Rarefaction pulses for the Nonlinear Schrödinger Equation in the transonic limit.

David Chiron, Mihai Maris

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D. Chiron* & M. Mariş†

Abstract

We investigate the properties of finite energy travelling waves to the nonlinear Schrödinger equation with nonzero conditions at infinity for a wide class of nonlinearities. In space dimension two and three we prove that travelling waves converge in the transonic limit (up to rescaling) to ground states of the Kadomtsev-Petviashvili equation. Our results generalize an earlier result of F. Béthuel, P. Gravejat and J-C. Saut for the two-dimensional Gross-Pitaevskii equation, and provide a rigorous proof to a conjecture by C. Jones and P. H. Roberts about the existence of an upper branch of travelling waves in dimension three.

Keywords. Nonlinear Schrödinger equation, Gross-Pitaevskii equation, Kadomtsev-Petviashvili equation, travelling waves, ground state.

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Contents

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1 Introduction

We consider the nonlinear Schrödinger equation in $\mathbb{R}^N$

$$i\frac{\partial \Psi}{\partial t} + \Delta \Psi + F(|\Psi|^2)\Psi = 0 \tag{NLS}$$

with the condition $|\Psi(t,x)| \to r_0$ as $|x| \to \infty$, where $r_0 > 0$ and $F(r_0^2) = 0$. This equation arises as a relevant model in many physical situations, such as the theory of Bose-Einstein condensates, superfluidity (see [19], [23], [24], [26], [25] and the surveys [37], [1]) or as an approximation of the Maxwell-Bloch system in Nonlinear Optics (cf. [29], [30]). When $F(\varrho) = 1 - \varrho$, the corresponding (NLS) equation is called the Gross-Pitaevskii equation and is a common model for Bose-Einstein condensates. The so-called "cubic-quintic" (NLS), where

$$F(\varrho) = -\alpha_1 + \alpha_3 \varrho - \alpha_5 \varrho^2$$

for some positive constants α_1, α_3 and α_5 and F has two positive roots, is also of high interest in Physics (see, e.g., [2]). In Nonlinear Optics, the nonlinearity F can take various forms (cf. [29]), for instance

$$F(\varrho) = -\alpha\varrho^\nu - \beta\varrho^{2\nu}, \qquad F(\varrho) = -\alpha\Big(1 - \frac{1}{(1+\frac{\varrho}{\varrho_0})^\nu}\Big), \qquad F(\varrho) = -\alpha\varrho\Big(1+\gamma\tanh(\frac{\varrho^2-\varrho_0^2}{\sigma^2})\Big), \qquad \text{etc.,} \tag{1}$$

where α, β, γ, ν, $\sigma > 0$ are given constants (the second formula, for instance, was proposed to take into account saturation effects). It is therefore important to allow the nonlinearity to be as general as possible.

The travelling wave solutions propagating with speed c in the x_1-direction are the solutions of the form $\Psi(x,t) = U(x_1 - ct, x_2, \ldots, x_N)$. The profile U satisfies the equation

$$-ic\partial_{x_1} U + \Delta U + F(|U|^2)U = 0. \tag{TW$_c$}$$

They are supposed to play an important role in the dynamics of (NLS). Since (U,c) is a solution of (TW$_c$) if and only if $(\overline{U}, -c)$ is also a solution, we may assume that $c \geq 0$. The nonlinearities we consider are general, and we will merely make use of the following assumptions:

(A1) The function F is continuous on $[0,+\infty)$, of class $\mathcal{C}^1$ near r_0^2, $F(r_0^2) = 0$ and $F'(r_0^2) < 0$.

(A2) There exist $C > 0$ and $p_0 \in [1, \frac{2}{N-2})$ ($p_0 < \infty$ if $N = 2$) such that $|F(\varrho)| \leq C(1 + \varrho^{p_0})$ for all $\varrho \geq 0$.

(A3) There exist $C_0 > 0$, $\alpha_0 > 0$ and $\varrho_0 > r_0$ such that $F(\varrho) \leq -C_0\varrho^{\alpha_0}$ for all $\varrho \geq \varrho_0$.

Assumptions (A1) and ((A2) or (A3)) are sufficient to guarantee the existence of travelling waves. However, in order to get some sharp results we will need sometimes more information about the behavior of F near r_0^2, so we will replace (A1) by

(A4) The function F is continuous on $[0,+\infty)$, of class $\mathcal{C}^2$ near r_0^2, with $F(r_0^2) = 0$, $F'(r_0^2) < 0$ and

$$F(\varrho) = F(r_0^2) + F'(r_0^2)(\varrho - r_0^2) + \frac{1}{2}F''(r_0^2)(\varrho - r_0^2)^2 + \mathcal{O}((\varrho - r_0^2)^3) \qquad \text{as} \qquad \varrho \to r_0^2.$$

If F is $\mathcal{C}^2$ near r_0^2, we define, as in [17],

$$\Gamma = 6 - \frac{4r_0^4}{\mathfrak{c}_s^2}F''(r_0^2). \tag{2}$$

The coefficient Γ is positive for the Gross-Pitaevskii nonlinearity ($F(\varrho) = 1 - \varrho$) as well as for the cubic-quintic Schrödinger equation. However, for the nonlinearity $F(\varrho) = b\mathrm{e}^{-\varrho/\alpha} - a$, where $\alpha > 0$ and $0 < a < b$ (which arises in nonlinear optics and takes into account saturation effects, see [29]), we have $\Gamma = 6 + 2\ln(a/b)$, so that Γ can take any value in $(-\infty, 6)$, including zero. The coefficient Γ may also vanish for some polynomial nonlinearities (see [16] for some examples and for the study of travelling waves in dimension one in that case). In this paper we shall be concerned only with the nondegenerate case $\Gamma \neq 0$.

Notation and function spaces. For $x = (x_1, x_2, \ldots, , x_N) \in \mathbb{R}^N$, we denote $x = (x_1, x_\perp)$, where $x_\perp = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$. Given a function f defined on $\mathbb{R}^N$, we denote $\nabla_{x_\perp} f = (\frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_N})$. We will write $\Delta_{x_\perp} = \frac{\partial^2}{\partial x^2} + \cdots + \frac{\partial^2}{\partial x^N}$. By "$f(t) \sim g(t)$ as $t \to t_0$" we mean $\lim_{t \to t_0} \frac{f(t)}{g(t)} = 1$.

We denote by $\mathscr{F}$ the Fourier transform, defined by $\mathscr{F}(f)(\xi) = \int_{\mathbb{R}^N} \mathrm{e}^{-ix.\xi} f(x)\, dx$ whenever $f \in L^1(\mathbb{R}^N)$.

Unless otherwise stated, the L^p norms are computed on the whole space $\mathbb{R}^N$.

