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# A simple commutativity condition for block decimators and expanders

Didier Pinchon and Pierre Siohan

## Abstract

Commutativity rules play an essential role when building multirate signal processing systems. In this letter, we focus on the interchangeability of block decimators and expanders. We, formally, prove that commutativity between these two operators is possible if and only if the data blocks are of an equal length corresponding to the greatest common divisor of the integer decimation and expansion factors.

## Index Terms

Block sampling, Decimation, Expansion, Commutativity.

## I. INTRODUCTION

The polyphase decomposition, introduced by Bellanger et al. in 1976 [1], is a key element in multirate signal processing to reduce the computational complexity of digital equipments. Its implementation involves decimation and expansion operators. For an input sequence  $x[n]$ , a conventional decimator, with integer decimation factor  $p_1$ , only retains the samples at time multiple of  $p_1$ . A conventional expander, with integer expansion factor  $p_2$ , inserts  $p_2 - 1$  samples between each pair of consecutive  $x[n]$  samples. It is well-known that conventional decimators and expanders can commute if and only if  $p_1$  and  $p_2$  are coprime. The important question of commutativity of these basic operators also occurs when designing either multidimensional [2], [3] or block processing systems [4], [5]. Block samplers are of a particular interest to deal with the class of incompatible nonuniform filter banks [6]. Block decimators (resp. expanders) are defined by parameters that, in addition to the decimation (resp. expansion) factors, include the block length. In [5] the authors consider a block decimator  $\downarrow(q_1, p_1)$  and an expander  $\uparrow(q_2, p_2)$ , with  $q_1, q_2$  the block sizes and  $p_1, p_2$  the decimation and expansion factors, respectively. Assuming  $q_1$  and  $q_2$  are not necessarily equal and  $p_1, p_2$  non necessarily integers, they give three joint conditions that are necessary and sufficient conditions to insure the commutativity of  $\downarrow(q_1, p_1)$  and  $\uparrow(q_2, p_2)$ . However no examples are given with unequal block lengths and/or rational non integer sampling ratios where the three conditions are satisfied. In [4], the commutativity of up and down sampling is studied when the sampling ratios are integers but with unequal block lengths. Again, the authors do not provide any example where the commutativity is obtained with unequal block lengths.

Our notations slightly differ from those in [5]. For integers  $q, p$  such that  $1 \leq q < p$ , let us denote by  $D(q, p)$  the decimator with block length  $q$  and sampling ratio  $p/q$ . Such a decimator receives a sequence of  $p$  consecutive input symbols of a signal, keeps the first  $q$  ones and discards the last  $p - q$  symbols. For integers  $q, p$  such that  $1 \leq q < p$ ,  $E(q, p)$  denotes the expander with block length  $q$  and sampling ratio  $p/q$ : each block with length  $q$  of the input signal is transmitted with the addition of  $p - q$  zero taps.

In this letter, the following theorem is proved.

**Theorem.** *Let  $q_1, p_1, q_2, p_2$  be integers such that  $1 \leq q_1 < p_1$  and  $1 \leq q_2 < p_2$ . Then  $D(q_1, p_1)$  and  $E(q_2, p_2)$  commute if and only if  $q_1 = q_2 = \gcd(p_1, p_2)$ .*

For  $q_1 = q_2 = 1$ , it is already well known that, when  $q_1 = q_2 = \gcd(p_1, p_2)$ , then  $D(q_1, p_1)$  and  $E(q_2, p_2)$  commute (see for example [7], pages 119 and 179). The extension of this result to arbitrary equal block lengths

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corresponds to the easiest part of our theorem. However, as we could not find any proof of it, in order to provide a self contained paper, we demonstrate it in our Lemmas 2 and 3.

## II. PROOF OF THE THEOREM

Before proving the theorem, let us introduce some notations and definitions.

