Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations

Stefan Le Coz, Tai-Peng Tsai

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Abstract

We look for solutions to generic nonlinear Schrödinger equations build upon solitons and kinks. Solitons are localized solitary waves and kinks are their non localized counterparts. We prove the existence of *infinite soliton trains*, i.e. solutions behaving at large time as the sum of infinitely many solitons. We also show that one can attach a kink at one end of the train. Our proofs proceed by fixed point arguments around the desired profile. We present two approaches leading to different results, one based on a combination of $L^p - L^{p'}$ dispersive estimates and Strichartz estimates, the other based only on Strichartz estimates.

Keywords: soliton train, multi-soliton, multi-kink, nonlinear Schrödinger equations.

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1 Introduction

We consider the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + f(u) = 0, \tag{NLS}$$

where $u = u(t,x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^d$, $d \geq 1$.

Our goal in this paper is push forward a study initiated in [9] on the existence of exotic solutions to (NLS). We look for *infinite soliton trains*, i.e. solutions which behave asymptotically as the sum of infinitely many solitons, possibly attached to a kink at one end. We want to show that such a behavior is possible for general nonlinearities under mild hypotheses. A typical nonlinearity example is the double-power nonlinearity

$$f(u) = |u|^{\alpha} u - |u|^{\beta} u, \quad 0 < \alpha < \beta < \alpha_{\max}. \tag{1.1}$$

Here and thereafter we denote the critical exponent by $\alpha_{\max} = +\infty$ for $d = 1, 2$ and $\alpha_{\max} = \frac{4}{d-2}$ for $d \geq 3$.

Let us shortly review some results on multi-solitons, i.e. solutions to (NLS) behaving at large time as a finite sum of solitons. The inverse scattering transform provides a convenient way to build multi-solitons (see e.g. [15]), however it is limited to integrable equations (for

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Schrödinger equations, only the 1D cubic case is integrable). For non-integrable Schrödinger equations, one of the first result of existence of multi-solitons was obtained by Merle [13] for L^2-critical equations, triggering a series of work on multi-solitons. For energy-subcritical nonlinearities, Côte, Martel and Merle [6, 11] obtained the existence of multi-solitons build upon ground states, while the excited states case was treated by Côte and Le Coz [5] under a high speed assumption. Stability/instability results have been obtained by Côte and Le Coz [5], Martel, Merle, Tsai [12] and Perelman [14]. However, stability of multi-solitons for power-type nonlinearities is still an open issue.

The existence of objects like infinite soliton trains is of importance as they usually provide examples of extreme phenomena in the asymptotic behavior of solutions of nonlinear dispersive equations. For example, for the Korteweg-de Vries equation, an infinite train of solitons was used in [10] as a counter example to show the optimality of an asymptotic stability statement. For nonlinear Schrödinger equation, the asymptotic stability results usually hold under assumptions (typically in weighted spaces) excluding the infinite train behavior. To our knowledge, our previous work [9] was the first one to establish the existence of infinite soliton trains for non-integrable Schrödinger equations (for the integrable 1D cubic nonlinear Schrödinger equation, the existence of infinite soliton trains may be obtained via the inverse-scattering transform, see [8]).

Before stating our main results, let us give some preliminaries. To work in an energy subcritical context, we first assume the following.

Assumption (F0). *Let* $d \geq 1$. *Suppose* $f(u) = g(|u|^2)u$ *where* $g \in C^0([0,\infty),\mathbb{R}) \cap C^2((0,\infty),\mathbb{R})$, $g(0) = 0$ *and*

$$|sg'(s)| + |s^2 g''(s)| \leq C_0(s^{\alpha_1/2} + s^{\alpha_2/2}), \quad \forall s > 0,$$

where $0 < \alpha_1 \leq \alpha_2 < \alpha_{\max}$ *and* $C_0 > 0$.

A *bound state* is a nontrivial solution $\phi \in H^1(\mathbb{R}^d)$ of the elliptic equation

$$\Delta\phi + f(\phi) = \omega\phi \tag{1.2}$$

for some frequency $\omega > 0$. We shall sometimes denote a bound state along with its frequency (ϕ, ω) to emphasize the dependency of ϕ on ω. Any bound state ϕ with frequency ω and parameters $x^0 \in \mathbb{R}^d$ (position), $v \in \mathbb{R}^d$ (velocity) and $\gamma \in \mathbb{R}$ (phase) corresponds to a *solitary wave* solution (*soliton*) of (NLS),

$$R_{\phi,\omega,x^0,v,\gamma}(t,x) = e^{i(\omega t + \frac{1}{2}vx - \frac{1}{4}|v|^2 t + \gamma)}\phi(x - x^0 - vt). \tag{1.3}$$

The profile of an infinite soliton train is a sum of the form

$$R_\infty = \sum_{j=1}^{\infty} R_j, \quad R_j(t,x) = R_{\phi_j,\omega_j,x_j^0,v_j,\gamma_j}(t,x), \quad j \in \mathbb{N}, \tag{1.4}$$

where $(R_j)_j$ are given solitons with bound states profiles (ϕ_j, ω_j) and parameters $x_j^0, v_j \in \mathbb{R}^d$ and $\gamma_j \in \mathbb{R}$. A solution $u(t)$ is called an *infinite soliton train* if, for some profile R_∞,

$$u(t) - R_\infty(t) \to 0 \quad \text{as} \quad t \to \infty$$

in some space-time norm.

Constructing a solution to (NLS) around an infinite train profile as (1.4) is much trickier than when the profile is made with a finite number of solitons. First of all, we need to make sure that the profile is well defined, as the addition of infinitely many solitons may very well be infinite. We also have to take into account that it is very likely that the profile will not belong to the same functional spaces as the solitons. In order to deal with these issues we need a control on the growth of the solitons' profiles (see (1.5)) and also to guarantee some space integrability of the train (see (1.6)).

We will assume the following for our infinite train.

Assumption (T1). *For $0 < \alpha_1 < \alpha_{\max}$ given, the sequence of bound states $\{(\phi_j, \omega_j) : j \in \mathbb{N}\}$ satisfies, for some $0 < a < 1$ and D_a independent of j,*

$$|\phi_j(x)| + \omega_j^{-1/2}|\nabla\phi_j(x)| \leq D_a \omega_j^{1/\alpha_1} e^{-a\omega_j^{1/2}|x|}, \quad \forall x \in \mathbb{R}^d, \ \forall j \in \mathbb{N}, \tag{1.5}$$

and, for some $r_0 \geq 1$, $\frac{d\alpha_1}{2} < r_0 < 2 + \alpha_1$,

$$A_1 := \sum_{j\in\mathbb{N}} \omega_j^{\frac{1}{\alpha_1} - \frac{d}{2r_0}} < \infty. \tag{1.6}$$

We say a nonlinearity f satisfies (T1) if such an infinite sequence $(\phi_j, \omega_j)_j$ exists for some r_0. Examples of such nonlinearities will be given in Section 2.

Note that the set $[1, \infty) \cap (\frac{d\alpha_1}{2}, 2 + \alpha_1)$ for r_0 is nonempty since $0 < \alpha_1 < \alpha_{\max}$. The condition $r_0 > \frac{d\alpha_1}{2}$ ensures that the exponent $\frac{1}{\alpha_1} - \frac{d}{2r_0} > 0$. Thus $\omega_j \to 0$ as $j \to \infty$, and (1.6) is a condition on how fast ω_j goes to 0. The existence of sequences of bound states satisfying Assumption (T1) is guaranteed by Proposition 2.1, where bound states with small frequencies are constructed as bifurcation from 0 along a fixed radial bound state Q of the equation $\Delta Q + |Q|^{\alpha_1}Q = Q$ together with the estimate (1.5). Note that the ϕ_j may be arbitrary excited states solutions of (1.2); in particular they may be sign-changing, non-radial, or complex-valued. Also note that we do not need the bound for $\omega^{-1/2}|\nabla\phi_\omega(x)|$ in (1.5) for Theorems 1.2 and 1.11 below, but we assume it for all theorems for simplicity of presentation. For the same reason, we shall also set all initial positions x_j to 0. Our assumption includes the finite multi-soliton case by setting $(\phi_j, \omega_j) = (0, 0)$ for j sufficiently large.

We have followed two independent approaches for the study of this problem, leading to two different types of results with different assumptions and conclusions. Before stating our main results, we need a preliminary lemma which will be proved in Section 4.

Lemma 1.1. *Let $d \geq 1$. For any $0 < \alpha_1 < \alpha_2 < \alpha_{\max}$ satisfying $\frac{\alpha_2}{2+\alpha_2} \leq \alpha_1$, one can choose r_0 so that the following conditions hold.*

$$\max(1, \frac{d\alpha_1}{2}) < r_0 < 2 + \alpha_1, \tag{1.7}$$

$$\frac{1}{2} \leq \frac{\alpha_1}{r_0} + \frac{1}{r_2}, \tag{1.8}$$

$$1 < \frac{\alpha_1 + 1}{r_0} + \frac{1}{r_2}, \tag{1.9}$$

where $r_2 = 2 + \alpha_2$. Furthermore, if $\alpha_1 < 4/d$, we can choose $r_0 \leq 2$.

1.1 Infinite soliton trains

We now state our two results on the existence of infinite soliton trains. The first approach of the first theorem is based on L^p-L^q decay estimates for $e^{it\Delta}$. The Strichartz space $S([t,\infty))$ will be defined in Section 3.

