Well-posedness for a model of individual clustering

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November 13, 2012

Abstract. We study the well-posedness of a model of individual clustering. Given $p > N \geq 1$ and an initial condition in $W^{1,p}(\Omega)$, the local existence and uniqueness of a strong solution is proved. We next consider two specific reproduction rates and show global existence if $N = 1$, as well as, the convergence to steady states for one of these rates.

Keywords: Elliptic system, local existence, global existence, steady state, compactness method.

1 Introduction

In [5], a model for the dispersal of individuals with an additional aggregation mechanism is proposed. More precisely, classical models for the spatial dispersion of biological populations read

$$\partial_t u = \Delta(\Phi(u)) + f(u,t,x). \tag{1}$$

where $u(t,x)$ denotes the population density at location x and time t, and $f(u,t,x)$ represents the population supply, due to births and deaths. The dispersal of individuals is either due to random motion with $\Phi(u) = u$ or rests on the assumption that individuals disperse to avoid crowding and Φ satisfies

$$\Phi(0) = 0, \quad \text{and } \Phi'(u) > 0, \quad \text{for } u > 0. \tag{2}$$

No aggregation mechanism is present in this model though, as discussed in [5], the onset of clustering of individuals in a low density region might balance the death and birth rates and guarantee the survival of the colony. To account for such a phenomenon, a modification of the population balance (1) is proposed in [5] and reads

$$\partial_t u = -\nabla \cdot (u\ \boldsymbol{V}(u,t,x)) + u\ E(u,t,x). \tag{3}$$

where $\boldsymbol{V}$ is the average velocity of individuals, and E is the net rate of reproduction per individual at location x and time t. To complete the model, we must specify how $\boldsymbol{V}$ is related to u and E. Following [5], we assume that each individual disperses randomly with probability $\delta \in (0,1)$ and disperses deterministically with an average velocity $\boldsymbol{\omega}$ so as to increase his expected rate of reproduction with probability $1-\delta$. The former is accounted for by a usual Fickian diffusion $\frac{\nabla u}{u}$ while the latter should be in the direction of increasing $E(u,t,x)$, say, of the form $\lambda\ \nabla E(u,t,x)$ with $\lambda > 0$. A slightly different choice is made in [5] and results in the following system

$$\begin{aligned} \partial_t u &= \delta\ \Delta u - (1-\delta)\ \nabla \cdot (u\ \boldsymbol{\omega}) + u\ E(u,t,x) \\ -\varepsilon\ \Delta \boldsymbol{\omega} + \boldsymbol{\omega} &= \lambda \nabla E(u,t,x). \end{aligned} \tag{4}$$

After a suitable rescaling, and assuming that the environment is homogeneous, (4) becomes

$$\begin{aligned} \partial_t u &= \delta\ \Delta u - \nabla\cdot(u\ \boldsymbol{\omega}) + r\ u\ E(u) \\ -\varepsilon\ \Delta\boldsymbol{\omega} + \boldsymbol{\omega} &= E'(u)\ \nabla u, \end{aligned} \tag{5}$$

for $x \in \Omega$ and $t \geq 0$, where Ω is an open bounded domain of $\mathbb{R}^N$, $1 \leq N \leq 3$. We supplement (5) with no-flux boundary conditions

$$\boldsymbol{n}\cdot\nabla u = \boldsymbol{n}\cdot\boldsymbol{\omega} = 0 \quad x \in \partial\Omega, t \geq 0, \tag{6}$$

as suggested in [5]. However, the previous boundary conditions (6) are not sufficient for the well-posedness of the elliptic system verified by $\boldsymbol{\omega}$ in several space dimensions and we must impose the following additional condition given in [3, 4, 8]:

$$\partial_n\boldsymbol{\omega}\times n = 0, \quad x \in \partial\Omega, t > 0. \tag{7}$$

As usual, $v \times \omega$ is the number $v_1\ \omega_1 + v_2\ \omega_2$ if $N = 2$ and the vector field $(v_2\ \omega_3 - v_3\ \omega_2, -v_1\ \omega_3 + v_3\ \omega_1, v_1\ \omega_2 - v_2\ \omega_1)$ if $N = 3$. We note that the boundary condition (7) is useless if $N = 1$.

Summarizing, given a sufficiently smooth function E, parameters $\delta > 0$, $\varepsilon \geq 0$ and $r \geq 0$, our aim in this paper is to look for $(u, \boldsymbol{\omega})$ solving the problem

$$\begin{cases} \partial_t u &= \delta\ \Delta u - \nabla\cdot(u\ \boldsymbol{\omega}) + r\ u\ E(u), & x\in\Omega, t>0 \\ -\varepsilon\ \Delta\boldsymbol{\omega} + \boldsymbol{\omega} &= \nabla E(u), & x\in\Omega, t>0 \\ \partial_n u = 0 &, \quad \boldsymbol{\omega}\cdot n = 0, & x\in\partial\Omega, t>0 \\ \partial_n\boldsymbol{\omega}\times n &= 0, & x\in\partial\Omega, t>0 \\ u(0,x) &= u_0(x), & x\in\Omega. \end{cases} \tag{8}$$

In the first part of this paper, we show that, for $p > N$, the system (8) has a maximal solution u in the sense of Definition 2.1 where $u \in C\left([0, T_{\max}), W^{1,p}(\Omega)\right) \cap C\left((0, T_{\max}), W^{2,p}(\Omega)\right)$, see Theorem 2.2.

In the second part, we turn to the global existence issue and focus on space dimension 1, and two specific forms of E suggested in [5]: the "bistable case" where $E(u) = (1-u)(u-a)$ for some $a \in (0, 1)$, see Theorem 2.3, and the "monostable case" $E(u) = 1 - u$. In both cases, we prove the global existence of solution. In addition, in the monostable case, i.e $E(u) = 1 - u$, thanks to the Liapunov functional

$$L(u) = \int_{-1}^{1} (u\ \log u - u + 1)\ dx,$$

we can study the asymptotic behaviour of solutions for t large, and show that the solution u converges, when t goes to ∞, to a steady state in $L^2(-1, 1)$, see Theorem 2.4.

In the third part, we investigate the limiting behaviour as $\varepsilon \to 0$. Heuristically, when ε goes to zero, the velocity $\boldsymbol{\omega}$ becomes sensitive to extremely local fluctuations in $E(u)$, and the system (8) reduces to the single equation

$$\partial_t u = \nabla\cdot\left((\delta - u\ E'(u))\ \nabla u\right) + r\ u\ E(u). \tag{9}$$

Clearly (9) is parabolic only if $\delta - u\ E'(u) \geq 0$ for all $u > 0$. This is in particular the case when $E(u) = 1 - u$, see Theorem 2.6. But this limit is not well-posed in general. As a result the population distribution may become discontinuous when neighbouring individuals decide to disperse in opposite direction, that is in particular the case when $E(u) = (1 - u)(u - a)$.

2 Main results

Throughout this paper and unless otherwise stated, we assume that

$$E \in C^2(\mathbb{R}),\ \delta > 0,\ \varepsilon > 0,\ r \geq 0.$$

We first define the notion of solution to (8) to be used in this paper.

Definition 2.1. *Let $T > 0$, $p > N$, and an initial condition $u_0 \in W^{1,p}(\Omega)$. A strong solution to* (8) *on* $[0,T)$ *is a function*

$$u \in C\left([0,T), W^{1,p}(\Omega)\right) \cap C\left((0,T), W^{2,p}(\Omega)\right),$$

such that

$$\begin{cases} \partial_t u &= \delta\ \Delta u - \nabla \cdot (u\ \boldsymbol{\omega}_u) + r\ u\ E(u), & \text{a.e. in } [0,T) \times \Omega \\ u(0,x) &= u_0(x), & \text{a.e. in } \Omega \\ \partial_n u &= 0, & \text{a.e. on } [0,T) \times \partial\Omega, \end{cases} \tag{10}$$

where, for all $t \in [0,T)$, $\boldsymbol{\omega}_u(t)$ is the unique solution in $W^{2,p}(\Omega)$ of

$$\begin{cases} -\varepsilon \Delta \boldsymbol{\omega}_u(t) + \boldsymbol{\omega}_u(t) &= \nabla E(u(t)) & \text{a.e. in } \Omega \\ \boldsymbol{\omega}_u(t) \cdot n = \partial_n \boldsymbol{\omega}_u(t) \times n &= 0 & \text{a.e. on } \partial\Omega \end{cases} \tag{11}$$

Our first result gives the existence and uniqueness of a maximal solution of (8) in the sense of Definition 2.1.

Theorem 2.2. *Let $p > N$ and a nonnegative function $u_0 \in W^{1,p}(\Omega)$. Then there is a unique maximal solution $u \in C\left([0,T_{\max}), W^{1,p}(\Omega)\right) \cap C\left((0,T_{\max}), W^{2,p}(\Omega)\right)$ to* (8) *in the sense of Definition 2.1, for some $T_{\max} \in (0,\infty]$. In addition, u is nonnegative.*
Moreover, if for each $T > 0$, there is $C(T)$ such that

$$||u(t)||_{W^{1,p}} \leq C(T), \quad \text{for all } t \in [0,T] \cap [0,T_{\max}),$$

then $T_{\max} = \infty$.

The proof of the previous theorem relies on a contraction mapping argument.

We now turn to the global existence issue and focus on the one dimensional case, where $E(u)$ has the structure suggested in [5].
In the following theorem we give the global existence of solution to (8) in the bistable case, that is when $E(u) = (1-u)(u-a)$, for some $a \in (0,1)$.

Theorem 2.3. *Assume that u_0 is a nonnegative function in $W^{1,2}(-1,1)$, and $E(u) = (1-u)(u-a)$ for some $a \in (0,1)$. Then* (8) *has a global nonnegative solution u in the sense of Definition 2.1.*

The proof relies on a suitable cancellation of the coupling terms in the two equations which gives an estimate for u in $L^\infty(L^2)$ and for $\boldsymbol{\omega}$ in $L^2\left(W^{1,2}\right)$.

Next, we can prove the global existence of a solution to (8) in the monostable case, that is, when $E(u) = 1-u$, and we show that the solution converges as $t \to \infty$ to a steady state. More precisely, we have the following theorem

Theorem 2.4. *Assume that u_0 is a nonnegative function in $W^{1,2}(-1,1)$, and $E(u) = 1 - u$. There exists a global nonnegative solution u of* (8) *in the sense of Definition 2.1 which belongs to $L^\infty\left([0,\infty); W^{1,2}(-1,1)\right)$.*
In addition, if $r = 0$,

$$\lim_{t\to\infty} \left|\left| u(t) - \frac{1}{2}\int_{-1}^{1} u_0 \, dx \right|\right|_2 = 0,$$

and if $r > 0$ the solution $u(t)$ converges either to 0 or to 1 in $L^2(-1,1)$ as $t \to \infty$.

In contrast to the bistable case, it does not seem to be possible to begin the global existence proof with a $L^\infty(L^2)$ estimate on u. Nevertheless, there is still a cancellation between the two equations which actually gives us an $L^\infty(L \log L)$ bound on u and a L^2 bound on $\partial_x \sqrt{u}$.

