Preemption Operators

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Abstract

We introduce a family of operators for belief change that aim at making a new piece of information to be preemptive so that any former belief subsuming it is given up. That is, the current belief base is to be altered even in the case that it is logically consistent with the new piece of information. Existing operators for belief revision are inadequate for this purpose because they amount to set-theoretic union in a contradiction-free case. We propose a series of postulates for such preemption operators. We show that a preemption operator can be defined as a multiple contraction followed by an expansion, drawing on operators from belief revision.

Introduction

Formalizing belief change is a major topic in Artificial Intelligence. Belief revision is dedicated to the special case that a new piece of information must be taken into account, as a statement to be inserted in the belief base. Should the current belief base be contradicted by the new piece of information, then the current belief base must undergo some modifications before it can simply be unioned with the new piece of information, resulting in a new current belief base.

The AGM setting gives a logic-based characterization of revision operators via a list of postulates that a "rational" revision operator is meant to satisfy (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988). Two of the postulates, *vacuity* and *inclusion*, when taken together, enforce the property that the belief base is simply supplemented with the new piece of information in the case that the latter is logically consistent with the belief base: in such a case, no information is to be expelled from the belief base (see Appendix). However, should the new piece of information be preemptive in a belief base that it can be deduced from, then some information must be taken out –this may happen to be necessary even though the current belief base is logically consistent with the new piece of information.

An illustration is as follows. Assume that the current belief base expresses "Paul is in his office or at home". Consider the situation that the information "Paul is in his office or at home or at his club" is then provided. In some respects, if the information "Paul is in his office or at home or at his club" is now at hand, it presumably should take precedence over the former information. That is, "Paul is in his office or at home" should no longer be deduced from the belief base. The new piece of information "Paul is in his office or at home or at his club" conveys some uncertainty that Paul's office or home are where he is right now. Yet, from a purely logical viewpoint, the new piece of information "Paul is in his office or at home or at his club" does not contradict the current belief base. Moreover, "Paul is in his office or at home or at his club" provides per se no means to obviate "Paul is in his office or at home" from which it can be deduced.

Here is another illustration. Assume that "If Dana agrees then we begin tomorrow" is in the current belief base. Presumably, a new, incoming, belief stating that "If Dana and Alexander agree then we begin tomorrow" is meant to rule out the former belief although they do not form a contradiction in terms of classical logic.

There recently has been some work dealing with this, in a classical logic setting (Besnard, Grégoire, and Ramon 2011a) and in a non-monotonic logic setting (Besnard, Grégoire, and Ramon 2011b). It is shown there that expelling from the belief base every piece of information f entailing (possibly through other information from the belief base) the preemptive information g is not enough. The way the problem is addressed in (Besnard, Grégoire, and Ramon 2011a; 2011b) is to apply contraction of the current belief base by $g \Rightarrow f$, for prime implicants f of g, then add g.

This paper is a first attempt at providing postulates for such preemption operators. It is also shown that a preemption operator can be alternatively defined as multiple contraction (Fuhrmann and Hansson 1994) (of appropriate formulas) followed by expansion. Finally, a related work, Hansson's replacement operator (Hansson 2009), is discussed.

We consider classical logic throughout. We assume a propositional language $\mathcal{L}$ defined using a finite list of propositional variables and the usual connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\Rightarrow$ (material implication). $\bot$ stands for a contradiction and $\top$ stands for a tautology. A literal is a propositional variable or its negation. A *clausal formula* (called a *clause*) is a finite disjunction of literals. Lowercase letters denote formulas of $\mathcal{L}$ whereas uppercase letters denote sets of formulas, these being called *belief bases*.