Theorem 1.2 (Infinite train of solitons (i))**.** *Let $d \geq 1$ and assume Assumption (F0) and*

$$\frac{\alpha_2}{2+\alpha_2} \leq \alpha_1. \tag{1.10}$$

Let $r_2 = 2 + \alpha_2$ and take any r_0 verifying (1.7)*,* (1.8)*, and* (1.9)*. Let $(\phi_j, \omega_j)_{j\in\mathbb{N}}$ be a sequence of bound states satisfying Assumption (T1) with the chosen r_0. There exist constants $c_1 > 0$ and $v_\sharp \gg 1$ such that, for any infinite soliton train profile R_∞ given as in* (1.4) *with parameters $v_j \in \mathbb{R}^d$, $x_j^0 = 0$, $\gamma_j \in \mathbb{R}$ satisfying*

$$v_* = \inf_{j,k\in\mathbb{N}, j\neq k} \sqrt{\omega_j}\,|v_k - v_j| \geq v_\sharp, \tag{1.11}$$

there exists a solution u to (NLS) *on $[0,\infty)$ satisfying*

$$\|(u - R_\infty)(t)\|_{L^{r_2}} + \|u - R_\infty\|_{S([t,\infty))} \leq e^{-c_1 v_* t}, \quad \forall t \geq 0. \tag{1.12}$$

It is unique in the class of solutions satisfying the above estimate.

Remark 1.3 (L^2-solutions). By (1.12) and Hölder inequality,

$$\|(u - R_\infty)(t)\|_{L^r} \leq e^{-c_1 v_* t}, \quad \forall t \geq 0, \quad \forall r \in [2, r_2].$$

As we will show that $R_\infty \in L^\infty(0,\infty; L^{r_0} \cap L^\infty(\mathbb{R}^d))$ in (4.1), we have $u \in L^\infty(0,\infty; L^{r_1} \cap L^\infty(\mathbb{R}^d))$ where $r_1 = \max(2, r_0)$. In the case $\alpha_1 < 4/d$, we can choose $r_0 \leq 2$ by Lemma 1.1, and thus $u \in L^\infty(0,\infty; L^2(\mathbb{R}^d))$.

Remark 1.4 (Comparison to previous results). Theorem 1.2 contains the pure power case $f(u) = |u|^\alpha u$ by writing $f(u) = |u|^\alpha u - 0|u|^{\alpha+\epsilon} u$ for some small $\epsilon > 0$. It also includes the finite soliton train (multi-soliton) case by taking $(\phi_j, \omega_j) = (0, 0)$ for j sufficiently large. In addition the range of exponents is larger than in [9, Theorem 6.4]. Hence Theorem 1.2 extends Theorems 1.1, 1.7, 6.3 and 6.4 in [9] in a unified approach (except that [9, Theorem 6.3] does not require (1.10)).

Remark 1.5 (L^2-subcritical nonlinearities). If we use a pure Strichartz norm approach and do not use L^{r_2} norm, we can construct infinite soliton trains for all L^2-subcritical or critical exponents $0 < \alpha_1 < \alpha_2 \leq 4/d$ as in [9, Theorem 6.3], without the restriction (1.10).

In our second main result, we also control the train at the gradient level. The approach is based solely on Strichartz estimates.

Theorem 1.6 (Infinite train of solitons (ii))**.** *Let $d \geq 1$ and assume Assumption (F0) with $0 < \alpha_1 < \frac{4}{d+2}$. Let $(\phi_j, \omega_j)_{j\in\mathbb{N}}$ be a sequence of bound states satisfying Assumption (T1) for some r_0. There exist constants $C > 0$, $c_1 > 0$, $c_2 > 0$, and $v_\sharp \gg 1$ such that, for any infinite soliton train profile R_∞ given as in* (1.4) *with parameters $v_j \in \mathbb{R}^d$, $x_j^0 = 0$, $\gamma_j \in \mathbb{R}$ satisfying*

$$v_* := \inf_{j,k\in\mathbb{N}, j\neq k} \sqrt{\omega_j}\,|v_k - v_j| \geq v_\sharp, \tag{1.13}$$

and

$$V_* := \sum_{j\in\mathbb{N}} \langle v_j\rangle \omega_j^{\frac{1}{\alpha_1}-\frac{d}{4}} < \infty, \tag{1.14}$$

there exists a unique solution u to (NLS) *satisfying, for some $T_0 = T_0(V_*) \gg 1$,*

$$e^{c_1 v_* t}\|u - R_\infty\|_{S([t,\infty))} + e^{c_2 v_* t}\|\nabla(u - R_\infty)\|_{S([t,\infty))} \le C, \quad \forall t \ge T_0. \tag{1.15}$$

Remark 1.7 (Examples of parameters choices). Condition (1.13) requires sufficiently large relative speed, while condition (1.14) puts an upper bound on the growth of $\langle v_j\rangle$. By (1.14) we may assume $r_0 \le 2$. One possible choice of parameters is

$$\omega_j = 4^{-j}, \quad v_j = 2^{j+1}\bar{v}, \quad |\bar{v}| \gg 1. \tag{1.16}$$

Condition (1.14) can be satisfied ($V_* \lesssim \sum_j (4^{-j})^{-\frac{1}{2}+\frac{1}{\alpha_1}-\frac{d}{4}} < \infty$) thanks to the assumption $\alpha_1 < \frac{4}{d+2}$ (note this implies $\alpha_1 < 1$ unless $d = 1$).

In the above choice V_* and v_* grow linearly in $|\bar{v}|$. In the following choice $V_* = O(h(|\bar{v}|)|\bar{v}|)$ while $v_* = C|\bar{v}|$ for any function $h > 1$:

$$\omega_j = 4^{-j}, \quad v_j = \begin{cases} 2^{j+1}h(|\bar{v}|)\bar{v}, & \text{if } j \text{ is odd} \\ -2^{j+1}\bar{v}, & \text{if } j \text{ is even} \end{cases}, \quad |\bar{v}| \gg 1. \tag{1.17}$$

Remark 1.8 (Infinite train starting at time 0). We use large T_0 to off-set the contribution of large V_*. If we impose that V_* grows sub-exponentially in v_*, e.g., $V_* \le C(1 + v_*)^M$ for some $M \ge 1$ (e.g. $h(s) = (1 + s)^{M-1}$ in (1.17)), we may take $T_0 = 0$ as in [9, Theorem 6.1].

Remark 1.9 (Existence of infinite trains under (F0) and (T1)). The proof of Theorem 1.2 uses a combination of L^{r_2} norm and Strichartz norm. To estimate $|\eta|^{\alpha_1+1}$ in L^{r_2} using $L^{r'}$-L^r decay estimates, a restriction like (1.10) is needed to avoid the limiting case $\alpha_1 = 0+$ and $\alpha_2 = \alpha_{\max}-$. However, we claim that exponents excluded by (1.10) are covered by Theorem 1.6 above. Indeed, let $\bar{\alpha} = \sup_{0<\alpha<\alpha_{\max}} \frac{\alpha}{2+\alpha}$. We have $\bar{\alpha} = 1$ for $d = 1, 2$ and $\bar{\alpha} = 2/d$ for $d \ge 3$. One then verifies that $\bar{\alpha} \le \frac{4}{d+2}$ for all dimensions.

Hence *we can construct infinite soliton trains for all energy-subcritical nonlinearities satisfying* Assumptions (F0) and (T1).

Remark 1.10 (Comparison between Theorems 1.2 and 1.6). Theorem 1.2 applies for nonlinearities whose general form is not far from a power type nonlinearity, no matter what this power is (α_1 can be any H^1-subcritical power). Theorem 1.6 applies for nonlinearities that are sufficiently strong at 0 (α_1 has to be small), but with any kind of growth possible away from 0. For the choice of the profile, Theorem 1.2 is more flexible as it requires only some weak integrability condition (1.6), whereas Theorem 1.6 requires L^2-integrability of the profile (one take $r_0 = 2$ in (T1)) *and* its first derivative (1.14).

1.2 Infinite kink-soliton trains

In our next couple of theorems we let $d = 1$ and consider in $\mathbb{R}$ a train of the form

$$W = K + R_\infty$$

where R_∞ is as in (1.4), and K is a kink solution of (NLS) given by the same formula (1.3) but with the profile $\phi = \phi_K$ now being a half-kink satisfying the same equation (1.2) ($\phi'' = \omega\phi - f(\phi)$), $0 < \phi_K(s) < b$ for some $b > 0$, and

$$\lim_{s\to-\infty} \phi_K(s) = b, \quad \phi_K'(s) < 0 \quad \forall s \in \mathbb{R}, \quad \phi_K'(0) = \min \phi_K', \quad \lim_{s\to+\infty} \phi_K(s) = 0. \tag{1.18}$$

A solution which converges to a profile W as above at positive time infinity will be called an *infinite kink-soliton train.* We are going to give two results of existence of infinite kink-soliton trains. Note that such object was never exhibited before, even in integrable cases.

In addition to Assumption (F0), we make the following assumption, which in particular ensure the existence of a half-kink satisfying (1.18) (see Proposition 5.3).

Assumption (F1). *For some $\omega_0 > 0$, there is a first $b > 0$ such that for $h(s) = \omega_0 s - f(s)$,*

$$h(b) = 0, \qquad \int_0^b h(s)ds = 0. \tag{1.19}$$

Moreover, $h'(b) > 0$, and for some $\tilde{\alpha} \in [0, \alpha_2]$,

$$|f'(b+s)| + |s||f''(b+s)| \le C|s|^{\tilde{\alpha}} + C|s|^{\alpha_2}, \quad \forall s \in \mathbb{R}. \tag{1.20}$$

Note that the nonlinearity (1.1) admits a half-kink when $d = 1$. See Example 5.2.