Remark 2.5. *We note that when $N = 1$, there is a relation between our model when $E(u) = 1 - u$ and $r = 0$, and the following chemorepulsion model studied in [2]*

$$\begin{cases} \partial_t u &= \delta\, \partial^2_{xx} u + \partial_x(u\, \partial_x \psi), & \text{in } (0,\infty)\times(-1,1) \\ -\varepsilon\, \partial^2_{xx}\psi + \psi &= u, & \text{in } (0,\infty)\times(-1,1) \\ \partial_x u(t,\pm1) &= \partial_x\psi(t,\pm1) = 0 & \text{on } (0,\infty). \end{cases} \tag{12}$$

Indeed, define $\varphi = -\partial_x \psi$, and substitute it into (12)*. Then differentiating the second equation in* (12) *we find*

$$\begin{cases} \partial_t u &= \delta\, \partial^2_{xx} u - \partial_x(u\, \varphi), & \text{in } (0,\infty)\times(-1,1) \\ -\varepsilon\, \partial^2_{xx}\varphi + \varphi &= -\partial_x u = \partial_x E(u), & \text{in } (0,\infty)\times(-1,1) \\ \partial_x u(t,\pm1) &= \varphi(t,\pm1) = 0 & \text{on } (0,\infty). \end{cases} \tag{13}$$

So that u is a solution to our model.

When $E(u) = 1 - u$, the limit $\varepsilon \to 0$ is formally justified and (8) takes the qualitative form of (1) with $\Phi(u) = \delta u + \frac{1}{2}\, u^2$. In this example though, since $E' < 0$, the individuals dispersing so as to maximise E would seek isolation, and there is clearly no mechanism capable of producing aggregation of individuals. This observation is actually consistent with Remark 2.5.

Theorem 2.6. *Assume that u_0 is a nonnegative function in $W^{1,2}(-1,1)$, and that $E(u) = 1 - u$. For $\varepsilon > 0$ let u_ε be the global solution to* (8) *given by Theorem 2.4. Then, for all $T > 0$,*

$$\lim_{\varepsilon\to 0} \int_0^T ||u_\varepsilon(t) - u(t)||_2^2 \, dt = 0, \tag{14}$$

where u is the unique solution to

$$\begin{cases} \partial_t u &= \partial^2_{xx}(\delta\, u + \frac{1}{2}\, u^2) + r\, u\,(1-u), & x \in (-1,1), t > 0, \\ u(0,x) &= u_0(x), & x \in (-1,1), \\ \partial_x u(t,\pm1) &= 0, & t > 0. \end{cases} \tag{15}$$

Since $\delta + u > 0$ for $u \geq 0$, the previous equation (15) is uniformly parabolic and has a unique solution u, see [6] for instance.
The proof of Theorem 2.6 is performed by a compactness method.

3 Preliminaries

We first recall some properties of the following system,

$$\begin{cases} -\varepsilon\, \Delta\boldsymbol{\omega} + \boldsymbol{\omega} &= f, \quad \text{in } \Omega, \\ \boldsymbol{\omega}\cdot n &= 0, \quad \text{on } \partial\Omega, \\ \partial_n \boldsymbol{\omega} \times n &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $f \in (L^p(\Omega))^N$ and Ω is a bounded open subset of $\mathbb{R}^N$, $N = 2, 3$. Let us first consider weak solutions of (16). For that purpose, we define

$$W_1 = \{\mathbf{v} \in (H^2(\Omega))^N; \mathbf{v}\cdot n = 0, \text{ and } \partial_n \mathbf{v} \times n = 0 \text{ on } \partial\Omega\}$$

and take W as the closure of W_1 in $(H^1(\Omega))^N$.
If $f \in (L^2(\Omega))^N$, the weak formulation for (16) is

$$\begin{cases} \varepsilon \displaystyle\int_\Omega \nabla\boldsymbol{\omega}\cdot\nabla\mathbf{v} + \int_\Omega \boldsymbol{\omega}\cdot\mathbf{v} = \int_\Omega f\cdot\mathbf{v}, \quad \text{for all } \mathbf{v}\in W \\ \boldsymbol{\omega} \in W \end{cases} \tag{17}$$

where $\nabla\boldsymbol{\omega}\cdot\nabla\mathbf{v} = \sum_i \nabla\omega_i\cdot\nabla v_i$ and $\boldsymbol{\omega}\cdot\mathbf{v} = \sum_i \omega_i\, v_i$.

We recall some results about the existence, regularity and uniqueness of solution for (17), see [3, 4].

Theorem 3.1. *If $f \in (L^2(\Omega))^N$, (17) has a unique solution in W and there is $C = C(\Omega, N)$ such that*

$$||\boldsymbol{\omega}||_W \le C\, ||f||_2.$$

We next consider strong solutions of (16), that is, solutions solving (16) a.e. in Ω. In this direction the existence and uniqueness of the strong solution to (16) is proved in [8]:

Theorem 3.2. *If $f \in (L^p(\Omega))^N$ with $1 < p < \infty$, (16) has a unique solution in $(W^{2,p}(\Omega))^N$ and*

$$||\boldsymbol{\omega}||_{W^{2,p}} \le \frac{K(p)}{\varepsilon}\, ||f||_p, \tag{18}$$

where $K(p) = K(p, \Omega, N)$.

In other words, the strong solution has the same regularity as elliptic equations with classical boundary conditions.

We finally recall some functional inequalities: in several places we shall need the following version of Poincaré's inequality

$$||u||_{W^{1,p}} \le C\, \left(||\nabla u||_p + ||u||_q\right), \quad u \in W^{1,p}(\Omega) \tag{19}$$

with arbitrary $p \ge 1$ and $q \in [1, p]$. Also, we will frequently use the Gagliardo-Nirenberg inequality

$$||u||_p \le C\, ||u||^{\theta}_{W^{1,2}}\, ||u||_q^{1-\theta}, \quad \text{with } \theta = \frac{\frac{N}{q} - \frac{N}{p}}{1 - \frac{N}{2} + \frac{N}{q}}, \; u \in W^{1,2}(\Omega) \tag{20}$$

which holds for all $p \ge 1$ satisfying $p\,(N-2) < 2\,N$ and $q \in [1, p)$.

4 Local well-posedness

Throughout this section, we assume that

$$E \in C^2(\mathbb{R}), \text{ and set } \tilde{E}(z) = z\, E(z) \text{ for } z \in \mathbb{R}. \tag{21}$$

Proof of Theorem 2.2. We fix $p > N$, $R > 0$, and define for $T \in (0,1)$ the set

$$X_R(T) := \left\{ u \in C\left([0,T]; W^{1,p}(\Omega)\right),\ \sup_{t\in[0,T]} ||u(t)||_{W^{1,p}} \le R \right\},$$

which is a complete metric space for the distance

$$d_X(u,v) = \sup_{t\in[0,T]} ||u(t) - v(t)||_{W^{1,p}}, \quad (u,v) \in X_R(T) \times X_R(T).$$

For $u \in X_R(T)$, and $t \in [0,T]$, the embedding of $W^{1,p}(\Omega)$ in $L^\infty(\Omega)$ ensures that $\nabla E(u(t)) \in L^p(\Omega)$ so that (16) with $f = \nabla E(u)$ has a unique solution $\boldsymbol{\omega}_u \in \left(W^{2,p}(\Omega)\right)^N$. We then define $\Lambda(u)$ by

$$\Lambda u(t,x) = (e^{t\ (\delta\ \Delta)}\ u_0)(x) + \int_0^t e^{(t-s)\ (\delta\ \Delta)} \left[-\nabla \cdot (u\ \boldsymbol{\omega}_u) + r\ \tilde{E}(u) \right] (s,x)\ ds, \tag{22}$$

for $(t,x) \in [0,T] \times \Omega$, where $\left(e^{t\ (\delta\ \Delta)}\right)$ denotes the semigroup generated in $L^p(\Omega)$ by $\delta\ \Delta$ with homogeneous Neumann boundary conditions. We now aim at showing that Λ maps $X_R(T)$ into itself, and is a strict contraction for T small enough. In the following, $(C_i)_{i\ge1}$ and C denote positive constants depending only on Ω, δ, r, ε, E, p and R.

- Step 1. Λ maps $X_R(T)$ into itself.

  We first recall that there is $C_1 > 0$ such that

$$||v||_\infty \le C_1 ||v||_{W^{1,p}}, \tag{23}$$

  and

$$||e^{t(\delta\ \Delta)}\ v||_{W^{1,p}} \le C_1\ ||v||_{W^{1,p}}, \text{ and } ||\nabla e^{t\ (\delta\ \Delta)}\ v||_p \le C_1\ \delta^{-\frac{1}{2}}\ t^{-\frac{1}{2}}\ ||v||_p, \tag{24}$$

  for all $v \in W^{1,p}(\Omega)$. Indeed, (23) follows from the continuous embedding of $W^{1,p}(\Omega)$ in $L^\infty(\Omega)$ due to $p > N$ while (24) is a consequence of the regularity properties of the heat semigroup.
  Consider $u \in X_R(T)$, and $t \in [0,T]$. It follows from (24) that

$$\begin{aligned} ||\Lambda u(t)||_p &\le C_1\ ||u_0||_p + \int_0^t ||\nabla e^{(t-s)(\delta\ \Delta)}\ (u\ \boldsymbol{\omega}_u)(s)\ ||_p\ ds \\ &+ r \int_0^t ||e^{(t-s)(\delta\ \Delta)}\ \tilde{E}(u)(s)||_p\ ds \\ &\le C_1\ ||u_0||_p + C_1\ \delta^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}}\ ||u\ \boldsymbol{\omega}_u(s)||_p\ ds \\ &+ C_1\ r \int_0^t ||\tilde{E}(u)(s)||_p\ ds. \end{aligned}$$

Thanks to (23), we have

$$||u(t)||_\infty \leq C_1 \; ||u(t)||_{W^{1,p}} \leq R \; C_1 \leq C_2. \tag{25}$$

Therefore, using elliptic regularity (see Theorem 3.2) and (25), we obtain

$$||\boldsymbol{\omega}_u(t)||_{W^{2,p}} \leq \frac{K(p)}{\varepsilon} \; ||\nabla E(u(t))||_p \leq C \; ||E'||_{L^\infty(-C_2,C_2)} \; R \leq C. \tag{26}$$

Using again (25) along with (26) we find

$$\begin{aligned}
||\Lambda u(t)||_p &\leq C_1 \; ||u_0||_p + C \int_0^t (t-s)^{-\frac{1}{2}} ||u(s)||_\infty \; ||\boldsymbol{\omega}_u(s)||_p \; ds \\
&+ r \int_0^t ||u(s)||_p \; ||E(u(s))||_\infty \; ds \\
&\leq C_1 \; ||u_0||_p + C \int_0^t (t-s)^{-\frac{1}{2}} \; ds + r \; T \; R \; ||E||_{L^\infty(-C_2,C_2)} \\
&\leq C_1 \; ||u_0||_p + C \; t^{\frac{1}{2}} + C \; T \\
&\leq C_1 \; ||u_0||_p + C_3 \; T^{\frac{1}{2}} \qquad (27)
\end{aligned}$$

