Robust Measurement Feedback Control of an Inclined Cable

Lucie Baudouin, Simon Neild, Aude Rondepierre, David Wagg

Robust Measurement Feedback Control of an Inclined Cable

Lucie Baudouin [*] Simon Neild [**] Aude Rondepierre [***]
David Wagg [**]

[*] *CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France, Univ de Toulouse, LAAS, F-31400 Toulouse, France. (e-mail: baudouin@laas.fr).*
[**] *Faculty of Engineering, University of Bristol, Queens Building, University Walk, Bristol BS8 1TR, UK. (e-mail: simon.neild/david.wagg@bristol.ac.uk)*
[***] *Institut de Mathématiques de Toulouse, INSA de Toulouse, 135 avenue de Rangueil 31077 Toulouse, France. (e-mail: aude.rondepierre@math.univ-toulouse.fr).*

Abstract: Considering the partial differential equation model of the vibrations of an inclined cable, we are interested in applying robust control technics to stabilize the system with measurement feedback when it is submitted to external disturbances. This paper focuses indeed on the construction of a standard linear infinite dimensional state space system and an H_∞ feedback control of vibrations with partial observation of the state. The control and observation are performed using an active tendon.

Keywords: partial differential equations, robust control, inclined cable, state space model, measurement feedback.

1. INTRODUCTION

Inclined cables are critical components in many structures used in civil engineering - for instance in cable stayed bridges. Nonetheless, cable structures are not restricted to this range of applications and can also be found e.g. in telescopes or spacecraft, Smrz et al. (2011). One of the major issues related to such structures involving cables is the control of vibrations induced by any exterior perturbation.

Since cables are very flexible and lightly damped, cable structure systems usually have a range of dynamic problems. Their modelling is therefore very important in predicting and controlling the response to excitation. A good review about vibration suppression in civil structures and many references on this topic can be found in Song et al. (2006) and Preumont (1997).

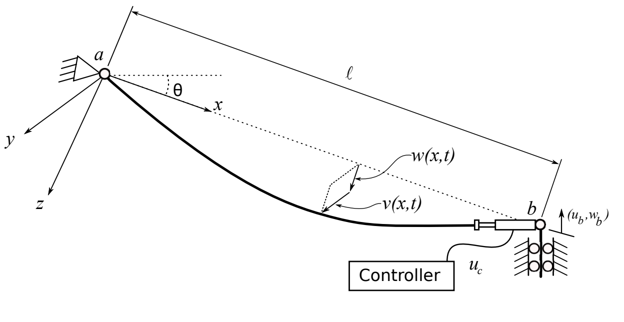

Fig. 1. Inclined Cable

This paper is devoted to the design of control laws for a vibrating system composed of an inclined cable connected at its bottom end to an active control input. We will present a linearized model using partial differential equations (PDE) and we aim at designing robust measurement feedback control laws for this kind of vibrating system.

In Section 2, we will first recall a linear PDE modeling of an inclined cable before detailing the standard state space model of infinite dimension describing the system, where the control and measurement of the vibrations of the inclined cable come from and active tendon connected to the cable, see Figure 1. Using a support motion in the cable axial direction is a natural choice of active control since the installation of the proper device can be done with small modification of the cable anchorage (see Fujino and Susumpow (1994)). We hope to obtain good results when considering robust control with partial observation using an active tendon since the collocation of actuator and sensor has proved great effectiveness in active damping of cables, as in for example Bossens and Preumont (2001), Song et al. (2006). In a second step, we will detail the state space model of infinite dimension we can derive from the PDE model and that we can use to perform the robust control of the inclined cable.

The H_∞-control question for an infinite dimensional system with measurement feedback will be described in Section 3. In mechanics, the usual study of the vibrations of inclined cables is made through the consideration of a few structural modes. An important aspect of this work lies in the fact that we consider, as far as we can, the complete PDE model of the in-plane vibrations. Here, we will show that we can apply a robust control strategy based on modern control tools for distributed parameter systems, presented in Bensoussan and Bernhard (1993) and van Keulen (1993). We will indeed prove the H_∞-control of the inclined cable, robust with respect to specific exterior perturbations, by the means of the possible resolution of Ricatti equations.

Among the numerous possibilities for modelling the motions of inclined cables with small sag, we adopt the PDE presented in Wagg and Neild (2010) using the derivation from Warnitchai et al. (1995).

The cable, which is of length ℓ, is supported at end points a and b and the direction of the chord line from a to b is defined as x, see Figure 1. The angle of inclination of the chord line relative to the horizontal is denoted θ and we set ρ the density of the cable, $\mathcal{A}$ the cross-sectional area, E Young's modulus and g gravity. We then define $\varrho = \rho g \cos\theta$ as the distributed weight perpendicular to the cable cord. The cable equilibrium sag position and the chord line both lie in the xz plane.

