u|^2\,\mathrm{d}x + (r-1)\int_\Omega u^{r-1}\,\nabla\varphi\cdot\nabla u\,\mathrm{d}x.
\end{aligned}$$

By Young's inequality and Lemma 3.7,

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}||u||_r^r &\leq \frac{-4m(r-1)}{(r+m-1)^2}\int_\Omega |\nabla u^{\frac{r+m-1}{2}}|^2\,\mathrm{d}x + \frac{2(r-1)\,||\nabla\varphi||_\infty}{(r+m-1)}\int_\Omega u^{\frac{r-m+1}{2}}\,|\nabla u^{\frac{r+m-1}{2}}|\,\mathrm{d}x \\
&\leq \frac{-2m(r-1)}{(r+m-1)^2}||\nabla u^{\frac{r+m-1}{2}}||_2^2 + \frac{r-1}{2m}||\nabla\varphi||_\infty^2\int_\Omega u^{r-m+1}\,\mathrm{d}x \\
&\leq \frac{-2m(r-1)}{(r+m-1)^2}||\nabla u^{\frac{r+m-1}{2}}||_2^2 + C(T)\,r\int_\Omega u^{r-m+1}\,\mathrm{d}x.
\end{aligned}$$

Using Hölder and Young inequalities and Lemma 3.3 we obtain

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}||u||_r^r &\leq \frac{-2m\,(r-1)}{(r+m-1)^2}||\nabla u^{\frac{r+m-1}{2}}||_2^2 + r\,C(T)\,||u||_1^{\frac{m-1}{r-1}}\,||u||_r^{\frac{r(r-m)}{r-1}} \\
&\leq \frac{-2m\,(r-1)}{(r+m-1)^2}||\nabla u^{\frac{r+m-1}{2}}||_2^2 + C^r + r^2||u||_r^r, \qquad (24)
\end{aligned}$$

where we have used that $r^{\frac{r-1}{r-m}} \leq r^2$ for $r \geq 4$.
By Lemma 3.8, we have for $r \geq 4$

$$||u||_r^r \leq C_1^{\frac{2\ r\ \theta}{r+m-1}}\ ||u||_{\frac{r}{4}}^{r(1-\theta)}\ ||u^{\frac{r+m-1}{2}}||_{H^1}^{\frac{2\ r\ \theta}{r+m-1}}, \qquad (25)$$

where

$$\theta = \frac{3\,N(r+m-1)}{(3N+2)r+4N(m-1)} < 1.$$

Therefore, Young inequality and (25) yield that

$$\begin{aligned}
2\,r^2\,||u||_r^r &\leq 2\,r^2\,C_1^{\frac{2\ r\ \theta}{r+m-1}}\ ||u||_{\frac{r}{4}}^{r(1-\theta)}||\,u^{\frac{r+m-1}{2}}||_{H^1}^{\frac{2\ r\ \theta}{r+m-1}} \\
&\leq \frac{\theta\,r}{r+m-1}\ \frac{m\,(r-1)}{(r+m-1)^2}\ \frac{r+m-1}{\theta\,r\,C_2(1)}\ ||u^{\frac{r+m-1}{2}}||_{H^1}^2 \\
&+ \frac{r+m-1-\theta r}{r+m-1}\left(C_2(1)\ \frac{\theta(r+m-1)r}{m(r-1)}\right)^{\frac{\theta r}{r(1-\theta)+m-1}} \\
&\times\ (2\,r^2)^{\frac{(r+m-1)}{r(1-\theta)+m-1}}\ C_1^{\frac{2\ \theta r}{r(1-\theta)+m-1}}\ ||u||_{\frac{r}{4}}^{(1-\theta)r\frac{(r+m-1)}{r(1-\theta)+m-1}},
\end{aligned}$$

where $C_2(1)$ is the Poincaré constant defined in Lemma 3.6. Then we obtain

$$\begin{aligned}
2\,r^2\,||u||_r^r &\leq \frac{m\,(r-1)}{C_2(1)\,(r+m-1)^2}||u^{\frac{r+m-1}{2}}||_{H^1}^2 \\
&+ C_1^{\frac{\theta\ r}{r(1-\theta)+m-1}}\ 2^{\frac{(r+m-1)}{r(1-\theta)+m-1}}\ r^{\frac{2(r+m-1)+\theta r}{r(1-\theta)+m-1}}\ ||u||_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}.
\end{aligned}$$