(recall that $T \leq 1$). On another hand, by (24) we have

$$\begin{aligned}
||\nabla \Lambda u(t)||_p &\leq C_1 \; ||\nabla u_0||_p + \int_0^t ||\nabla e^{(t-s) \; (\delta \; \Delta)} \; \nabla \cdot (u \; \boldsymbol{\omega}_u)(s)||_p \; ds \\
&+ r \int_0^t ||e^{(t-s)(\delta \; \Delta)} \; \nabla (u \; E(u))(s)||_p \; ds \\
&\leq C_1 \; ||\nabla u_0||_p + \delta^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \; ||(\nabla u \cdot \boldsymbol{\omega}_u + u \; \nabla \cdot \boldsymbol{\omega}_u)(s)||_p \; ds \\
&+ r \int_0^t ||\nabla (\tilde{E}(u))(s)||_p \; ds.
\end{aligned}$$

Since $u \in X_R(T)$, using (25) we can see that

$$r \; ||\nabla(\tilde{E}(u))||_p \leq r \; ||\tilde{E}'(u) \; \nabla u||_p \leq r \; ||\tilde{E}'||_{L^\infty(-C_2,C_2)} \; ||\nabla u||_p \leq C_4,$$

which gives that

$$\begin{aligned}
||\nabla \Lambda u(t)||_p &\leq C_1 \; ||\nabla u_0||_p + \delta^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} ||\boldsymbol{\omega}_u||_\infty \; ||\nabla u||_p \; ds \\
&+ \delta^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (||\nabla \cdot \boldsymbol{\omega}_u||_p \; ||u||_\infty) \; ds + C_4 \; T.
\end{aligned}$$

Since

$$||\boldsymbol{\omega}_u||_\infty \leq C_1 \; ||\boldsymbol{\omega}_u||_{W^{1,p}} \leq C_5, \tag{28}$$

by (26) and (23), we use once more (25) and obtain that

$$\begin{aligned}
||\nabla \Lambda u(t)||_p &\leq C_1 \; ||\nabla u_0||_p + C \int_0^t (t-s)^{-\frac{1}{2}} \; ds + C \int_0^t (t-s)^{-\frac{1}{2}} \; ds + C_4 \; T \\
&\leq C_1 \; ||\nabla u_0||_p + C_6 \; T^{\frac{1}{2}}. \qquad (29)
\end{aligned}$$

Combining (27) and (29) we get

$$\sup_{t\in[0,T]} ||\Lambda u(t)||_{W^{1,p}} \leq C_1 \; ||u_0||_{W^{1,p}} + C_7 \; T^{\frac{1}{2}}.$$

Choosing $R = 2 \; C_1 \; ||u_0||_{W^{1,p}}$ and $T \in (0,1)$ such that

$$C_1 \; ||u_0||_{W^{1,p}} + C_7 \; T^{\frac{1}{2}} \leq R,$$

we obtain that

$$\sup_{t\in[0,T]} ||\Lambda u(t)||_{W^{1,p}} \leq R.$$

It follows that Λ maps $X_R(T)$ into itself.

- Step 2. We next show that Λ is a strict contraction for T small enough.

Let u and v be two functions in $X_R(T)$. Using (24) we have

$$\begin{aligned} ||\Lambda u(t) - \Lambda v(t)||_p &\leq \int_0^t \left|\left|\nabla e^{(t-s)\;(\delta\;\Delta)}\;[-u\;\boldsymbol{\omega}_u + v\;\boldsymbol{\omega}_v]\right|\right|_p\;ds \\ &+ \; r\int_0^t \left|\left|e^{(t-s)(\delta\;\Delta)}\;[\tilde{E}(u) - \tilde{E}(v)]\right|\right|_p\;ds \\ &\leq C_1\;\delta^{-\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}\;||-u\;\boldsymbol{\omega}_u + v\;\boldsymbol{\omega}_v||_p\;ds \\ &+ \; r\;C_1\int_0^t \left|\left|\tilde{E}(u) - \tilde{E}(v)\right|\right|_p\;ds. \end{aligned} \tag{30}$$

Note that, by (25) and (28), we have

$$\begin{aligned} ||u\;\boldsymbol{\omega}_u - v\;\boldsymbol{\omega}_v||_p &\leq ||u\;\boldsymbol{\omega}_u - u\;\boldsymbol{\omega}_v - v\;\boldsymbol{\omega}_v + u\;\boldsymbol{\omega}_v||_p \\ &\leq ||u||_\infty\;||\boldsymbol{\omega}_u - \boldsymbol{\omega}_v||_p + ||\boldsymbol{\omega}_v||_\infty\;||u-v||_p \\ &\leq C\;||\boldsymbol{\omega}_u - \boldsymbol{\omega}_v||_p + C\;||u-v||_p, \end{aligned} \tag{31}$$

and it follows from Theorem 3.2 and (25) that

$$\begin{aligned} ||\boldsymbol{\omega}_u - \boldsymbol{\omega}_v||_{W^{2,p}} &\leq C\;||\nabla E(u) - \nabla E(v)||_p \\ &\leq C\;||E'(u)\;\nabla u - E'(u)\;\nabla v - E'(v)\;\nabla v + E'(u)\;\nabla v||_p \\ &\leq C\;||E'||_{L^\infty(-C_2,C_2)}\;||\nabla u - \nabla v||_p \\ &+ C||E'(v) - E'(u)||_\infty\;||\nabla v||_p \\ &\leq C\;||\nabla v - \nabla u||_p + C\;||E''||_{L^\infty(-C_2,C_2)}\;d_X(u,v). \\ &\leq C_8\;d_X(u,v). \end{aligned} \tag{32}$$

Combining (32) and (31) we obtain

$$\begin{aligned} &\int_0^t (t-s)^{-\frac{1}{2}}\;||-u\;\boldsymbol{\omega}_u + v\;\boldsymbol{\omega}_v||_p\;ds \\ \leq\;& C\;T^{\frac{1}{2}}\;C_8\;d_X(u,v) + T^{\frac{1}{2}}\;C_9\;d_X(u,v) \\ \leq\;& T^{\frac{1}{2}}\;C_{10}\;d_X(u,v). \end{aligned} \tag{33}$$

Since u and v are bounded by (25), we have

$$r\ ||\tilde{E}(u)-\tilde{E}(v)||_p \quad \le \quad C\ ||u-v||_p.$$

Then, we get

$$\int_0^t r\ ||\tilde{E}(u)-\tilde{E}(v)||_p(s)\ ds \quad \le \quad C_{11}\ T\ d_X(u,v)$$

Substituting (33) and the above inequality in (30) we conclude that

$$||\Lambda u(t)-\Lambda v(t)||_p \le C_{12}\ T^{\frac{1}{2}}\ d_X(u,v).$$

Using again (24), we have

$$\begin{aligned} ||\nabla \Lambda u(t)-\nabla \Lambda v(t)||_p &\le \int_0^t \left|\left|\nabla e^{(t-s)\ (\delta\ \Delta)}\ [-\nabla\cdot(u\ \boldsymbol{\omega}_u)+\nabla\cdot(v\ \boldsymbol{\omega}_v)]\right|\right|_p\ ds \\ &+\ r\int_0^t \left|\left|\nabla e^{(t-s)\ (\delta\ \Delta)}\ (\tilde{E}(u)-\tilde{E}(v))\right|\right|_p\ ds \\ &\le \delta^{-\frac{1}{2}}\ C_1\ \int_0^t (t-s)^{-\frac{1}{2}}\ ||-\nabla\cdot(u\ \boldsymbol{\omega}_u)+\nabla\cdot(v\ \boldsymbol{\omega}_v)||_p\ ds \\ &+\ r\ C_1\ \int_0^t \left|\left|\nabla\left(\tilde{E}(u)-\tilde{E}(v)\right)\right|\right|_p (s)\ ds. \end{aligned} \tag{34}$$

Since the mapping

$$\begin{array}{ccc} W^{1,p}(\Omega)\times W^{1,p}(\Omega) & \longrightarrow & W^{1,p}(\Omega) \\ u,v & \longmapsto & u\ v \end{array}$$

is bilinear and continuous due to $p>N$, we deduce from (26) and (32) that

$$\begin{aligned} \delta^{-\frac{1}{2}}\ C_1\ ||-\nabla\cdot(u\ \boldsymbol{\omega}_u)+\nabla\cdot(v\ \boldsymbol{\omega}_v)||_p &\le ||u\ \boldsymbol{\omega}_u - v\ \boldsymbol{\omega}_v||_{W^{1,p}} \\ &\le C\ ||u||_{W^{1,p}}\ ||\boldsymbol{\omega}_u-\boldsymbol{\omega}_v||_{W^{1,p}} \\ &+ C\ ||\boldsymbol{\omega}_v||_{W^{1,p}}\ ||u-v||_{W^{1,p}} \\ &\le C\ d_X(u,v)+C\ ||u-v||_{W^{1,p}} \\ &\le C_{13}\ d_X(u,v) \end{aligned}$$

Thus,

$$\delta^{-\frac{1}{2}}\ C_1\ \int_0^t (t-s)^{-\frac{1}{2}}\ ||-\nabla\cdot(u\ \boldsymbol{\omega}_u)+\nabla\cdot(v\ \boldsymbol{\omega}_v)||_p\ ds \le T^{\frac{1}{2}}\ C_{13}\ d_X(u,v).$$

On the other hand, due to (25) and the embedding of $W^{1,p}(\Omega)$ in $L^\infty(\Omega)$

$$\begin{aligned} ||\nabla(\tilde{E}(u)-\tilde{E}(v))||_p &= ||\tilde{E}'(u)\ \nabla u-\tilde{E}'(v)\ \nabla v||_p \\ &\le C\ ||\tilde{E}'(u)||_\infty\ ||\nabla u-\nabla v||_p + C\ ||\tilde{E}'(u)-\tilde{E}'(v)||_\infty\ ||\nabla v||_p, \\ &\le C\ ||\tilde{E}'||_{L^\infty(-C_2,C_2)}\ ||\nabla u-\nabla v||_p + C\ ||\tilde{E}''||_{L^\infty(-C_2,C_2)}\ ||u-v||_\infty \\ &\le C\ ||u-v||_{W^{1,p}}. \end{aligned}$$

Then

$$r\ C_1\ ||\nabla(\tilde{E}(u)-\tilde{E}(v))||_p \le C_{14}\ ||u-v||_{W^{1,p}}.$$

Therefore,

$$||\nabla\Lambda u(t) - \nabla\Lambda v(t)||_p \le T^{\frac{1}{2}}\ C_{13}\ d_X(u,v) + T\ C_{14}\ d_X(u,v) \le T^{\frac{1}{2}}\ C_{15}\ d_X(u,v).$$

Finally we get

$$d_X(\Lambda u(t) - \Lambda v(t)) \le T^{\frac{1}{2}}\ C_{16}\ d_X(u,v).$$

Choosing $T \in (0,1)$ such that $T^{\frac{1}{2}}\ C_{16} < 1$ we obtain that Λ is indeed a strict contraction in $X_R(T)$ and thus has a unique fixed point u.

Furthermore, since $u \in C\left([0,T], W^{1,p}(\Omega)\right)$ and $p > N$, we have $\nabla E(u) \in C\left([0,T], L^p(\Omega)\right)$ and we infer from Theorem 3.2 that $\boldsymbol{\omega}_u \in C\left([0,T], W^{2,p}(\Omega)\right)$. Combining this property with the fact that $W^{1,p}(\Omega)$ is an algebra, we realize that both $\nabla\cdot(u\ \boldsymbol{\omega}_u)$ and $u\ E(u)$ belong to $C\left([0,T], L^p(\Omega)\right)$. Classical regularity properties of the heat equation then guarantee that $u \in C\left((0,T], W^{2,p}(\Omega)\right)$ and is a strong solution to (10).