We now state our second set of results on the existence of infinite kink-soliton trains. Recall $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Theorem 1.11 (An infinite kink-soliton train (i))**.** *Let $d = 1$ and assume Assumptions (F0), (F1) and*

$$\frac{\alpha_2}{2+\alpha_2} \le \alpha_1. \tag{1.21}$$

Let $r_2 = 2 + \alpha_2$. Then we can find r_0 satisfying (1.7)–(1.9)*. Assume that $\tilde{\alpha}$ is such that*

$$\frac{1}{2} \le \frac{\tilde{\alpha}}{r_0} + \frac{1}{r_2}, \qquad 1 < \frac{\tilde{\alpha}+1}{r_0} + \frac{1}{r_2}. \tag{1.22}$$

Assume there is a sequence of bound states $(\phi_j, \omega_j)_{j\in\mathbb{N}}$ satisfying Assumption (T1) with the chosen r_0. Let $\phi_0 = \phi_K$ be the kink profile to be given in Proposition 5.3. There exist constants $c_1 > 0$, and $v_\sharp \gg 1$ such that, for the infinite kink-soliton profile $W = K + R_\infty$, given as in (1.4)*, with any parameters $v_j \in \mathbb{R}$, $v_j < v_{j+1}$, $x_j^0 = 0$, $\gamma_j \in \mathbb{R}$ for $j \in \mathbb{N}_0$ satisfying*

$$v_* = \inf_{j,k\in\mathbb{N}_0, j\ne k} \sqrt{\omega_j}\,|v_k - v_j| \ge v_\sharp,$$

there exists a unique solution u to (NLS) *for $t \ge 0$ satisfying*

$$\|(u-W)(t)\|_{L^{r_2}} + \|u - W\|_{S([t,\infty))} \le e^{-c_1 v_* t}, \quad \forall t \ge 0. \tag{1.23}$$

Theorem 1.12 (An infinite kink-soliton train (ii))**.** *Let $d = 1$ and assume Assumptions (F0) and (F1) with $0 < \alpha_1 < 4/3$. Let (ϕ_j, ω_j), $j \in \mathbb{N}$ be given and satisfying Assumption (T1) for some r_0 which further satisfies*

$$r_0(\alpha_1 + 1) < (\tilde{\alpha} + 1)(\alpha_1 + 2). \tag{1.24}$$

Let $\phi_0 = \phi_K$ be the kink profile to be given in Proposition 5.3. There exist constants $C > 0$, $c_1 > 0$, $c_2 > 0$, $T_0 \gg 1$ and $v_\sharp \gg 1$ such that, for the kink-soliton train profile $W = K + R_\infty$ given as in (1.4) *with any parameters $v_j \in \mathbb{R}$, $v_j > v_0$, $x_j^0 = 0$, $\gamma_j \in \mathbb{R}$ for $j \in \mathbb{N}_0$ and sufficiently large relative speed*

$$v_* = \inf_{j\in\mathbb{N}, k\in\mathbb{N}_0, j\ne k} \sqrt{\omega_j}\,|v_k - v_j| \ge v_\sharp, \tag{1.25}$$

$$V_* := \sum_{j\in\mathbb{N}} \langle v_j\rangle \omega_j^{\frac{1}{\alpha_1} - \frac{d}{4}} < \infty, \tag{1.26}$$

there exists a unique solution u to (NLS) *for $t \ge T_0$ satisfying*

$$e^{c_1 v_* t}\|u - W\|_{S([t,\infty))} + e^{c_2 v_* t}\|\nabla(u - W)\|_{S([t,\infty))} \le C, \quad \forall t \ge T_0. \tag{1.27}$$

Remark 1.13. In Theorems 1.11 and 1.12, the kink K is on the left in the profile and its velocity is less than the velocity of any soliton. This picture can be reversed by the symmetry $u(x,t) \to \tilde{u}(x,t) = u(-x,t)$.

Remark 1.14. In Theorem 1.12 we require upper bound $\alpha_1 < 4/3$ and lower bound (1.24) on $\tilde{\alpha}$. The bound (1.24) is redundant if we choose a smaller r_0, e.g. $r_0 = 1$, but is nontrivial if we take $r_0 = 2$.

The rest of the paper is organized as follows: In Section 2 we give an example of nonlinearity for which Assumption (T1) is satisfied. In Section 3 we give the general scheme of our proofs. In Section 4 we prove Theorems 1.2 and 1.6. In Section 5 we give Examples 5.1 and 5.2 for nonlinearities verifying Assumption (F1) and we prove Theorems 1.11 and 1.12.

2 Existence of a family of bound states satisfying (T1)

Assumption (T1) is satisfied for the nonlinearity f if, for example, f satisfies Assumption (F2) below.

Assumption (F2). *Suppose $f(u) = f_1(u) + f_2(u)$ where $f_1(u) = |u|^\alpha u$, $f_2(u) = g_2(|u|^2)u$, $g_2 \in C^0([0,\infty),\mathbb{R}) \cap C^2((0,\infty),\mathbb{R})$, $g_2(0) = 0$ and*

$$|sg_2'(s)| + |s^2 g_2''(s)| \le C_0(s^{\beta_1/2} + s^{\beta_2/2}), \quad \forall s > 0,$$

where $0 < \alpha < \beta_1 \le \beta_2 < \alpha_{\max}$ and $C_0 > 0$.

This assumption is more specific about the small u behavior of $f(u)$ than those in Assumption (F0) so that we can have more control on the bound states with respect to their frequencies. In particular, we do not consider $f_1(u)$ with opposite sign.

The following proposition gives an existence result of bound states with small frequencies, obtained as the bifurcation from the radial ground state Q of the pure power nonlinearity, together with uniform estimates.

Proposition 2.1 (Bifurcation of solitons)**.** *Let $d \ge 1$ and assume Assumption (F2). Let $Q(x)$ be the unique positive radial solution of $\Delta Q + |Q|^\alpha Q = Q$ in $\mathbb{R}^d$. There is a small $\omega_* = \omega_*(d,\alpha,\beta_1,\beta_2,C_0) > 0$ so that for all $0 < \omega < \omega_*$ there is a solution $\phi = \phi_\omega$ of* (1.2) *of the form*

$$\phi_\omega(x) = \omega^{1/\alpha}[Q(\omega^{1/2}x) + \xi_\omega(\omega^{1/2}x)], \tag{2.1}$$

where $\|\xi_\omega\|_{H^2} \le C\omega^m$ with $m = \frac{\beta_1/\alpha - 1}{\min(1,\alpha)} > 0$. Moreover, for any $0 < a < 1$ there is a constant $D_a > 0$ such that

$$|\phi_\omega(x)| + \omega^{-1/2}|\nabla\phi_\omega(x)| \le D_a \omega^{1/\alpha} e^{-a\omega^{1/2}|x|}, \quad \forall x \in \mathbb{R}^d,\ \forall \omega \in (0,\omega_*). \tag{2.2}$$

Note that we could allow Q to be any radial excited state, provided we knew its non-degeneracy, i.e invertibility of L_+ in the proof below (such a result should be a consequence of the classifications results [3, 4], however we did not pursue in that direction).

Before proving Proposition 2.1, we recall without proof the following classical lemma.

Lemma 2.2. *Suppose* $f(u) = g(|u|^2)u$, $g \in C^0([0,\infty),\mathbb{R})$, $f(0) = 0$ *and*

$$|sg'(s)| \le C(s^{\alpha_1/2} + s^{\alpha_2/2}), \quad \forall s > 0.$$

For $W, \eta \in \mathbb{C}$ *we have*

$$|f(W+\eta) - f(W)| \lesssim |\eta|(|W|^{\alpha_1} + |W|^{\alpha_2}) + |\eta|^{1+\alpha_1} + |\eta|^{1+\alpha_2}.$$

Proof of Proposition 2.1. Since Q is real and radial, we will look for real and radial ξ_ω. For the sake of simplicity in notation, we drop the subscript ω during the proof. Denoting $y = \omega^{1/2}x$ and substituting (2.1) in (1.2), we get

$$(-\Delta_y + 1)\xi = \omega^{-\frac{1}{\alpha}-1} f(\omega^{1/\alpha}(Q+\xi)) - |Q|^\alpha Q.$$

It can be rewritten as

$$L_+\xi = N(\xi) = N_1(\xi) + N_2(\xi), \tag{2.3}$$

where

$$\begin{aligned} L_+ &= -\Delta_y + 1 - (1+\alpha)|Q|^\alpha \\ N_1(\xi) &= f_1(Q+\xi) - f_1(Q) - (1+\alpha)|Q|^\alpha \xi \\ N_2(\xi) &= \omega^{-\frac{1}{\alpha}-1} f_2(\omega^{1/\alpha}(Q+\xi)). \end{aligned}$$

In the special case $f_2(u) = -|u|^\beta u$, we have $N_2(\xi) = -\omega^{\frac{\beta}{\alpha}-1}|Q+\xi|^\beta(Q+\xi)$.

Let $X = H^2_{rad}(\mathbb{R}^d)$. The properties of L_+ are well-known (see e.g [2]). It has one negative eigenvalue, its kernel in $L^2(\mathbb{R}^d)$ is spanned by $(\partial_{y_j}Q)_j$ and the rest of its spectrum is positive away from 0. Hence for radial functions $L_+ : X \to L^2_{rad}$ is invertible and we have

$$C_3 := \|(L_+)^{-1}\|_{\mathcal{B}(L^2_{rad};X)} < \infty.$$

We have

$$|N_1(\xi)| \lesssim 1_{\alpha>1}|Q|^{\alpha-1}|\xi|^2 + |\xi|^{1+\alpha} \tag{2.4}$$

$$|N_1(\xi_1) - N_1(\xi_2)| \lesssim 1_{\alpha>1}|Q|^{\alpha-1}(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2| + (|\xi_1| + |\xi_2|)^\alpha |\xi_1 - \xi_2|. \tag{2.5}$$

We also have, by Assumption (F2) and Lemma 2.2,

$$|N_2(\xi)| \lesssim \omega^{-\frac{1}{\alpha}-1} \textstyle\sum_{j=1}^2 |\omega^{1/\alpha}(Q+\xi)|^{1+\beta_j} = \sum_{j=1}^2 \omega^{\frac{\beta_j}{\alpha}-1}|Q+\xi|^{1+\beta_j}. \tag{2.6}$$

$$|N_2(\xi_1) - N_2(\xi_2)| \lesssim \textstyle\sum_{j=1}^2 \omega^{\frac{\beta_j}{\alpha}-1}(|Q| + |\xi_1| + |\xi_2|)^{\beta_j}|\xi_1 - \xi_2|. \tag{2.7}$$

Denote $B_r = \{\xi \in X : \|\xi\|_X \le r\}$ for $0 < r < 1$ and let $0 < \omega < 1$. Because X is imbedded in $L^{2+2\alpha} \cap L^{2+2\beta_2}$ for any dimension d, we have, for some C_4,

$$\|N(\xi_1) - N(\xi_2)\|_{L^2} \le C_4 \left((\|\xi_1\|_X + \|\xi_2\|_X)^{\min(1,\alpha)} + \omega^{\frac{\beta_1}{\alpha}-1} \right) \|\xi_1 - \xi_2\|_X, \tag{2.8}$$

for any $\xi_1, \xi_2 \in B_r$. Thus the map $\xi \mapsto (L_+)^{-1}N(\xi)$ is a contraction map in $B_r \subset X$ for any $\omega \in (0, \omega_*)$ if we choose $(2r)^{\min(1,\alpha)} = \omega_*^{\frac{\beta_1}{\alpha}-1} < (4C_3C_4 + 1)^{-1}$.