Now, since $N > 2$, which gives $4N \geq 3N+2$, we find the following upper bound for θ

$$\theta \leq \frac{3N}{3N+2} \tag{26}$$

In addition,

$$\frac{\theta\ r}{r(1-\theta)+m-1} \leq \frac{\theta}{1-\theta} = -1 + \frac{1}{1-\theta} \leq \frac{3N}{2}, \tag{27}$$

$$\frac{r+m-1}{r(1-\theta)+m-1} \leq \frac{r+m-1}{(1-\theta)(r+m-1)} \leq \frac{1}{1-\theta} \leq \frac{3N+2}{2}, \tag{28}$$

and

$$\frac{2(r+m-1)+\theta r}{r(1-\theta)+m-1} \leq \frac{2+\theta}{1-\theta} \leq 9N+4. \tag{29}$$

As $C_1 \geq 1$ and $r \geq 1$, we get

$$2\ r^2\ ||u||_r^r \ \leq \ \frac{m\ (r-1)}{C_2(1)\ (r+m-1)^2} ||u^{\frac{r+m-1}{2}}||_{H^1}^2 + C\ r^{9N+4}\ ||u||_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}. \tag{30}$$

Using Lemma 3.6 we have

$$||u^{\frac{r+m-1}{2}}||_{H^1}^2 \leq C_2(1)\ \left(||\nabla u^{\frac{r+m-1}{2}}||_2^2 + ||u^{\frac{r+m-1}{2}}||_1^2\right). \tag{31}$$

Using Hölder inequality, Young inequality and Lemma 3.3, we get

$$||u^{\frac{r+m-1}{2}}||_1^2 = ||u||_{\frac{r+m-1}{2}}^{r+m-1} \leq ||u||_r^{r\ \frac{r+m-3}{r-1}}\ ||u||_1^{\frac{r-m+1}{r-1}} \leq ||u||_r^{r\ \frac{r+m-3}{r-1}}\ ||u_0||_1^{\frac{r-m+1}{r-1}},$$

then,

$$\begin{aligned}
\frac{m\ (r-1)}{(r+m-1)^2}||u^{\frac{r+m-1}{2}}||_1^2 \ &\leq\ (r-1)^{\frac{r-1}{r+m-3}}\ \frac{r+m-3}{r-1}\ ||u||_r^r \\
&+\ \frac{2-m}{r-1}\ \left(\frac{m}{(r+m-1)^2}||u_0||_1^{\frac{r+m-1}{r-1}}\right)^{\frac{r-1}{2-m}} \\
&\leq\ r^2||u||_r^r + \left(\frac{m}{(r+m-1)^2}\ ||u_0||_1^{\frac{r+m-1}{r-1}}\right)^{\frac{r-1}{2-m}} \\
&\leq\ r^2||u||_r^r + C_4^r.
\end{aligned} \tag{32}$$

Now substituting (32) and (31) into (30) we get

$$\begin{aligned}
2\ r^2\ ||u||_r^r \ &\leq\ \frac{m\ (r-1)}{(r+m-1)^2}\ \left(||\nabla u^{\frac{r+m-1}{2}}||_2^2 + ||u^{\frac{r+m-1}{2}}||_1^2\right) + C\ r^{9N+4}\ ||u||_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}} \\
&\leq\ \frac{m\ (r-1)}{(r+m-1)^2}\ ||\nabla u^{\frac{r+m-1}{2}}||_2^2 + r^2||u||_r^r + C_4^r + C\ r^{9N+4}\ ||u||_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}},
\end{aligned}$$

hence

$$r^2\ ||u||_r^r \leq \frac{m\ (r-1)}{(r+m-1)^2}\ ||\nabla u^{\frac{r+m-1}{2}}||_2^2 + C_4^r + C\ r^{9N+4}\ ||u||_{\frac{r}{4}}^{\frac{r\ (1-\theta)(r+m-1)}{r(1-\theta)+m-1}}.$$