- Step 3. Thanks to the analysis performed in Steps 1 and 2, the existence and uniqueness of a maximal solution follows by classical argument, see [1] for instance.
- Step 4. Since 0 clearly solves (10), and $u_0 \ge 0$, the positivity of u follows from the comparison principle.

□

5 Global existence

From now on we choose $N = 1$, $\Omega = (-1,1)$, $p = 2$ and we set $\varphi = \boldsymbol{\omega}_u$ to simplify the notation.

5.1 The bistable case: $E(u) = (1-u)(u-a)$.

In this case, the system (8) now reads

$$\left\{\begin{array}{lcll} \partial_t u & = & \delta\ \partial^2_{xx}u - \partial_x(u\ \varphi) + r\ u\ (u-a)(1-u), & x \in (-1,1), t > 0 \\ -\varepsilon\ \partial^2_{xx}\varphi + \varphi & = & (-2u + (a+1))\ \partial_x u, & x \in (-1,1), t > 0 \\ \partial_x u(t,\pm 1) & = & \varphi(t,\pm 1) = 0, & t > 0, \\ u(0,x) & = & u_0(x), & x \in (-1,1), \end{array}\right. \tag{35}$$

for a some $a \in (0,1)$.

Since $E \in C^2(\mathbb{R})$, Theorem 2.2 ensures that there is a maximal solution u of (35) in $C\left([0,T_{\max}), W^{1,p}((-1,1))\right) \cap C\left((0,T_{\max}), W^{2,p}(-1,1)\right)$.

To prove Theorem 2.3 we show that, for all $T > 0$ and $t \in [0,T] \cap [0,T_{\max})$, $u(t)$ is bounded in $W^{1,2}(-1,1)$.
We begin the proof by the following lemmas which give some estimates on u and φ.

Lemma 5.1. *Let the same assumptions as that of Theorem 2.3 hold, and u be the non-negative maximal solution of* (35)*. Then for all $T > 0$ there exists $C_1(T)$, such that u and φ satisfy the following estimates*

$$||u(t)||_2 \le C_1(T), \quad \text{for all } t \in [0,T] \cap [0,T_{\max}), \tag{36}$$

$$\int_0^t ||\partial_x u||_2^2 \; dt \le C_1(T) \quad \text{for all } t \in [0,T] \cap [0, T_{\max}), \tag{37}$$

and

$$\int_0^t ||\varphi(t)||_{W^{1,2}}^2 \; dt \le C_1(T) \quad \text{for all } t \in [0,T] \cap [0, T_{\max}). \tag{38}$$

Proof. Multiplying the first equation in (35) by $u(t)$ and integrating it over $(-1,1)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 |u|^2 \; dx = -2 \; \delta \int_{-1}^1 |\partial_x u|^2 \; dx + 2 \; \int_{-1}^1 \; u \; \varphi \; \partial_x u \; dx + 2 \; r \; \int_{-1}^1 u^2 \; E(u) \; dx. \tag{39}$$

Multiplying now the second equation in (35) by φ and integrating it over $(-1,1)$ we obtain

$$\varepsilon \; \int_{-1}^1 |\partial_x \; \varphi|^2 \; dx + \int_{-1}^1 |\varphi|^2 \; dx = -2 \; \int_{-1}^1 u \; \varphi \; \partial_x u \; dx + (a+1) \int_{-1}^1 \partial_x u \; \varphi \; dx. \tag{40}$$

At this point we notice that the cubic terms on the right hand side of (39) and (40) cancel one with the other, and summing (40) and (39) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} ||u||_2^2 + \varepsilon \; ||\partial_x \varphi||_2^2 + ||\varphi||_2^2 + 2 \; \delta \; ||\partial_x u||_2^2 = 2 \; r \int_{-1}^1 u^2 \; E(u) \; dx + (a+1) \int_{-1}^1 \partial_x u \; \varphi \; dx. \tag{41}$$

We integrate by parts and use Cauchy-Schwarz inequality to obtain

$$(a+1) \; \int_{-1}^1 \partial_x u \; \varphi \; dx = -(a+1) \int_{-1}^1 u \; \partial_x \varphi \; dx \le \frac{(a+1)^2}{2 \; \varepsilon} \; ||u||_2^2 + \frac{\varepsilon}{2} \; ||\partial_x \varphi||_2^2.$$

On the other hand, $u^2 \; E(u) \le 0$ if $u \notin (a,1)$ so that

$$\int_{-1}^1 u^2 \; E(u) \; dx \le 2 \; (1-a)$$

The previous inequalities give that

$$\frac{\mathrm{d}}{\mathrm{d}t} ||u||_2^2 + \frac{\varepsilon}{2} \; ||\partial_x \varphi||_2^2 + ||\varphi||_2^2 + 2 \; \delta \; ||\partial_x u||_2^2 \le \frac{(a+1)^2}{2 \; \varepsilon} \; ||u||_2^2 + 4 \; r \; (1-a).$$

Therefore, for all $T > 0$ there exists $C_1(T)$ such that (36), (37) and (38) hold. ☐

Lemma 5.2. *Let the same assumptions as that of Theorem 2.3 hold, and u be the non-negative maximal strong solution of* (35). *For all* $T > 0$, *there is* $C_\infty(T)$ *such that*

$$||u(t)||_\infty \le C_\infty(T), \quad \text{for all } t \in [0,T] \cap [0, T_{\max}). \tag{42}$$

Proof. The estimates (36) and (37) and the Gagliardo-Nirenberg inequality (20) yield that there exists $C_2(T)$ such that

$$\int_0^t ||u \; \partial_x u||_2 \; ds \le C \; \int_0^t ||u||_2^{\frac{1}{2}} \; ||\partial_x u||_2^{\frac{3}{2}} \; ds \le C_2(T) \quad \text{for all } t \in [0,T] \cap [0, T_{\max}).$$

The second equation in (35) and classical elliptic regularity theory ensure that there exists $C(T)$ such that

$$\int_0^t ||\varphi||_{W^{2,2}} \; ds \le C(T) \quad \text{for all } t \in [0,T] \cap [0, T_{\max}),$$

which gives in particular, since $W^{2,2}(-1,1)$ is embedded in $W^{1,\infty}(-1,1)$,

$$\int_0^t ||\partial_x \varphi||_\infty \; ds \leq C_3(T) \;\; \text{for all } t \in [0,T] \cap [0, T_{\max}). \tag{43}$$

Now, we multiply the first equation in (35) by $q\; u^{q-1}$ where $q > 1$ and integrate it over $(-1,1)$ to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 |u|^q \; dx &= -4 \frac{\delta\; (q-1)}{q} \int_{-1}^1 |\partial_x u^{\frac{q}{2}}|^2 \; dx + (q-1) \int_{-1}^1 \varphi \; \partial_x u^q \; dx \\ &+ \; r\; q \int_{-1}^1 u^q \; (u-a)\; (1-u) \; dx \\ &\leq -(q-1) \int_{-1}^1 \partial_x \varphi \; u^q \; dx + r\; q\; 2\; (1-a) \end{aligned}$$

Using Hölder's inequality , we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} ||u||_q^q \leq (q-1)\; ||\partial_x \varphi||_\infty \; ||u||_q^q + 2\; q\; r. \tag{44}$$

Introducing

$$\phi(t) = \int_0^t ||\partial_x \varphi(s)||_\infty \; ds \leq C_3(T) \text{ for all } t \in [0,T] \cap [0, T_{\max}),$$

the bound being a sequence of (43), we integrate (44) and find

$$\begin{aligned} ||u(t)||_q^q &\leq ||u_0||_q^q \; e^{(q-1)\; \phi(t)} + 2\; q\; r \int_0^t e^{(q-1)\; (-\phi(s)+\phi(t))} \; ds \\ &\leq (||u_0||_q^q + 2\; q\; r)\; T\; e^{q\; C(T)}, \\ ||u(t)||_q &\leq \big((||u_0||_q^q + 2\; q\; r)\; T \big)^{\frac{1}{q}} \; e^{C(T)} \;\; \text{for all } t \in [0,T] \cap [0, T_{\max}). \end{aligned}$$

Consequently, by letting q tend to ∞, we see that there exists $C_\infty(T)$ such that

$$||u(t)||_\infty \leq C_\infty(T), \text{ for all } t \in [0,T] \cap [0, T_{\max}).$$

□

Lemma 5.3. *Let the same assumptions as that of Theorem 2.3 hold, and u be the non-negative maximal strong solution of* (35)*. For all $T > 0$, there is $C_4(T)$ such that*

$$||\partial_x u(t)||_2 \leq C_4(T) \;\; \text{for all } t \in [0,T] \cap [0, T_{\max}). \tag{45}$$

Proof. We multiply the first equation in (35) by $(-\partial_{xx}^2 u)$ and integrate it over $(-1,1)$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 |\partial_x u|^2 \; dx &= -\delta \int_{-1}^1 |\partial_{xx}^2 u|^2 \; dx + \int_{-1}^1 \partial_x (u\; \varphi)\; \partial_{xx}^2 u \; dx \\ &+ \; r \int_{-1}^1 u\; (1-u)\; (u-a)\; (-\partial_{xx}^2 u)\; dx \\ &= -\delta \int_{-1}^1 |\partial_{xx}^2 u|^2 \; dx + \int_{-1}^1 (u\; \partial_x \varphi + \partial_x u\; \varphi)\; \partial_{xx}^2 u \; dx \\ &+ \; r \int_{-1}^1 u\; (-u^3 + (a+1)u^2 - a)\; \partial_{xx}^2 u \; dx. \end{aligned}$$

Using Cauchy-Schwarz inequality and Lemma 5.2 we obtain,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}|\partial_x u|^2\,dx &\le -\delta\int_{-1}^{1}|\partial_{xx}^2 u|^2\,dx+\frac{\delta}{3}\int_{-1}^{1}|\partial_{xx}^2 u|^2\,dx+C\,||u||_\infty^2\int_{-1}^{1}|\partial_x\varphi|^2\,dx\\ &+\frac{\delta}{3}\int_{-1}^{1}|\partial_{xx}^2 u|^2\,dx+C\,||\varphi||_\infty^2\int_{-1}^{1}|\partial_x u|^2\,dx\\ &+\frac{\delta}{3}\int_{-1}^{1}|\partial_{xx}^2 u|^2\,dx+C\,||(u\,(-u^3+(a+1)u^2-a))||_\infty^2\\ &\le C(T)\int_{-1}^{1}|\partial_x\varphi|^2\,dx+C\,||\varphi(t)||_\infty^2\int_{-1}^{1}|\partial_x u|^2\,dx+C(T). \qquad (46)\end{aligned}$$

Using (38) and Sobolev embedding theorem we obtain the following estimate

$$\int_0^t ||\varphi||_\infty^2\,ds\le C(T)\ \text{ for all } t\in[0,T]\cap[0,T_{\max}). \qquad (47)$$

Since (38) and (47) hold, then it follows from (46) after integration that

$$||\partial_x u(t)||_2^2\le C_4(T),\ \text{ for all } t\in[0,T]\cap[0,T_{\max}).$$

□

It remains to prove Theorem 2.3.