For  $n \in \mathbb{Z}$  and  $m > 0$  two integers, the quotient  $a$  and the remainder  $b$  of the euclidean division of  $n$  by  $m$  are denoted by  $a = \text{quo}(n, m)$ ,  $b = \text{rem}(n, m)$ , which may be also written in a condensed form  $(a, b) = \text{div}(n, m)$ .

Let  $1 \leq q_1 < p_1$ . Applied to a discrete-time input signal, named in short a sequence,  $x = (x[n], n \in \mathbb{Z})$ , the decimator  $D(q_1, p_1)$  returns an output sequence  $t = (t[k], k \in \mathbb{Z})$  obtained by

$$(a, b) = \text{div}(n, p_1), (a \in \mathbb{Z}, 0 \leq b < p_1), \quad (1)$$

$$k = aq_1 + b, \text{ if } b < q_1, \quad (2)$$

$$t[k] = x[n], \quad (3)$$

When  $q_1 \leq b < p_1$ , the  $x[n]$  sample is discarded.

For  $1 \leq q_2 < p_2$ , the expander  $E(q_2, p_2)$  is now applied to the input sequence  $t = (t[k], k \in \mathbb{Z})$ , producing a sequence  $y_1 = (y_1[m], m \in \mathbb{Z})$  defined by

$$(\alpha, \beta) = \text{div}(k, q_2), (\alpha \in \mathbb{Z}, 0 \leq \beta < q_2) \quad (4)$$

$$y_1[\alpha p_2 + \beta] = t[k]. \quad (5)$$

and the unassigned symbols in the sequence  $y_1 = (y_1[m], m \in \mathbb{Z})$  are set to zero. The transformation to go from  $x = (x[n], n \in \mathbb{Z})$  to  $y_1 = (y_1[m], m \in \mathbb{Z})$  is denoted by  $E(q_2, p_2)D(q_1, p_1)$ .

Equations (1)-(5) allow us to define a function  $f_1(q_1, p_1, q_2, p_2; \bullet)$ , depending on parameters  $q_1, p_1, q_2, p_2$ , and defined on integers  $n \in \mathbb{Z}$  that are expressed in an intuitive algorithmic language in (6) such that, when  $f_1(q_1, p_1, q_2, p_2; n) \neq -\infty$ ,  $y_1[f_1(q_1, p_1, q_2, p_2; n)] = x[n]$ . When  $f_1(q_1, p_1, q_2, p_2; n) = -\infty$ , the symbol  $x[n]$  is discarded and  $y_1[m] = 0$  when  $m \in \mathbb{Z}$  is not in the image of  $f_1(q_1, p_1, q_2, p_2; \bullet)$ .

$$\left[ \begin{array}{l} f_1 = \text{proc}(q_1, p_1, q_2, p_2; n) \\ \text{local } a, b, \alpha, \beta \\ (a, b) = \text{div}(n, p_1) \\ \text{if } b < q_1 \text{ then} \\ \quad (\alpha, \beta) = \text{div}(aq_1 + b, q_2) \\ \quad \text{return } \alpha p_2 + \beta \\ \text{else return } -\infty \\ \text{end} \end{array} \right. \quad (6)$$

The action of  $E(q_2, p_2)$  on the input sequence  $x = (x[n], n \in \mathbb{Z})$  produces a sequence  $z = (z[l], l \in \mathbb{Z})$  defined by

$$(c, d) = \text{div}(n, q_2) (c \in \mathbb{Z}, 0 \leq d < q_2), \quad (7)$$

$$l = cp_2 + d, \quad (8)$$

$$z[l] = x[n]. \quad (9)$$

Then, applying the decimator  $D(q_1, p_1)$  to the input sequence  $z = (z[l], l \in \mathbb{Z})$  produces the sequence  $y_2 = (y_2[m], m \in \mathbb{Z})$ , such that

$$(\gamma, \delta) = \text{div}(l, p_1) (\gamma \in \mathbb{Z}, 0 \leq \delta < p_1), \quad (10)$$

$$y_2[\gamma q_1 + \delta] = z[l], \text{ if } \delta < q_1. \quad (11)$$

Again unassigned samples in the sequence  $y_2 = (y_2[m], m \in \mathbb{Z})$  are set to zero. The overall transformation is denoted by  $D(q_1, p_1)E(q_2, p_2)$ .