We assume that there is no significant dynamic response along the x-axis (the axial vibrations are usually excluded from models since the frequency of oscillations is much faster and of smaller amplitude than that in other directions) and that the sag is small in comparison to the length of the cable.

Each dynamic variable is the sum of a quasi-static component ($_q$) and a modal component ($_m$):

- $u = u_q(x,t) + u_m(x,t)$ is the dynamic axial displacement of the cable ;
- v is the dynamic out-of-plane transverse displacement;
- $w = w_q(x,t) + w_m(x,t)$ is the dynamic in-plane transverse displacement ;
- T_s is the static tension of the cable and is assumed to be constant (w.r.t. x and t) ;
- $T_d = T_q + T_m$ is the dynamic tension of the cable and is assumed not to depend on x ;
- $w_s(x) = \dfrac{\varrho\mathcal{A}}{2T_s}\left(\ell x - x^2\right)$ is the static in-plane displaced shape of the cable.

As one can read (and find details about the modeling of the inclined cable movement) in Wagg and Neild (2010), since we neglect the axial inertial force ($\partial_{tt}u = 0$) we have the following nonlinear equation about the dynamic tension:

$$T_d = \mathcal{A}E\left[\partial_x u + \frac{1}{2}(\partial_x v)^2 + \frac{1}{2}(\partial_x w)^2 + \frac{dw_s}{dx}\partial_x w\right].$$

In this current derivation, because of the intrinsic linearity of the setting we have to fit with, we will not consider the non-linearities that arise usually in modelling the vibrations of an inclined cable. For example, Warnitchai et al. (1995) use a detailed non-linear PDE model which is decomposed into the first few vibration modes from which the precise non-linear coupling between in-plane and out-of-plane vibrations can be seen. Neither do we consider a finite element modelling approach as in Preumont (1997), where one can find an introduction to active tendon control of cables.

We will actually work with a linearized equation of T_d:

$$T_d(t) = \mathcal{A}E\left[\partial_x u(x,t) + \frac{dw_s}{dx}(x)\partial_x w(x,t)\right].$$

Considering that this tension due to the dynamics, is small compared to the static tension (*i.e.* $T_d << T_s$), the equation of motion for the dynamic analysis of this inclined cable, are then given by, for all (x,t) in $(0,\ell)\times(0,\infty)$,

$$\begin{cases} \rho\mathcal{A}\partial_{tt}v(x,t) = T_s\partial_{xx}v(x,t), \\ \rho\mathcal{A}\partial_{tt}w(x,t) = T_s\partial_{xx}w(x,t) + T_d(t)\dfrac{d^2w_s}{dx^2}. \end{cases}$$

2. MODELING OF A CONTROLLED INCLINED CABLE

We are interested in the modeling of the robust feedback control of the vibrations of the cable subjected to perturbations that take the form of in-plane oscillations, using an active tendon as a control/measurement device. We use an infinite dimensional state space approach of the robust control, and it implies a linear model of the system. It is the reason why we linearized the dynamic tension T_d. Nevertheless, it also means that we loose the (nonlinear) coupling between v and w. Therefore, the fact that the control and perturbations will only act in the sag plane does not allow us to consider the out-of-plane motion v anymore. It won't appear in the construction of our state space model since it satisfies a conservative wave equation that could only be influenced by coupling nonlinearities we don't deal with here.

2.1 Partial differential equation model of the inclined cable

One can read in Wagg and Neild (2010) the details about how the decomposition of each displacement (u,v,w) into quasi-static and modal components helps to simplify the analysis of the inclined cable vibration. Anyway, imposing the boundary conditions corresponding to the support motion

$$\begin{cases} u_q(0,t) = 0, \quad u_q(\ell,t) = u_b(t), \ \forall t\in(0,\infty), \\ w_q(0,t) = 0, \quad w_q(\ell,t) = w_b(t), \ \forall t\in(0,\infty), \end{cases}$$

we can calculate the following quasi-static components corresponding to the motion of the cable without taking into account any dynamic response. We obtain

$$\begin{cases} u_q(x,t) = \dfrac{E_q}{E}u_b(t)\dfrac{x}{\ell} - \dfrac{\varrho\mathcal{A}\ell}{2T_s}w_b(t)\left[\dfrac{x}{\ell} - \left(\dfrac{x}{\ell}\right)^2\right] \\ \qquad\qquad + \dfrac{\lambda^2E_q}{4E}u_b(t)\left[\dfrac{x}{\ell} - 2\left(\dfrac{x}{\ell}\right)^2 + \dfrac{4}{3}\left(\dfrac{x}{\ell}\right)^3\right] \\ w_q(x,t) = w_b(t)\dfrac{x}{\ell} - \dfrac{\varrho E_q\ell\mathcal{A}^2}{2T_s^2}u_b(t)\left[\dfrac{x}{\ell} - \left(\dfrac{x}{\ell}\right)^2\right] \\ T_q(t) = \dfrac{\mathcal{A}E_q}{\ell}u_b(t) \end{cases} \tag{1}$$

for $(x,t)\in(0,\ell)\times(0,\infty)$, $E_q = E/(1+\lambda^2/12)$ being the equivalent modulus of the cable and $\lambda^2 = E\varrho^2\ell^2\mathcal{A}^3/T_s^3$, Irvine's parameter.