We apply Young inequality again to the last term of the above inequality. It is easy to see that

$$\frac{2}{3N+2} \leq 1-\theta \leq \frac{(1-\theta)\ (r+m-1)}{r(1-\theta)+m-1} = \frac{(1-\theta)r + (1-\theta)(m-1)}{r(1-\theta)+m-1} < 1,$$

so that

$$r^2||u||_r^r \le \frac{m\ (r-1)}{(r+m-1)^2}||\nabla u^{\frac{r+m-1}{2}}||_2^2 + C_4^r + 1 + \left(C\ r^{9N+4}\right)^{3N+1}||u||_{\frac{r}{4}}^r, \tag{33}$$

for any $r \in [4, \infty)$.

Substituting (33) into (24) we end up with

$$\frac{\mathrm{d}}{\mathrm{dt}}||u||_r^r \le r\ C_4^r + r + r\ \left(C\ r^{9N+4}\right)^{3N+1}||u||_{\frac{r}{4}}^r \le C_5^r + Cr^\alpha\ ||u||_{\frac{r}{4}}^r, \tag{34}$$

for any $r \in [4, \infty)$, where $\alpha = (9N+4)(3N+1)+1$. After integrating (34) from 0 to t, we obtain the L^r estimate for u as follows:

$$\sup_{0<t<T}||u(t)||_r^r \le ||u_0||_r^r + T\ C_5^r + C\ r^\alpha\ T \sup_{0<t<T}||u(t)||_{\frac{r}{4}}^r. \tag{35}$$

Since

$$||u_0||_r \le ||u_0||_\infty^{\frac{r-1}{r}}\ ||u_0||_1^{\frac{1}{r}} \le C_6,$$

then

$$\sup_{0<t<T}||u(t)||_r^r \le C_7(T)\ r^\alpha \max\left\{C_6, \sup_{0<t<T}||u(t)||_{\frac{r}{4}}\right\}^r, \tag{36}$$

and we obtain for $r \ge 4$

$$\sup_{0<t<T}||u(t)||_r \le C_7(T)^{\frac{1}{r}}\ r^{\frac{\alpha}{r}} \max\left\{C_6, \sup_{0<t<T}||u(t)||_{\frac{r}{4}}\right\}. \tag{37}$$

We are now in a position to derive the claimed L^∞ estimate. To this end, we set

$$\alpha_p := \max\left\{C_6, \sup_{0<t<T}||u(t)||_{4^p}\right\}$$

for $p \ge 0$. Then we take $r = 4^p$ with $p \ge 0$ in (37) which reads

$$\begin{aligned} \alpha_p &\le 4^{\frac{\alpha\ p}{4^p}}\ C_7(T)^{\frac{1}{4^p}}\ \max\left\{C_6, \sup_{0<t<T}||u(t)||_{4^{p-1}}\right\}, \\ &\le 4^{\frac{\alpha}{2^p}}\ C_7(T)^{\frac{1}{4^p}}\ \alpha_{p-1} \end{aligned}$$

since $p \le 2^p$ for $p \ge 1$. Arguing by induction we conclude that

$$\alpha_p \le 4^{\alpha \sum_{k=1}^p 2^{-k}}\ C_7(T)^{\sum_{k=1}^p 4^{-k}}\ \alpha_0.$$