In a similar way, a function  $f_2(q_1, p_1, q_2, p_2; \bullet)$ , depending on parameters  $q_1, p_1, q_2, p_2$ , and defined for  $n \in \mathbb{Z}$  is defined by (12). For the values of  $n$  such that  $f_2(q_1, p_1, q_2, p_2; n) \neq -\infty$ , we have  $y_2[f_2(q_1, p_1, q_2, p_2; n)] = x[n]$ . When  $f_2(n) = -\infty$ , the symbol  $x[n]$  is discarded and  $y_2[m] = 0$  when  $m$  is not in the image of  $f_2$ .

$$\left[ \begin{array}{l} f_2 = \text{proc}(q_1, p_1, q_2, p_2; n) \\ \quad \text{local } c, d, \gamma, \delta \\ \quad (c, d) = \text{div}(n, q_2) \\ \quad (\gamma, \delta) = \text{div}(cp_2 + d, p_1) \\ \quad \text{if } \delta < q_1 \text{ then} \\ \quad \quad \text{return } \gamma q_1 + \delta \\ \quad \text{else return } -\infty \\ \quad \text{end} \end{array} \right. \quad (12)$$

Inverting equations (1)–(5) (resp. (7)–(11)) for given parameters  $q_1, p_1, q_2, p_2$ , we may introduce the function  $g_1(q_1, p_1, q_2, p_2; \bullet)$  (resp.  $g_2(q_1, p_1, q_2, p_2; \bullet)$ ) defined for  $m \in \mathbb{Z}$  such that  $y_1[m] = 0$  when  $g_1(q_1, p_1, q_2, p_2; m) = -\infty$  and  $y_1[m] = x[g_1(q_1, p_1, q_2, p_2; m)]$  otherwise (resp.  $y_2[m] = 0$  when  $g_2(q_1, p_1, q_2, p_2; m) = -\infty$  and  $y_2[m] = x[g_2(q_1, p_1, q_2, p_2; m)]$  otherwise).

$$\left[ \begin{array}{l} g_1 = \text{proc}(q_1, p_1, q_2, p_2; m) \\ \quad \text{local } \alpha, \beta, a, b \\ \quad (\alpha, \beta) = \text{div}(m, p_2) \\ \quad \text{if } \beta < q_2 \text{ then} \\ \quad \quad (a, b) = \text{div}(\alpha q_2 + \beta, q_1) \\ \quad \quad \text{return } ap_1 + b \\ \quad \quad \text{else return } -\infty \\ \quad \text{end} \end{array} \right. \quad (13)$$

$$\left[ \begin{array}{l} g_2 = \text{proc}(q_1, p_1, q_2, p_2; m) \\ \quad \text{local } \gamma, \delta, c, d \\ \quad (\gamma, \delta) = \text{div}(m, q_1) \\ \quad (c, d) = \text{div}(\gamma p_1 + \delta, p_2) \\ \quad \text{if } d < q_2 \text{ then} \\ \quad \quad \text{return } cq_2 + d \\ \quad \quad \text{else return } -\infty \\ \quad \text{end} \end{array} \right. \quad (14)$$

It is now obvious that the following properties are equivalent

- $D(q_1, p_1)$  and  $E(q_2, p_2)$  commute,
- $f_1(q_1, p_1, q_2, p_2; \bullet) = f_2(q_1, p_1, q_2, p_2; \bullet)$ ,
- $g_1(q_1, p_1, q_2, p_2; \bullet) = g_2(q_1, p_1, q_2, p_2; \bullet)$ .