Then, assuming that the modal axial displacement is small ($u_m = 0$), the modal dynamic tension satisfies

$$T_m = \frac{\varrho\mathcal{A}^2E}{2T_s}\left(\ell - 2x\right)\partial_x w_m$$

and the in-plane modal displacement is solution of the following PDE on $(0,\ell)\times(0,\infty)$:

$$\rho\mathcal{A}\partial_{tt}(w_q + w_m) = T_s\partial_{xx}w_m - T_m\frac{\varrho\mathcal{A}}{T_s},$$

submitted to homogeneous Dirichlet boundary conditions

$$w_m(0,t) = 0, \quad w_m(\ell,t) = 0, \forall t \in (0,\infty)$$

and initial conditions equal to zero. Since can calculate easily $\partial_{tt} w_q$ from (1), we obtain the self-contained equation:

$$\begin{aligned}\partial_{tt} w_m = & \frac{T_s}{\rho\mathcal{A}}\partial_{xx} w_m - \frac{\varrho^2\mathcal{A}^2 E}{2\rho T_s^2}(\ell - 2x)\,\partial_x w_m \\ & -\frac{x}{\ell} w_b'' + \frac{\varrho E_q \ell \mathcal{A}^2}{2T_s^2}\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] u_b''.\end{aligned} \tag{2}$$

2.2 Measurement and control of the cable

We are considering an inclined cable as in Figure 1 perturbed by in-plane oscilations (u_b, w_b) and connected at its bottom end with an active tendon. As one can read in Preumont and Bossens (2000), an active tendon consists of a displacement actuator (e.g. piezoelectric) collocated with a force sensor. Here, a small discussion is necessary. A tendon is principally meant to have an axial movement, that corresponds to a control u_c which is then an additive displacement term to the perturbation u_b in equation (2). But we can also prove that if we only consider this *inertial* control, we only have access to the symmetric modes of vibration. And because of the linear framework, we loose the complementary *parametric* control (see Preumont (1997)). To overcome this, we will consider that the control also acts through the in-plane bottom displacement as a term w_c which will be an added to the perturbation w_b.

Besides, the force sensor allows to define the observation we will measure, in order to build our feedback, as the dynamic tension T_d.

Let us now incorporate these facts into the PDE model. Since the perturbations of the system u_b and w_b will be proportional to $\cos(\omega t)$, we can assume that $u_b''(t) = -\omega_u^2 u_b(t)$ and $w_b''(t) = -\omega_w^2 w_b(t)$ and obtain the following state equation

$$\begin{aligned}\partial_{tt} w_m = & \frac{T_s}{\rho\mathcal{A}}\partial_{xx} w_m - \frac{\varrho^2\mathcal{A}^2 E}{2\rho T_s^2}(\ell - 2x)\,\partial_x w_m \\ & + \frac{\varrho E_q \ell \mathcal{A}^2}{2T_s^2}\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] u_c'' - \frac{x}{\ell} w_c'' \\ & - \omega_u^2 \frac{\varrho E_q \ell \mathcal{A}^2}{2T_s^2}\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] u_b + \omega_w^2 \frac{x}{\ell} w_b,\end{aligned} \tag{3}$$

with the measurement output

$$T_d = T_q + T_m = \frac{\mathcal{A}E_q}{\ell} u_b + \frac{\varrho\mathcal{A}^2 E}{2T_s}(\ell - 2x)\,\partial_x w_m. \tag{4}$$

2.3 State space model of an inclined cable's vibrations

A linear infinite-dimensional state space model derived from the PDE model presented above will be used in the sequel. It will take the usual shape (Doyle et al. (1996))

$$\begin{cases} X'(t) &= AX(t) + B_1 W(t) + B_2 U(t), \qquad \forall t \geq 0 \\ X(0) &= 0, \\ Z(t) &= C_1 X(t) + D_{12} U(t), \\ Y(t) &= C_2 X(t) + D_{21} W(t). \end{cases} \tag{5}$$

In order to fit in this formalism, the following notations are introduced:

- The state vector is $X = (w_m, \partial_t w_m)^\top$;
- The exogenous disturbance is $W = (W_{\text{mod}}, u_b, w_b)^\top$ where W_{mod} gathers uncertainty on the model (e.g. the neglected nonlinearities) ;
- The control vector $U = (u_c'', w_c'')^\top$ is the acceleration of the active tendon actuator ;
- The measurement vector $Y = T_d$ is the dynamic tension of the cable;
- The controlled vector $Z = (w_m, U)^\top$ gathers the in-plane vibrations and the control input vector;