Finally, standard argument for exponential decay (see [1] or [7, Appendix]) shows that for any $a \in (0,1)$

$$|\xi(x)| + |\nabla\xi(x)| \le o(1)e^{-a|x|}, \quad |Q(x)| + |\nabla Q(x)| \le Ce^{-a|x|},$$

using the uniform bound $\|\xi\|_{H^2} \ll 1$. We get (2.2) after rescaling. □

3 The perturbation argument

We recall the definition of the Strichartz spaces $S([t,\infty))$ and $N([t,\infty))$ and the well known dispersive and Strichartz estimates. A pair of exponents (q,r) is said to be *(Schrödinger)-admissible* if

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2},\quad 2\le q,r\le+\infty,\quad (d,q,r)\neq(2,2,+\infty).$$

Given a time $t\in\mathbb{R}$, the Strichartz space $S([t,\infty))$ is defined via the norm

$$\|u\|_{S([t,\infty))}=\sup_{\substack{(q,r)\text{ admissible}\\ r\le r_{\mathrm{Str}}}}\|u\|_{L_t^qL_x^r([t,+\infty)\times\mathbb{R}^d)}.$$

Above $r_{\mathrm{Str}}=\infty$ for $d\neq 2$, but we choose $\alpha_2+2<r_{\mathrm{Str}}<\infty$ when $d=2$ to stay away from the forbidden endpoint. We denote the dual space by $N([t,\infty))=S([t,\infty))^*$. Hence for any (q,r) admissible, its norm verifies

$$\|u\|_{N([t,\infty))}\le\|u\|_{L_t^{q'}L_x^{r'}([t,+\infty)\times\mathbb{R}^d)}$$

where q',r' are the conjugate exponents of q and r.

Let us recall the standard *dispersive inequality*

$$\|e^{it\Delta}u\|_p\lesssim|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_{p'}\quad\text{for } t\neq 0,\ 2\le p\le+\infty$$

from which one can deduce the usual Strichartz estimate:

$$\|u\|_{S([t_0,+\infty))}\lesssim\|u_0\|_{L^2}+\|F\|_{N([t_0,+\infty))}$$

where for $u_0\in L^2(\mathbb{R})$ u solves on $[t_0,\infty)$ the following equation

$$iu_t+\Delta u=F,\quad u(t_0)=u_0.$$

For the proof of the main theorems with a profile $W=R_\infty$ or $W=K+R_\infty$, we will consider the error term $\eta=u-W$, which satisfies

$$i\partial_t\eta+\Delta\eta=-[f(W+\eta)-f(W)]-H,\quad H=f(W)-\sum_{j\in\mathbb{N}_0}f(R_j).\tag{3.1}$$

Above $R_0=0$ if $W=R_\infty$ and $R_0=K$ if $W=K+R_\infty$. In Duhamel form,

$$\eta(t)=-i\int_t^\infty e^{i(t-s)\Delta}[f(W+\eta)-f(W)+H](s)\,ds.\tag{3.2}$$

The proofs of Theorems 1.2 and 1.11 given in Sections 4 and 5 are self contained. For the proofs of Theorems 1.6 and 1.12, we rely on the following generic result proved in [9, Proposition 2.4].

Proposition 3.1. *Let $d\ge 1$ and assume Assumption (F0). Let $H=H(t,x):[0,\infty)\times\mathbb{R}^d\to\mathbb{C}$, $W=W(t,x):[0,\infty)\times\mathbb{R}^d\to\mathbb{C}$ be given functions which satisfy for some $C_1>0$, $C_2>0$, $\lambda>0$, $T_0\ge 0$:*

$$\begin{aligned}&\|W(t)\|_\infty+e^{\lambda t}\|H(t)\|_2\le C_1,\qquad\forall t\ge T_0;\\&\|\nabla W(t)\|_2+\|\nabla W(t)\|_\infty+e^{\lambda t}\|\nabla H(t)\|_2\le C_2,\qquad\forall t\ge T_0.\end{aligned}\tag{3.3}$$

Consider the equation (3.2). *There exists a constant* $\lambda_* = \lambda_*(d, \alpha_1, \alpha_2, C_1) > 0$ *independent of* C_2, *and a time* $T_* = T_*(d, \alpha_1, \alpha_2, C_1, C_2) > 0$ *sufficiently large such that if* $\lambda \geq \lambda_*$ *and* $T_0 \geq T_*$, *then there exists a unique solution* η *to* (3.2) *on* $[T_0, +\infty) \times \mathbb{R}^d$ *satisfying*

$$e^{\lambda t}\|\eta\|_{S([t,\infty))} + e^{\lambda c_1 t}\|\nabla\eta\|_{S([t,\infty))} \leq 1, \qquad \forall t \geq T_0. \tag{3.4}$$

Here $c_1 > 0$ *is a constant depending only on* (α_1, d).

4 Construction of infinite soliton trains

4.1 Proof of Theorem 1.2

In this section we prove Theorem 1.2 and construct infinite soliton trains in $\mathbb{R}^d$, $d \geq 1$. Note that (1.6) in Assumption (T1) implies $A_2 := \sum_{j\in\mathbb{N}} \omega_j^{\frac{1}{\alpha_1}} < \infty$, and

$$\|R_\infty(t)\|_{L^\infty\cap L^{r_0}} \leq \sum_{j\in\mathbb{N}}\|R_j(t)\|_{L^\infty\cap L^{r_0}} \lesssim \sum_{j\in\mathbb{N}}(\omega_j^{\frac{1}{\alpha_1}} + \omega_j^{\frac{1}{\alpha_1}-\frac{d}{2r_0}}) = A_2 + A_1. \tag{4.1}$$

We first show the existence of the exponent r_0 and prove Lemma 1.1.

Proof of Lemma 1.1. The idea is to choose $r_0 = \max(1, \frac{d\alpha_1}{2}) + \epsilon$ for some $0 < \epsilon \ll 1$. Clearly $r_0 < 2 + \alpha_1$ for sufficiently small $\epsilon > 0$ since $\alpha_1 < \alpha_{\max}$. So (1.7) is satisfied.

In the case $\frac{d\alpha_1}{2} \geq 1$, we claim

$$\frac{\alpha_1}{\frac{d\alpha_1}{2}} + \frac{1}{r_2} > \frac{1}{2}, \quad \frac{\alpha_1 + 1}{\frac{d\alpha_1}{2}} + \frac{1}{r_2} > 1.$$

Both are clear if $d \leq 2$. For $d \geq 3$, both left sides become strictly smaller if α_1 is replaced by $\alpha_{\max} = \frac{4}{d-2}$ and r_2 is replaced by $2 + \alpha_{\max}$, but are no less than the right sides by direct computation. Thus (1.8) and (1.9) are satisfied for sufficiently small $\epsilon > 0$.

In the case $\frac{d\alpha_1}{2} < 1$, we claim

$$\frac{\alpha_1}{1} + \frac{1}{r_2} > \frac{1}{2}, \quad \frac{\alpha_1 + 1}{1} + \frac{1}{r_2} > 1.$$

The first inequality is a consequence of the assumption $\alpha_1 \geq \alpha_2/(\alpha_2 + 2)$, while the second is trivial. Thus (1.8) and (1.9) are satisfied for sufficiently small $\epsilon > 0$.

Suppose $\alpha_1 < 4/d$. In the case $\frac{d\alpha_1}{2} \geq 1$, since $\frac{d\alpha_1}{2} < 2$, $r_0 = \frac{d\alpha_1}{2} + \epsilon < 2$ for sufficiently small $\epsilon > 0$. In the case $\frac{d\alpha_1}{2} < 1$, $r_0 = 1 + \epsilon < 2$. The proof of the lemma is complete. □

Remark 4.1. Although we chose $r_0 = \max(1, \frac{d\alpha_1}{2}) + \epsilon$ in the proof of Lemma 1.1, it is not necessary for Theorem 1.2. We only need r_0 to satisfy (1.7)–(1.9).

We next estimate the source term in the equation for the error.