Proof of Theorem 2.3. For all $T>0$, Lemma 5.3 and the estimate (36) ensure that

$$||u(t)||_{W^{1,2}}\le C(T),\quad \text{for all } t\in[0,T]\cap[0,T_{\max}),$$

which guarantees that u cannot explode in $W^{1,2}(-1,1)$ in finite time and thus that $T_{\max}=\infty$. □

5.2 The monostable case: $E(u)=1-u$

For this choice of E, system (8) now reads

$$\begin{cases}\partial_t u &= \delta\,\partial_{xx}^2 u-\partial_x(u\,\varphi)+r\,u\,(1-u) & x\in(-1,1),\ t>0\\ -\varepsilon\,\partial_{xx}^2\varphi+\varphi &= -\,\partial_x u, & x\in(-1,1),\ t>0\\ \partial_x u(t,\pm1)=\varphi(t,\pm1) &= 0 & t>0,\\ u(0,x) &= u_0(x) & x\in(-1,1),\end{cases} \qquad (48)$$

Since $E\in C^2(\mathbb{R})$, Theorem 2.2 ensures that there is a maximal solution u of (48) in $C\left([0,T_{\max}),W^{1,p}(-1,1)\right)\cap C\left((0,T_{\max}),W^{2,p}(-1,1)\right)$.

In contrast to the previous case, it does not seem to be possible to begin the global existence proof with an $L^\infty(L^2)$ estimate on u. Nevertheless, there is still a cancellation between the two equations which actually gives us an $L^\infty(L\log L)$ bound on u and a L^2 bound on $\partial_x\sqrt{u}$. Integrating (48) over $(0,t)\times(-1,1)$ and using the nonnegativity of u, we first observe that,

$$||u(t)||_1\le||u_0||_1+2\,r\,t,\quad \text{for all } t\in[0,T_{\max}). \qquad (49)$$

To prove Theorem 2.4 we need to prove the following lemmas:

Lemma 5.4. *Let the same assumptions as that of Theorem 2.4 hold, and let u be the maximal solution of* (48)*. Then for all $T > 0$, there exists a constant $C_1(T)$ such that*

$$\int_0^t ||\partial_x \sqrt{u}||_2^2 \; ds \leq C_1(T), \quad \text{for all } t \in [0,T] \cap [0, T_{\max}), \tag{50}$$

and

$$\int_0^t ||\varphi||_{W^{1,2}}^2 \; ds \leq C_1(T), \quad \text{for all } t \in [0,T] \cap [0, T_{\max}). \tag{51}$$

Proof. The proof goes as follows. On the one hand, we multiply the first equation in (48) by $(\log u + 1)$ and integrate it over $(-1, 1)$. Since $u\ (1-u)\ \log u \leq 0$ and $u\ (1-u) \leq 1$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 u \ \log u \; dx &= -\int_{-1}^1 (\delta \ \partial_x u - u \ \varphi) \ (\frac{1}{u} \ \partial_x u) \; dx + r \int_{-1}^1 u \ (1-u) \ (\log u + 1) \; dx \\ &\leq -\int_{-1}^1 \frac{\delta}{u} \ (\partial_x u)^2 \; dx + \int_{-1}^1 \varphi \ \partial_x u \; dx + 2\ r. \end{aligned} \tag{52}$$

On the other hand, we multiply the second equation in (48) by φ and integrate it over $(-1, 1)$ to obtain

$$\varepsilon \int_{-1}^1 |\partial_x \varphi|^2 \; dx + \int_{-1}^1 |\varphi|^2 \; dx = -\int_{-1}^1 \partial_x u \ \varphi \; dx. \tag{53}$$

Adding (53) and (52) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 u \ \log u \; dx + \varepsilon \ ||\partial_x \varphi||_2^2 + ||\varphi||_2^2 \leq -4\ \delta \int_{-1}^1 |\partial_x \sqrt{u}|^2 \; dx + 2\ r. \tag{54}$$

Finally, (50) and (51) are obtained by a time integration of (54). □

Lemma 5.5. *Let the same assumptions as that of Theorem 2.4 hold, and let u be the maximal solution of* (48)*. Then for all $T > 0$, there exists a constant $C_2(T)$ such that*

$$||u(t)||_2 \leq C_2(T), \quad \text{for all } t \in [0,T] \cap [0, T_{\max}), \tag{55}$$

and

$$\int_0^t ||\partial_x u||_2^2 \; ds \leq C_2(T), \quad \text{for all } t \in [0,T] \cap [0, T_{\max}). \tag{56}$$

Proof. A simple computation shows that, since $u^2\ (1-u) \leq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 u^2 \; dx \leq -2\ \delta \int_{-1}^1 |\partial_x u|^2 \; dx + 2 \int_{-1}^1 u \ \varphi \ \partial_x u \; dx + 4\ r. \tag{57}$$

Using Cauchy-Schwarz inequality, Gagliardo-Nirenberg inequality (20), Young inequality and (49) we obtain that for all $T > 0$,

$$\begin{aligned} 2 \int_{-1}^1 u \ \varphi \ \partial_x u \; dx &= -\int_{-1}^1 u^2 \ \partial_x \varphi \; dx \leq ||u||_4^2 \ ||\partial_x \varphi||_2 \\ &\leq C \left(||u||_{W^{1,2}}^{\frac{1}{2}} \ ||u||_1^{\frac{1}{2}} \right)^2 ||\partial_x \varphi||_2 \leq C(T) \ ||\partial_x \varphi||_2 \ ||u||_{W^{1,2}} \\ &\leq \delta \ ||u||_{W^{1,2}}^2 + C(T) \ ||\partial_x \varphi||_2^2 \end{aligned}$$

We substitute the previous inequality in (57) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}||u||_2^2 \le -2\ \delta\ ||\partial_x u||_2^2 + \delta\ ||u||_{W^{1,2}}^2 +\ C(T)\ ||\partial_x \varphi||_2^2 + 4\ r. \tag{58}$$

Integrating (58) in time, and using (51) yield that there exists $C_3(T)$ such that (55) and (56) hold. □

Now we are in a position to show the global existence of solution to (48).

Proof of Theorem 2.4 (global existence).
By elliptic regularity, and the continuous embedding of $W^{2,2}(-1,1)$ in $W^{1,\infty}(-1,1)$, we have

$$||\partial_x \varphi(t)||_\infty \le C\ ||\varphi(t)||_{W^{2,2}} \le C\ ||E'(u)\ \partial_x u||_2 \le C\ ||\partial_x u||_2,$$

which together with (56), implies that

$$\int_0^t ||\partial_x \varphi(s)||_\infty^2\ ds \le C\ \int_0^t ||\partial_x u(s)||_2^2\ ds \le C(T), \quad \text{for all } t \in [0,T] \cap [0, T_{\max}).$$

Thanks to this estimate, we now argue as in the proof of Lemma 5.2 and Lemma 5.3 to get that

$$||u(t)||_\infty + ||\partial_x u(t)||_2 \le C(T), \quad \text{for all } t \in [0,T] \cap [0, T_{\max}).$$

Thus, the maximal solution u of (48) cannot explode in finite time. □

To complete the proof of Theorem 2.4, it remains to prove the asymptotic behaviour of u when $t \to \infty$. We note that we have the following lemma which controls the $L^1(-1,1)$ norm of u. For $f \in L^1(-1,1)$, we set

$$< f >= \frac{1}{2} \int_{-1}^1 f(x)\ dx$$

Lemma 5.6. *Let the same assumptions as that of Theorem 2.4 hold, and let u be the nonnegative global solution of (48). For $r > 0$, there exists a constant $C_0 > 0$ such that*

$$0 \le < u(t) > \le C_0, \quad t \in (0, \infty), \tag{59}$$

and if $r = 0$

$$< u(t) > = < u_0 > = \frac{1}{2}||u_0||_1, \quad t \in (0, \infty). \tag{60}$$

Proof. We note that if $r = 0$, $\frac{\mathrm{d}}{\mathrm{d}t} < u > = 0$, so that

$$< u(t) > = \frac{1}{2}\ ||u(t)||_1 = \frac{1}{2}\ ||u_0||_1.$$

If $r > 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} < u(t) > &= r\ < u(t) > - r\ < u^2(t) > \\ &\le r\ < u(t) > - r\ < u(t) >^2, \end{aligned}$$

whence $< u(t) > \le \max\{1, < u_0 >\}$. □

Next we turn to the existence of a Liapunov functional for (48) which is the cornerstone of our analysis.

Lemma 5.7. *Let the same assumptions as of that Theorem 2.4 hold, and let u be the nonnegative global solution of* (48). *There exists a constant C_1 such that*

$$r \int_0^\infty \int_{-1}^1 u\ (u-1)\ \log u\ dx\ dt + \int_0^\infty \left(\ ||\partial_x \sqrt{u}||_2^2 + \varepsilon\ ||\partial_x \varphi||_2^2 + ||\varphi||_2^2\ \right)\ dt \le C_1, \quad (61)$$

and

$$\int_0^\infty ||\varphi||_\infty^2\ ds \le C_1. \quad (62)$$

Proof. Let us define the following functional L

$$L(u) = \int_{-1}^1 (u\ \log u - u + 1)\ dx \ge 0,$$

and show that it is a Liapunov functional. Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t} L(u) = -\delta \int_{-1}^1 \frac{|\partial_x u|^2}{u}\ dx + \int_{-1}^1 \varphi\ \partial_x u\ dx + r \int_{-1}^1 u\ \log u\ (1-u)\ dx. \quad (63)$$

Combining (63) and (53) we obtain that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} L(u) &= -4\ \delta \int_{-1}^1 |\partial_x \sqrt{u}|^2\ dx - \varepsilon \int_{-1}^1 |\partial_x \varphi|^2\ dx - \int_{-1}^1 |\varphi|^2\ dx + r \int_{-1}^1 u\ \log u\ (1-u)\ dx. \\ &= -D(u,\varphi) \le 0, \end{aligned} \quad (64)$$

since $u\ \log u\ (1-u) \le 0$. Then for all $t \ge 0$

$$L(u(t)) + \int_0^t D(u(s),\varphi(s))\ ds \le L(u_0).$$

Since u_0 and u are nonnegative , we have

$$L(u_0) \ \le\ \int_{-1}^1 \left(u_0^2 + 1\right)\ dx \le ||u_0||_2^2 + 2,$$

and

$$L(u(t)) \ge -\frac{2}{e} - \int_{-1}^1 u_0\ dx,$$

so that

$$\int_0^t D(u(s),\varphi(s))\ ds \le 1 + ||u_0||_2^2 + \frac{2}{e} + 2\ < u_0 >, \quad t \ge 0. \quad (65)$$

Therefore, (65) yields there exists $C_1 > 0$ such that

$$\int_0^\infty D(u,\varphi)\ dt \le C_1. \quad (66)$$

From (66), we see that (61) holds true. In addition, inequality (61) together with Sobolev's embedding theorem give (62). □

In the following lemma we show that $\{u(t):\ t \ge 0\}$ is bounded in $W^{1,2}(-1,1)$.