Moreover, A should be the infinitesimal generator of a C_0-semigroup $T(t) = e^{At}$ on a real separable Hilbert space $\mathcal{X}$ (see Pazy (1983)). Let be the following linear bounded operators $B_1 \in \mathcal{L}(\mathcal{W},\mathcal{X})$, $B_2 \in \mathcal{L}(\mathcal{U},\mathcal{X})$, $C_1 \in \mathcal{L}(\mathcal{X},\mathcal{Z})$, $C_2 \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $D_{12} \in \mathcal{L}(\mathcal{U},\mathcal{Z})$ and $D_{21} \in \mathcal{L}(\mathcal{W},\mathcal{Y})$ where $\mathcal{U}$ (space of controls), $\mathcal{W}$ (space of perturbations), $\mathcal{Z}$ and $\mathcal{Y}$ (space of observations) are also real separable Hilbert spaces.

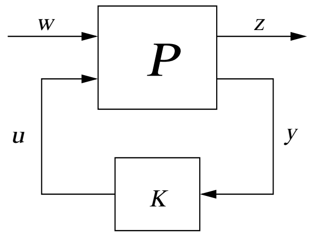

Fig. 2. Closed-loop system. Plant $\mathcal{P}$, controller $\mathcal{K}$.

We consider the system (5) where the state $X(t)$ belongs to $\mathcal{X}$, and where the control input $U(t) \in \mathcal{U}$, the disturbance input $W(t) \in \mathcal{W}$, the measured output $Y(t) \in \mathcal{Y}$ and the "to be controlled output" $Z(t) \in \mathcal{Z}$ are formally linked in a closed-loop system sketched by the standard description in Figure 2. Using (3) and (4), the operator matrices involved in the system (5) are therefore given by:

$$A = \begin{pmatrix} 0 & I \\ \frac{T_s}{\rho\mathcal{A}}\partial_{xx} - \frac{\varrho^2\mathcal{A}^2 E}{2\rho T_s^2}(\ell - 2x)\,\partial_x & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ d_1 & -\omega_u^2 \frac{\varrho E_q \ell \mathcal{A}^2}{2T_s^2}\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] & \omega_w^2 \frac{x}{\ell} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ \frac{\varrho E_q \ell \mathcal{A}^2}{2T_s^2}\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] & -\frac{x}{\ell} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} \frac{\varrho\mathcal{A}^2 E}{2T_s}(\ell - 2x)\,\partial_x & 0 \end{pmatrix},$$

$$D_{12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_{21} = \begin{pmatrix} d_2 & \frac{\mathcal{A}E_q}{\ell} & 0 \end{pmatrix},$$

with the tuning parameters $d_1, d_2 \in L^2(0,\ell)$. The appropriate functional Hilbert spaces associated with the infinite-dimensional model are the state space

$$\mathcal{X} = H_0^1(0,\ell) \times L^2(0,\ell)$$

and the input or output spaces $\mathcal{U} = \mathbb{R}^2$, $\mathcal{W} = \mathbb{R}^3$, $\mathcal{Y} = L^2(0,\ell)$, $\mathcal{Z} = L^2(0,\ell) \times \mathbb{R}^2$.

In order to prove that A is the infinitesimal generator of a C_0-semigroup on $\mathcal{X}$, one can rely on the classical theory of semi-groups as in Pazy (1983) or refer to the book Tucsnak and Weiss (2009) or the article Tadmor (1990). Since equation (2) is essentially a damped wave equation with a source term $-\partial_{tt} w_q$, we do not give details of the proof here. This equation can also be seen as a non-homogeneous wave equation perturbed by a bounded operator (see Tucsnak and Weiss (2009)). Either way, one can prove existence, uniqueness and regularity of the solution when the initial conditions are $w(t=0) = w_0 \in H_0^1(0,\ell)$ and $\partial_t w(t=0) = w_1 \in L^2(0,\ell)$ and $\partial_{tt} w_q \in L^\infty(0,\infty; H^2(0,\ell))$ (obvious with $u_b, w_b \in W^{2,\infty}(0,\infty)$): then there exists a unique solution w of (2), that belongs to $C([0,\infty), H_0^1(0,\ell)) \cap C^1([0,\infty), L^2(0,\ell))$. This regularity also explains why the operator $C_2 \in \mathcal{L}(H_0^1(0,\ell) \times L^2(0,\ell), L^2(0,\ell))$ is a bounded one.