Then by using Lemma 3.3 we get

$$\sup_{0<t<T}||u(t)||_{4^p} \le 4^\alpha\ C_7(T)\ \alpha_0 \le C_8(T).$$

Consequently, by letting p tend to ∞, we see that $u \in L^\infty((0,T)\times\Omega)$ and

$$\sup_{0<t<T}||u(t)||_\infty \le C_8(T). \tag{38}$$

Since the right hand side is independent of δ, we have proved the lemma. □

Lemma 3.10. *Let the same assumptions as that in Theorem 3.1 hold, and $(u_\delta, \varphi_\delta)$ be the solution to (7). Then for all $T > 0$ there is $C_9(T)$ such that the solution u_δ satisfies the following derivation estimate*

$$\int_0^T ||\partial_t u_\delta^m||_{(W^{1,N+1})'}\ dt \leq C_9(T).$$

Proof. Consider $\psi \in W^{1,N+1}(\Omega)$ and $t \in (0,T)$, we have

$$\begin{aligned}
&\left|\int_\Omega m\ u_\delta^{m-1}(t)\ \partial_t u_\delta(t)\ \psi\ dx\right| \\
= &\ m\left|\int_\Omega \nabla(u_\delta^{m-1}\ \psi)\cdot(\nabla u_\delta^m - u_\delta\ \nabla\varphi_\delta)\ dx\right| \\
= &\ m\left|\int_\Omega (u_\delta^{m-1}\ \nabla\psi + \psi\ \nabla u_\delta^{m-1})\cdot(\nabla u_\delta^m - u_\delta\ \nabla\varphi_\delta)\ dx\right| \\
\leq &\ m\int_\Omega \big[u_\delta^{m-1}\ |\nabla u_\delta^m|\ |\nabla\psi| + u_\delta^m\ |\nabla\psi|\ |\nabla\varphi_\delta| \\
&+ |\psi|\ m(m-1)\ u_\delta^{2m-3}\ |\nabla u_\delta|^2 + |\psi|(m-1)u_\delta^{m-1}\ |\nabla u_\delta|\ |\nabla\varphi_\delta|\big]\ dx \\
\leq &\ m\ \Big[||u_\delta||_\infty^{m-1}\ ||\nabla u_\delta^m||_2\ ||\nabla\psi||_2 + ||\nabla\psi||_2\ ||u_\delta||_\infty^m||\nabla\varphi_\delta||_\infty|\Omega|^{\frac{1}{2}} \\
&+ ||\psi||_\infty\ \frac{4m(m-1)}{(2m-1)^2}||\nabla u_\delta^{m-\frac{1}{2}}||_2^2 + ||\psi||_2\frac{m-1}{m}||\nabla u_\delta^m||_2\ ||\nabla\varphi_\delta||_\infty\Big].
\end{aligned}$$

Using Lemma 3.8, Lemma 3.9, and the embedding of $W^{1,N+1}(\Omega)$ in $L^\infty(\Omega)$, we end up with

$$|< \partial_t u_\delta^m(t), \psi >| \leq C(T)\left(||\nabla u_\delta(t)^m||_2 + ||\nabla u_\delta^{m-\frac{1}{2}}(t)||_2^2 + 1\right)\ ||\psi||_{W^{1,N+1}},$$

and a duality argument gives

$$||\partial_t u_\delta^m(t)||_{(W^{1,N+1})'} \leq C(T)\left(||\nabla u_\delta^m(t)||_2 + ||\nabla u_\delta^{m-\frac{1}{2}}(t)||_2^2 + 1\right).$$

Integrating the above inequality over $(0,T)$ and using Lemma 3.4 with $p = 2$ and $p = m$ give Lemma 3.10. □

4 Proof of Theorem 2.2

4.1 Existence

In this section, we assume that u_0 is a nonnegative function in $L^\infty(\Omega)$ satisfying (5). For $\delta \in (0,1)$, $(u_\delta, \varphi_\delta)$ denotes the solution to $(KS)_\delta$ constructed in Section 3. To prove existence of a weak solution, we use a compactness method. For that purpose, we first study the compactness properties of $(u_\delta, \varphi_\delta)_\delta$.