This is clearly stated in Theorem 1 of [4] which amounts to say that the up and down block sampling with integer sampling ratios commute if and only if  $g_1(q_1, p_1, q_2, p_2; \bullet) = g_2(q_1, p_1, q_2, p_2; \bullet)$ .

The method of our proof to prove that  $D(q_1, p_1)$  and  $E(q_2, p_2)$  do not commute for a given subset of parameters  $q_1, p_1, q_2, p_2$  will be to find a particular value of  $n$ , depending on  $q_1, p_1, q_2, p_2$ , such that  $f_1(q_1, p_1, q_2, p_2; n) \neq f_2(q_1, p_1, q_2, p_2; n)$  or  $g_1(q_1, p_1, q_2, p_2; n) \neq g_2(q_1, p_1, q_2, p_2; n)$ .

*Notations.* In a context where parameters  $q_1, p_1, q_2, p_2$  are fixed,  $f_1(q_1, p_1, q_2, p_2; n)$  will be denoted simply by  $f_1(n)$ . In the evaluation of  $f_1(q_1, p_1, q_2, p_2; n)$  following (6), the value assigned to a local variable like  $a$  will be denoted by  $a(q_1, p_1, q_2, p_2; n)$ , but only by  $a(n)$  in the context of fixed values for the parameters  $q_1, p_1, q_2, p_2$ , and even simply  $a$  when a given fixed value of  $n$  is considered. The same notation simplification will apply also for function  $f_2$  defined by (12) and for functions  $g_1$  and  $g_2$  defined by (13) and (14).

The following exchange property will be useful to restrict the number of cases to study on parameters  $q_1, p_1, q_2, p_2$ .

**Lemma 1.** *Let  $q_1, p_1, q_2, p_2$  be integers with  $1 \leq q_1 \leq p_1$ ,  $1 \leq q_2 \leq p_2$ . Then  $D(q_1, p_1)$  and  $E(q_2, p_2)$  commute if and only if  $D(q_2, p_2)$  and  $E(q_1, p_1)$  commute.*

*Proof.*– For any set of parameters  $q_1, p_1, q_2, p_2$ , exchanging  $(q_1, p_1)$  and  $(q_2, p_2)$  in the definition (6) and changing the name of the local variables  $(a, b, \alpha, \beta)$  by  $(\alpha, \beta, a, b)$  gives the definition (13), and thus

$$f_1(q_1, p_1, q_2, p_2; n) = g_1(q_2, p_2, q_1, p_1; n), \quad n \in \mathbb{Z}. \quad (15)$$

In a similar way

$$f_2(q_1, p_1, q_2, p_2; n) = g_2(q_2, p_2, q_1, p_1; n), \quad n \in \mathbb{Z}. \quad (16)$$

The lemma is proved by using afterwards as a commutativity criterion the equality of functions  $f_1$  and  $f_2$  or the equality of  $g_1$  and  $g_2$ .  $\square$

$D(1, p_1)$  corresponds to the traditional decimator of factor  $p_1$  while  $E(1, p_2)$  is the traditional expander of factor  $p_2$ . The following lemma is a classical result reobtained using our own notations.

**Lemma 2.** *Let  $p_1 > 1$  and  $p_2 > 1$ .  $D(1, p_1)$  and  $E(1, p_2)$  commute if and only if  $p_1$  and  $p_2$  are relatively prime integers.*

*Proof.*– For  $n \in \mathbb{Z}$ , we get from (6) and (12)

- If  $n$  is a multiple of  $p_1$  i.e.  $n = ap_1$ , then  $f_1(n) = ap_2$ , otherwise  $f_1(n) = -\infty$ ,
- If  $np_2$  is a multiple of  $p_1$  i.e.  $np_2 = \gamma p_1$ , then  $f_2(n) = \gamma$ , otherwise  $f_2(n) = -\infty$ .