We fix an odd function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(s) = s$ for $0 \le s \le 2r_0$, $\chi(s) = 3r_0$ for $s \ge 4r_0$ and $0 \le \chi' \le 1$ on $\mathbb{R}_+$. As usually, we denote $\dot{H}^1(\mathbb{R}^N) = \{h \in L^1_{loc}(\mathbb{R}^N) \mid \nabla h \in L^2(\mathbb{R}^N)\}$. We define the Ginzburg-Landau energy of a function $\psi \in \dot{H}^1(\mathbb{R}^N)$ by

$$E_{\mathrm{GL}}(\psi) = \int_{\mathbb{R}^N} |\nabla \psi|^2 + (\chi^2(|\psi|) - r_0^2)^2 \, dx.$$

We will use the function space

$$\mathcal{E} = \left\{ \psi \in \dot{H}^1(\mathbb{R}^N) \;\middle|\; \chi^2(|\psi|) - r_0^2 \in L^2(\mathbb{R}^N) \right\} = \left\{ \psi \in \dot{H}^1(\mathbb{R}^N) \;\middle|\; E_{\mathrm{GL}}(\psi) < \infty \right\}.$$

The basic properties of this space have been discussed in the Introduction of [17]. We will also consider the space

$$\mathcal{X} = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \;\middle|\; \chi^2(|r_0 - u|) - r_0^2 \in L^2(\mathbb{R}^N) \right\},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ for the norm $\|u\|_{\mathcal{D}^{1,2}} = \|\nabla u\|_{L^2(\mathbb{R}^N)}$. If $N \ge 3$ it can be proved that $\mathcal{E} = \{\alpha(r_0 - u) \mid u \in \mathcal{X},\ \alpha \in \mathbb{C},\ |\alpha| = 1\}$.

Hamiltonian structure. The flow associated to (NLS) formally preserves the energy

$$E(\psi) = \int_{\mathbb{R}^N} |\nabla \psi|^2 + V(|\psi|^2) \, dx,$$

where V is the antiderivative of $-F$ which vanishes at r_0^2, that is $V(s) = \int_s^{r_0^2} F(\varrho)\, d\varrho$, as well as the momentum. The momentum (with respect to the direction of propagation x_1) is a functional Q defined on $\mathcal{E}$ (or, alternatively, on $\mathcal{X}$) in the following way. Denoting by $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{C}$, it has been proven in [17] and [36] that for any $\psi \in \mathcal{E}$ we have $\langle i \frac{\partial \psi}{\partial x_1}, \psi \rangle \in \mathcal{Y} + L^1(\mathbb{R}^N)$, where $\mathcal{Y} = \{ \frac{\partial h}{\partial x_1} \mid h \in \dot{H}^1(\mathbb{R}^N)\}$ and $\mathcal{Y}$ is endowed with the norm $\|\partial_{x_1} h\|_{\mathcal{Y}} = \|\nabla h\|_{L^2(\mathbb{R}^N)}$. It is then possible to define the linear continuous functional L on $\mathcal{Y} + L^1(\mathbb{R}^N)$ by

$$L\left(\frac{\partial h}{\partial x_1} + \Theta \right) = \int_{\mathbb{R}^N} \Theta(x)\, dx \qquad \text{for any } \frac{\partial h}{\partial x_1} \in \mathcal{Y} \text{ and } \Theta \in L^1(\mathbb{R}^N).$$

The momentum (with respect to the direction x_1) of a function $\psi \in \mathcal{E}$ is $Q(\psi) = L\left(\langle i \frac{\partial \psi}{\partial x_1}, \psi \rangle \right)$.
If $\psi \in \mathcal{E}$ does not vanish, it can be lifted in the fom $\psi = \rho \mathrm{e}^{i\phi}$ and we have

$$Q(\psi) = \int_{\mathbb{R}^N} (r_0^2 - \rho^2) \frac{\partial \phi}{\partial x_1} \, dx. \tag{3}$$

Any solution $U \in \mathcal{E}$ of (TW$_c$) is a critical point of the functional $E_c = E + cQ$ and satisfies the standard Pohozaev identities (see Proposition 4.1 p. 1091 in [34]):

$$\begin{cases} P_c(U) = 0, \qquad \text{where } P_c(U) = E(U) + cQ(U) - \dfrac{2}{N-1} \displaystyle\int_{\mathbb{R}^N} |\nabla_{x_\perp} U|^2 \, dx, \qquad \text{and} \\ E(U) = 2 \displaystyle\int_{\mathbb{R}^N} |\partial_{x_1} U|^2 \, dx. \end{cases} \tag{4}$$

We denote

$$\mathscr{C}_c = \{ \psi \in \mathcal{E} \mid \psi \text{ is not constant and } P_c(\psi) = 0 \}. \tag{5}$$

Using the Madelung transform $\Psi = \sqrt{\varrho} \mathrm{e}^{i\theta}$ (which makes sense in any domain where $\Psi \neq 0$), equation (NLS) can be put into a hydrodynamical form. In this context one may compute the associated speed of sound at infinity (see, for instance, the introduction of [34]):

$$\mathfrak{c}_s = \sqrt{-2r_0^2 F'(r_0^2)} > 0.$$

Under general assumptions it was proved that finite energy travelling waves to (NLS) with speed c exist if and only if $|c| < \mathfrak{c}_s$ (see [34, 36]).

Let us recall the existence results of nontrivial travelling waves that we use.

Theorem 1 ([17]) *Let $N = 2$ and assume that the nonlinearity F satisfies (A2) and (A4)and that $\Gamma \neq 0$.*

(a) Suppose moreover that V is nonnegative on $[0, \infty)$. Then for any $q \in (-\infty, 0)$ there exists $U \in \mathcal{E}$ such that $Q(U) = q$ and

$$E(U) = \inf\{E(\psi) \mid \psi \in \mathcal{E},\ Q(\psi) = q\}.$$

(b) Without any assumption on the sign of V, there is $q_\infty > 0$ such that for any $q \in (-q_\infty, 0)$ there is $U \in \mathcal{E}$ satisfying $Q(U) = q$ and

$$E(U) = \inf\Big\{E(\psi) \mid \psi \in \mathcal{E},\ Q(\psi) = q,\ \int_{\mathbb{R}^2} V(|\psi|^2)\, dx > 0\Big\}.$$

For any U satisfying (a) or (b) there exists $c = c(U) \in (0, \mathfrak{c}_s)$ such that U is a nonconstant solution to $(\mathrm{TW}_{c(U)})$. Moreover, if $Q(U_1) < Q(U_2) < 0$ we have $0 < c(U_1) < c(U_2) < \mathfrak{c}_s$ and $c(U) \to \mathfrak{c}_s$ as $q \to 0$.

Theorem 2 ([17]) *Let $N = 2$. Assume that the nonlinearity F satisfies (A2) and (A4) and that $\Gamma \neq 0$. Then there exists $0 < k_\infty \leq \infty$ such that for any $k \in (0, k_\infty)$, there is $\mathcal{U} \in \mathcal{E}$ such that $\int_{\mathbb{R}^2} |\nabla \mathcal{U}|^2\, dx = k$ and*

$$\int_{\mathbb{R}^2} V(|\mathcal{U}|^2)\, dx + Q(\mathcal{U}) = \inf\left\{\int_{\mathbb{R}^2} V(|\psi|^2)\, dx + Q(\psi) \;\Big|\; \psi \in \mathcal{E},\ \int_{\mathbb{R}^2} |\nabla \psi|^2\, dx = k\right\}.$$

For any such $\mathcal{U}$ there exists $c = c(\mathcal{U}) \in (0, \mathfrak{c}_s)$ such that the function $U(x) = \mathcal{U}(x/c)$ is a solution to (TW_c). Moreover, if $\mathcal{U}_1$, $\mathcal{U}_2$ are as above and $\int_{\mathbb{R}^2} |\nabla \mathcal{U}_1|^2\, dx < \int_{\mathbb{R}^2} |\nabla \mathcal{U}_2|^2\, dx$, then $\mathfrak{c}_s > c(\mathcal{U}_1) > c(\mathcal{U}_2) > 0$ and we have $c(\mathcal{U}) \to \mathfrak{c}_s$ as $k \to 0$.