Lemma 4.1. *There are functions u and φ and a sequence $(\delta_n)_{n\geq 1}$, $\delta_n \to 0$, such that, for all $T > 0$ and $p \in (1,\infty)$,*

$$u_{\delta_n} \longrightarrow u,\ \text{in } L^p((0,T)\times\Omega) \text{ as } \delta_n \to 0, \tag{39}$$

$$\varphi_{\delta_n} \longrightarrow \varphi,\ \text{in } L^p((0,T); W^{2,p}(\Omega)) \text{ as } \delta_n \to 0. \tag{40}$$

In addition, $u \in L^\infty((0,T)\times\Omega)$ for all $T > 0$ and is nonnegative.

Proof. Thanks to Lemma 3.4 and Lemma 3.9, $(u_\delta^m)_\delta$ is bounded in $L^2((0,T);H^1(\Omega))$ while $(\partial_t u_\delta^m)_\delta$ is bounded in $L^1((0,T);(W^{1,N+1})'(\Omega))$ by Lemma 3.10.
Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and $L^2(\Omega)$ is continuously embedded in $(W^{1,N+1})'(\Omega)$, it follows from [19, corollary 4] that (u_δ^m) is compact in $L^2((0,T)\times\Omega)$ for all $T>0$. Since $r \longmapsto r^{\frac{1}{m}}$ is $\frac{1}{m}$-Hölder continuous, it is easy to check that the previous compactness property implies that (u_δ) is compact in $L^{2m}((0,T)\times\Omega)$ for all $T>0$. There are thus a function $u\in L^{2m}((0,T)\times\Omega)$ for all $T>0$ and a sequence $(\delta_n)_{n\geq 1}$ such that

$$u_{\delta_n} \longrightarrow u \text{ in } L^{2m}((0,T)\times\Omega) \text{ as } \delta_n\to 0, \tag{41}$$

for all $T>0$, owing to Lemma 3.9, we may also assume that

$$u_{\delta_n} \overset{*}{\rightharpoonup} u \text{ in } L^\infty((0,T)\times\Omega) \text{ as } \delta_n\to 0. \tag{42}$$

for all $T>0$. It readily follows from (41) and (42), and Hölder inequality that (39) holds true. Since elliptic regularity ensure that

$$||\varphi_{\delta_k}-\varphi_{\delta_n}||_{W^{2,p}} \leq C(p)\ ||u_{\delta_k}-u_{\delta_n}||_p,$$

for all $k\geq 1$, $n\geq 1$, and $p\in(1,\infty)$, a straightforward consequence of (39) is that $(\varphi_{\delta_n})_{n\geq 1}$ is a Cauchy sequence in $L^p((0,T);W^{2,p}(\Omega))$ and thus converges to some function φ in that space. Finally, the nonnegativity of u follows easily from that of u_{δ_n} by (39). □

Proof of Theorem 2.2 (existence). It remains to identify the equations solved by the limit (u,φ) of $(u_{\delta_n},\varphi_{\delta_n})_{n\geq 1}$ constructed in Lemma 4.1. To this end we first note that , owing to (39) and the boundedness of $(u_{\delta_n})_n$ and u in $L^\infty((0,T)\times\Omega)$, we have

$$u_{\delta_n}^m \longrightarrow u^m \text{ in } L^p((0,T)\times\Omega) \text{ as } \delta_n\to 0, \tag{43}$$

for all $T>0$. Since $(\nabla(u_{\delta_n}+\delta_n)^{\frac{m+1}{2}})_{n\geq 1}$ and $(\nabla u_{\delta_n}^m)_{n\geq 1}$ are bounded in $L^2((0,T)\times\Omega)$ for all $T>0$ by Lemma 3.4 with $p=2$ and $p=m+1$, we may extract a further subsequence (not relabeled) such that

$$\nabla(u_{\delta_n}+\delta_n)^{\frac{m+1}{2}} \rightharpoonup \nabla u^{\frac{m+1}{2}} \text{ in } L^2((0,T)\times\Omega), \tag{44}$$