If  $p_1$  and  $p_2$  are not relatively primes, then  $p_1 = dp'_1, p_2 = dp'_2$  with  $d > 1$ . Choosing  $n = p'_1$ , we get  $f_1(p'_1) = -\infty$  because  $p'_1$  is not multiple of  $p_1$ . But  $np_2 = p'_1 dp'_2 = p'_2 p_1$  and thus  $f_2(p'_1) = p'_2$ . This proves that  $f_1 \neq f_2$  meaning that  $D(1, p_1)$  and  $E(1, p_2)$  do not commute.

If  $p_1$  and  $p_2$  are relatively primes, then if  $np_2$  is a multiple of  $p_1$  if and only if  $n$  is a multiple of  $p_1$ , and  $f_1 = f_2$ , that is  $D(1, p_1)$  and  $E(1, p_2)$  commute.  $\square$

The following lemma allows us to multiply the parameters  $q_1, p_1, q_2, p_2$  by a same positive integer which is an already well known result.

**Lemma 3.** *Let  $q_1, p_1, q_2, p_2$  be integers with  $1 \leq q_1 \leq p_1$ ,  $1 \leq q_2 \leq p_2$  and  $d > 1$  an integer. Then  $D(q_1, p_1)$  and  $E(q_2, p_2)$  commute if and only if  $D(dq_1, dp_1)$  and  $E(dq_2, dp_2)$  commute.*

*Proof.*– For  $n \in \mathbb{Z}$ , define  $n' \in \mathbb{Z}$  and  $0 \leq \gamma < d$  by  $n = n'd + \gamma$ . In the evaluation of  $f_1(q_1, p_1, q_2, p_2; n')$ , we get  $n' = a(q_1, p_1, q_2, p_2; n')p_1 + b(q_1, p_1, q_2, p_2; n')$ ,  $a(q_1, p_1, q_2, p_2; n') \in \mathbb{Z}$ ,  $0 \leq b(q_1, p_1, q_2, p_2; n') < p_1$ . So

$$\begin{aligned} n &= a(q_1, p_1, q_2, p_2; n')dp_1 + b(q_1, p_1, q_2, p_2; n')d + \gamma, \\ &= a(dq_1, dp_1, dq_2, dp_2; n)dp_1 + b(dq_1, dp_1, dq_2, dp_2; n). \end{aligned}$$

As  $0 \leq b(q_1, p_1, q_2, p_2; n') \leq p_1 - 1$  and  $0 \leq \gamma < d$ , we get  $dp_1 + b(q_1, p_1, q_2, p_2; n')d + \gamma < dp_1$ , and thus

$$a(dq_1, dp_1, dq_2, dp_2; n) = a(q_1, p_1, q_2, p_2; n'), \quad (17)$$

$$b(dq_1, dp_1, dq_2, dp_2; n) = b(q_1, p_1, q_2, p_2; n')d + \gamma. \quad (18)$$

If  $b(q_1, p_1, q_2, p_2; n') < q_1$  then from (18)

$$b(dq_1, dp_1, dq_2, dp_2; n) \leq d(q_1 - 1) + \gamma < dq_1,$$

and, since

$$a(q_1, p_1, q_2, p_2; n')q_1 + b(q_1, p_1, q_2, p_2; n') = \alpha(q_1, p_1, q_2, p_2; n')q_2 + \beta(q_1, p_1, q_2, p_2; n'), \quad (19)$$

with  $0 \leq \beta(q_1, p_1, q_2, p_2; n') < q_2$ , from (17) and (18), we may write

$$a(dq_1, dp_1, dq_2, dp_2; n)dq_1 + b(dq_1, dp_1, dq_2, dp_2; n) \quad (20)$$

$$\begin{aligned} &= a(q_1, p_1, q_2, p_2; n')dq_1 + b(q_1, p_1, q_2, p_2; n')d + \gamma, \\ &= [\alpha(q_1, p_1, q_2, p_2; n')q_2 + \beta(q_1, p_1, q_2, p_2; n')]d + \gamma. \end{aligned} \quad (21)$$