Theorem 3 ([36]) *Assume that $N \geq 3$ and the nonlinearity F satisfies (A1) and (A2). Then for any $0 < c < \mathfrak{c}_s$ there exists a nonconstant $\mathcal{U} \in \mathcal{E}$ such that $P_c(\mathcal{U}) = 0$ and $E(\mathcal{U}) + cQ(\mathcal{U}) = \inf_{\psi \in \mathscr{C}_c} (E(\psi) + cQ(\psi))$. If $N \geq 4$, any such $\mathcal{U}$ is a nontrivial solution to (TW_c). If $N = 3$, for any $\mathcal{U}$ as above there exists $\sigma > 0$ such that $U(x) = \mathcal{U}(x_1, \sigma x_\perp) \in \mathcal{E}$ is a nontrivial solution to (TW_c).*

If (A3) holds it was proved that there is $C_0 > 0$, depending only on F, such that for any $c \in (0, \mathfrak{c}_s)$ and for any solution $U \in \mathcal{E}$ to (TW_c) we have $|U| \leq C_0$ in $\mathbb{R}^N$ (see Proposition 2.2 p. 1079 in [34]). If (A3) is satisfied but (A2) is not, one can modify F in a neighborhood of infinity in such a way that the modified nonlinearity $\tilde{F}$ satisfies (A2) and (A3) and $F = \tilde{F}$ on $[0, 2C_0]$. Then the solutions of (TW_c) are the same as the solutions of (TW_c) with F replaced by $\tilde{F}$. Therefore all the existence results above hold if (A2) is replaced by (A3); however, the minimizing properties hold only if we replace throughout F and V by $\tilde{F}$ and $\tilde{V}$, respectively, where $\tilde{V}(s) = \int_s^{r_0^2} \tilde{F}(\tau)\, d\tau$.

The above results provide, under various assumptions, travelling waves to (NLS) with speed close to the speed of sound $\mathfrak{c}_s$. We will study the behavior of travelling waves in the transonic limit $c \to \mathfrak{c}_s$ in each of the previous situations.

1.1 Convergence to ground states for (KP-I)

In the transonic limit, the travelling waves are expected to be rarefaction pulses close, up to a rescaling, to ground states of the Kadomtsev-Petviashvili I (KP-I) equation. We refer to [26] in the case of the Gross-Pitaevskii equation ($F(\varrho) = 1 - \varrho$) in space dimension $N = 2$ or $N = 3$, and to [29], [28], [30] in the context of Nonlinear Optics. In our setting, the (KP-I) equation associated to (NLS) is

$$2\partial_\tau \zeta + \Gamma \zeta \partial_{z_1} \zeta - \frac{1}{\mathfrak{c}_s^2}\, \partial_{z_1}^3 \zeta + \Delta_{z_\perp} \partial_{z_1}^{-1} \zeta = 0, \tag{KP-I}$$

where $\Delta_{z_\perp} = \sum_{j=2}^{N} \partial_{z_j}^2$ and the coefficient Γ is related to the nonlinearity F by (2).

The (KP-I) flow preserves (at least formally) the L^2 norm

$$\int_{\mathbb{R}^N} \zeta^2\, dz$$

and the energy

$$\mathscr{E}(\zeta) = \int_{\mathbb{R}^N} \frac{1}{\mathfrak{c}_s^2} (\partial_{z_1}\zeta)^2 + |\nabla_{z_\perp}\partial_{z_1}^{-1}\zeta|^2 + \frac{\Gamma}{3}\zeta^3 \, dz.$$

A solitary wave of speed $1/(2\mathfrak{c}_s^2)$, moving to the left in the z_1 direction, is a particular solution of (KP-I) of the form $\zeta(\tau, z) = \mathcal{W}(z_1 + \tau/(2\mathfrak{c}_s^2), z_\perp)$. The profile $\mathcal{W}$ solves the equation

$$\frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\mathcal{W} + \Gamma\mathcal{W}\partial_{z_1}\mathcal{W} - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3\mathcal{W} + \Delta_{z_\perp}\partial_{z_1}^{-1}\mathcal{W} = 0. \tag{SW}$$

Equation (SW) has no nontrivial solution in the degenerate linear case $\Gamma = 0$ or in space dimension $N \geq 4$ (see Theorem 1.1 p. 214 in [20] or the begining of section 2). If $\Gamma \neq 0$, since the nonlinearity is homogeneous, one can construct solitary waves of any (positive) speed just by using the scaling properties of the equation. The solutions of (SW) are critical points of the associated action

$$\mathscr{S}(\mathcal{W}) = \mathscr{E}(\mathcal{W}) + \frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}^N} \mathcal{W}^2 \, dz.$$

The natural energy space for (KP-I) is $\mathscr{Y}(\mathbb{R}^N)$, which is the closure of $\partial_{z_1}\mathcal{C}_c^\infty(\mathbb{R}^N)$ for the (squared) norm

$$\|\mathcal{W}\|^2_{\mathscr{Y}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \frac{1}{\mathfrak{c}_s^2}\mathcal{W}^2 + \frac{1}{\mathfrak{c}_s^2}(\partial_{z_1}\mathcal{W})^2 + |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 \, dz.$$

From the anisotropic Sobolev embeddings (see [7], p. 323) it follows that $\mathscr{S}$ is well-defined and is a continuous functional on $\mathscr{Y}(\mathbb{R}^N)$ for $N = 2$ and $N = 3$. Here we are not interested in arbitrary solitary waves for (KP-I), but only in *ground states*. A ground state of (KP-I) with speed $1/(2\mathfrak{c}_s^2)$ (or, equivalently, a ground state of (SW)) is a nontrivial solution of (SW) which minimizes the action $\mathscr{S}$ among all solutions of (SW). We shall denote $\mathscr{S}_{\min}$ the corresponding action:

$$\mathscr{S}_{\min} = \inf\Big\{\mathscr{S}(\mathcal{W}) \;\Big|\; \mathcal{W} \in \mathscr{Y}(\mathbb{R}^N)\setminus\{0\}, \; \mathcal{W} \text{ solves (SW)}\Big\}.$$

The existence of ground states (with speed $1/(2\mathfrak{c}_s^2)$) for (KP-I) in dimensions $N = 2$ and $N = 3$ follows from Lemma 2.1 p. 1067 in [21]. In dimension $N = 2$, we may use the variational characterization provided by Lemma 2.2 p. 78 in [22]:

Theorem 4 ([22]) *Assume that $N = 2$ and $\Gamma \neq 0$. There exists $\mu > 0$ such that the set of solutions to the minimization problem*

$$\mathscr{M}(\mu) = \inf\left\{\mathscr{E}(\mathcal{W}) \;\Big|\; \mathcal{W} \in \mathscr{Y}(\mathbb{R}^2), \; \int_{\mathbb{R}^2} \frac{1}{\mathfrak{c}_s^2}\mathcal{W}^2 \, dz = \mu\right\}, \tag{6}$$

is precisely the set of ground states of (KP-I) *and it is not empty. Moreover, any sequence $(\mathcal{W}_n)_{n\geq 1} \subset \mathscr{Y}(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \frac{1}{\mathfrak{c}_s^2}\mathcal{W}_n^2 \, dz \to \mu$ and $\mathscr{E}(\mathcal{W}_n) \to \mathscr{M}(\mu)$ contains a convergent subsequence in $\mathscr{Y}(\mathbb{R}^2)$ (up to translations). Finally, we have*

$$\mu = \frac{3}{2}\mathscr{S}_{\min} \qquad \textit{and} \qquad \mathscr{M}(\mu) = -\frac{1}{2}\mathscr{S}_{\min}.$$