Lemma 4.2. *Under the assumptions of Theorem 1.2, the source term* $H = f(R_\infty) - \sum_{j\in\mathbb{N}} f(R_j)$ *satisfies, for some* $c_1 \in (0, a/2)$,

$$\|H(\cdot, t)\|_{L^\infty\cap L^{r_2'}} \leq Ce^{-c_1 v_* t}.$$

Proof. Fix $t > 0$. For any $x \in \mathbb{R}^d$, choose $m = m(x) \in \mathbb{N}$ so that ϕ_m is a nearest soliton, i.e.

$$|x - v_m t| = \min_{j\in\mathbb{N}} |x - v_j t|.$$

For $j \neq m$, we have

$$|x - v_j t| \geq \frac{1}{2}|v_j t - v_m t| = \frac{t}{2}|v_j - v_m|. \tag{4.2}$$

Thus, by (1.5), we have

$$|(R_\infty - R_m)(x,t)| \leq \sum_{j\neq m} |R_j(x,t)| \leq \delta_m(x,t) := \sum_{j\neq m} D_a \omega_j^{\frac{1}{\alpha_1}} e^{-a\omega_j^{1/2}|x-v_j t|}. \tag{4.3}$$

Hence, by (1.6), the definition of v_* (1.11) and (4.2), we have

$$\delta_m(x,t) \leq \sum_{j\neq m} D_a \omega_j^{\frac{1}{\alpha_1}} e^{-\frac{1}{2}a v_* t} = D_a A_2 e^{-\frac{1}{2}a v_* t}. \tag{4.4}$$

Denote $A_3 = \sup_{0<s<\|R_\infty\|_{L^\infty}} |f'(s)|$. By Lemma 2.2 and (4.1), we have

$$\begin{aligned} |H(t,x)| &\leq |f(R_\infty) - f(R_m)| + \sum_{j\neq m} |f(R_j)| \\ &\leq A_3|R_\infty - R_m| + \textstyle\sum_{j\neq m} A_3|R_j| \leq 2A_3 \textstyle\sum_{j\neq m}|R_j| \leq 2A_3\delta_m(t,x). \end{aligned}$$

In particular,

$$\|H(t)\|_{L^\infty} \leq 2D_a A_2 A_3 e^{-\frac{1}{2}a v_* t}. \tag{4.5}$$

Condition (1.9) is equivalent to $\frac{1}{r_2'} < \frac{1+\alpha_1}{r_0}$. Thus we can choose s so that

$$\frac{1+\alpha_1}{r_0} > \frac{1}{s} > \frac{1}{r_2'}, \quad s > 1. \tag{4.6}$$

The first inequality of (4.6) ensures that

$$\frac{\alpha_1+1}{\alpha_1} - \frac{d}{2s} > \frac{1}{\alpha_1} - \frac{d}{2r_0},$$

and hence, using (1.6),

$$\sum_{j\in\mathbb{N}} \|f(R_j)\|_{L^s} \lesssim \sum_{j\in\mathbb{N}} \||R_j|^{\alpha_1+1} + |R_j|^{\alpha_2+1}\|_{L^s} \lesssim \sum_{j\in\mathbb{N}} \omega_j^{\frac{\alpha_1+1}{\alpha_1} - \frac{d}{2s}} < C < \infty.$$

Since $r_0 < s(1+\alpha_1) < s(1+\alpha_2) < \infty$ by (4.6), we have by (4.1)

$$\|f(R_\infty)\|_{L^s} \lesssim \|R_\infty\|_{L^\infty\cap L^{r_0}}^{1+\alpha_1} + \|R_\infty\|_{L^\infty\cap L^{r_0}}^{1+\alpha_2} < C < \infty.$$

Thus

$$\|H(t)\|_{L^s} < \|f(R_\infty)\|_{L^s} + \sum_{j\in\mathbb{N}} \|f(R_j)\|_{L^s} < C < \infty. \tag{4.7}$$

By Hölder inequality between L^∞ and L^s using (4.5) and (4.7), we have

$$\|H(t)\|_{L^r} \leq Ce^{-(1-s/r)\frac{a}{2}v_* t}, \quad \forall r \in (s,\infty).$$

Since $s < r_2' < \infty$ by (4.6), we get the desired conclusion. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. The existence of r_0 has been shown in Lemma 1.1. We now fix such a choice. The difference $\eta = u - R_\infty$ satisfies equation (3.2) with $W = R_\infty$ and $H = f(R_\infty) - \sum_{j\in\mathbb{N}} f(R_j)$. Denote the right side of (3.2) as $\Phi\eta$. We will show it is a contraction mapping and has a unique fixed point $\eta = \Phi\eta$ in the class

$$\|\eta(t)\|_{L^{r_2}} + \|\eta\|_{S([t,\infty))} \le e^{-c_1 v_* t}, \quad \forall t \ge 0. \tag{4.8}$$

We first show boundedness and suppose η satisfies (4.8). By Hölder inequality,

$$\|\eta(t)\|_{L^r} \le e^{-c_1 v_* t}, \quad \forall t \ge 0, \quad \forall r \in [2, r_2].$$

We have

$$\|\Phi\eta(t)\|_{L^{r_2}} \le C \int_t^\infty |t-\tau|^{-\theta} \left(\|f(W+\eta) - f(W)\|_{L^{r_2'}} + \|H(\tau)\|_{L^{r_2'}} \right) d\tau,$$

where $\theta = d(\frac{1}{2} - \frac{1}{r_2})$, and $0 < \theta < 1$ since $2 < r_2 < 2 + \alpha_{\max}$.

By Lemma 4.2 we have $\|H(\tau)\|_{L^{r_2'}} \le Ce^{-c_1 v_* \tau}$. By Lemma 2.2,

$$\|f(W+\eta) - f(W)\|_{L^{r_2'}} \lesssim \||\eta|(|W|^{\alpha_1} + |W|^{\alpha_2})\|_{L^{r_2'}} + \||\eta|^{\alpha_1+1} + |\eta|^{\alpha_2+1}\|_{L^{r_2'}}. \tag{4.9}$$

The first term on the right side is bounded by Hölder inequality

$$\||\eta|(|W|^{\alpha_1} + |W|^{\alpha_2})\|_{L^{r_2'}} \le (1 + \|W\|_{L^\infty}^{\alpha_2-\alpha_1}) \|W\|_{L^{r_0}\cap L^\infty}^{\alpha_1} \|\eta\|_{L^2\cap L^{r_2}} \le Ce^{-c_1 v_* t}$$

if

$$\frac{\alpha_1}{\infty} + \frac{1}{r_2} \le \frac{1}{r_2'} \le \frac{\alpha_1}{r_0} + \frac{1}{2}.$$

The first inequality is always true since $r_2' \le 2 \le r_2$. The second inequality is correct if (1.8) holds. Thus this term can be estimated.

The last term of (4.9) is bounded by

$$\||\eta|^{\alpha_1+1} + |\eta|^{\alpha_2+1}\|_{L^{r_2'}} \lesssim \|\eta\|_{L^{r_2'(\alpha_1+1)}}^{\alpha_1+1} + \|\eta\|_{L^{r_2'(\alpha_2+1)}}^{\alpha_2+1},$$

which is bounded by $Ce^{-c_1 v_* t}$ since

$$2 \le r_2'(\alpha_1+1) < r_2'(\alpha_2+1) \le r_2,$$

due to (1.10) and $r_2 = 2 + \alpha_2$.

Combining the above we have, assuming (4.8),

$$\|\Phi\eta(t)\|_{L^{r_2}} \le \int_t^\infty |t-\tau|^{-\theta} Ce^{-c_1 v_* \tau} d\tau \le Cv_*^{-1+\theta} e^{-c_1 v_* t}$$

for all $t \ge 0$, which is bounded by $\frac{1}{4}e^{-c_1 v_* t}$ if v_* is sufficiently large.

For the Strichartz estimate, since $(2/\theta, r_2)$ is admissible, we have with $a = (2/\theta)'$

$$\begin{aligned}\|\Phi\eta\|_{S([t,\infty))} &\lesssim \|f(W+\eta) - f(W) + H\|_{L^a(t,\infty;L^{r_2'})} \\ &\lesssim \|e^{-c_1 v_* \tau}\|_{L^a(t,\infty)} \lesssim v_*^{-1/a} e^{-c_1 v_* t},\end{aligned}$$

for all $t \geq 0$, which is bounded by $\frac{1}{4}e^{-c_1 v_* t}$ if v_* is sufficiently large.

Consider now the difference estimate. Suppose both η_1 and η_2 satisfy (4.8). Denote $\eta = \eta_1 - \eta_2$ and

$$\delta = \sup_{t>0} e^{c_1 v_* t} \left(\|\eta(t)\|_{L^{r_2}} + \|\eta\|_{S([t,\infty))} \right) \leq 2.$$

We have

$$\|(\Phi\eta_1 - \Phi\eta_2)(t)\|_{L^{r_2}} \leq C \int_t^\infty |t-\tau|^{-\theta} \|f(W+\eta_1) - f(W+\eta_2)\|_{L^{r_2'}}(\tau)\, d\tau.$$

By Lemma 2.2 again with W replaced by $W + \eta_2$,

$$\begin{aligned}\|f(W+\eta_1) - f(W+\eta_2)\|_{L^{r_2'}} &\lesssim \||\eta|(|W+\eta_2|^{\alpha_1} + |W+\eta_2|^{\alpha_2})\|_{L^{r_2'}} + \||\eta|^{\alpha_1+1} + |\eta|^{\alpha_2+1}\|_{L^{r_2'}} \\ &\lesssim \||\eta|(|W|^{\alpha_1} + |W|^{\alpha_2})\|_{L^{r_2'}} + \||\eta|(E^{\alpha_1} + E^{\alpha_2})\|_{L^{r_2'}} \qquad (4.10)\end{aligned}$$

where $E = |\eta_1| + |\eta_2|$. The first term is already bounded above

$$\||\eta|(|W|^{\alpha_1} + |W|^{\alpha_2})\|_{L^{r_2'}} \leq C\|\eta\|_{L^2 \cap L^{r_2}} \leq C\delta e^{-c_1 v_* t}.$$

The last term of (4.10) is bounded similarly as above

$$\||\eta|(E^{\alpha_1} + E^{\alpha_2})\|_{L^{r_2'}} \leq \|\eta\|_{L^2\cap L^{r_2}} (\|E\|^{\alpha_1}_{L^2\cap L^{r_2}} + \|E\|^{\alpha_2}_{L^2\cap L^{r_2}}) \leq C\delta e^{-c_1(1+\alpha_1) v_* t}.$$

Thus

$$\begin{aligned}\|(\Phi\eta_1 - \Phi\eta_2)(t)\|_{L^{r_2}} &\leq \int_t^\infty |t-\tau|^{-\theta} C\delta e^{-c_1 v_* \tau} d\tau \\ &\leq C\delta v_*^{-1+\theta} e^{-c_1 v_* t}\end{aligned}$$

for all $t \geq 0$, which is bounded by $\frac{1}{4}\delta e^{-c_1 v_* t}$ if v_* is sufficiently large.