As  $\beta(q_1, p_1, q_2, p_2; n')d + \gamma < (q_2 - 1)d + \gamma < dq_2$ , we get from (21),

$$\begin{aligned}\alpha(dq_1, dp_1, dq_2, dp_2; n) &= \alpha(q_1, p_1, q_2, p_2; n'), \\ \beta(dq_1, dp_1, dq_2, dp_2; n) &= d\beta(q_1, p_1, q_2, p_2; n') + \gamma,\end{aligned}$$

from which it follows that

$$f_1(dq_1, dp_1, dq_2, dp_2; n) = \alpha(dq_1, dp_1, dq_2, dp_2; n)dp_2 + d\beta(q_1, p_1, q_2, p_2; n') + \gamma, \quad (22)$$

$$f_1(dq_1, dp_1, dq_2, dp_2; n) = df_1(q_1, p_1, q_2, p_2; n') + \gamma. \quad (23)$$

If  $b(q_1, p_1, q_2, p_2; n') \geq q_1$  then, from (18),  $b(dq_1, dp_1, dq_2, dp_2; n) \geq dq_1$  and thus  $f_1(dq_1, dp_1, dq_2, dp_2; n) = f_1(q_1, p_1, q_2, p_2; n') = -\infty$  and the relation (23) still holds.

In a similar way, it is easy to prove the relation

$$f_2(dq_1, dp_1, dq_2, dp_2; n) = df_2(q_1, p_1, q_2, p_2; n') + \gamma. \quad (24)$$

Relations (23) and (24) imply that  $f_1(q_1, p_1, q_2, p_2; \bullet)$  and  $f_2(q_1, p_1, q_2, p_2; \bullet)$  are equal if and only if  $f_1(dq_1, dp_1, dq_2, dp_2; \bullet)$  and  $f_2(dq_1, dp_1, dq_2, dp_2; \bullet)$  are equal, which proves the lemma.  $\square$

**Lemma 4.** *Let  $q_1, p_1, q_2, p_2$  be integers with  $1 \leq q_1 \leq p_1$ ,  $1 \leq q_2 \leq p_2$ . If  $q_1 > q_2$  then  $f_1(q_1 - 1) > f_2(q_1 - 1)$ .*

*Proof.*— According to our notation convention  $f_1(q_1 - 1)$  and  $f_2(q_1 - 1)$  stands here for  $f_1(q_1, p_1, q_2, p_2; q_1 - 1)$  and  $f_2(q_1, p_1, q_2, p_2; q_1 - 1)$ .

With  $n = q_1 - 1$  in function  $f_1$ , we get  $a = 0$  and  $b = q_1 - 1$ . As  $b < q_1$ ,  $\alpha$  and  $\beta$  are such that  $\alpha q_2 + \beta = q_1 - 1$ ,  $0 \leq \beta < q_2$  and  $\alpha > 0$  because  $q_1 > q_2$ . then  $f_1(q_1 - 1) = \alpha p_2 + \beta$ .

On the other hand, in function  $f_2$  for  $n = q_1 - 1$ ,  $c = \alpha$  and  $d = \beta$ . Then  $\alpha p_2 + \beta = \gamma p_1 + \delta$  with  $0 \leq \delta < p_1$ .

- If  $\gamma = 0$ , then  $\delta = \alpha p_2 + \beta > \alpha q_2 + \beta = q_1 - 1$  and thus  $f_2(q_1 - 1) = -\infty$ .
- If  $\gamma > 0$  and  $\delta \geq q_1$  then  $f_2(q_1 - 1) = -\infty$ .
- If  $\gamma > 0$  and  $\delta < q_1$  then  $f_2(q_1 - 1) = \gamma q_1 + \delta < \gamma p_1 + \delta = \alpha p_2 + \beta = f_1(q_1 - 1)$ .