Cn denotes deductive closure: $Cn(A)$ denotes the set of all deductive consequences of A. A *theory* A is a deductively closed set of formulas, $A = Cn(A)$. Alternatively, $p \in Cn(A)$ can also be written $A \vdash p$. $\vdash p$ means that p is tautologous and $\vdash \neg p$ that p is a contradiction. Two formulas p and q are logically equivalent, written $p \equiv q$, iff $p \in Cn(\{q\})$ and $q \in Cn(\{p\})$. Throughout, it is assumed that belief bases are deductively closed unless stated otherwise. As usual, a set of formulas A is consistent iff $\perp \notin Cn(A)$. $K_\perp$ is the trivial belief base, i.e., it consists of all formulas of $\mathcal{L}$. $K_\top$ is the tautologous belief base, i.e., it consists of the tautologous formulas of $\mathcal{L}$. The concept of strict implicant is central to this paper: f is a strict implicant of g iff $f \vdash g$ and $g \not\vdash f$.

Postulates

Let K be a consistent belief base and g a clause. Let $+$ be an AGM expansion operator (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988) (see Appendix). Preemption by g over K is denoted $K \circledast g$. Here is a tentative list of postulates for $\circledast$.

$(\mathbf{K} \circledast \mathbf{1})$ $K \circledast g$ is a theory. **(closure)**

I.e., similarly with revision, the output of preemption is required to be deductived closed.

$(\mathbf{K} \circledast \mathbf{2})$ $g \in K \circledast g$. **(success of insertion)**

I.e., similarly with revision, the new piece of information is meant to be part of the resulting belief base.

$(\mathbf{K} \circledast \mathbf{3})$ $f \notin K \circledast g$ for all clausal strict implicants f of g. **(success of preemption)**

$(\mathbf{K} \circledast \mathbf{3})$ is in contrast with revision. Here, no clausal strict implicant f of g is allowed in the resulting belief base. If g is a contradiction, no formula of $\mathcal{L}$ is a strict implicant of g according to classical logic, hence the postulate vacuously holds despite Property 1 below.

Observe that $(\mathbf{K} \circledast \mathbf{3})$ cannot be extended to all strict implicants of g because, together with $(\mathbf{K} \circledast \mathbf{1})$ and $(\mathbf{K} \circledast \mathbf{2})$, it would entail that $K \circledast g$ be logically equivalent with g (see Property 3).

$(\mathbf{K} \circledast \mathbf{4})$ $K \circledast g \subseteq K + g$. **(inclusion)**

In other words, preempting never introduces beliefs beyond those in (the deductive closure of) the expansion of K by g.

$(\mathbf{K} \circledast \mathbf{5})$ If $(g \Rightarrow f) \notin K$ for all clausal strict implicants f of g then $K + g \subseteq K \circledast g$. **(vacuity)**

I.e., if no $g \Rightarrow f$ is in K, whatever f clausal strict implicant of g, then preempting amounts to expanding (K by g).

$(\mathbf{K} \circledast \mathbf{6})$ If $g \equiv h$ then $K \circledast g = K \circledast h$. **(extensionality)**

That is, similarly with revision, the outcome of preempting does not depend on the "syntax" of g.

The following property shows that the only way a trivial belief base results from preempting is by means of preempting by a contradiction.

Property 1. *Let $\circledast$ satisfy $(\mathbf{K} \circledast \mathbf{1})$, $(\mathbf{K} \circledast \mathbf{2})$ and $(\mathbf{K} \circledast \mathbf{3})$. Then, $K \circledast g = K_\perp$ iff $\vdash \neg g$.*

The following property states that if g is tautologous, then the outcome of preempting is a tautologous belief base.

Property 2. *Let $\circledast$ satisfy $(\mathbf{K} \circledast \mathbf{2})$ and $(\mathbf{K} \circledast \mathbf{3})$. Then, if $\vdash g$ then $K \circledast g = K_\top$.*

The property below formally states the case mentioned in the comment following *(success of preemption)*.