We emphasize that this characterization of ground states is specific to the two-dimensional case. Indeed, since $\mathscr{E}$ and the L^2 norm are conserved by (KP-I), it implies the orbital stability of the set of ground states for (KP-I) if $N = 2$ (cf. [22]). On the other hand, it is known that this set is orbitally unstable if $N = 3$ (see [33]). In the three-dimensional case we need the following result, which shows that ground states are minimizers of the action under a Pohozaev type constraint. Notice that any solution of (SW) in $\mathscr{Y}(\mathbb{R}^N)$ satisfies the Pohozaev identity

$$\int_{\mathbb{R}^N} \frac{1}{\mathfrak{c}_s^2}(\partial_{z_1}\mathcal{W})^2 + |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 + \frac{\Gamma}{3}\mathcal{W}^3 + \frac{1}{\mathfrak{c}_s^2}\mathcal{W}^2 \, dz = \frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 \, dz,$$

which is (formally) obtained by multiplying (SW) by $z_\perp \cdot \nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}$ and integrating by parts (see Theorem 1.1 p. 214 in [20] for a rigorous justification). Taking into account how travelling wave solutions to (NLS) are constructed in Theorem 3 above, in the case $N = 3$ we consider the minimization problem

$$\mathscr{S}_* = \inf\Big\{\mathscr{S}(\mathcal{W}) \;\Big|\; \mathcal{W} \in \mathscr{Y}(\mathbb{R}^3)\setminus\{0\}, \; \mathscr{S}(\mathcal{W}) = \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 \, dz\Big\}. \tag{7}$$

Our first result shows that in space dimension $N = 3$ the ground states (with speed $1/(2\mathfrak{c}_s^2)$) of (KP-I) are the solutions of the minimization problem (7).

Theorem 5 *Assume that $N = 3$ and $\Gamma \neq 0$. Then $\mathscr{S}_* > 0$ and the problem (7) has minimizers. Moreover, $\mathcal{W}_0$ is a minimizer for the problem (7) if and only if there exist a ground state $\mathcal{W}$ for* (KP-I) *(with speed $1/(2\mathfrak{c}_s^2)$) and $\sigma > 0$ such that $\mathcal{W}_0(z) = \mathcal{W}(z_1, \sigma z_\perp)$. In particular, we have $\mathscr{S}_* = \mathscr{S}_{\min}$.*

Furthermore, let $(\mathcal{W}_n)_{n\geq 1} \subset \mathscr{Y}(\mathbb{R}^3)$ be a sequence satisfying:

(i) There are positive constants m_1, m_2 such that $m_1 \leq \int_{\mathbb{R}^3} \mathcal{W}_n^2 + (\partial_{z_1}\mathcal{W}_n)^2\, dz \leq m_2$.

(ii) $\int_{\mathbb{R}^3} \frac{1}{\mathfrak{c}_s^2}\mathcal{W}_n^2 + \frac{1}{\mathfrak{c}_s^2}(\partial_{z_1}\mathcal{W}_n)^2 + \frac{\Gamma}{3}\mathcal{W}_n^3\, dz \to 0$ as $n \to \infty$.

(iii) $\liminf_{n\to\infty} \mathscr{S}(\mathcal{W}_n) \leq \mathscr{S}_*$.

Then there exist $\sigma > 0$, $\mathcal{W} \in \mathscr{Y}(\mathbb{R}^3) \setminus \{0\}$, a subsequence $(\mathcal{W}_{n_j})_{j\geq 0}$, and a sequence $(z^j)_{j\geq 0} \subset \mathbb{R}^3$ such that $z \mapsto \mathcal{W}(z_1, \sigma^{-1}z_\perp)$ is a ground state for (KP-I) *with speed $1/(2\mathfrak{c}_s^2)$ and*

$$\mathcal{W}_{n_j}(\cdot - z^j) \to \mathcal{W} \qquad \text{in} \quad \mathscr{Y}(\mathbb{R}^3).$$

We will study the behavior of travelling waves to (TW_c) in the transonic limit $c \nearrow \mathfrak{c}_s$ in space dimension $N = 2$ and $N = 3$ under the assumption $\Gamma \neq 0$ (so that (KP-I) has nontrivial solitary waves). For $0 < \varepsilon < \mathfrak{c}_s$, we define $c(\varepsilon) > 0$ by

$$c(\varepsilon) = \sqrt{\mathfrak{c}_s^2 - \varepsilon^2}.$$

As already mentioned, in this asymptotic regime the travelling waves are expected to be close to "the" ground state of (KP-I) (to the best of our knowledge, the uniqueness of this solution up to translations has not been proven yet). Let us give the formal derivation of this result, which follows the arguments given in [26] for the Gross-Pitaevskii equation in dimensions $N = 2$ and $N = 3$. We insert the ansatz

$$U(x) = r_0\Big(1 + \varepsilon^2 A_\varepsilon(z)\Big)\exp\big(i\varepsilon\varphi_\varepsilon(z)\big) \qquad \text{where } z_1 = \varepsilon x_1, \quad z_\perp = \varepsilon^2 x_\perp \tag{8}$$

in ($\mathrm{TW}_{c(\varepsilon)}$), cancel the phase factor and separate the real and imaginary parts to obtain the system

$$\begin{cases} -c(\varepsilon)\partial_{z_1}A_\varepsilon + 2\varepsilon^2\partial_{z_1}\varphi_\varepsilon\partial_{z_1}A_\varepsilon + 2\varepsilon^4\nabla_{z_\perp}\varphi_\varepsilon\cdot\nabla_{z_\perp}A_\varepsilon + (1+\varepsilon^2 A_\varepsilon)\Big(\partial_{z_1}^2\varphi_\varepsilon + \varepsilon^2\Delta_{z_\perp}\varphi_\varepsilon\Big) = 0 \\ -c(\varepsilon)\partial_{z_1}\varphi_\varepsilon + \varepsilon^2(\partial_{z_1}\varphi_\varepsilon)^2 + \varepsilon^4|\nabla_{z_\perp}\varphi_\varepsilon|^2 - \dfrac{1}{\varepsilon^2}F\Big(r_0^2(1+\varepsilon^2A_\varepsilon)^2\Big) - \varepsilon^2\dfrac{\partial_{z_1}^2A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon^2A_\varepsilon} = 0. \end{cases} \tag{9}$$