$\square$

**Lemma 5.** *Let  $q_1, p_1, q_2, p_2$  be integers with  $1 \leq q_1 \leq p_1$ ,  $1 \leq q_2 \leq p_2$ . If  $q_1 < q_2$  then  $g_1(q_2 - 1) > g_2(q_2 - 1)$ .*

*Proof.*— As  $q_2 > q_1$ , the set of parameters  $q_2, p_2, q_1, p_1$  satisfy the condition given by Lemma 4 implying  $f_1(q_2, p_2, q_1, p_1; q_2 - 1) > f_2(q_2, p_2, q_1, p_1; q_2 - 1)$ .

Using Lemma 1, we get  $g_1(q_1, p_1, q_2, p_2; q_2 - 1) > g_2(q_1, p_1, q_2, p_2; q_2 - 1)$  i.e.  $g_1(q_2 - 1) > g_2(q_2 - 1)$ .  $\square$

Let us consider now the case of equal block lengths  $q_1 = q_2 = q$ .

**Lemma 6.** *Let  $q, p_1, q_2, p_2$  be integers,  $1 \leq q \leq p_2 < p_1$  such that  $p_1 = kq + r, p_2 = lq + s$  with  $0 \leq r < s < q$ . Then  $f_1(p_1 + q - s) = p_2 + q - s$  and  $f_2(p_1 + q - s) = -\infty$ .*

*Proof.*— In the evaluation of  $f_1(p_1 + q - s)$ , we get  $a = 1$ ,  $b = q - s$  since  $0 < q - s < q$ . Now  $aq + b = 2q - s$  and thus  $\alpha = 1$  and  $\beta = q - s$ .

Finally  $f_1(p_1 + q - s) = \alpha p_2 + \beta = p_2 + q - s$ .

From  $f_2(p_1 + q - s) = f_2(kq + r + q - s)$ , we get  $c = k$  and  $d = r + q - s$  since  $0 \leq r \leq r + q - s < q$ . Then

$$cp_2 + d = k(lq + s) + r + q - s = l(kq + r) + (k - 1)s - (l - 1)r + q,$$

The conditions  $p_1 > p_2$  and  $r < s$  imply that  $k > l$  and thus  $(k - 1)s - (l - 1)r + q > q$ .

As  $s < q$ ,  $(k - 1)s - (l - 1)r + q < kq < p_1$ , we get  $cp_2 + d = \gamma p_1 + \delta$  with  $\gamma = l$  and  $\delta = (k - 1)s - (l - 1)r + q > q$ . So  $f_2(p_1 + q - s) = -\infty$ .  $\square$

**Lemma 7.** Let  $q, p_1, q_2, p_2$  be integers,  $1 \leq q \leq p_2 < p_1$  such that  $p_1 = kq + r, p_2 = lq + s$  with  $r > 0$  and  $0 \leq s \leq r < q$ . Then  $f_1(p_1 + q - 1) > f_2(p_1 + q - 1)$ .

*Proof.*– In the evaluation of  $f_1(p_1 + q - 1)$ , we get  $a = 1, b = q - 1$ . From  $aq + b = 2q - 1$ , it comes  $\alpha = 1$  and  $\beta = q - 1$  and then  $f_1(p_1 + q - 1) = p_2 + q - 1 = (l + 1)q + s - 1$ .

In the evaluation of  $f_2(p_1 + q - 1)$ , we get  $p_1 + q - 1 = (k + 1)q + r - 1$  and since  $r > 0, c = k + 1, d = r - 1$ . Then  $\gamma$  and  $\delta$  are determined by

$$\gamma p_1 + \delta = cp_2 + d = (k + 1)(lq + s) + r - 1, \quad (25)$$

and

- If  $q \leq \delta < p_1, f_2(p_1 + q - 1) = -\infty$  and the lemma is proved.
- If  $0 \leq \delta < q$ , first prove the inequation

$$\gamma p_1 + \delta < (l + 1)p_1 + s - 1. \quad (26)$$

Using (25), we get

$$\begin{aligned} \Delta &= (l + 1)p_1 + s - 1 - (\gamma p_1 + \delta) = (l + 1)(kq + r) + s - 1 - ((k + 1)(lq + s) + r - 1) \\ &= k(q - s) - l(q - r). \end{aligned}$$