3. H_∞ CONTROL WITH MEASUREMENT-FEEDBACK

This section is devoted to recalling the H_∞-robust control theorem proved in Bensoussan and Bernhard (1993) and/or van Keulen (1993) that we want to apply on the PDE model we derived. This result gives an equivalence between the H_∞-robust control with measurement-feedback of an infinite-dimensional system and the solvability of two Ricatti equations. We will explain later how this theorem can be applied to the model we consider, following the assumptions recalled below. For a survey of the H_∞-control theory with state-feedback in the infinite-dimensional case, one can also read van Keulen et al. (1993). The main results in all these articles are a generalization of the finite-dimensional regular H_∞-control problem (see for instance Doyle et al. (1996) and Tadmor (1990)) presented in a standard state-space approach.

Let us now make precise what is meant by H_∞-optimal control (or robust control) with measurement feedback. The description (5) of the system we consider implies that, as in Figure 2, the plant $\mathcal{P}$ is to be controlled under the cost function (related to the output Z)

$$J_0(U,W) = \int_0^\infty \left(|C_1 X|^2 + |D_{12} U|^2\right) dt$$

and the partial observation $Y = C_2 X + D_{21} W$. The point is to construct a feedback controller $\mathcal{K} = (L, M, N, R)$ of the form

$$\begin{cases} \Phi'(t) &= (A+M)\Phi(t) + NY(t), \\ \Phi(0) &= 0, \\ U(t) &= L\Phi(t) + RY(t). \end{cases} \quad (6)$$

where Φ, the adjoint state, depends on the measurement Y and leads to the control U. The coupled system is then as follows:

$$\begin{cases} X' &= (A + B_2 R C_2) X + B_2 L \Phi + B_1 W \\ \Phi' &= (A+M)\Phi + N C_2 X + N D_{21} W \end{cases}$$

and introduces an operator

$$\Lambda = \begin{pmatrix} A + B_2 R C_2 & B_2 L \\ N C_2 & A + M \end{pmatrix}.$$

The goal is to find a dynamic measurement-feedback controller $\mathcal{K}$ that exponentially stabilizes this system (meaning that Λ is exponentially stable, and also yield to a finite cost $J_0(L\Phi + RY, W)$) and ensures that the influence of the disturbances on the "to be controlled output", *i.e.* the ratio

$$\rho(\mathcal{K}) = \sup_{W \in \mathcal{W}} \frac{J_0(L\Phi + RY, W)}{\|W\|^2_{\mathcal{W}}}$$

is smaller than some specific bound.

The main result we apply to the dynamic measurement feedback control of an inclined cable is the following.

Theorem 1. [Proof to be read in Bensoussan and Bernhard (1993) or van Keulen (1993)] Let $\gamma > 0$ and assume that the pair A, B_1 is stabilizable and that A, C_1 is detectable. The following assertions are equivalent:

(i) The γ^2-robustness property with partial observation hold for the system (5) ;
(ii) There exists an exponentially stabilizing dynamic output-feedback controller $\mathcal{K}$ of the form (6) such that Λ is exponentially stable and $\rho(\mathcal{K}) < \gamma^2$;
(iii) There exist two nonnegative definite symmetric operators $P, \Sigma \in \mathcal{L}(\mathcal{X})$ solutions of the Riccati and compatibility equations:

- $\forall X \in D(A)$, $PX \in D(A^*)$,
$$\left(PA + A^*P + P(B_2B_2^* - \gamma^{-2}B_1B_1^*)P + C_1^*C_1\right)X = 0 \quad (7)$$
and $A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$ generates an exponentially stable semigroup ;
- $\forall X \in D(A^*)$, $PX \in D(A)$,
$$\left(\Sigma A^* + A\Sigma + \Sigma(C_2^*C_2 - \gamma^{-2}C_1^*C_1)\Sigma + B_1B_1^*\right)X = 0 \quad (8)$$
and $A^* - (C_2^*C_2 - \gamma^{-2}C_1^*C_1)\Sigma$ generates an exponentially stable semigroup ;
- $I - \gamma^{-2}P\Sigma$ is invertible and
$$\Pi = \Sigma\left(I - \gamma^{-2}P\Sigma\right)^{-1} \geq 0. \quad (9)$$

Moreover, if the conditions (7)-(9) hold, then the feedback controller $\mathcal{K}$ given by

$$\begin{array}{ll} M &= -(B_2B_2^* - \gamma^{-2}B_1B_1^*)P - \Pi C_2^*C_2, \\ N &= \Pi C_2^*, \quad L = -B_2^*P, \quad R = 0 \end{array} \quad (10)$$

give an exponentially stable operator Λ and guarantees that $\rho(\mathcal{K}) < \gamma^2$. Finally, if the solutions to the Riccati equations exists, then they are unique. □

One can notice that the feedback controller $\mathcal{K}$ given in (10) is actually sub-optimal and known as the central controller.

Since B_1, B_2, C_1, C_2, D_{12} and D_{21} are bounded operators well defined in the appropriate spaces we will be allowed to apply Theorem 1 if we can confirm that (A, B_1) is stabilizable and (A, C_1) is detectable.