Formally, if $A_\varepsilon \to A$ and $\varphi_\varepsilon \to \varphi$ as $\varepsilon \to 0$ in some reasonable sense, then to the leading order we obtain $-\mathfrak{c}_s\partial_{z_1}A + \partial_{z_1}^2\varphi = 0$ for the first equation in (9). Since F is of class $\mathcal{C}^2$ near r_0^2, using the Taylor expansion

$$F\Big(r_0^2(1+\varepsilon^2A_\varepsilon)^2\Big) = F(r_0^2) - \mathfrak{c}_s^2\varepsilon^2A_\varepsilon + \mathcal{O}(\varepsilon^4)$$

with $F(r_0^2) = 0$ and $\mathfrak{c}_s^2 = -2r_0^2F'(r_0^2)$, the second equation in (9) implies $-\mathfrak{c}_s\partial_{z_1}\varphi + \mathfrak{c}_s^2A = 0$. In both cases, we obtain the constraint

$$\mathfrak{c}_sA = \partial_{z_1}\varphi. \tag{10}$$

We multiply the first equation in (9) by $c(\varepsilon)/\mathfrak{c}_s^2$ and we apply the operator $\frac{1}{\mathfrak{c}_s^2}\partial_{z_1}$ to the second one, then we add the resulting equalities. Using the Taylor expansion

$$F\Big(r_0^2(1+\alpha)^2\Big) = -\mathfrak{c}_s^2\alpha - \Big(\frac{\mathfrak{c}_s^2}{2} - 2r_0^4F''(r_0^2)\Big)\alpha^2 + F_3(\alpha), \qquad \text{where } F_3(\alpha) = \mathcal{O}(\alpha^3) \text{ as } \alpha \to 0,$$

we get

$$\begin{aligned} &\frac{\mathfrak{c}_s^2 - c^2(\varepsilon)}{\varepsilon^2\mathfrak{c}_s^2}\,\partial_{z_1}A_\varepsilon \; - \frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1}\Big(\frac{\partial_{z_1}^2A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon^2A_\varepsilon}\Big) + \frac{c(\varepsilon)}{\mathfrak{c}_s^2}(1+\varepsilon^2A_\varepsilon)\Delta_{z_\perp}\varphi_\varepsilon \\ &\qquad + \Big\{2\frac{c(\varepsilon)}{\mathfrak{c}_s^2}\partial_{z_1}\varphi_\varepsilon\partial_{z_1}A_\varepsilon + \frac{c(\varepsilon)}{\mathfrak{c}_s^2}A_\varepsilon\partial_{z_1}^2\varphi_\varepsilon + \frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1}\big((\partial_{z_1}\varphi_\varepsilon)^2\big) + \Big[\frac{1}{2} - 2r_0^4\frac{F''(r_0^2)}{\mathfrak{c}_s^2}\Big]\partial_{z_1}(A_\varepsilon^2)\Big\} \\ &\qquad = -2\varepsilon^2\frac{c(\varepsilon)}{\mathfrak{c}_s^2}\nabla_{z_\perp}\varphi_\varepsilon\cdot\nabla_{z_\perp}A_\varepsilon - \frac{\varepsilon^2}{\mathfrak{c}_s^2}\partial_{z_1}\big(|\nabla_{z_\perp}\varphi_\varepsilon|^2\big) - \frac{1}{\mathfrak{c}_s^2\varepsilon^4}\,\partial_{z_1}\big(F_3(\varepsilon^2A_\varepsilon)\big). \end{aligned} \tag{11}$$

If $A_\varepsilon \to A$ and $\varphi_\varepsilon \to \varphi$ as $\varepsilon \to 0$ in a suitable sense, we have $\mathfrak{c}_s^2 - c^2(\varepsilon) = \varepsilon^2$ and $\partial_{z_1}^{-1} A = \varphi/\mathfrak{c}_s$ by (10), and then (11) gives

$$\frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1} A - \frac{1}{\mathfrak{c}_s^2}\,\partial_{z_1}^3 A + \Gamma A \partial_{z_1} A + \Delta_{z_\perp} \partial_{z_1}^{-1} A = 0,$$

which is (SW).

The main result of this paper is as follows.

Theorem 6 *Let $N \in \{2,3\}$ and assume that the nonlinearity F satisfies (A2) and (A4) with $\Gamma \neq 0$. Let $(U_n, c_n)_{n\geq 1}$ be any sequence such that $U_n \in \mathcal{E}$ is a nonconstant solution of (TW_{c_n}), $c_n \in (0, \mathfrak{c}_s)$ and $c_n \to \mathfrak{c}_s$ as $n \to \infty$ and one of the following situations occur:*

(a) $N = 2$ and U_n minimizes E under the constraint $Q = Q(U_n)$, as in Theorem 1 (a) or (b).

(b) $N = 2$ and $U_n(c_n \cdot)$ minimizes the functional $I(\psi) := Q(\psi) + \int_{\mathbb{R}^N} V(|\psi|^2)\,dx$ under the constraint $\int_{\mathbb{R}^N} |\nabla \psi|^2\,dx = \int_{\mathbb{R}^N} |\nabla U_n|^2\,dx$, as in Theorem 2.

(c) $N = 3$ and U_n minimizes $E_{c_n} = E + c_n Q$ under the constraint $P_{c_n} = 0$, as in Theorem 3.

Then there exists $n_0 \in \mathbb{N}$ such that $|U_n| \geq r_0/2$ in $\mathbb{R}^N$ for all $n \geq n_0$ and, denoting $\varepsilon_n = \sqrt{\mathfrak{c}_s^2 - c_n^2}$ (so that $c_n = c(\varepsilon_n)$), we have

$$E(U_n) \sim -\mathfrak{c}_s Q(U_n) \sim r_0^2 \mathfrak{c}_s^4 (7-2N) \mathscr{S}_{\min} \Big(\mathfrak{c}_s^2 - c_n^2\Big)^{\frac{5-2N}{2}} = r_0^2 \mathfrak{c}_s^4 (7-2N) \mathscr{S}_{\min} \varepsilon_n^{5-2N} \tag{12}$$

and

$$E(U_n) + c_n Q(U_n) \sim \mathfrak{c}_s^2 r_0^2 \mathscr{S}_{\min} \varepsilon_n^{7-2N} \qquad \text{as } n \to \infty. \tag{13}$$

Moreover, U_n can be written in the form

$$U_n(x) = r_0 \Big(1 + \varepsilon_n^2 A_n(z)\Big) \exp\big(i \varepsilon_n \varphi_n(z)\big), \qquad \text{where} \quad z_1 = \varepsilon_n x_1, \quad z_\perp = \varepsilon_n^2 x_\perp,$$

and there exist a subsequence $(U_{n_k}, c_{n_k})_{k\geq 1}$, a ground state $\mathcal{W}$ of (KP-I) *and a sequence $(z^k)_{k\geq 1} \subset \mathbb{R}^N$ such that, denoting $\tilde{A}_k = A_{n_k}(\cdot - z^k)$, $\tilde{\varphi}_k = \varphi_{n_k}(\cdot - z^k)$, for any $1 < p < \infty$ we have*

$$\tilde{A}_k \to \mathcal{W}, \qquad \partial_{z_1} \tilde{A}_k \to \partial_{z_1} \mathcal{W}, \qquad \partial_{z_1} \tilde{\varphi}_k \to \mathfrak{c}_s \mathcal{W} \quad \text{and} \quad \partial_{z_1}^2 \tilde{\varphi}_k \to \mathfrak{c}_s \partial_{z_1} \mathcal{W} \quad \text{in} \quad W^{1,p}(\mathbb{R}^N) \text{ as } k \to \infty.$$

As already mentioned, if F satisfies (A3) and (A4) it is possible to modify F in a neighborhood of infinity such that the modified nonlinearity $\tilde{F}$ also satisfies (A2) and (TW_c) has the same solutions as the same equation with $\tilde{F}$ instead of F. Then one may use Theorems 1, 2 and 3 to construct travelling waves for (NLS). It is obvious that Theorem 6 above also applies to the solutions constructed in this way.