- If  $k = l$  then  $r > s$  because  $p_1 > p_2$ , and then  $\Delta = k(r - s) > 0$ ,
- If  $k > l$  and because  $q - s \geq q - r > 0, k(q - s) > l(q - r)$  and  $\Delta > 0$ ,

which proves that (26) is satisfied.

Since the application  $d(q, p_1; \bullet)$  defined on  $\{n, n \in \mathbb{Z}, \text{rem}(n, p_1) < q\}$  by  $d(q, p_1; n) = \text{quo}(n, p_1)q + \text{rem}(n, p_1)$  is a strictly increasing function, relation (26) implies

$$f_2(p_1 + q - 1) = \gamma q + \delta < (l + 1)q + s - 1 = f_1(p_1 + q - 1).$$

□

*Proof of the theorem.*– When  $q_1 \neq q_2$  Lemmas 4 and 5 prove that  $D(q_1, p_1)$  and  $E(q_2, p_2)$  cannot commute.

If  $q_1 = q_2 = q$  and  $p_2 < p_1$ , the only case for  $(r, s)$  not considered in lemmas 6 and 7, as shown in Figure 1, is the case where  $r = s = 0$ , i.e.  $p_1 = kq$  and  $p_2 = lq$  with  $l < k$ , and thus  $D(q_1, p_1)$  and  $E(q_2, p_2)$  cannot commute when  $p_1$  or  $p_2$  are not multiples of  $q$ .

Using the exchange property given by Lemma 1, we obtain a similar result for  $p_2 > p_1$ .

So, if  $q_1 = q_2 = q$  and  $p_1 \neq p_2, D(q_1, p_1)$  and  $E(q_2, p_2)$  cannot commute unless  $p_1 = qp'_1$  and  $p_2 = qp'_2$  for some  $p'_1 > 1$  and  $p'_2 > 1$ . Using Lemma 3 and Lemma 2, if  $D(q_1, p_1)$  and  $E(q_2, p_2)$  commute then  $\text{gcd}(p'_1, p'_2) = 1$ , which is equivalent to  $\text{gcd}(p_1, p_2) = q$ .

The only case not yet considered is the case where  $q_1 = q_2 = q$  and  $p_1 = p_2 = p$  with  $q < p$ .  $D(q, p)$  do not commute with  $E(q, p)$  because  $D(q, p)E(q, p)$  is the identity while in  $E(q, p)D(q, p)$  the  $x[q]$  sample is discarded, i.e.  $f_1(q) = -\infty$  while  $f_2(q) = q$ .

This achieves the proof of the direct part of the theorem. The converse part of the theorem results immediately from Lemmas 3 and 2. □

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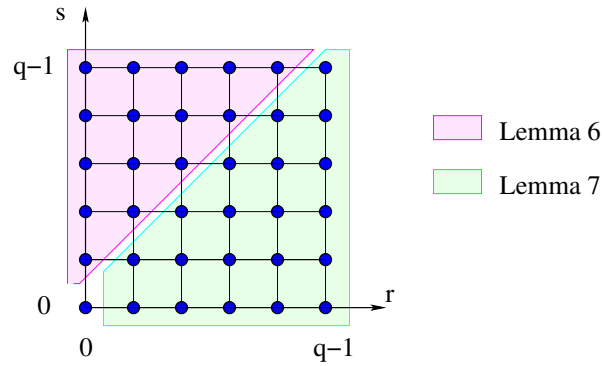


Fig. 1. Cases for  $(r, s)$  studied in Lemmas 6 and 7 for  $q_1 = q_2 = q$  and  $q < p_2 < p_1$ .

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