Property 3. *Let $\circledast$ satisfy $(\mathbf{K} \circledast \mathbf{1})$ and $(\mathbf{K} \circledast \mathbf{2})$. Then, $f \notin K \circledast g$ for all strict implicants f of g iff $K \circledast g$ is logically equivalent with g.*

The next property shows why *(vacuity)* does not require a proviso about the negation of g not to be in K (please observe that such a proviso occurs in the corresponding postulate for revision operators).

Property 4. *If $g \Rightarrow f \notin K$ for all clausal strict implicants f of g then $\neg g \notin K$.*

As by Property 5, it is otiose to check in *(vacuity)* that disjunctions of $g \Rightarrow f_i$ (for distinct clausal strict implicants f_i's of g) are not in K.

Property 5. *Let K and g be such that $(g \Rightarrow f) \notin K$ for all clausal strict implicants f of g. Then, there exist no clausal strict implicants i and j of g such that $(i \vee j) \not\equiv g$ and $(g \Rightarrow i) \vee (g \Rightarrow j) \in K$.*

Characterization

According to (Besnard, Grégoire, and Ramon 2011a; 2011b), similarly to Levi's identity (Gärdenfors 1988) defining revision as contraction followed by expansion, a preemption operation could be captured as multiple contraction followed by expansion. As given by Fuhrmann and Hansson (Fuhrmann and Hansson 1994), multiple contraction permits to contract a belief base K by a set of information Λ, written $K \ominus \Lambda$, so that no information of Λ can be inferred from $K \ominus \Lambda$ (see Appendix).

Definition 1 ($|||$ operator)**.** *Let $\{f_1, f_2, \ldots, f_n, \ldots\}$ be the set of all clausal strict implicants of g.*

$$K \, ||| \, g = (K \ominus \{g \Rightarrow f_i\}_{i=1,2,..}) + g.$$

Theorem 1. *If $\ominus$ satisfies $(K \ominus 1) - (K \ominus 4)$ and $(K \ominus 6)$, and if $+$ satisfies $(K + 1) - (K + 6)$, then $|||$ satisfies $(K \circledast 1) - (K \circledast 6)$.*

Theorem 2. *Every $\circledast$ operator satisfying $(K \circledast 1) - (K \circledast 6)$ can be written as an $|||$ operator s.t. $\ominus$ satisfies $(K \ominus 1) - (K \ominus 4)$ and $(K \ominus 6)$, and $+$ satisfies $(K + 1) - (K + 6)$.*

Related work: Hansson's replacement operator

As given by Hansson in (Hansson 2009), replacement permits to replace in a belief base K a proposition p by a proposition q, written $K|^p_q$. Similarly to Levi's identity (Gärdenfors 1988), in (Hansson 2009) it is shown that a replacement operation could be captured as a contraction by $q \Rightarrow p$ followed by an expansion by q.

Should there exist a multiple replacement operator extending Hansson's, applying it to our preemption problem would be as follows. The above p would rather be a set of formulas Δ (consisting here of the clausal strict implicants of g and q would be g). The desired outcome would be that

$K|_q^\Delta \cap \Delta = \emptyset$. Of course, $q \in K|_q^\Delta$ would still be enforced. Now, the question is: Can we find a way such that the desired outcome w.r.t. our preemption problem could still be obtained through Hansson's operator? At this stage, a useful observation is that g is a clause. Hence, the set of clausal strict implicants of g is finite. Although Δ definitely cannot be captured via identifying p with the disjunction $\bigvee_{\delta \in \Delta}$ (it is equivalent with g), it seems possible to get the same outcome in our preemption problem by means of

$$K|_q^\Delta = \bigcap_{p \in \Delta} K|_q^p.$$

In particular, set-theoretic intersection applies only finitely many times as Δ is the (finite) set of clausal strict implicants of g. Also, please observe that in contrast to maxichoice contraction $\bigcap(K \perp p)$ yielding (the deductive closure of) p, no such problem arises here as set-theoretic intersection is not applied to all maximal subsets consistent with $\neg p$, only some of them through Hansson's operator. In any case, consider the following operator, based on Hansson's replacement operator:

$$K \,|||_H\, g = \bigcap_{f \in \{f_1, f_2, \ldots, f_n, \ldots\}} K|_g^f$$

where $\{f_1, f_2, \ldots, f_n, \ldots\}$ is the (finite) set of all clausal strict implicants of g. It follows from the properties of Hansson's replacement operator that the desired features $g \in K \,|||_H\, g$ and $K \,|||_H\, g \cap \{f_1, f_2, \ldots, f_n, \ldots\} = \emptyset$ are enforced. However, a concern remains. Even in the event that $|||_H$ be a preemption operator, it does not necessarily mean that Hansson's operator is sufficient. It might be the case that some preemption operators cannot be written as a $|||_H$ operator. Intuitively, the reason is that a *fixed* Hansson's operator might turn out to be short of capturing an arbitrary preemption operator.

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AGM Operators

Let K be a consistent base, Λ be a set of formulas and g be a formula.

Expansion *(Alchourrón, Gärdenfors, and Makinson 1985)*
The postulates for the expansion of K by g, denoted $K + g$, are:

$(\mathbf{K+1})$ *$K + g$ is a theory.* ***(closure)***
$(\mathbf{K+2})$ *$g \in K + g$.* ***(success)***
$(\mathbf{K+3})$ *$K \subseteq K + g$.* ***(inclusion)***
$(\mathbf{K+4})$ *If $g \in K$ then $K + g = K$.* ***(vacuity)***
$(\mathbf{K+5})$ *If $K \subseteq H$ then $K + g \subseteq H + g$.* ***(monotony)***
$(\mathbf{K+6})$ *$K + g$ is the smallest set satisfying $(K+1)$ to $(K+5)$.* ***(minimality)***

Revision *(Alchourrón, Gärdenfors, and Makinson 1985)*
The postulates for the revision of K by g, denoted $K * g$, are:

$(\mathbf{K*1})$ *$K * g$ is a theory.* ***(closure)***
$(\mathbf{K*2})$ *$g \in K * g$.* ***(success)***
$(\mathbf{K*3})$ *$K * g \subseteq K + g$.* ***(inclusion)***
$(\mathbf{K*4})$ *If $\neg g \notin K$ then $K + g \subseteq K * g$.* ***(vacuity)***
$(\mathbf{K*5})$ *$K * g = K_\perp$ iff $\vdash \neg g$.* ***(consistent)***
$(\mathbf{K*6})$ *If $g \equiv h$ then $K * g = K * h$.* ***(extensionality)***
$(\mathbf{K*7})$ *$K * (g \wedge h) \subseteq (K * g) + h$.* ***(conjunctive inclusion)***
$(\mathbf{K*8})$ *If $\neg h \notin K * g$ then $(K * g) + h \subseteq K * (g \wedge h)$.* ***(conjunctive vacuity)***

$(K*7)$-$(K*8)$ are additional postulates devoted to minimal change.

Multiple contraction *(Fuhrmann and Hansson 1994)*
The postulates for the multiple contraction of K by Λ, denoted $K \ominus \Lambda$, are:

$(\mathbf{K \ominus 1})$ *$K \ominus \Lambda$ is a theory.* ***(closure)***
$(\mathbf{K \ominus 2})$ *$K \ominus \Lambda \subseteq K$.* ***(inclusion)***
$(\mathbf{K \ominus 3})$ *If $\Lambda \cap K = \emptyset$ then $K \ominus \Lambda = K$.* ***(vacuity)***
$(\mathbf{K \ominus 4})$ *If $\Lambda \cap Cn(\emptyset) = \emptyset$ then $\Lambda \cap (K \ominus \Lambda) = \emptyset$.* ***(success)***
$(\mathbf{K \ominus 5})$ *$K \subseteq Cn((K \ominus \Lambda) \cup \Lambda)$.* ***(recovery)***
$(\mathbf{K \ominus 6})$ *If $\Lambda \cong \Theta$ then $K \ominus \Lambda = K \ominus \Theta$.* ***(extensionality)***
$(\mathbf{K \ominus 7})$ *$(K \ominus \Lambda) \cap (K \ominus \Theta) \subseteq K \ominus (\Lambda \cap \Theta)$.* ***(intersection)***
$(\mathbf{K \ominus 8})$ *If $\varphi \notin K \ominus \Theta$ then $K \ominus \Theta \subseteq K \ominus (\Theta \cup \{\varphi\})$.* ***(non-deterioration)***
$(\mathbf{K \ominus 9})$ *If $\Lambda \cap (K \ominus \Theta) = \emptyset$ then $K \ominus \Theta \subseteq K \ominus (\Lambda \cup \Theta)$.* ***(conjonction)***

$\Lambda \cong \Theta$ means that for every element of Λ there exists a logically equivalent element of Θ, and vice versa*. Also, $(K \ominus 7)$, $(K \ominus 8)$ and $(K \ominus 9)$ are additional postulates devoted to minimal change.*

Proofs

Property 1 *Let $\circledast$ satisfy $(\mathbf{K \circledast 1})$, $(\mathbf{K \circledast 2})$ and $(\mathbf{K \circledast 3})$. Then, $K \circledast g = K_\perp$ iff $\vdash \neg g$.*

Proof of Property 1 ($\leftarrow$) Assume $\vdash \neg g$. That is, $g \equiv \bot$. According to $(K\circledast 2)$, $g \in K\circledast g$, and, by $(K\circledast 1)$, $K\circledast g = K_\bot$. ($\rightarrow$) Assume $K \circledast g = K_\bot$. Of course, $\bot \in K_\bot$, and $\bot \in K \circledast g$ trivially ensues. By $(K \circledast 1)$, it then follows that $f \in K \circledast g$ for every clausal strict implicant f of g. However, $(K \circledast 3)$ means that $f \notin K \circledast g$ for every clausal strict implicant f of g. Therefore, g has no clausal strict implicant. The only clause that has no clausal strict implicant is the empty clause. Hence, $g \equiv \bot$ and $\vdash \neg g$. ■

Property 2 *Let $\circledast$ satisfy $(\mathbf{K} \circledast \mathbf{2})$ and $(\mathbf{K} \circledast \mathbf{3})$. Then, if $\vdash g$ then $K \circledast g = K_\top$.*

Proof of Property 2 Let g be tautologous. Thus, all non-tautologous clausal formulas of $\mathcal{L}$ are strict implicants of g. By the conjunctive normal form theorem, each non-tautologous formula is an implicant of a non-tautologous clausal formula. Hence, $(K \circledast 3)$ and $(K \circledast 2)$ yield that only tautologous formulas are in $K \circledast g$. ■

Property 3 *Let $\circledast$ satisfy $(\mathbf{K} \circledast \mathbf{1})$ and $(\mathbf{K} \circledast \mathbf{2})$. Then, $f \notin K \circledast g$ for all strict implicants f of g iff $(K \circledast g) \equiv g$.*

Proof of Property 3 ($\rightarrow$) Let $h \in K \circledast g$. By $(\mathbf{K} \circledast \mathbf{1})$ and $(\mathbf{K} \circledast \mathbf{2})$, $g \wedge h \in K \circledast g$. Of course, $g \wedge h \vdash g$. Should $g \nvdash g \wedge h$, then $g \wedge h$ would be a strict implicant of g, and the assumption that $f \notin K \circledast g$ for all strict implicants f of g would be contradicted. Therefore, $g \vdash g \wedge h$. So, $g \vdash h$. ($\leftarrow$) A strict implicant f of g is such that $g \nvdash f$. Since $(K \circledast g) \equiv g$, it follows that $(K \circledast g) \nvdash f$. By $(\mathbf{K} \circledast \mathbf{1})$, $f \notin K \circledast g$. ■