$$\nabla u_{\delta_n}^m \rightharpoonup \nabla u^m \text{ in } L^2((0,T)\times\Omega), \tag{45}$$

for all $T>0$. Then if $\psi\in L^4((0,T)\times\Omega;\mathbb{R}^N)$,

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \psi\cdot[\nabla(u_{\delta_n}+\delta_n)^m-\nabla u^m]\ dxds\right|\\
&=\frac{2}{m+1}\left|\int_0^T\int_\Omega \psi\cdot\left[(u_{\delta_n}+\delta_n)^{\frac{m-1}{2}}\nabla(u_{\delta_n}+\delta_n)^{\frac{m+1}{2}}-u^{\frac{m-1}{2}}\nabla u^{\frac{m+1}{2}}\right]\ dxds\right|\\
&\leq\frac{2}{m+1}\left|\int_0^T\int_\Omega \psi\cdot\nabla(u_{\delta_n}+\delta_n)^{\frac{m+1}{2}}((u_{\delta_n}+\delta_n)^{\frac{m-1}{2}}-u^{\frac{m-1}{2}})\ dxds\right|\\
&+\frac{2}{m+1}\left|\int_0^T\int_\Omega u^{\frac{m-1}{2}}\ \psi\cdot\left(\nabla(u_{\delta_n}+\delta_n)^{\frac{m+1}{2}}-\nabla u^{\frac{m+1}{2}}\right)\ dxds\right|\\
&\leq\frac{2}{m+1}||\psi||_4\ ||\nabla(u_{\delta_n}+\delta_n)^{\frac{m+1}{2}}||_2\ ||(u_{\delta_n}+\delta_n)^{\frac{m-1}{2}}-u^{\frac{m-1}{2}}||_4\\
&+\frac{2}{m+1}\left|\int_0^T\int_\Omega u^{\frac{m-1}{2}}\ \psi\cdot\left(\nabla(u_{\delta_n}+\delta_n)^{\frac{m+1}{2}}-\nabla u^{\frac{m+1}{2}}\right)\ dxds\right|.
\end{aligned}$$

Since $u^{\frac{m-1}{2}}\ \psi\in L^2((0,T)\times\Omega)$, we deduce from (39) and (44) that the right-hand side of the above inequality converges to zero as $n\longrightarrow\infty$. In other words,

$$\nabla(u_{\delta_n}+\delta_n)^m \rightharpoonup \nabla u^m \text{ in } L^{\frac{4}{3}}((0,T)\times\Omega), \tag{46}$$

for all $T > 0$.

Now, we are going to show that (u, φ) in Lemma 4.1 is the desired weak solution in Theorem 2.2. Let $T > 0$ and $\psi \in C^1([0,T] \times \overline{\Omega})$ with $\psi(T) = 0$. The solution of (7) satisfies

$$\int_0^T \int_\Omega [\nabla(u_{\delta_n} + \delta_n)^m \cdot \nabla\psi - u_{\delta_n}\ \nabla\varphi_{\delta_n} \cdot \nabla\psi - u_{\delta_n}\ \partial_t\psi]\ \mathrm{d}x\mathrm{d}t = \int_\Omega u_0\ \psi(0,x)\ \mathrm{d}x, \quad (47)$$

and,

$$\int_0^T \int_\Omega [\nabla\varphi_{\delta_n} \cdot \nabla\psi + M\ \psi - u_{\delta_n}\ \psi]\ \mathrm{d}x\mathrm{d}t = 0. \quad (48)$$

From (46) we see that

$$\int_0^T \int_\Omega \nabla(u_{\delta_n} + \delta_n)^m \cdot \nabla\psi\ \mathrm{d}x\mathrm{d}t \longrightarrow \int_0^T \int_\Omega \nabla u^m \cdot \nabla\psi\ \mathrm{d}x\mathrm{d}t \text{ as } \delta_n \to 0.$$