The proof that (A, B_1) is controllable (implying stabilizability) relies on the study of the controllability through u_b and w_b of the wave equation

$$\partial_{tt} w_m - \frac{T_s}{\rho\mathcal{A}} \partial_{xx} w_m + \frac{\varrho^2 \mathcal{A}^2 E}{2\rho T_s^2} (\ell - 2x)\,\partial_x w_m \\ = -\omega_u^2 \frac{\varrho E_q \ell \mathcal{A}^2}{2T_s^2} \left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] u_b + \omega_w^2 \frac{x}{\ell} w_b. \quad (11)$$

The main difficulty comes from the fact that the control input has a prescribed shape in space. The control $u_b(t)$ alone is not sufficient because of the symmetry property of the function in x in front of it (see the α_i in section 4). We do not wish to give the proof here, but the additional control $w_c(t)$ is used to overcome this restriction. One can consult Tucsnak and Weiss (2009) for details about observability results for the wave equation (even with a first order term), and to see the proof that observability is also true with a control acting only on the boundary of the domain. Therefore, we have (A, B_1) stabilizable. Finally, since $C_1 X = w_m$ on the whole domain, one proves easily an observability estimate and we obtain that (A, C_1) is detectable. This closes the verification of the assumptions of Theorem 1.

Considering that we have now a well-posed robust control problem in infinite dimension, we would like to perform some numerical experiments to illustrate the results we can obtain.

4. A TRUNCATED MODEL FOR NUMERICAL DESIGN

4.1 Model of finite dimension

The goal of this section is to define an appropriate finite-dimensional model of the PDE system (5) whose state representation can be written as:

$$\begin{cases} X_N'(t) &= A_N X_N(t) + B_{2,N} U(t) + B_{1,N} W(t), \\ X_N(0) &= 0 \\ Y_N(t) &= C_{2,N} X_N(t) + D_{21,N} W(t) \\ Z_N(t) &= C_{1,N} X_N(t) + D_{12,N} U(t), \end{cases} \tag{12}$$

where the operators of system (5) are replaced by real-valued matrices computed on a truncated basis of the N first eigenfunctions precisely defined below. $X_N \in \mathbb{R}^{2N}$ is the state vector, $Y_N \in \mathbb{R}^N$ is the measurement output vector, $Z_N \in \mathbb{R}^{N+2}$ is the to-be-controlled output vector, $W \in \mathbb{R}^3$ and $U \in \mathbb{R}^2$ are still the exogenous perturbation and the control. The truncation of the PDE system can be seen as a way of coming back to the structural vibrations of the system.

In order to compute these objects, we choose to use everywhere the Hermitian base of $L^2(0, \ell)$ given by the eigenfunctions of the (compact self-adjoint) operator $\frac{T_s}{\rho \mathcal{A}} \partial_{xx}$. The orthonormal base $(\phi_i)_{i \in \mathbb{N}^*}$ is indeed defined by:

$$\phi_i(x) = \sqrt{\frac{2}{\ell}} \sin\left(i\pi \frac{x}{\ell}\right), \quad \omega_i = \frac{i\pi}{\ell}\sqrt{\frac{T_s}{\rho\mathcal{A}}}$$

and satisfies for all $x \in (0, \ell)$ and $i \in \mathbb{N}^*$,

$$\frac{T_s}{\rho\mathcal{A}} \partial_{xx} \phi_i(x) = -\omega_i^2 \phi_i(x).$$

This approach meets the Galerkin method used in (Wagg and Neild, 2010, chap 7) and the point is that every $y \in L^2(0, \ell)$ can be written

$$y(x) = \sum_{i \geq 1} y_i \phi_i(x),$$

$(y_i)_{i \in \mathbb{N}^*}$ being a sequence of real numbers satisfying

$$y_i = \langle y, \phi_i \rangle := \int_0^\ell y(x)\phi_i(x)\, dx \text{ and } \sum_{i \geq 1} y_i^2 < \infty.$$

Given $N \in \mathbb{N}$, we compute A_N, $B_{1,N}$, $B_{2,N}$, $C_{1,N}$, $C_{2,N}$, $D_{12,N}$ and $D_{21,N}$ using the truncated basis $\{\phi_1, \ldots, \phi_N\}$. We make the assumption that the tuning parameters $d = (d_i)$ are vectors of real numbers and recall that it is a weighting function of the disturbance signal W_{mod} that corresponds for instance to the "forgotten" non-linearities. We obtain:

$$A_N = \text{block}_{i,j} \left(\begin{bmatrix} 0 & \delta_{ij} \\ -\omega_i^2 \delta_{ij} - \dfrac{\varrho}{\rho T_s} a_{ij} & 0 \end{bmatrix} \right),$$

where δ_{ij} is the Kronecker symbol and

$$a_{ij} = \left\langle \frac{\varrho \mathcal{A}^2 E}{2T_s} (\ell - 2x)\, \partial_x \phi_j, \phi_i \right\rangle;$$