Property 4 *If $g \Rightarrow f \notin K$ for all clausal strict implicants f of g then $\neg g \notin K$.*

Proof of Property 4 Assume $\neg g \in K$ while $(g \Rightarrow f) \notin K$ for all clausal strict implicants f of g. Since $\neg g \in K$ and K is a theory, $(\neg g \vee x) \in K$ for all non-tautologous clause x of $\mathcal{L}$. Each clausal strict implicant f of g is clearly such an x. I.e., $\neg g \vee f \in K$ for all clausal strict implicants f of g, contradicting the assumption. ■

Property 5 *Let K and g be such that $(g \Rightarrow f) \notin K$ for all clausal strict implicants f of g. Then, there exist no clausal strict implicants i and j of g such that $(i \vee j) \not\equiv g$ and $(g \Rightarrow i) \vee (g \Rightarrow j) \in K$.*

Proof of Property 5 Assume that such an i and such a j exist. $((g \Rightarrow i) \vee (g \Rightarrow j)) \in K$ yields $(g \Rightarrow (i \vee j)) \in K$ by virtue of classical logic since K is a theory. Now, the disjunction of two strict implicants of g is either equivalent with g or a strict implicant of g. In the former case, $(i \vee j) \equiv g$, contradicting the assumption. In the latter case, $(g \Rightarrow h) \in K$ where $h = i \vee j$ is a strict implicant of g, contradicting the statement about K and g. ■

Theorem 1 *If $\ominus$ obeys $(K \ominus 1) - (K \ominus 4)$ and $(K \ominus 6)$, and if $+$ obeys $(K + 1) - (K + 6)$, then $|||$ satisfies $(K \circledast 1) - (K \circledast 6)$.*

Proof of Theorem 1 Trivially, if $(K \ominus 1)$ and $(K + 1)$ are satisfied, $(K \circledast 1)$ is satisfied, too.

In the same way, $(K \circledast 2)$ is trivially satisfied if $(K + 2)$ is.

Regarding $(K \circledast 3)$, if $(K \ominus 4)$ holds, then, for all clausal strict implicants f of g, $(g \Rightarrow f) \notin K \ominus \{g \Rightarrow f_i\}_{i=1,2,..}$. Assume that there exists $f \in K \circledast g$ such that f is a clausal strict implicant of g. From $K\circledast g = (K\ominus\{g \Rightarrow f_i\}_{i=1,2,..})+ g$, $(K + 1) - (K + 2)$ together with $(K + 6)$ imply $g \Rightarrow f \in K \ominus \{g \Rightarrow f_i\}_{i=1,2,..}$. Then, a contradiction arises, and $(K \circledast 3)$ is satisfied.

As to $(K \circledast 4)$, if $(K \ominus 2)$ is satisfied, $K \ominus \{g \Rightarrow f_i\}_{i=1,2,..} \subseteq K$. Applying $(K + 5)$, it follows that $(K \circledast 4)$ is satisfied.

Regarding $(K \circledast 5)$, by $(K \ominus 3)$, if for all clausal strict implicants f of g, $(g \Rightarrow f) \notin K$ then $K \ominus \{g \Rightarrow f_i\}_{i=1,2,..} = K$. Hence, $K \subseteq (K \ominus \{g \Rightarrow f_i\}_{i=1,2,..})$. Applying $(K + 3)$, $K + g \subseteq K \circledast g$ and $(K \circledast 5)$ is satisfied.