We also have (recall $a = (2/\theta_1)'$)

$$\begin{aligned}\|\Phi\eta_1 - \Phi\eta_2\|_{S([t,\infty))} &\lesssim \|f(W+\eta_1) - f(W+\eta_2)\|_{L^a(t,\infty; L^{r_2'})} \\ &\lesssim \|\delta e^{-c_1 v_* \tau}\|_{L^a(t,\infty)} \lesssim \delta v_*^{-1/a} e^{-c_1 v_* t},\end{aligned}$$

for all $t \geq 0$, which is bounded by $\frac{1}{4}\delta e^{-c_1 v_* t}$ if v_* is sufficiently large.

We have shown that Φ is a contraction mapping and hence has a unique fixed point in the set (4.8). The proof of Theorem 1.2 is complete. □

Remark 4.3. The assumption (1.10) is used to estimate $L^{r_2'}$. To estimate $|\eta|^{\alpha_1+1}$ in L^{r_2} using $L^{r'}$-L^r decay estimates, a restriction like (1.10) is needed to avoid the limiting case $\alpha_1 = 0+$ and $\alpha_2 = \alpha_{\max}-$.

The condition (1.8) is used to bound the linear term in η, while (1.9) is used to bound the source term (it ensures the existence of s in the proof of Lemma 4.2).

In (1.7), we need $r_0 \geq 1$ for (4.1). We need $r_0 > \frac{d\alpha_1}{2}$ so that the exponent in (1.6) is positive. The condition $r_0 < \alpha_1 + 2$ in (1.7) is redundant and follows from (1.9).

4.2 Proof of Theorem 1.6

In this section we prove Theorem 1.6 and construct infinite soliton trains in $\mathbb{R}^d$, $d \geq 1$. All along this section, we assume that we are under the assumptions of Theorem 1.6, in particular we suppose that we are given a sequence of bound states (ϕ_j, ω_j) for $j \in \mathbb{N}$ satisfying assumptions (T1), (1.13) (with $v_\sharp$ to be determined later) and (1.14).

We first prove the following lemma.

Lemma 4.4. *Let $a \in (0,1)$ be given by Assumption (T1). For $\lambda = a \min(1, 2a) v_*/4 > 0$, we have*

$$\begin{aligned} &\|R_\infty(t)\|_\infty + e^{\lambda t}\|H(t)\|_2 \leq C, \qquad \forall\, t \geq 0; \\ &\|\nabla R_\infty(t)\|_2 + \|\nabla R_\infty(t)\|_\infty + e^{\lambda t}\|\nabla H(t)\|_2 \leq C(1+V_*), \qquad \forall\, t \geq 0. \end{aligned} \tag{4.11}$$

where H is the source term defined by $H = f(R_\infty) - \sum_{j\in\mathbb{N}} f(R_j)$.

Proof. Equation (1.6) in Assumption (T1) implies $A_2 := \sum_{j\in\mathbb{N}} \omega_j^{\frac{1}{\alpha_1}} < \infty$, and

$$\|R_\infty(t)\|_{L^\infty\cap L^{r_0}} \leq \sum_{j\in\mathbb{N}} \|R_j(t)\|_{L^\infty\cap L^{r_0}} \lesssim \sum_{j\in\mathbb{N}} (\omega_j^{\frac{1}{\alpha_1}} + \omega_j^{\frac{1}{\alpha_1} - \frac{d}{2r_0}}) = A_2 + A_1.$$

We also have for $1 \leq r \leq \infty$

$$\|\nabla R_\infty(t)\|_{L^r} \lesssim \sum_{j\in\mathbb{N}} \|\nabla R_j(t)\|_{L^r} \lesssim \sum_{j\in\mathbb{N}} \omega_j^{\frac{1}{\alpha_1} + \frac{1}{2} - \frac{d}{2r}} + \sum_{j\in\mathbb{N}} |v_j| \omega_j^{\frac{1}{\alpha_1} - \frac{d}{2r}}. \tag{4.12}$$

If we take $r = 2$, we have $\frac{1}{\alpha_1} + \frac{1}{2} - \frac{d}{2r} \geq \frac{1}{\alpha_1} - \frac{d}{2r_0}$ for all dimensions since $r_0 < 2 + \alpha_{\max}$. Thus the first sum of the right hand side of (4.12) is finite for $r \in [2, \infty]$ by (1.6). The second sum is also finite for $r \in [2, \infty]$ by (1.14). Thus

$$\|\nabla R_\infty(t)\|_{L^2\cap L^\infty} \lesssim A_1 + V_*.$$

We next consider the estimates of $H = f(R_\infty) - \sum_{j\in\mathbb{N}} f(R_j)$. Fix $t > 0$. As in the proof of Lemma 4.2, take any $x \in \mathbb{R}^d$ and choose $m = m(x) \in \mathbb{N}$ so that ϕ_m is a nearest soliton, i.e.

$$|x - v_m t| = \min_{j\in\mathbb{N}} |x - v_j t|.$$

Since $\alpha_1 < \alpha_{\max}$ and $r_0 < 2 + \alpha_1$, there exists $s = \frac{\alpha_1 + 2 - \epsilon}{\alpha_1 + 1}$ with $0 < \epsilon \ll 1$ such that

$$r_0 < 2 + \alpha_1 - \epsilon, \qquad \frac{\alpha_1 + 1}{\alpha_1} - \frac{d}{2} \cdot \frac{1}{s} \geq \frac{1}{\alpha_1} - \frac{d}{2r_0}.$$

From arguments identical to those of the proof of Lemma 4.2, we have

$$\|H(t)\|_{L^r} \leq C e^{-c(1-s/r)v_* t}, \quad \forall r \in (s, \infty),$$

with acceptable r including $\frac{\alpha_1+2}{\alpha_1+1}$ and 2.

To estimate $\|\nabla H(t)\|_{L^2}$, recall that by the Chain Rule we have

$$\begin{aligned} \nabla H &= \nabla(f(R_\infty)) - \sum_{j\in\mathbb{N}} \nabla(f(R_j)) \\ &= \sum_{j\in\mathbb{N}} (f_z(R_\infty) - f_z(R_j)) \nabla R_j + \sum_{j\in\mathbb{N}} (f_{\bar{z}}(R_\infty) - f_{\bar{z}}(R_j)) \overline{\nabla R_j}. \end{aligned} \tag{4.13}$$

Here, we denoted $f_z = \frac{\partial}{\partial z} f$ and $f_{\bar{z}} = \frac{\partial}{\partial \bar{z}} f$ the Wirtinger derivatives of f. Thus (here x and $m = m(x)$ are still as above), we have

$$\begin{aligned}
|\nabla H(t,x)| &\lesssim \sum_{j\neq m} |\nabla R_j| + \big(|f_z(R_\infty) - f_z(R_m)| + |f_{\bar{z}}(R_\infty) - f_{\bar{z}}(R_m)|\big)\, |\nabla R_m| \\
&\lesssim \sum_{j\neq m} \langle v_j \rangle \omega_j^{1/\alpha_1} e^{-a\omega_j^{1/2}|x-v_jt|} + (\delta_m(t,x))^{\min(1,\alpha_1)} \langle v_m \rangle \omega_m^{1/\alpha_1} e^{-a\omega_m^{1/2}|x-v_mt|} \\
&\lesssim \sum_{j\neq m} \langle v_j \rangle \omega_j^{1/\alpha_1} e^{-\frac{1}{2}a\omega_j^{1/2}|x-v_jt|} e^{-\frac{a}{4}v_* t} + e^{-\frac{a}{2}\min(1,\alpha_1)v_* t} \langle v_m \rangle \omega_m^{1/\alpha_1} e^{-a\omega_m^{1/2}|x-v_mt|} \\
&\lesssim e^{-\lambda t} \sum_{j\in\mathbb{N}} \langle v_j \rangle \omega_j^{1/\alpha_1} e^{-\frac{1}{2}a\omega_j^{1/2}|x-v_jt|}
\end{aligned}$$

where $\delta_m(t,x)$ is defined and estimated in (4.3)–(4.4), and $\lambda = \frac{a}{4}\min(1, 2\alpha_1)v_*$. Thus

$$\|\nabla H(t)\|_{L^2} \lesssim e^{-\lambda t} \sum_{j\in\mathbb{N}} \langle v_j \rangle \omega_j^{1/\alpha_1 - \frac{d}{4}} \lesssim V_*$$

by Assumption (1.14). The proof of Lemma 4.4 is complete. □

We now prove Theorem 1.6.

Proof of Theorem 1.6. By Lemma 4.4, there exists $v_\sharp$ such that if $v_* > v_\sharp$, then the hypothesis (3.3) of Proposition 3.1 is satisfied under the assumptions of Theorem 1.6, with $W = R_\infty$ and $H = f(R_\infty) - \sum_{j\in\mathbb{N}} f(R_j)$. By Proposition 3.1, there exist T_0 large enough and $\eta \in C([T_0,\infty), H^1)$ with $\|\langle\nabla\rangle\eta\|_{S([t,\infty))}$ (in particular $\|\eta(t)\|_{H^1}$) decaying exponentially in t. □

Remark 4.5. One may tend to relax the exponent 2 in the norm $\|\nabla W\|_{L^2}$ so that ∇W is not that localized. However, $\|\nabla W\|_{L^{2+\beta_1}}$ with $\beta_1 < 0.01$ is used in the proof of Proposition 3.1. It would not gain much trying to optimize it.

5 Construction of infinite kink-soliton trains

In this section we prove Theorems 1.11 and 1.12, and construct a train made of infinitely many solitons and a half-kink for space dimension 1.

We first examine Assumption (F1) and give some examples. Estimate (1.20) is natural since f' is Hölder continuous. If $f'(b) \neq 0$, we can only take $\tilde{\alpha} = 0$. Otherwise, we may take $\tilde{\alpha} = 1$ if f is locally $C^{1,1}$ near b. For certain $f(s)$ we have $\tilde{\alpha} > 1$.