Lemma 5.8. *Let the same assumptions as that of Theorem 2.4 hold, and let u be the nonnegative global solution of* (48). *Then u belongs to $L^\infty\left((0,\infty); W^{1,2}(-1,1)\right)$.*

Proof. First,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\ ||u- < u > ||_2^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\ ||u||_2^2 - \int_{-1}^{1} \partial_t u\ dx\ \int_{-1}^{1} u\ dx \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\ ||u||_2^2 - 4\ r\ \ < u >^2 + 4\ r\ \ < u > < u^2 > \\
&\leq \frac{\mathrm{d}}{\mathrm{d}t}\ ||u||_2^2 + 2\ r\ C_0\ ||u||_2^2. \qquad (67)
\end{aligned}$$

Multiplying the first equation in (48) by 2 u, integrating it over $(-1, 1)$, and using the Cauchy-Schwarz inequality and the fact that $u \geq 0$ we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\ ||u||_2^2 &\leq -2\ \delta \int_{-1}^{1} |\partial_x u|^2\ dx - \int_{-1}^{1} u^2\ \partial_x \varphi\ dx + 2\ r \int_{-1}^{1} u^2\ (1-u)\ dx \\
&\leq -2\ \delta\ ||\partial_x u||_2^2 + ||u||_4^2\ ||\partial_x \varphi||_2 + 2\ r. \qquad (68)
\end{aligned}$$

Gagliardo-Nirenberg inequality (20) together with the Poincaré inequality (19) and (59) give

$$\begin{aligned}
||u||_4^2 \leq C\ ||u||_{W^{1,2}}\ ||u||_1 &\leq C\ \left(\ ||\partial_x u||_2 + ||u||_1\ \right) \\
&\leq C\left(||\partial_x u||_2 + 1\right). \qquad (69)
\end{aligned}$$

Thus

$$||u||_4^2\ ||\partial_x \varphi||_2 \leq C\ \left(||\partial_x u||_2 + 1\right)\ ||\partial_x \varphi||_2. \qquad (70)$$

Substituting (70), (69) and (68) into (67), and using Young and Hölder inequalities to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\ ||u- < u > ||_2^2 &\leq -2\ \delta\ ||\partial_x u||_2^2 + C\ \left(||\partial_x u||_2 + 1\right)\ ||\partial_x \varphi||_2 + 2r \\
&+\ 2r\ C_0\ ||u||_1^{\frac{2}{3}}\ ||u||_4^{\frac{4}{3}} \\
&\leq -2\ \delta\ ||\partial_x u||_2^2 + C\ ||\partial_x \varphi||_2^2 + \delta\ ||\partial_x u||_2^2 + C.
\end{aligned}$$

Using Poincaré's inequality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\ ||u(t)- < u(t) > ||_2^2 + \alpha\ ||u(t)- < u(t) > ||_2^2\ \leq\ C\ ||\partial_x \varphi(t)||_2^2 + C,$$

for some $\alpha > 0$ independent of t. Integrating this differential inequality gives

$$\begin{aligned}
e^{\alpha\ t}\ ||u(t)- < u(t) > ||_2^2 &\leq ||u_0- < u_0 > ||_2^2 + \int_0^t e^{\alpha\ s}\left(C\ ||\partial_x \varphi(s)||_2^2 + C\right)\ ds \\
||u(t)- < u(t) > ||_2^2 &\leq C\ e^{-\alpha\ t} + C\ \int_0^t e^{\alpha\ (s-t)}\ \left(||\partial_x \varphi(s)||_2^2 + 1\right)\ ds
\end{aligned}$$

Since $e^{-\alpha\ t} \leq 1$, and $e^{\alpha\ (s-t)} \leq 1$ as $s \leq t$ we obtain

$$||u(t)- < u(t) > ||_2^2\ \leq\ C + C\ \int_0^t ||\partial_x \varphi(s)||_2^2\ ds + \frac{1}{\alpha},$$

Using (61) we end up with

$$||u(t)- < u(t) > ||_2^2 \leq C, \qquad (71)$$

where C is independent of t. Therefore, u belongs to $L^\infty((0,\infty);L^2(-1,1))$.

It remains to show that $\partial_x u$ is in $L^\infty((0,\infty);L^2(-1,1))$. We multiply the first equation in (48) by $-\partial^2_{xx}u$ and integrate it over $(-1,1)$. Since $u \geq 0$ we use Cauchy-Schwarz and Young inequalities and (71) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial_x u||_2^2 &\leq -\delta\ ||\partial^2_{xx}u||_2^2 + \int_{-1}^1 \partial^2_{xx}u\ (\partial_x u\ \varphi + \partial_x\varphi\ u) - r\int_{-1}^1 u\ (1-u)\ \partial^2_{xx}u\ dx \\
&\leq -\delta\ ||\partial^2_{xx}u||_2^2 - \frac{1}{2}\int_{-1}^1 \partial_x\varphi\ |\partial_x u|^2\ dx + \frac{\delta}{8}\ ||\partial^2_{xx}u||_2^2 \\
&+ \ C\int_{-1}^1 |u|^2\ |\partial_x\varphi|^2\ dx - r\int_{-1}^1 u\ \partial^2_{xx}u\ dx - 2\ r\ \int_{-1}^1 u\ |\partial_x u|^2\ dx \\
&\leq -\delta\ ||\partial^2_{xx}u||_2^2 + \frac{1}{2}\ ||\partial_x\varphi||_2\ ||\partial_x u||_4^2 + \frac{\delta}{8}\ ||\partial^2_{xx}u||_2^2 \\
&+ \ C\ ||u||_\infty^2\ ||\partial_x\varphi||_2^2 + C + \frac{\delta}{8}\ ||\partial^2_{xx}u||_2^2. \qquad (72)
\end{aligned}$$

Since $\partial_x u \in W_0^{1,2}(-1,1)$, using the Gagliardo-Nirenberg inequality (20) and the classical Poincaré inequality (19), we obtain

$$||\partial_x u||_4 \leq C\ ||\partial^2_{xx}u||_2^{\frac{1}{2}}\ ||\partial_x u||_1^{\frac{1}{2}}. \qquad (73)$$

Then, we substitute (73) into (72), and by Young inequality, the Sobolev embedding, (59) and (71), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial_x u||_2^2 &\leq -\frac{3\ \delta}{4}\ ||\partial^2_{xx}u||_2^2 + C\ ||\partial_x\varphi||_2\ ||\partial^2_{xx}u||_2\ ||\partial_x u||_1 + C\ ||u||_\infty^2\ ||\partial_x\varphi||_2^2 + C \\
&\leq -\frac{\delta}{2}\ ||\partial^2_{xx}u||_2^2 + C\ ||\partial_x\varphi||_2^2\ ||\partial_x u||_1^2 + C\ ||u||_{W^{1,2}}^2\ ||\partial_x\varphi||_2^2 + C \\
&\leq -\frac{\delta}{2}\ ||\partial^2_{xx}u||_2^2 + C\ ||\partial_x u||_2^2\ ||\partial_x\varphi||_2^2 + C\ (||\partial_x\varphi||_2^2 + 1). \qquad (74)
\end{aligned}$$

Since $\partial_x u \in W_0^{1,2}(-1,1)$, we use once more the classical Poincaré inequality to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}||\partial_x u||_2^2 + \beta||\partial_x u||_2^2 \leq C\ ||\partial_x u||_2^2\ ||\partial_x\varphi||_2^2 + C\ (||\partial_x\varphi||_2^2 + 1),$$

for some $\beta > 0$ independent of t.

Define

$$\phi(t) = \int_0^t ||\partial_x\varphi(s)||^2\ ds, \quad t \geq 0,$$

and notice that , since $||\partial_x\varphi||_2^2$ belongs to $L^1(0,\infty)$ by (61),

$$0 \leq \phi(t) \leq \phi_\infty = \int_0^\infty ||\partial_x\varphi(s)||_2^2\ ds.$$

Integrating the previous differential inequality we find

$$\begin{aligned}
||\partial_x u(t)||_2^2 &\leq ||\partial_x u_0||_2^2 \ e^{C\phi(t)-\beta\ t} \\
&+ C\int_0^t (1+\phi'(s))\ e^{\beta(s-t)+C\phi(t)-C\phi(s)}\ ds \\
&\leq ||\partial_x u_0||_2^2\ e^{C\ \phi_\infty} + C\ e^{C\ \phi_\infty} \int_0^t [e^{\beta(s-t)}+\phi'(s)]\ ds \\
&\leq ||\partial_x u_0||_2^2\ e^{C\ \phi_\infty} + C\ e^{C\ \phi_\infty} \left(\frac{1}{\beta}+\phi_\infty\right).
\end{aligned}$$

Therefore, $\partial_x u$ belongs to $L^\infty\left((0,\infty), L^2(-1,1)\right)$, and Lemma 5.8 is proved. ☐

Lemma 5.9. *Let the same assumptions as that of Theorem 2.4 hold, and let u be the nonnegative global solution of* (48)*. There is C_2 such that*

$$\int_0^\infty ||\partial_t u||_2^2\ dt \leq C_2. \tag{75}$$

Proof. We multiply the first equation in (48) by $\partial_t u$ and integrate it over $(-1,1)$ to obtain

$$\begin{aligned}
\int_{-1}^1 (\partial_t u)^2\ dx &= \delta \int_{-1}^1 \partial_{xx}^2 u\ \partial_t u\ dx - \int_{-1}^1 \partial_x(u\ \varphi)\ \partial_t u\ dx + r\int_{-1}^1 u\ (1-u)\ \partial_t u\ dx \\
&= -\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial_x u||_2^2 - \int_{-1}^1 (\partial_x u\ \varphi + u\ \partial_x\varphi)\ \partial_t u\ dx + r\int_{-1}^1 (u-u^2)\ \partial_t u\ dx.
\end{aligned}$$

Using Young and Cauchy-Schwarz inequalities we obtain

$$\begin{aligned}
||\partial_t u||_2^2 &\leq -\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial_x u||_2^2 + \int_{-1}^1 |\partial_x u|^2\ |\varphi|^2\ dx + \frac{1}{4}||\partial_t u||_2^2 + \int_{-1}^1 |u|^2\ |\partial_x \varphi|^2\ dx + \frac{1}{4}||\partial_t u||_2^2 \\
&+ r\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^1 \left(\frac{u^2}{2}-\frac{u^3}{3}\right)\ dx,
\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta}{2}\ ||\partial_x u||_2^2 + \int_{-1}^1 F(u)\ dx\right) + \frac{1}{2}||\partial_t u||_2^2 \leq ||\partial_x u||_2^2\ ||\varphi||_\infty^2 + ||u||_\infty^2\ ||\partial_x\varphi||_2^2,$$

where $F(u) = r\left(-\frac{u^2}{2}+\frac{u^3}{3}\right) \geq -\frac{r}{6}$.
Next we integrate the above inequality in time, and use (62), (61) and Lemma 5.8 to obtain

$$-\frac{r}{3}+\frac{1}{2}\int_0^t ||\partial_t u||_2^2\ ds \leq C + C\int_0^t \left(||\varphi||_\infty^2 + ||\partial_x\varphi||_2^2\right)\ ds \leq C$$

for $t \geq 0$ where C is independent of t. We have thus proved (75). ☐

To end the proof of Theorem 2.4, our aim now is to look at the large time behaviour of the solution.