$$B_{1,N} = \text{vect}_i \left(\begin{bmatrix} 0 & 0 & 0 \\ d_i^1 & -\omega_u^2 \alpha_i & \omega_w^2 \beta_i \end{bmatrix} \right),$$

$$B_{2,N} = \text{vect}_i \left(\begin{bmatrix} 0 & 0 \\ \alpha_i & -\beta_i \end{bmatrix} \right),$$

where $\alpha_i = \left\langle \dfrac{\varrho E_q \ell \mathcal{A}^2}{2T_s^2} \left[\dfrac{x}{\ell} - \left(\dfrac{x}{\ell}\right)^2 \right], \phi_i \right\rangle$, $\beta_i = \left\langle \dfrac{x}{\ell}, \phi_i \right\rangle$;

$$C_{2,N} = \text{block}_{i,j} \left(\begin{bmatrix} a_{ij} & 0 \end{bmatrix} \right),$$

$$D_{21,N} = \text{vect}_i \left(\begin{bmatrix} d_i^2 & \dfrac{\mathcal{A} E_q}{\ell} \langle 1, \phi_i \rangle_{L^2} & 0 \end{bmatrix} \right).$$

When computing all the terms a_{ij}, one will observe that we get terms different from zero in the non diagonal blocks of the matrices A_N and $C_{2,N}$. This is due to the choice of the eigenfunctions of the operator $\frac{T_s}{\rho\mathcal{A}}\partial_{xx}$ which is different from the global operator of the damped wave equation in w_m. Besides, one can notice that the calculation of the term α_i describe the effective influence of the control u_c'' on the i-th vibration mode: since $\alpha_{2j} = 0$ only the symmetric modes are controllable (odd index i). Therefore, the terms $\beta_i \neq 0$ (thus the control w_c'') have a crucial importance to stabilize the other modes.

The numerical aim now is to find a dynamic measurement-feedback controller that exponentially stabilizes the system (12) and makes the influence of the perturbations W on the output vector Z_N small in a sense to define. In this short paper we choose to focus on the attenuation of the N first modes of vibrations with controls of small amplitude:

$$C_{1,N} = \left(\text{diag}_i \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \quad \mathbf{0} \quad \mathbf{0} \right)^\top, \quad D_{12,N} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4.2 Numerical simulations

In the numerical simulations we use the parameter values given in Table 1 as chosen in Gonzalez-Buelga et al. (2008) to approximately match a typical full-scale bridge cable inclined at $\theta = 20°$ to the horizontal, of length 400m, mass per unit length 130kg.m^{-1} and tension 8000kN:

Cable length	ℓ	1.98 m
Density	ρ	1.34×10^6 kg.m^{-3}
Cross sectional area	$\mathcal{A}$	0.5×10^{-6} m^3
Static tension	T_s	205 N
Steel Young's modulus	E	200×10^9 N.m^{-2}

Table 1. Cable characteristics

This gives the following parameters: $E_q = 174 \times 10^9$N.m^{-2}, $\lambda^2 = 1.74$ and $\varrho = \rho g \cos\theta = 12.35 \times 10^6$ kg.s^{-2}.m^{-2}. These values correspond to an inclined steel cable experiment that could be used in a next step to implement our controllers. We also choose the perturbation frequencies

$\omega_u = \omega_w = 55.4\text{rad.s}^{-1}$, i.e. close to the second linear in-plane natural frequency mode where the studies usually focus when studying the effect of the nonlinear modal coupling (not considered here).

We can illustrate the results of the closed loop control based on this truncated state model of the PDE modelling in several ways. Through the singular values of the frequency response, Figure 3 presents the attenuation of the first modes of vibrations for the case $N = 5$, with respect to the uncontrolled open loop. The H_∞ optimal controller is computed using the *hinfsyn* Matlab function.

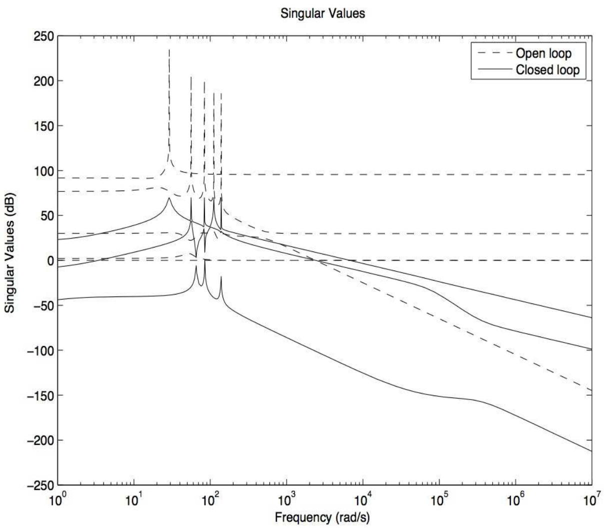

Fig. 3. Singular values of the frequency respond of the truncated model in open and closed loop

5. DISCUSSION AND CONCLUSION

The goal of this article was to study the robust measurement feedback control of an infinite-dimensional state space model of an inclined cable. After dealing with several modelling issues, we were able to ensure the H_∞-control of the system under the condition of solvability of two Riccati equations. This allowed us to perform some numerical computations on a truncated model in order to illustrate the action of the robust controller. We examined the possibility of connecting the inclined cable to an active tendon in order to bring active damping into the cable structure and as far as we know, there exists no such study of the robust control of an inclined cable when the partial differential equation model is used.