Example 5.1. Let $f(s) = s - \frac{\sin|s|}{|s|}s$. If we write $f(s) = f_1(s) + f_2(s)$ with $f_1(s) = \frac{1}{3}|s|^2 s$ and $f_2(s) = s - \frac{\sin|s|}{|s|}s - \frac{1}{3}|s|^2 s = O(s^5)$, f satisfies Assumptions (F0) and (F2) with $\alpha_1 = 2$ and $\alpha_2 = 4$. We can choose $r_0 = 1 + \epsilon$, $0 < \epsilon \ll 1$, for Assumption (T1). The function $f(s)$ also satisfies Assumption (F1) with $\omega = 1$, $b = 2\pi$, $h(s) = \sin s$ and $h'(b) = 1$. Moreover,

$$f(2\pi) = 2\pi \neq 0, \quad |f'(2\pi + s)| = |1 - \cos s| \leq C s^{\tilde{\alpha}}, \quad \tilde{\alpha} = 2.$$

Hence conditions (1.21)-(1.22) are satisfied. Thus we can construct infinite kink-soliton trains using Theorem 1.11. Since $\alpha_1 > 4/3$, Theorem 1.12 does not apply to this example. □

Example 5.2. let $f(s) = |s|^\alpha s - |s|^\beta s$, $0 < \alpha < \beta < \infty$. Clearly f satisfies Assumptions (F0) and (F2) with $\alpha_1 = \alpha$ and $\alpha_2 = \beta$. The conditions $h(b) = 0 = \int_0^b h(s)ds$ in Assumption (F1) give

$$\omega = b^\alpha - b^\beta = \frac{2}{2+\alpha} b^\alpha - \frac{2}{2+\beta} b^\beta.$$

Thus

$$b^{\beta-\alpha} = \frac{\alpha(2+\beta)}{(2+\alpha)\beta} \in \Big(\frac{\alpha}{\beta}, 1\Big), \quad \omega = b^\alpha(1 - b^{\beta-\alpha}) > 0,$$

and

$$h'(b) = \omega - (1+\alpha)b^\alpha + (1+\beta)b^\beta = -\alpha b^\alpha + \beta b^\beta > 0.$$

Thus (1.19) can be always satisfied by unique $\omega > 0$ and $b > 0$. For (1.20), we have $\tilde\alpha = 0$ for most pair (α, β). Theorem 1.11 is not applicable in those cases. The exception is when $0 = f'(b) = (1+\alpha)b^\alpha - (1+\beta)b^\beta$, hence $b^{\beta-\alpha} = \frac{\alpha(2+\beta)}{(2+\alpha)\beta} = \frac{1+\alpha}{1+\beta}$, or $\alpha\beta = 2$. Thus the exceptional case is

$$\tilde\alpha = 1 \quad \text{if} \quad 0 < \alpha < \sqrt{2}, \quad \beta = \frac{2}{\alpha}.$$

Since $d\alpha/2 < 1$, we can take $r_0 = 1$. Conditions (1.21)-(1.22) imply

$$\frac{\sqrt{5}-1}{2} \le \alpha < \sqrt{2}, \quad \beta = \frac{2}{\alpha}. \tag{5.1}$$

Thus for α satisfying (5.1), using Theorem 1.11 we can construct infinite kink-soliton trains for the nonlinearity $f(u) = (|u|^\alpha - |u|^{2/\alpha})u$. On the other hand, by Theorem 1.12 we can construct infinite kink-soliton trains if

$$0 < \alpha < 4/3, \quad \alpha < \beta < \infty. \tag{5.2}$$

We do not need $\alpha\beta = 2$. Indeed, since $d\alpha/2 < 1$ for $\alpha < 4/3$, we can take $r_0 = 1$, and condition (1.24) is satisfied for any $\tilde\alpha \ge 0$. We can choose ω_j and v_j as in (1.16) or (1.17). In comparison, Theorem 1.12 covers more exponents than Theorem 1.11 except when $4/3 \le \alpha < \sqrt{2}$ and $\beta = 2/\alpha$. □

The existence of half-kink profiles is guaranteed by the following result.

Proposition 5.3. *Let $d = 1$ and assume Assumptions (F0) and (F1). There is a solution $\phi_K(s)$ of*

$$\phi_K'' = \omega_0 \phi_K - f(\phi_K)$$

such that $0 < \phi_K(s) < b$,

$$\lim_{s\to-\infty} \phi_K(s) = b, \quad \phi_K'(s) < 0 \quad \forall s \in \mathbb{R}, \quad \phi_K'(0) = \min \phi_K', \quad \lim_{s\to+\infty} \phi_K(s) = 0,$$

and that, for any $0 < a < \min(\omega_0, h'(b))$, there is $D_a > 0$ so that

$$\mathbf{1}_{s<0}(b - \phi_K(s)) + \mathbf{1}_{s\ge0}\, \phi_K(s) + |\phi_K'(s)| \le D_a e^{-a|s|}, \quad \forall s \in \mathbb{R}.$$

Proposition 5.3 can be easily proved using classical ordinary differential equations techniques (see e.g. [9, Proposition 1.12]). As mentioned in Section 1, a kink solution of (NLS) with parameters (v_0, γ) is (setting the spatial translation to $x_0 = 0$)

$$K(t, x) = \phi_K(x - v_0 t) e^{i(\omega_0 t + \frac{1}{2} v_0 x - \frac{1}{4} v_0^2 t + \gamma)}.$$

For notational simplicity, we denote $K = R_0$ and we consider the kink-soliton train profile

$$W = K + R_\infty = \sum_{j=0}^{\infty} R_j$$

where R_∞ and R_j, $j > 0$, are given in (1.4).

5.1 Proof of Theorem 1.11

We will solve the difference $\eta = u - W$ in the class (1.23).

To start the proof, we note that, because $\tilde{\alpha}$ satisfies the same conditions as α_1, we can choose r_0 as in Lemma 1.1 to satisfy (1.22) in addition to (1.7)–(1.9). From now on we fix r_0.

We start by estimating the source term.

Lemma 5.4. *Under the assumptions of Theorem 1.11, the source term* $H = f(W) - \sum_{j=0}^{\infty} f(R_j)$ *satisfies, for some* $c_1 > 0$,

$$\|H(\cdot,t)\|_{L^\infty \cap L^{r_2'}} \le Ce^{-c_1 v_* t}.$$

Proof. By (1.22), we have $\frac{\tilde{\alpha}+1}{r_0} > \frac{1}{r_2'}$. We can choose s as in the proof of Lemma 4.2 to satisfy (4.6) and $\frac{\tilde{\alpha}+1}{r_0} > \frac{1}{s} > \frac{1}{r_2'}$.

For $x \ge \frac{1}{2}(v_0 + v_1)t$, the contribution from R_0 is the same as if R_0 were a soliton. Thus the estimate follows from Lemma 4.2.

For $x \le \frac{1}{2}(v_0 + v_1)t$, we have $H = (f(W) - f(K)) - \sum_{j=1}^{\infty} f(R_j)$. In the proof of Lemma 4.2 we have shown

$$|R_\infty(t,x)| \le Ce^{-\frac{1}{2}av_* t}, \quad \|R_\infty\|_{L^{r_0}} \le C, \tag{5.3}$$
$$\Big|\sum_{j=1}^{\infty} f(R_j)(t,x)\Big| \le Ce^{-\frac{1}{2}av_* t}, \quad \Big\|\sum_{j=1}^{\infty} f(R_j)\Big\|_{L^s} \le C.$$

For simplicity in notations, we assume now that $v_0 = \gamma_0 = 0$. This causes no loss of generality since (NLS) is invariant under a Galilean transform and it guarantees that the left part of the kink is approximately b without correction by a phase factor containing $e^{i\frac{1}{2}v_0 x}$. By Assumption (F1), the mean value theorem, and since $K, R_\infty \in L^\infty$, we have

$$|f(W) - f(K)| = |f(b + K - b + R_\infty) - f(b + K - b)| \lesssim (||K| - b| + |R_\infty|)^{\tilde{\alpha}} |R_\infty|.$$

We first derive

$$|f(W) - f(K)| \le Ce^{-\frac{1}{2}av_* t}.$$

Because $r_0 < (1 + \tilde{\alpha})s$,

$$\begin{aligned}\|f(W) - f(K)\|_{L^s} &\le \|||K| - b|^{\tilde{\alpha}} |R_\infty|\|_{L^s} + \||R_\infty|^{\tilde{\alpha}+1}\|_{L^s} \\ &\le \||K| - b\|^{\tilde{\alpha}}_{L^{r_0} \cap L^\infty} \|R_\infty\|_{L^{r_0} \cap L^\infty} + \|R_\infty\|^{\tilde{\alpha}+1}_{L^{r_0} \cap L^\infty} \le C.\end{aligned}$$

Summing these estimates, we have

$$\|H(t)\|_{L^\infty} \le Ce^{-\frac{1}{2}av_* t}, \quad \|H(t)\|_{L^s} \le C.$$

The lemma follows by Hölder inequality between L^∞ and L^s. □

Proof of Theorem 1.11. Fix a choice of r_0 satisfying (1.7)–(1.9) and (1.22). Let $\chi_1 = \chi_1(x,t) = \mathbf{1}_{x\le \frac{1}{2}(v_0+v_1)t}$ and $\chi_2 = 1-\chi_1$. Using (5.3), we have

$$\|\chi_1(W-b)\|_{L^{r_0}\cap L^\infty} + \|\chi_2 W\|_{L^{r_0}\cap L^\infty} \lesssim 1.$$

Assume

$$\|\eta(t)\|_{L^{r_2}} + \|\eta\|_{S([t,\infty))} \le e^{-c_1 v_* t}, \quad \forall t\ge 0. \tag{5.4}$$

Note

$$|f(W+\eta)-f(W)| \lesssim \chi_1|W-b|^{\tilde\alpha}|\eta| + \chi_2|W|^{\alpha_1}|\eta| + |\eta|^{\tilde\alpha+1} + |\eta|^{\alpha_1+1} + |\eta|^{\alpha_2+1}.$$

Thus by (1.8) and (1.22) we have

$$\begin{aligned}\|f(W+\eta)-f(W)\|_{L^{r_2'}}(\tau) &\le (1+\|\chi_1(W-b)\|^{\tilde\alpha}_{L^{r_0}\cap L^\infty} + \|\chi_2 W\|^{\alpha_1}_{L^{r_0}\cap L^\infty})\|\eta\|_{L^2\cap L^{r_2}}\\ &\quad + \|\eta\|^{\tilde\alpha+1}_{L^2\cap L^{r_2}} + \|\eta\|^{\alpha_1+1}_{L^2\cap L^{r_2}} + \|\eta\|^{\alpha_2+1}_{L^2\cap L^{r_2}}\\ &\le Ce^{-c_1 v_*\tau}.\end{aligned}$$

Denote the right side of (3.2) as $\Phi\eta$. The same argument as for Theorem 1.2 shows that

$$\|\Phi\eta(t)\|_{L^{r_2}} \le Cv_*^{-1+\theta}e^{-c_1v_*t}, \quad \|\Phi\eta\|_{S([t,\infty))} \le Cv_*^{-1+\theta/2}e^{-c_1v_*t}.$$

Thus $\|\Phi\eta(t)\|_{L^{r_2}} + \|\Phi\eta\|_{S([t,\infty))} \le e^{-c_1v_*t}$ for v_* sufficiently large.