Regarding $(K \circledast 6)$, if $g \equiv h$ then the set Λ of all the clausal strict implicants of g is exactly the set Θ of all the clausal strict implicants of h. Then, $\Lambda \cong \Theta$ (see Appendix). By $(K \ominus 6)$, $K \ominus \Lambda = K \ominus \Theta$. By $(K + 5)$, $K \circledast g = K \circledast h$. I.e., $(K \circledast 6)$ is satisfied. ■

Theorem 2 *Every $\circledast$ operator satisfying $(K \circledast 1) - (K \circledast 6)$ can be written as an $|||$ operator s.t. $\ominus$ satisfies $(K \ominus 1) - (K \ominus 4)$ and $(K \ominus 6)$, and $+$ satisfies $(K + 1)$-$(K + 6)$.*

Proof of Theorem 2 We use the following usual definitions. (Fuhrmann and Hansson 1994)

A selection function γ must be such that $\gamma(K\bot\Lambda) = \{K\}$ if $K\bot\Lambda = \emptyset$ and $\emptyset \neq \gamma(K\bot\Lambda) \subseteq K\bot\Lambda$ if $K\bot\Lambda \neq \emptyset$, where for all deductively closed K, K', K'',

$$K\bot\Lambda \stackrel{\text{def}}{=} \left\{ K' \subseteq K \;\middle|\; \begin{array}{l} K' \cap \Lambda = \emptyset \\ K' \subset K'' \subseteq K \Rightarrow K'' \cap \Lambda \neq \emptyset \end{array} \right\}$$

If for all clausal strict implicants f_i of g, $g \Rightarrow f_i \notin K$, then $(K \circledast 4)$ and $(K \circledast 5)$ together yield $K \circledast g = K + g$ and it is easy to see that any $\ominus$ satisfying $(K \ominus 3)$ would do.
Let then assume $g \Rightarrow f \in K$ for some clausal strict implicant f of g. By $(K \circledast 4)$, $K \circledast g \subseteq K + g = Cn(K \cup \{g\})$ (applying a theorem due to Gärdenfors (Gärdenfors 1988) stating that $Cn(K \cup \{\cdot\})$ is the unique expansion operator satisfying $(K + 1) - (K + 6)$). Due to $(K \circledast 2)$, $K \circledast g$ is then logically equivalent with $K' \cup \{g\}$ for some $K' \subseteq K$. Define

$$\mathcal{K} = \left\{ X \;\middle|\; \begin{array}{l} X \in K\bot\{g \Rightarrow f_i\}_{i=1,2,..} \\ K' \subseteq X \\ \exists x \in K \setminus K' \text{ s.t. } x \Rightarrow (g \Rightarrow f) \in X \end{array} \right\}.$$

By $(K \circledast 3)$, for all clausal strict implicants f_i of g, $K' \nvdash g \Rightarrow f_i$. Also, it is clear that, for all $x \in K \setminus K'$, there exists $X \in \mathcal{K}$ such that $x \notin X$. Consequently,

$$\bigcap \mathcal{K} = Cn(K').$$

Now, $\mathcal{K} \subseteq K\bot\{g \Rightarrow f_i\}_{i=1,2,..}$. Therefore, there exists a selection function γ such that

$$\gamma(K\bot\{g \Rightarrow f_i\}_{i=1,2,..}) = Cn(K').$$

Take

$$K \ominus \{g \Rightarrow f_i\}_{i=1,2,..} = \bigcap \gamma(K\bot\{g \Rightarrow f_i\}_{i=1,2,..}).$$

By the latter, $(K \ominus 1) - (K \ominus 6)$ hold (cf representation theorem in (Fuhrmann and Hansson 1994)). By construction, too, $K \ominus \{g \Rightarrow f_i\}_{i=1,2,..} = Cn(K')$. $(K \ominus \{g \Rightarrow f_i\}_{i=1,2,..}) \cup \{g\}$ is therefore logically equivalent to $K' \cup \{g\}$ and $K \circledast g = Cn((K \ominus \{g \Rightarrow f_i\}_{i=1,2,..}) \cup \{g\})$ then easily ensues. ■