Let us mention that in the case of the Gross-Pitaevskii nonlinearity $F(\varrho) = 1 - \varrho$ and in dimension $N = 2$, F. Béthuel, P. Gravejat and J-C. Saut proved in [8] the same type of convergence for the solutions constructed in [9]. Those solutions are global minimizers of the energy with prescribed momentum, which allows to derive *a priori* bounds: for instance, their energy is small. In fact, if V is nonnegative and $N = 2$, Theorem 1 provides travelling wave solutions with speed $\simeq \mathfrak{c}_s$ for $|q|$ small and the proof of Theorem 6 is quite similar to [8], and therefore we will focus on the other cases. However, if the potential V achieves negative values, the minimization of the energy under the constraint of fixed momentum on the whole space $\mathcal{E}$ is no longer possible, hence the approach in Theorem 2 or the local minimization approach in Theorem 1 (*b*). In dimension $N = 3$ (even for the Gross-Pitaevskii nonlinearity $F(\varrho) = 1 - \varrho$), the travelling waves we deal with have high energy and momentum and are *not* minimizers of the energy at fixed momentum (which are the vortex rings, see [13]). In particular, we have to show that the U_n's are vortexless ($|U_n| \geq r_0/2$). For the Gross-Pitaevskii nonlinearity, Theorem 6 provides a rigorous proof to the existence of the upper branch in the so-called Jones-Roberts curve in dimension three ([26]). This upper branch was conjectured by formal expansions and numerical simulations (however limited to not so large momentum). In dimension $N = 3$, the solutions on this upper branch are expected to be unstable (see [5]), and these rarefaction pulses should evolve by creating vortices (cf. [3]).

It is also natural to investigate the one dimensional case. Firstly, the (KP-I) equation has to be replaced by the (KdV) equation

$$2\partial_\tau \zeta + \Gamma \zeta \partial_z \zeta - \frac{1}{\mathfrak{c}_s^2}\,\partial_z^3 \zeta = 0, \tag{KdV}$$

and (SW) becomes

$$\frac{1}{\mathfrak{c}_s^2}\,\partial_z \mathcal{W} + \Gamma \mathcal{W} \partial_z \mathcal{W} - \frac{1}{\mathfrak{c}_s^2}\,\partial_z^3 \mathcal{W} = 0.$$

If $\Gamma \neq 0$, the only nontrivial travelling wave for (KdV) (up to space translations) is given by

$$\mathrm{w}(z) = -\frac{3}{\mathfrak{c}_s^2 \Gamma \cosh^2(z/2)},$$

and there holds

$$\mathscr{S}(\mathrm{w}) = \int_{\mathbb{R}} \frac{1}{\mathfrak{c}_s^2}\,(\partial_z \mathrm{w})^2 + \frac{\Gamma}{3}\,\mathrm{w}^3 \; dz + \frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}} \mathrm{w}^2 \; dz = \int_{\mathbb{R}} \frac{2}{\mathfrak{c}_s^2}\,(\partial_z \mathrm{w})^2 \; dz = \frac{48}{5\mathfrak{c}_s^6 \Gamma^2}.$$

The following result, which corresponds to Theorem 6 in dimension $N = 1$, was proved in [16] by using ODE techniques.

Theorem 7 ([16]) *Let $N = 1$ and assume that F satisfies (A4) with $\Gamma \neq 0$. Then, there are $\delta > 0$ and $0 < \mathfrak{c}_0 < \mathfrak{c}_s$ with the following properties. For any $\mathfrak{c}_0 \leq c < \mathfrak{c}_s$, there exists a solution U_c to* (TW$_c$) *satisfying $\| |U_c| - r_0 \|_{L^\infty(\mathbb{R})} \leq \delta$. Moreover, for $\mathfrak{c}_0 \leq c < \mathfrak{c}_s$ any nonconstant solution u of* (TW$_c$) *verifying $\| |u| - r_0 \|_{L^\infty(\mathbb{R})} \leq \delta$ is of the form $u(x) = \mathrm{e}^{i\theta} U_c(x - \xi)$ for some $\theta \in \mathbb{R}$ and $\xi \in \mathbb{R}$. The map U_c can be written in the form*

$$U_c(x) = r_0 \left(1 + \varepsilon^2 A_\varepsilon(z)\right) \exp(i \varepsilon \varphi_\varepsilon(z)), \qquad \text{where } z = \varepsilon x \quad \text{and} \quad \varepsilon = \sqrt{\mathfrak{c}_s^2 - c^2}$$

and for any $1 \leq p \leq \infty$,

$$\partial_z \varphi_\varepsilon \to \mathfrak{c}_s \mathrm{w} \qquad \text{and} \qquad A_\varepsilon \to \mathrm{w} \quad \text{in} \quad W^{1,p}(\mathbb{R}) \qquad \text{as} \quad \varepsilon \to 0.$$

Finally, as $\varepsilon \to 0$,

$$E(U_{c(\varepsilon)}) \sim -\mathfrak{c}_s Q(U_{c(\varepsilon)}) \sim 5 r_0^2 \mathfrak{c}_s^4 \mathscr{S}(\mathrm{w}) \Big(\mathfrak{c}_s^2 - c^2(\varepsilon)\Big)^{\frac{3}{2}} = \varepsilon^3 \frac{48 r_0^2}{\mathfrak{c}_s^2 \Gamma^2}$$

and

$$E(U_{c(\varepsilon)}) + c(\varepsilon) Q(U_{c(\varepsilon)}) \sim \mathfrak{c}_s^2 r_0^2 \mathscr{S}(\mathrm{w}) \varepsilon^5 = \frac{48 r_0^2}{5 \mathfrak{c}_s^4 \Gamma^2} \varepsilon^5.$$

Remark 8 In the one-dimensional case it can be easily shown that the mapping $(\mathfrak{c}_0, \mathfrak{c}_s) \ni c \mapsto (A_c - r_0, \partial_z \phi) \in W^{1,p}(\mathbb{R})$, where $U_c = A_c \exp(i\phi)$, is continuous for every $1 \leq p \leq \infty$.

A natural question is to investigate the dynamical counterparts of Theorems 6 and 7. If Ψ_ε^0 is an initial datum for (NLS) of the type

$$\Psi_\varepsilon^0(x) = r_0 \Big(1 + \varepsilon^2 A_\varepsilon^0(z)\Big) \exp\Big(i \varepsilon \varphi_\varepsilon^0(z)\Big),$$

with $z = (z_1, z_\perp) = (\varepsilon x_1, \varepsilon^2 x_\perp)$ and $\mathfrak{c}_s A_\varepsilon^0 \simeq \partial_{z_1} \varphi_\varepsilon^0$, we use for Ψ_ε the ansatz at time $t > 0$, for some functions A_ε, φ_ε depending on (τ, z),

$$\Psi_\varepsilon(t, x) = r_0 \Big(1 + \varepsilon^2 A_\varepsilon(\tau, z)\Big) \mathrm{e}^{i \varepsilon \varphi_\varepsilon(\tau, z)}, \qquad \tau = \mathfrak{c}_s \varepsilon^3 t, \quad z_1 = \varepsilon(x_1 - \mathfrak{c}_s t), \quad z_\perp = \varepsilon^2 x_\perp.$$

Similar computations imply that, for times τ of order one (that is t of order ε^{-3}), we have $\mathfrak{c}_s A_\varepsilon \simeq \partial_{z_1} \varphi_\varepsilon$ and A_ε converges to a solution of the (KP-I) equation. This (KP-I) asymptotic dynamics for the Gross-Pitaevskii equation in dimension $N = 3$ is formally derived in [5] and is used to investigate the linear instability of the solitary waves of speed close to $\mathfrak{c}_s = \sqrt{2}$. The one-dimensional analogue, where the (KP-I) equation has to be replaced by the corresponding Korteweg-de Vries equation, can be found in [39] and [28]. The rigorous mathematical proofs of these regimes have been provided in [18] in arbitrary space dimension and for a general nonlinearity F (the coefficient Γ might even vanish), respectively in [11] for the one dimensional Gross-Pitaevskii equation by using the complete integrability of the equation (more precisely, the existence of sufficiently many conservation laws).