From (39) we get

$$\int_0^T \int_\Omega u_{\delta_n}\ \partial_t\psi\ \mathrm{d}x\mathrm{d}t \longrightarrow \int_0^T \int_\Omega u\ \partial_t\psi\ \mathrm{d}x\mathrm{d}t \text{ as } \delta_n \to 0.$$

From (39) and (40) we get

$$\int_0^T \int_\Omega u_{\delta_n}\ \nabla\varphi_{\delta_n} \cdot \nabla\psi\ \mathrm{d}x\mathrm{d}t \longrightarrow \int_0^T \int_\Omega u\ \nabla\varphi \cdot \nabla\psi\ \mathrm{d}x\mathrm{d}t \text{ as } \delta_n \to 0.$$

Thus we conclude that u satisfies

$$\int_0^T \int_\Omega (\nabla u^m \cdot \nabla\psi - u\nabla\varphi \cdot \nabla\psi - u \cdot \partial_t\psi)\ \mathrm{d}x\mathrm{d}t = \int_\Omega u_0(x) \cdot \psi(0,x)\ \mathrm{d}x.$$

Similarly, from (40) we see that

$$\int_0^T \int_\Omega \nabla\varphi_{\delta_n} \cdot \nabla\psi\ \mathrm{d}x\mathrm{d}t \longrightarrow \int_0^T \int_\Omega \nabla\varphi \cdot \nabla\psi\ \mathrm{d}x\mathrm{d}t \text{ as } \delta_n \to 0,$$

and from (39) we see that

$$\int_0^T \int_\Omega u_{\delta_n}\ \psi\ \mathrm{d}x\mathrm{d}t \longrightarrow \int_0^T \int_\Omega u\ \psi\ \mathrm{d}x\mathrm{d}t \text{ as } \delta_n \to 0.$$

Thus, we have constructed a weak solution (u, φ) of (KS). □

4.2 Uniqueness

In this section, we prove the uniqueness statement of Theorem 2.2 under the additionnal assumption (6) on φ. The proof relies on a classical duality technique, and on the method presented in [2]

Proof. The proof estimates the difference of weak solutions in dual space $H^1(\Omega)'$ of $H^1(\Omega)$, motivated by the fact that the nonlinear diffusion is monotone in this norm.

Assume that we have two different weak solutions (u_1,φ_1) and (u_2,φ_2) to equations (1) corresponding to the same initial conditions, and fix $T>0$. We put

$$(u,\varphi) = (u_1-u_2, \varphi_1-\varphi_2) \text{ in } [0,T]\times\Omega.$$

Then φ is the strong solution of

$$\begin{array}{rcll} -\Delta\varphi &=& u & \text{in } \Omega, \\ \partial_\nu\varphi &=& 0 & \text{on } \partial\Omega, \\ <\varphi> &=& 0. & \end{array} \tag{49}$$

Since $\partial_t u \in L^2((0,T);H^1(\Omega)')$, we have

$$-\Delta\partial_t\varphi = \partial_t u_1 - \partial_t u_2 = \partial_t u \ \text{ in } H^1(\Omega)',$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\nabla\varphi||_2^2 &= \int_\Omega \nabla\varphi\cdot\nabla\partial_t\varphi \ dx \\ &= -<\Delta\partial_t\varphi,\varphi>_{(H^1)',H^1} = <\partial_t u,\varphi>_{(H^1)',H^1} . \end{aligned} \tag{50}$$

Now it follows from (1) that u satisfies the equation

$$\left\{ \begin{array}{lcl} \partial_t u &=& \operatorname{div}(\nabla(u_1^m-u_2^m)) - \operatorname{div}(u_1\nabla\varphi + u\nabla\varphi_2) \\ \partial_\nu u &=& 0 \\ u(0,x) &=& 0. \end{array} \right. \tag{51}$$