Proof of Theorem 2.4, (large time behaviour). In this proof, we follow [7].

By Lemma 5.8, the family $\{u(t),\ t \geq 0\}$ is bounded in $W^{1,2}(-1,1)$. Since the embedding of $W^{1,2}(-1,1)$ in $L^2(-1,1)$ is compact then, there are a sequence of positive time (t_n), such that $t_n \to \infty$, and $z \in L^2(-1,1)$ such that

$$z = \lim_{n\to\infty} u(t_n) \text{ in } L^2(-1,1) \text{ and a.e. in } (-1,1).$$

Consider

$$U_n(s,x) = u(t_n + s, x), \quad x \in (-1,1),\ -1 < s < 1,\ n > 0,$$

and

$$\Phi_n(s,x) = \varphi(t_n + s, x), \quad -1 < s < 1.$$

We first prove that

$$U_n \longrightarrow z \ \text{ as } n \to \infty, \ \text{ in } C\left([-1,1]; L^2(-1,1)\right). \tag{76}$$

Indeed for each $s \in (-1,1)$

$$\int_{-1}^{1} |u(t_n+s,x) - u(t_n,x)|^2\ dx \leq \int_{-1}^{1} \int_{t_n-1}^{t_n+1} |\partial_t u|^2\ dt\ dx.$$

Hence

$$\sup_{s\in[-1,1]} ||U_n(s) - u(t_n)||_2 \leq \left[2 \int_{-1}^{1} \int_{t_n-1}^{\infty} |\partial_t u|^2\ dt\ dx\right]^{\frac{1}{2}}.$$

The right hand side goes to zero as $n \to \infty$ by Lemma 5.9. Letting $n \to \infty$ in the above inequality gives (76).

Next, using the definition of $D(u,\varphi)$ which is given in (64) we obtain that

$$\begin{aligned}
&\int_{-1}^{1} \left(r\ ||U_n\ \log U_n\ (U_n - 1)||_1 + ||\partial_x \sqrt{U_n}||_2^2 + ||\Phi_n||_2^2 + \varepsilon\ ||\partial_x \Phi_n||_2^2\right)\ ds \\
&\leq \int_{t_n-1}^{t_n+1} \left(r\ ||u(s)\ \log u(s)\ (u(s) - 1)||_1 + ||\partial_x \sqrt{u(s)}||_2^2 + ||\varphi(s)||_2^2 + \varepsilon\ ||\partial_x \varphi(s)||_2^2\right)\ ds \\
&\leq 2 \int_{t_n-1}^{\infty} D(u,\varphi)\ ds.
\end{aligned} \tag{77}$$

The right-hand side of (77) goes to zero as $n \to \infty$ by (66), so that

$$\Phi_n \longrightarrow 0 \ \text{ as } n \to \infty, \ \text{ in } L^2\left((-1,1); W^{1,2}(-1,1)\right).$$

In addition, using Cauchy-Schwarz inequality, (59) and (77) we obtain

$$\begin{aligned}
\int_{-1}^{1} ||\partial_x U_n(s)||_1^2\ ds &= 4 \int_{-1}^{1} \left(\int_{-1}^{1} \sqrt{U_n(s)}\ |\partial_x \sqrt{U_n(s)}|\ dx\right)^2\ ds \\
&\leq 4 \int_{-1}^{1} ||U_n(s)||_1\ ||\partial_x \sqrt{U_n(s)}||_2^2\ ds \\
&\leq C \int_{t_n-1}^{\infty} D(u,\varphi)\ ds.
\end{aligned}$$

Since the right-hand side goes to zero as $n \to \infty$ by (66), then we have

$$\partial_x U_n \longrightarrow 0 \ \text{ as } n \to \infty, \ \text{ in } L^2\left((-1,1); L^1(-1,1)\right). \tag{78}$$

Since the limit in the sense of distribution is unique, (76) and (78) yield that

$$\partial_x z = 0. \tag{79}$$

If $r = 0$, (79) together with (60) and (76) give that $z =< u_0 >$. We have thus shown that $< u_0 >$ is the only cluster point of $\{u(t),\ t \geq 0\}$. Since $\{u(t),\ t \geq 0\}$ is relatively compact in $L^2(-1,1)$ thanks to its boundedness in $W^{1,2}(-1,1)$ (see Lemma 5.8), we conclude that $u(t)$ converges to $< u_0 >$ in $L^2(-1,1)$ as $t \longrightarrow \infty$.

If $r > 0$, by (77) we have

$$\int_{-1}^{1} ||U_n\ \log U_n\ (U_n - 1)||_1\ ds \longrightarrow 0, \quad \text{as } n \to \infty, \tag{80}$$

Since (U_n) is bounded in $L^\infty((-1,1) \times (-1,1))$ thanks to the boundness of $\{u(t),\ t \geq 0\}$ in $W^{1,2}(-1,1)$ and the embedding of $W^{1,2}(-1,1)$ in $L^\infty(-1,1)$, we infer from (76), (79), (80) that $z\ \log z\ (z-1) = 0$, that is $z = 0$ or $z = 1$. Therefore 0 and 1 are the only two cluster points of $\{u(t),\ t \geq 0\}$ as $t \to \infty$. Since the ω-limit set of u is a compact connected subset of $L^2(-1,1)$, see [1, Theorem 9.1.8] for instance, we conclude that $u(t)$ converges either to 0 or to 1 in $L^2(-1,1)$ as $t \longrightarrow \infty$.

□

6 Limiting behaviour as $\varepsilon \to 0$

When $E(u) = 1 - u$, letting $\varepsilon \to 0$ in (48) formally leads to (9) which is well-posed since $E' < 0$ and $\delta > 0$. The purpose of this section is to justify rigorously this fact and prove Theorem 2.6. Let $T > 0$, $\delta > 0$, $r \geq 0$, $\varepsilon > 0$ and a nonnegative initial condition $u_0 \in W^{1,2}(-1,1)$. We discuss the limit as $\varepsilon \to 0$ of the unique solution u_ε of

$$\left\{ \begin{array}{lcll} \partial_t u_\varepsilon & = & \delta\ \partial^2_{xx} u_\varepsilon - \partial_x(u_\varepsilon\ \varphi_\varepsilon) + r\ u_\varepsilon\ (1 - u_\varepsilon) & \text{in } (0,T) \times (-1,1), \\ u_\varepsilon(0,x) & = & u_0(x) & \text{in } (-1,1), \\ \partial_x u_\varepsilon(t,\pm 1) & = & 0 & \text{on } (0,T), \end{array} \right. \tag{81}$$

given by Theorem 2.4, where φ_ε is the unique solution of

$$\left\{ \begin{array}{lcll} -\varepsilon\ \partial^2_{xx}\varphi_\varepsilon + \varphi_\varepsilon & = & -\partial_x u_\varepsilon & \text{in } (0,T) \times (-1,1), \\ \varphi_\varepsilon(t,\pm 1) & = & 0 & \text{on } (0,T). \end{array} \right.$$

6.1 Estimates

Lemma 6.1. *There is $C_1(T)$ independent of ε such that*

$$\int_0^T \left(\delta\ ||\partial_x \sqrt{u_\varepsilon}||_2^2 + \varepsilon\ ||\partial_x \varphi_\varepsilon||_2^2 + ||\varphi_\varepsilon||_2^2\right)\ dt \leq C_1(T), \tag{82}$$

Proof. By (54) see (the proof of Lemma 5.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^{1} u_\varepsilon\ \log u_\varepsilon\ dx + \varepsilon\ ||\partial_x \varphi_\varepsilon||_2^2 + ||\varphi_\varepsilon||_2^2 \leq -4\ \delta \int_{-1}^{1} |\partial_x \sqrt{u_\varepsilon}|^2\ dx + 2\ r,$$

from which (82) follows by a time integration. □

Using Gagliardo-Nirenberg inequality (20) we obtain the following estimate:

Lemma 6.2. *For $2 \leq p \leq 6$, there exists $C_2(T,p)$ independent of ε such that*

$$\int_0^T ||u_\varepsilon||_{\frac{p}{2}}^{\frac{p}{2}} \; dt \leq C_2(T,p). \tag{83}$$

Proof. For $t \in (0,T)$, thanks to (82), we can use Gagliardo-Nirenberg inequality (20) on $\sqrt{u_\varepsilon}$ and we obtain for all $p \in [2,\infty)$

$$||\sqrt{u_\varepsilon(t)}||_p \leq C \; ||\sqrt{u_\varepsilon(t)}||_{W^{1,2}}^{\theta} \; ||\sqrt{u_\varepsilon(t)}||_2^{1-\theta}, \tag{84}$$

where

$$\theta = \frac{p-2}{2\;p}.$$

Therefore

$$\begin{aligned} ||\sqrt{u_\varepsilon(t)}||_p^p &\leq C \; ||\sqrt{u_\varepsilon(t)}||_{W^{1,2}}^{\frac{p-2}{2}} \; ||\sqrt{u_\varepsilon(t)}||_2^{\frac{p+2}{2}} \\ ||u_\varepsilon(t)||_{\frac{p}{2}}^{\frac{p}{2}} &\leq C \; ||\sqrt{u_\varepsilon(t)}||_{W^{1,2}}^{\frac{p-2}{2}} \; ||u_\varepsilon(t)||_1^{\frac{p+2}{4}}. \end{aligned}$$

Since

$$||u_\varepsilon(t)||_1 \leq ||u_0||_1 + 2\; r\; T,$$

and $\frac{p-2}{2} \leq 2$ for $2 \leq p \leq 6$, the estimate (83) follows from (82) and the previous inequalities. □

Lemma 6.3. *There is $C_3(T)$ independent of ε such that*

$$\int_0^T ||\partial_x u_\varepsilon||_{\frac{3}{2}}^{\frac{3}{2}} \; dt \leq C_3(T). \tag{85}$$

Proof. Hölder and Young inequalities together with (82) and (83) with $p = 6$ yield

$$\begin{aligned} \int_0^T ||\partial_x u_\varepsilon||_{\frac{3}{2}}^{\frac{3}{2}} \; dt &\leq 2^{\frac{3}{2}} \int_0^T ||\sqrt{u_\varepsilon} \; \partial_x \sqrt{u_\varepsilon}||_{\frac{3}{2}}^{\frac{3}{2}} \; dt \\ &\leq 2^{\frac{3}{2}} \int_0^T ||u_\varepsilon||_3^{\frac{3}{4}} \; ||\partial_x \sqrt{u_\varepsilon}||_2^{\frac{3}{2}} \; dt \\ &\leq C \int_0^T \left(||u_\varepsilon||_3^3 + ||\partial_x \sqrt{u_\varepsilon}||_2^2\right) \; dt \leq C_3(T), \end{aligned}$$

which gives the result. □

Lemma 6.4. *There is $C_4(T)$ independent of ε such that*

$$\int_0^T ||\partial_t u_\varepsilon||_{(W^{2,\frac{3}{2}})'} \; dt \leq C_4(T).$$