1.2 Scheme of the proof of Theorem 6

In case (a) there is a direct proof of Theorem 6 which is quite similar to the one in [8]. Moreover, it follows from Proposition 5.12 in [17] that if (U_n, c_n) satisfies (a) then it also satisfies (b), so it suffices to prove Theorem 6 in cases (b) and (c).

The first step is to give sharp asymptotics for the quantities minimized in [17] and [36] in order to prove the existence of travelling waves, namely to estimate

$$I_{\min}(k) = \inf \Big\{ \int_{\mathbb{R}^2} V(|\psi|^2)\, dx + Q(\psi) \ \big| \ \psi \in \mathcal{E},\ \int_{\mathbb{R}^2} |\nabla \psi|^2\, dx = k \Big\} \qquad \text{as } k \to 0$$

and

$$T_c = \inf \Big\{ E(\psi) + cQ(\psi) \ \big| \ \psi \in \mathcal{E},\ \psi \text{ is not constant, } E(\psi) + cQ(\psi) = \int_{\mathbb{R}^3} |\nabla_{x_\perp} \psi|^2\, dx \Big\} \qquad \text{as } c \to \mathfrak{c}_s.$$

These bounds are obtained by plugging test functions with the ansatz (8) into the corresponding minimization problems, where $(A_\varepsilon, \varphi_\varepsilon) \simeq (A, \mathfrak{c}_s^{-1} \partial_{z_1}^{-1} A)$ and A is a ground state for (KP-I). A similar upper bound for $I_{\min}(k)$ was already a crucial point in [17] to rule out the dichotomy of minimizing sequences.

Proposition 9 *Assume that F satisfies (A2) and (A4) with $\Gamma \neq 0$. Then:*

(i) If $N = 2$, we have as $k \to 0$

$$I_{\min}(k) \leq -\frac{k}{\mathfrak{c}_s^2} - \frac{4k^3}{27 r_0^4 \mathfrak{c}_s^{12} \mathscr{S}_{\min}^2} + \mathcal{O}(k^5).$$

(ii) If $N = 3$, the following upper bound holds as $\varepsilon \to 0$ (that is, as $c(\varepsilon) \to \mathfrak{c}_s$):

$$T_{c(\varepsilon)} \leq \mathfrak{c}_s^2 r_0^2 \mathscr{S}_{\min} (\mathfrak{c}_s^2 - c^2(\varepsilon))^{\frac{1}{2}} + \mathcal{O}\Big((\mathfrak{c}_s^2 - c^2(\varepsilon))^{\frac{3}{2}} \Big) = \mathfrak{c}_s^2 r_0^2 \mathscr{S}_{\min} \varepsilon + \mathcal{O}(\varepsilon^3).$$

The second step is to derive upper bounds for the energy and the momentum. In space dimension three (case (c)) this is tricky. Indeed, if U_c is a minimizer of E_c under the constraint $P_c = 0$, the only information we have is about $T_c = \displaystyle\int_{\mathbb{R}^N} |\nabla_{x_\perp} U_c|^2\, dx$ (see the first identity in (4)). In particular, we have no *a priori* bounds on $\displaystyle\int_{\mathbb{R}^N} \Big| \frac{\partial U_c}{\partial x_1} \Big|^2 dx$, $Q(U_c)$ and the potential energy $\displaystyle\int_{\mathbb{R}^N} V(|U_c|^2)\, dx$. Using an averaging argument we infer that there is a sequence (U_n, c_n) for which we have "good" bounds on the energy and the momentum. Then we prove a rigidity property of "good sequences": any sequence (U_n, c_n) that satisfies the "good bounds" has a subsequence that satisfies the conclusion of Theorem 6. This rigid behavior combined with the existence of a sequence with "good bounds" and a continuation argument allow us to conclude that Theorem 6 holds for *any* sequence (U_n, c_n) with $c_n \to \mathfrak{c}_s$ (as in (c)). More precisely, we will prove:

Proposition 10 *Let $N \geq 3$ and assume that F satisfies (A1) and (A2). Then:*

(i) For any $c \in (0, \mathfrak{c}_s)$ and any minimizer U of E_c in $\mathscr{C}_c$ we have $Q(U) < 0$.

(ii) The function $(0, \mathfrak{c}_s) \ni c \longmapsto T_c \in \mathbb{R}_+$ is decreasing, thus has a derivative almost everywhere.

(iii) The function $c \longmapsto T_c$ is left continuous on $(0, \mathfrak{c}_s)$. If it has a derivative at c_0, then for any minimizer U_0 of E_{c_0} under the constraint $P_{c_0} = 0$, scaled so that U_0 solves (TW$_{c_0}$)*, there holds*

$$\frac{dT_c}{dc}_{|c=c_0} = Q(U_0).$$

(iv) Let $c_0 \in (0, \mathfrak{c}_s)$. Assume that there is a sequence $(c_n)_{n \geq 1}$ such that $c_n > c_0$, $c_n \to c_0$ and for any n there is a minimizer $U_n \in \mathcal{E}$ of E_{c_n} on $\mathscr{C}_{c_n}$ which solves (TW$_{c_n}$) and the sequence $(Q(U_n))_{n \geq 1}$ is bounded. Then $c \longmapsto T_c$ is continuous at c_0.

(v) Let $0 < c_1 < c_2 < \mathfrak{c}_s$. Let U_i be minimizers of E_{c_i} on $\mathscr{C}_{c_i}$, $i = 1, 2$, such that U_i solves (TW$_{c_i}$)*. Denote $q_1 = Q(U_1)$ and $q_2 = Q(U_2)$. Then we have*

$$\frac{T_{c_1}^2}{q_1^2} - c_1^2 \geq \frac{T_{c_2}^2}{q_2^2} - c_2^2.$$

(vi) If $N = 3$, F verifies (A4) and $\Gamma \neq 0$, there exist a constant $C > 0$ and a sequence $\varepsilon_n \to 0$ such that for any minimizer $U_n \in \mathcal{E}$ of $E_{c(\varepsilon_n)}$ on $\mathscr{C}_{c(\varepsilon_n)}$ which solves $(\mathrm{TW}_{c(\varepsilon_n)})$ *we have*

$$E(U_n) \leq \frac{C}{\varepsilon_n} \qquad \textit{and} \qquad |Q(U_n)| \leq \frac{C}{\varepsilon_n}.$$

Proposition 11 *Assume that $N = 3$, (A2) and (A4) hold and $\Gamma \neq 0$. Let $(U_n, \varepsilon_n)_{n \geq 1}$ be a sequence such that $\varepsilon_n \to 0$, U_n minimizes $E_{c(\varepsilon_n)}$ on $\mathscr{C}_{c(\varepsilon_n)}$, satisfies* $(\mathrm{TW}_{c(\varepsilon_n)})$ *and there exists a constant $C > 0$ such that*

$$E(U_n) \leq \frac{C}{\varepsilon_n} \qquad \textit{and} \qquad |Q(U_n)| \leq \frac{C}{\varepsilon_n} \qquad \textit{for all } n.$$

Then there is a subsequence of $(U_n, c(\varepsilon_n))_{n \geq 1}$ which satisfies the conclusion of Theorem 6.