We can mention several new directions that we intend to pursue as future extensions of this study. Some of them concern the modeling of the controlled inclined cable. First we would propose a more sophisticated modeling of the output Z_N (tuning the attenuation of the modes for instance) and of the H_∞ channel $W \to Z_N$ but also to compare the control results we manage to obtain applied to the non-linear model of 2 or more modes that one can read in Wagg and Neild (2010). Then the next challenging step of this preliminary study would be to propose a multi-objective design problem, where the feedback controller has to respond favorably to several performance specifications: typically we could consider the active tendon control as a mixed H_2/H_∞ synthesis problem, where the H_∞ channel is used to enhance the robustness and the H_2 channel guarantees good performance despite e.g. measurement noise or uncertainty on the cable dynamic. Last but not least, improvements in the modeling of the inclined cable itself could be studied. Indeed when linearizing the infinite dimensional model, we lost the parametric control and the nonlinear coupling between w_m and v_m and therefore needed another control w_c and neglected the out-of-plane motion v_m. Another objective would indeed be to find a way, maybe using a modified version of the dynamic tension that carries the coupling, to include some nonlinearities in the control loop in order to define a more complete system to stabilize. This will be considered in future studies.

REFERENCES

Bensoussan, A. and Bernhard, P. (1993). On the standard problem of H_∞-optimal control for infinite-dimensional systems. In *Identification and control in systems governed by partial differential equations (South Hadley, MA, 1992)*, 117–140. SIAM, Philadelphia, PA.

Bossens, F. and Preumont, A. (2001). Active tendon control of cable-stayed bridges: a large-scale demonstration. *Earthquake Engineering & Structural Dynamics*, 30(7), 961–979.

Doyle, J.C., Zhou, K., and Glover, K. (1996). *Robust and Optimal Control Robust and Optimal Control Robust and Optimal Control.* Prentice Hall.

Fujino, Y. and Susumpow, T. (1994). An experimental study on active control of in-plane cable vibration by axial support motion. *Earthquake Engineering and Structural Dynamics Structural Dynamics*, 23(12), 1283–1297.

Gonzalez-Buelga, A., Neild, S., Wagg, D., and Macdonald, J. (2008). Modal stability of inclined cables subjected to vertical support excitation. *Journal of Sound and Vibration*, 318(3), 565 – 579.

Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

Preumont, A. (1997). *Vibration control of active structures*, volume 50 of *Solid Mechanics and its Applications*. Kluwer Academic Publishers Group, Dordrecht.

Preumont, A.J. and Bossens, F. (2000). Active tendon control of cable-stayed bridges. *SPIE proceedings series*, Smart Structures and Materials, 188–198.

Smrz, M., Bastaits, R., and Preumont, A. (2011). Active damping of the camera support mast of a cherenkov gamma-ray telescope. *Nuclear Instruments and Methods in Physics Research Section A* , 635(1), 44 – 52.

Song, G., Sethi, V., and Li, H.N. (2006). Vibration control of civil structures using piezoceramic smart materials: A review. *Engineering Structures*, 28(11), 1513 – 1524.

Tadmor, G. (1990). Worst-case design in the time domain: the maximum principle and the standard H_∞ problem. *Math. Control Signals Systems*, 3(4), 301–324.

Tucsnak, M. and Weiss, G. (2009). *Observation and Control for Operator Semigroups*, volume XI of *Birkäuser Advanced Texts*. Springer.

van Keulen, B. (1993). H_∞-control with measurement-feedback for linear infinite-dimensional systems. *J. Math. Systems Estim. Control*, 3(4), 373–411.

van Keulen, B., Peters, M., and Curtain, R. (1993). H_∞-control with state-feedback: the infinite-dimensional case. *J. Math. Systems Estim. Control*, 3(1), 1–39.

Wagg, D. and Neild, S. (2010). *Nonlinear Vibration with Control: For Flexible and Adaptive Structures.* Solid mechanics and its applications. Springer.

Warnitchai, P., Fujino, Y., and Susumpow, T. (1995). A non-linear dynamic model for cables and its application to a cable structure system. *Journal of Sound and Vibration*, 187(4), 695 – 712.