Proof. Consider $\psi \in W^{2,\frac{3}{2}}(-1,1)$ and $t \in (0,T)$, we have

$$\begin{aligned} &\left|\int_{-1}^1 \partial_t u_\varepsilon \; \psi \; dx\right| \\ = &\left|\int_{-1}^1 \left[\partial_x\left(\delta \; \partial_x u_\varepsilon - u_\varepsilon \; \varphi_\varepsilon\right) + r\; u_\varepsilon \; E(u_\varepsilon)\right] \; \psi \; dx\right| \\ = &\left|\int_{-1}^1 \left(-\delta \; \partial_x u_\varepsilon \; \partial_x \psi + u_\varepsilon \; \varphi_\varepsilon \; \partial_x \psi + r \; u_\varepsilon \; (1-u_\varepsilon) \; \psi\right) \; dx\right| \\ \leq \; & \delta \; ||\partial_x \psi||_\infty \; ||\partial_x u_\varepsilon||_1 + ||\partial_x \psi||_\infty \; \; ||u_\varepsilon||_2 \; ||\varphi_\varepsilon||_2 + r \; ||u_\varepsilon(1-u_\varepsilon)||_1 \; ||\psi||_\infty. \end{aligned}$$

Using the embedding of $W^{2,\frac{3}{2}}(-1,1)$ in $W^{1,\infty}(-1,1)$, and Young's inequality, we end up with

$$\begin{aligned}\left|\int_{-1}^{1} \partial_t u_\varepsilon \ \psi \ dx\right| &\leq \left(\delta \ ||\partial_x u_\varepsilon||_1 + ||u_\varepsilon||_2 \ ||\varphi_\varepsilon||_2 + r \ ||u_\varepsilon||_1 + r \ ||u_\varepsilon||_2^2\right) \ ||\psi||_{W^{2,\frac{3}{2}}} \\ &\leq C \ \left(||\partial_x u_\varepsilon||_{\frac{3}{2}} + ||u_\varepsilon||_2^2 + ||\varphi_\varepsilon||_2^2 + 1\right) \ ||\psi||_{W^{2,\frac{3}{2}}},\end{aligned}$$

and a duality argument gives

$$||\partial_t u_\varepsilon(t)||_{(W^{2,\frac{3}{2}})'} \leq C \ \left(||\partial_x u_\varepsilon||_{\frac{3}{2}} + ||u_\varepsilon||_2^2 + ||\varphi_\varepsilon||_2^2 + 1\right).$$

Integrating the above inequality over $(0,T)$ and using Young's inequality we obtain

$$\int_0^T ||\partial_t u_\varepsilon(t)||_{(W^{2,\frac{3}{2}})'} \ dt \leq C(T) \ \int_0^T \left(||\partial_x u_\varepsilon||_{\frac{3}{2}}^{\frac{3}{2}} + ||u_\varepsilon||_2^2 + ||\varphi_\varepsilon||_2^2 + 1\right) \ dt.$$

By Lemma 6.1, Lemma 6.3 and Lemma 6.2 with $p = 4$ the right-hand side of the above inequality is bounded independently of ε and the proof of Lemma 6.4 is complete. □

6.2 Convergence

In this section we discuss the limit of u_ε as $\varepsilon \to 0$. For that purpose, we study the compactness properties of $(u_\varepsilon, \varphi_\varepsilon)$.

Proof of Theorem 2.6. Thanks to Lemma 6.2 and Lemma 6.3, $(u_\varepsilon)_\varepsilon$ is bounded in $L^{\frac{3}{2}}((0,T); W^{1,\frac{3}{2}}(-1,1))$ while $(\partial_t u_\varepsilon)_\varepsilon$ is bounded in $L^1((0,T); (W^{2,\frac{3}{2}})'(-1,1))$ by Lemma 6.4. Since $W^{1,\frac{3}{2}}(-1,1)$ is compactly embedded in $C([-1,1])$ and $C([-1,1])$ is continuously embedded in $(W^{2,\frac{3}{2}})'(-1,1)$, it follows from [9, Corollary 4] that $(u_\varepsilon)_\varepsilon$ is relatively compact in $L^{\frac{3}{2}}((0,T); C([-1,1]))$. Therefore, there are a sequence (ε_j) of positive real numbers, $\varepsilon_j \to 0$, and $u \in L^{\frac{3}{2}}\left((0,T); W^{1,\frac{3}{2}}\right)$ such that

$$u_{\varepsilon_j} \rightharpoonup u \ \text{ in } L^{\frac{3}{2}}\left((0,T); W^{1,\frac{3}{2}}(-1,1)\right), \tag{86}$$

and

$$u_{\varepsilon_j} \to u \text{ in } L^{\frac{3}{2}}\left((0,T); C[-1,1]\right) \text{ and a.e. in } (0,T) \times (-1,1). \tag{87}$$

Since $(u_\varepsilon)_\varepsilon$ is bounded in $L^\infty\left((0,T); L^1(-1,1)\right)$ by (59), it follows from (87)

$$\int_0^T ||u_{\varepsilon_j} - u||_2^3 \ dt \leq \int_0^T ||u_{\varepsilon_j} - u||_1^{\frac{3}{2}} \ ||u_{\varepsilon_j} - u||_\infty^{\frac{3}{2}} \ dt \leq \int_0^T ||u_{\varepsilon_j} - u||_\infty^{\frac{3}{2}} \ dt \to 0,$$

when $\varepsilon_j \to 0$. In particular, we have

$$u_{\varepsilon_j} \longrightarrow u, \ \text{ in } L^2\left((0,T) \times (-1,1)\right). \tag{88}$$

Observe that the nonnegativity of u follows easily from that of (u_{ε_j}) by (88).

Owing to Lemma 6.1, we may also assume that

$$\varphi_{\varepsilon_j} \rightharpoonup \varphi \text{ in } L^2((0,T) \times (-1,1)) \ \text{ as } \varepsilon_j \to 0, \tag{89}$$

$$\varepsilon_j \ \partial_x \varphi_{\varepsilon_j} \to 0 \text{ in } L^2((0,T) \times (-1,1)) \ \text{ as } \varepsilon_j \to 0. \tag{90}$$

It remains to identify the equations solved by the limit u of (u_{ε_j}). Let $\psi \in C^2([0,T] \times [-1,1])$ with $\psi(T) = 0$. Since

$$\int_0^T \int_{-1}^1 \partial_t u_{\varepsilon_j} \ \psi \ dxdt = \int_0^T \int_{-1}^1 \big((-\delta \ \partial_x u_{\varepsilon_j} + u_{\varepsilon_j} \ \varphi_{\varepsilon_j}) \ \partial_x \psi + r \ u_{\varepsilon_j} \ E(u_{\varepsilon_j}) \ \psi\big) \ \ dxdt \quad (91)$$

and

$$\varepsilon_j \int_0^T \int_{-1}^1 \partial_x \varphi_{\varepsilon_j} \ \partial_x \psi \ dxdt + \int_0^T \int_{-1}^1 \varphi_{\varepsilon_j} \ \psi \ \ dxdt = -\int_0^T \int_{-1}^1 \partial_x u_{\varepsilon_j} \ \psi \ dxdt. \quad (92)$$

Owing to (86), (89) and (90), it is straightforward to pass to the limit as $\varepsilon_j \to 0$ in (92) and find

$$\int_0^T \int_{-1}^1 \varphi \ \psi \ dxdt = -\int_0^T \int_{-1}^1 \partial_x u \ \psi \ dxdt,$$

which gives that

$$\varphi = -\partial_x u. \quad (93)$$

Next, by (87) and (86) we see that

$$\int_0^T \int_{-1}^1 \partial_t u_{\varepsilon_j} \ \psi \ dxdt \longrightarrow -\int_0^T \int_{-1}^1 u \ \partial_t \psi \ dxdt - \int_{-1}^1 u_0(x) \ \psi(0,x) \ dx, \quad \text{as } \varepsilon_j \to 0,$$

and

$$\int_0^T \int_{-1}^1 \partial_x u_{\varepsilon_j} \ \partial_x \psi \ dxdt \longrightarrow \int_0^T \int_{-1}^1 \partial_x u \ \partial_x \psi \ dxdt \ \ \text{as } \varepsilon_j \to 0.$$

From (88), (89) and (93) we see that

$$\int_0^T \int_{-1}^1 u_{\varepsilon_j} \ \varphi_{\varepsilon_j} \ \partial_x \psi \ dxdt \longrightarrow -\int_0^T \int_{-1}^1 u \ \partial_x u \ \partial_x \psi \ dxdt, \quad \text{as } \varepsilon_j \to 0.$$

From (88) we get

$$r \int_0^T \int_{-1}^1 u_{\varepsilon_j} \ E(u_{\varepsilon_j}) \ \psi \ dxdt \longrightarrow r \ \int_0^T \int_{-1}^1 u \ E(u) \ \psi \ dxdt, \quad \text{as } \varepsilon_j \to 0.$$

Thus we conclude that u satisfies

$$\int_0^T \langle \partial_t u, \psi \rangle \ dt = \int_0^T \int_{-1}^1 ((-\delta \ \partial_x u - u \ \partial_x u) \ \partial_x \psi + r \ u \ E(u) \ \psi) \ \ dx \ dt,$$

for all test functions ψ. Therefore, u is a weak solution of (15), and classical regularity results ensure that u is actually a classical solution of (15). Since it is unique and the only possible cluster point of $(u_\varepsilon)_\varepsilon$ in $L^2((0,T) \times (-1,1))$, we conclude that the whole family $(u_\varepsilon)_\varepsilon$ converges to u in $L^2((0,T) \times (-1,1))$ as $\varepsilon \to 0$. □

Acknowledgment

I thank Philippe Laurençot for his helpful advices and comments during this work.

References

[1] T. Cazenave. A. Haraux. An Introduction to semilinear evolution equations. *Oxford lecture series in mathematics and it applications, (2006).*

[2] T. Cieślak, Ph. Laurençot, C. Morales-Rodrigo. Global existence and convergence to steady states in a chemorepulsion system. Parabolic and Navier-Stokes equations. *Part 1, 105-117, Banach Center Publ., 81, Part 1, Polish Acad. Sci. Inst. Math., Warsaw, 2008.*

[3] J. P. Dias. A simplified variational model for the bidimensional coupled evolution equations of a nematic liquid crystal. *J. Math. Anal. Appl. 67 (1979), no. 2, 525-541.*

[4] J. P. Dias. Un problème aux limites pour un système d'équations non linéaires tridimensionnel. *Bolletino, U. M.I. (5) 16-B (1979), 22-31.*

[5] P. Grindrod. Models of individual aggregation or clustering in single and multi-species communities. *J. Math. Biol. (1988) 26:651-660.*

[6] O. A. Ladyzenskaja. V . A. Solonnikov. N. N. Uraltseva. Linear and quasi-linear equations of parabolic type. *Providence (R.I.) , American Mathematical Society, (1988).*

[7] M. Langlais, D. Phillips. Stabilization of solutions of nonlinear and degenerate evolution equations. *Nonlinear Anal. 9 (1985), no. 4, 321-333.*

[8] M. Schoenauer. Quelques résultats de régularité pour un système elliptique avec conditions aux limites couplées. *Annales de la Faculté des Sciences de Toulouse 5e série, tome 2, no. 2(1980), 125-135.*

[9] J. Simon, Compact sets in the space $L^p(0,T;B)$. *Annali di Mathematica Pura ed Applicata (IV), vol. CXLVI, (1987), 65-69.*