Substituting (51) in (50), we obtain

$$\frac{1}{2}\frac{d}{dt}||\nabla\varphi||_2^2 = \int_\Omega (u_1^m-u_2^m)\ \Delta\varphi \ \mathrm{d}x + \int_\Omega u_1\ |\nabla\varphi|^2 \mathrm{d}x + \int_\Omega u\ \nabla\varphi_2\cdot\nabla\varphi \ \mathrm{d}x. \tag{52}$$

The first integral on the right-hand side of (52) is nonnegative due to the fact that $z\mapsto z^m$ is an increasing function. The second integral on the right-hand side of (52) can be estimated by

$$\left|\int_\Omega u_1\ |\nabla\varphi|^2 \ \mathrm{d}x\right| \le ||u_1||_\infty \int_\Omega |\nabla\varphi|^2 \ \mathrm{d}x.$$

For the last integral, using an integration by parts we obtain

$$\begin{aligned} \int_\Omega u\ \nabla\varphi_2\cdot\nabla\varphi \ \mathrm{d}x &= -\int_\Omega \Delta\varphi\ \nabla\varphi_2\cdot\nabla\varphi \ dx \\ &= \int_\Omega \nabla\varphi\cdot\nabla(\nabla\varphi_2\cdot\nabla\varphi) \ dx \\ &= \sum_{i,j}\int_\Omega \partial_i\varphi\ \partial_{ij}^2\varphi_2\ \partial_j\varphi \ \mathrm{d}x + \sum_{i,j}\int_\Omega \partial_i\varphi\ \partial_j\varphi_2\ \partial_{ij}^2\varphi \ \mathrm{d}x. \end{aligned} \tag{53}$$

integrating by parts the second integral on the right-hand side of (53),

$$\begin{aligned} \sum_{i,j}\int_\Omega \partial_i\varphi\ \partial_j\varphi_2\ \partial_{ij}^2\varphi \ \mathrm{d}x &= \sum_{i,j}\frac{1}{2}\int_\Omega \partial_j\varphi_2\ \partial_j|\partial_i\varphi|^2 \ \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \nabla\varphi_2\cdot\nabla(|\nabla\varphi|^2) \ \mathrm{d}x \\ &= -\frac{1}{2}\int_\Omega \Delta\varphi_2\ |\nabla\varphi|^2 \ \mathrm{d}x \\ &\le C(T)\ ||\nabla\varphi||_2^2, \end{aligned}$$

since $-\Delta\varphi_2 = u_2 - < u_2 > \in L^\infty((0,T)\times\Omega)$. Together with (53) the previous inequality implies

$$\begin{aligned}\left|\int_\Omega u\ \nabla\varphi_2\cdot\nabla\varphi\ \mathrm{d}x\right| &\leq C(T)\int_\Omega (|D^2\varphi_2|+1)\ |\nabla\varphi|^2\ \mathrm{d}x.\\ &\leq C(T)\ (||\varphi_2||_{L^\infty((0,T);W^{2,\infty}(\Omega))}+1)\ \int_\Omega |\nabla\varphi|^2\ dx,\end{aligned}$$

provided that the $L^\infty((0,T);W^{2,\infty}(\Omega))$ norm of the function φ_2 is bounded. Thus, substituting the above estimates in (52), one finally obtains

$$\frac{d}{dt}\int_\Omega |\nabla\varphi|^2\ dx \leq C(T)\int_\Omega |\nabla\varphi|^2\ dx. \tag{54}$$

Notice that $||\nabla\varphi(0)||_2 = 0$ which follows from (49) and the property $u(0) = 0$. Thus, inequality (54) implies

$$||\nabla\varphi(t)||_2^2 \leq e^{C(T)\ t}\ ||\nabla\varphi(0)||_2^2 = 0.$$

Consequently, $\nabla\varphi(t) = 0$ for all $t\in[0,T]$ and, since $<\varphi(t)>= 0$, we have $\varphi(t) = 0$ for all $t\in[0,T]$. Using (49), we conclude that $u(t) = 0$ for all $t\in[0,T]$. Consequently $(u_1,\varphi_1) = (u_2,\varphi_2)$. □

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