For the difference estimate, for η_1 and η_2 satisfying (5.4), we use

$$|f(W+\eta_1)-f(W+\eta_2)| \lesssim \left(\chi_1|W-b|^{\tilde\alpha} + \chi_2|W|^{\alpha_1} + \sum_{j=1,2}\left(|\eta_j|^{\tilde\alpha} + |\eta_j|^{\alpha_1} + |\eta_j|^{\alpha_2}\right)\right)|\eta|$$

where $\eta=\eta_1-\eta_2$, and follow the same argument for Theorem 1.2 to derive, for v_* sufficiently large,

$$\|\Phi\eta_1 - \Phi\eta_2\| \le \frac{1}{2}\|\eta_1-\eta_2\|$$

where $\|\eta\| = \sup_{t>0} e^{c_1v_*t}\left(\|\eta(t)\|_{L^{r_2}} + \|\eta\|_{S([t,\infty))}\right)$. We have shown that Φ is a contraction mapping in the class (5.4). The proof of Theorem 1.11 is complete. ☐

5.2 Proof of Theorem 1.12

In this section we prove Theorem 1.12 and use Proposition 3.1 to construct a train of infinitely many solitons and a half-kink for space dimension 1.

We assume throughout this section that the assumptions of Theorem 1.12 hold. In particular, (ϕ_j,ω_j) for $j\in\mathbb{N}$ denote a sequence of bound states satisfying assumptions (T1), (1.25) (with $v_\sharp$ to be determined later) and $\phi_0=\phi_K$ is the kink profile given in Proposition 5.3 .

As in Section 4.2, our main task is to prove that the profile $W=K+R_\infty$ and the source term $H=f(W)-f(K)-\sum_{j\in\mathbb{N}} f(R_j)$ satisfy to the hypotheses of Proposition 3.1.

Lemma 5.5. *Let $a\in(0,1)$. For $\lambda = a\min(1,2a)v_*/4>0$, we have*

$$\begin{aligned}\|W(t)\|_\infty + e^{\lambda t}\|H(t)\|_2 &\le C_1, && \forall\, t\ge 0;\\ \|\nabla W(t)\|_2 + \|\nabla W(t)\|_\infty + e^{\lambda t}\|\nabla H(t)\|_2 &\le C(1+V_*), && \forall\, t\ge 0.\end{aligned}$$

Proof. Since R_∞ satisfies the same hypotheses as in Lemma 4.4, we only have to treat the addition of the kink. We have, by Lemma 4.4 and Proposition 5.3

$$\|W\|_\infty + \|\nabla W\|_\infty \le \|K\|_\infty + \|R_\infty\|_\infty + \|\nabla K\|_\infty + \|\nabla R_\infty\|_\infty \le C.$$

Note that by exponential decay $\nabla K \in L^2(\mathbb{R})$, therefore, combined with Lemma 4.4 this gives

$$\|\nabla W\|_2 \le \|\nabla K\|_2 + \|\nabla R_\infty\|_2 \le C.$$

We now estimate the source term H. As in the proof of Lemma 4.4, we fix $t > 0$, take any $x \in \mathbb{R}$ and choose $m = m(x)$ corresponding to the nearest profile, i.e.

$$|x - v_m t| = \min_{j\in\mathbb{N}} |x - v_j t|.$$

If $m \ge 1$, then as in the proof of Lemma 4.2, we still have

$$|(R_\infty - R_m)(t, x)| \le Ce^{-\frac{1}{2}av_* t},$$

and by Proposition 5.3 it holds

$$|K(t, x)| \le D_a e^{-a|x-v_0 t|} \le D_a e^{-\frac{1}{2}av_* t}.$$

Therefore, if $m \ge 1$ we have

$$H(t, x) \le |f(R_\infty) - \sum_{j\in\mathbb{N}} f(R_j)| + A_4|K| + |f(K)| \lesssim e^{-\frac{1}{2}av_* t},$$

where $A_4 = \max_{s\in[0,\|W\|_\infty]} f'(s)$. If $m = 0$, we replace the previous estimate by

$$H(t, x) \le A_4|R_\infty| + \sum_{j\in\mathbb{N}} |f(R_j)| \lesssim e^{-\frac{1}{2}av_* t}.$$

This implies that

$$\|H(t)\|_\infty \lesssim e^{-\frac{1}{2}av_* t}.$$

With x and m as above, if $m = 0$, we have (using a similar expression as (4.13))

$$|\nabla H(t, x)| \lesssim (|f_z(K + R_\infty) - f_z(K)| + |f_{\bar z}(K + R_\infty) - f_{\bar z}(K)|)|\nabla K| + \sum_{j\in\mathbb{N}} |\nabla R_j|$$

Since we are close to the kink ($m = 0$), the last sum will be small :

$$\sum_{j\in\mathbb{N}} |\nabla R_j| \lesssim e^{-\frac{a}{4}v_* t} \sum_{j\in\mathbb{N}} \langle v_j\rangle \omega_j^{1/\alpha_1} e^{-\frac{1}{2}\omega_j^{1/2}|x-v_j t|}.$$

In addition we have

$$(|f_z(K + R_\infty) - f_z(K)| + |f_{\bar z}(K + R_\infty) - f_{\bar z}(K)|) \lesssim |R_\infty| \lesssim e^{-\frac{a}{4}v_* t} \sum_{j\in\mathbb{N}} \omega_j^{1/\alpha_1} e^{-\frac{1}{2}\omega_j^{1/2}|x-v_j t|}.$$

Therefore

$$|\nabla H(t, x)| \lesssim e^{-\frac{a}{4}v_* t} \sum_{j\in\mathbb{N}} \langle v_j\rangle \omega_j^{1/\alpha_1} e^{-\frac{1}{2}\omega_j^{1/2}|x-v_j t|}.$$

The estimate for the case $m \geq 1$ is similar as in Lemma 4.4 and we can conclude by (1.26) that

$$\|\nabla H\|_2 \lesssim e^{-\lambda t} \sum_{j\in\mathbb{N}_0} \langle v_j \rangle \omega_j^{1/\alpha_1 - d/4} \leq e^{-\lambda t} V_*.$$

Let now s be defined as in the proof of Lemma 4.4. By (1.24), we can further assume

$$r_0 \leq s(\tilde{\alpha}+1). \tag{5.5}$$

For simplicity in notations, assume that the kink in not moving, i.e. $v_0 = 0$. Therefore the main contribution will come from the kink for $x < 0$ and the soliton train for $x > 0$. We have on the right

$$\begin{aligned}\|H\|_{L^s(x>0)} &\leq \|f(K+R_\infty)-f(R_\infty)\|_{L^s(x>0)} + \|f(K)\|_{L^s(x>0)} + \|f(R_\infty)\|_{L^s} + \sum_{j\in\mathbb{N}} \|f(R_j)\|_{L^s} \\ &\leq A_4 \|K\|_{L^s(x>0)} + C < +\infty,\end{aligned}$$

where the last inequality is due to exponential decay to 0 on the right for the kink. On the left, we have

$$\|H\|_{L^s(x<0)} \leq \|f(K+R_\infty) - f(K)\|_{L^s(x<0)} + \sum_{j\in\mathbb{N}} \|f(R_j)\|_{L^s}$$

The first term cannot be treated as previously (unless $R_\infty \in L^s(\mathbb{R})$, which is a priori not the case). Since f verifies (1.20), by the mean value theorem we have

$$|f(K+R_\infty) - f(K)| \lesssim \left((|K-b| + |R_\infty|)^{\tilde{\alpha}} + (|K-b| + |R_\infty|)^{\alpha_2}\right) |R_\infty|$$

Hence,

$$\|f(K+R_\infty) - f(K)\|_{L^s(x<0)} \lesssim \left(\|K-b\|_{L^1(x<0)}^{\tilde{\alpha}} + \|K-b\|_{L^1(x<0)}^{\alpha_2}\right) \|R_\infty\|_{L^\infty} + \|R_\infty\|_{L^{s(\tilde{\alpha}+1)}}^{1+\tilde{\alpha}}$$

The right hand side is finite since K converges exponentially to b and the $L^{s(\tilde{\alpha}+1)}$-norm of R_∞ is finite thanks to our choice of r_0 and (5.5). In conclusion,

$$\|H\|_{L^s} \leq \|H\|_{L^s(x<0)} + \|H\|_{L^s(x>0)} < +\infty.$$

By interpolation between $s < 2$ and ∞ we get

$$\|H\|_{L^2} \lesssim e^{-\lambda t}.$$

This concludes the proof.

□

Proof of Theorem 1.12. By Lemma 5.5, there exists $v_\sharp$ such that if $v_* > v_\sharp$, then the hypothesis (3.3) of Proposition 3.1 is satisfied under the assumptions of Theorem 1.12. The conclusion of the Theorem then follows immediately from the conclusion of Proposition 3.1. □

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