Proposition 12 *Let $N = 3$ and suppose that (A2) and (A4) hold with $\Gamma \neq 0$. There are $K > 0$ and $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$ and for any minimizer U of $E_{c(\varepsilon)}$ on $\mathscr{C}_{c(\varepsilon)}$ scaled so that U satisfies* $(\mathrm{TW}_{c(\varepsilon)})$ *we have*

$$E(U) \leq \frac{K}{\varepsilon} \qquad \textit{and} \qquad |Q(U)| \leq \frac{K}{\varepsilon}.$$

It is now obvious that the proof of Theorem 6 in the three-dimensional case follows directly from Propositions 11 and 12 above.

The most difficult and technical point in the above program is to prove Proposition 11. Let us describe our strategy to carry out that proof, as well as the proof of Theorem 6 in the two-dimensional case.

Once we have a sequence of travelling waves to (NLS) with "good bounds" on the energy and the momentum and speeds that tend to $\mathfrak{c}_s$, we need to show that those solutions do not vanish and can be lifted. We recall the following result, which is a consequence of Lemma 7.1 in [17]:

Lemma 13 ([17]) *Let $N \geq 2$ and suppose that the nonlinearity F satisfies (A1) and ((A2) or (A3)). Then for any $\delta > 0$ there is $M(\delta) > 0$ such that for all $c \in [0, \mathfrak{c}_s]$ and for all solutions $U \in \mathcal{E}$ of* (TW_c) *such that $\|\nabla U\|_{L^2(\mathbb{R}^N)} < M(\delta)$ we have*

$$\| |U| - r_0 \|_{L^\infty(\mathbb{R}^N)} \leq \delta.$$

In the two-dimensional case the lifting properties follow immediately from Lemma 13. However, in dimension $N = 3$, for travelling waves $U_{c(\varepsilon)}$ which minimize $E_{c(\varepsilon)}$ on $\mathscr{C}_{c(\varepsilon)}$ the quantity $\left\| \frac{\partial U_{c(\varepsilon)}}{\partial x_1} \right\|_{L^2}^2$ is large, of order $\simeq \varepsilon^{-1}$ as $\varepsilon \to 0$. We give a lifting result for those solutions, based on the fact that $\|\nabla_{x_\perp} U_{c(\varepsilon)}\|_{L^2}^2 = \frac{N-1}{2} T_{c(\varepsilon)}$ is sufficiently small.

Proposition 14 *We consider a nonlinearity F satisfying (A1) and ((A2) or (A3)). Let $U \in \mathcal{E}$ be a travelling wave to* (NLS) *of speed $c \in [0, \mathfrak{c}_s]$.*

(i) If $N \geq 3$, for any $0 < \delta < r_0$ there exists $\mu = \mu(\delta) > 0$ such that

$$\left\| \frac{\partial U}{\partial x_1} \right\|_{L^2(\mathbb{R}^N)} \cdot \|\nabla_{x_\perp} U\|_{L^2(\mathbb{R}^N)}^{N-1} \leq \mu(\delta) \qquad \textit{implies} \qquad \| |U| - r_0 \|_{L^\infty(\mathbb{R}^N)} \leq \delta.$$

(ii) If $N \geq 4$ and, moreover, (A3) holds or $\left\| \frac{\partial U}{\partial x_1} \right\|_{L^2(\mathbb{R}^N)} \cdot \|\nabla_{x_\perp} U\|_{L^2(\mathbb{R}^N)}^{N-1} \leq 1$, then for any $\delta > 0$ there is $m(\delta) > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla_{x_\perp} U|^2 \, dx \leq m(\delta) \qquad \textit{implies} \qquad \| |U| - r_0 \|_{L^\infty(\mathbb{R}^N)} \leq \delta.$$

As an immediate consequence, the three-dimensional travelling wave solutions provided by Theorem 3 have modulus close to r_0 (hence do not vanish) as $c \to \mathfrak{c}_s$:

Corollary 15 *Let $N = 3$ and consider a nonlinearity F satisfying (A2) and (A4) with $\Gamma \neq 0$. Then, the travelling wave solutions $U_{c(\varepsilon)}$ to (NLS) provided by Theorem 3 which satisfy an additional bound $E(U_{c(\varepsilon)}) \leq \frac{C}{\varepsilon}$ (with C independent on ε) verify*

$$\| |U_{c(\varepsilon)}| - r_0 \|_{L^\infty(\mathbb{R}^3)} \to 0 \qquad \textit{as} \qquad \varepsilon \to 0.$$

In particular, for ε sufficiently close to 0 we have $|U_{c(\varepsilon)}| \geq r_0/2$ in $\mathbb{R}^3$.

Proof. By the the second identity in (4) we have

$$\int_{\mathbb{R}^3} \Big| \frac{\partial U_{c(\varepsilon)}}{\partial x_1} \Big|^2 \, dx = \frac{1}{2} E(U_{c(\varepsilon)}) \leq \frac{C}{\varepsilon}.$$

Moreover, the first identity in (4) and Proposition 9 (ii) imply

$$\int_{\mathbb{R}^3} |\nabla_{x_\perp} U_{c(\varepsilon)}|^2 \, dx = E_{c(\varepsilon)}(U_{c(\varepsilon)}) = T_{c(\varepsilon)} \leq C\varepsilon.$$

Hence $\Big\| \frac{\partial U_{c(\varepsilon)}}{\partial x_1} \Big\|_{L^2(\mathbb{R}^3)} \|\nabla_{x_\perp} U_{c(\varepsilon)}\|^2_{L^2(\mathbb{R}^3)} \leq C\sqrt{\varepsilon}$ and the result follows from Proposition 14 (ii). ☐

We give now some properties of the two-dimensional travelling wave solutions provided by Theorem 2.

Proposition 16 *Let $N = 2$ and assume that F verifies (A2) and (A4) with $\Gamma \neq 0$. Then there exist constants C_1, C_2, C_3, $C_4 > 0$ and $0 < k_* < k_\infty$ such that all travelling wave solutions U_k provided by Theorem 2 with $0 < k = \int_{\mathbb{R}^2} |\nabla U_k|^2 \, dx < k_*$ satisfy $|U_k| \geq r_0/2$ in $\mathbb{R}^2$,*

$$C_1 k \leq -Q(U_k) \leq C_2 k, \qquad C_1 k \leq \int_{\mathbb{R}^2} V(|U_k|^2) \, dx \leq C_2 k, \qquad C_1 k \leq \int_{\mathbb{R}^2} (\chi^2(|U_k|) - r_0^2)^2 \, dx \leq C_2 k \quad (14)$$

and have a speed $c(U_k) = \sqrt{\mathfrak{c}_s^2 - \varepsilon_k^2}$ satisfying

$$C_3 k \leq \varepsilon_k \leq C_4 k. \quad (15)$$

At this stage, we know that the travelling waves provided by Theorems 2 and 3 do not vanish if their speed is sufficiently close to $\mathfrak{c}_s$. Using the above lifting results, we may write such a solution U_c in the form

$$U_c(x) = \rho(x) \mathrm{e}^{i\phi(x)} = r_0 \sqrt{1 + \varepsilon^2 \mathcal{A}_\varepsilon(z)} \; \mathrm{e}^{i\varepsilon\varphi_\varepsilon(z)}, \qquad \text{where} \quad \varepsilon = \sqrt{\mathfrak{c}_s^2 - c^2}, \quad z_1 = \varepsilon x_1, \; z_\perp = \varepsilon^2 x_\perp, \quad (16)$$

and we use the same scaling as in (8). The interest of writing the modulus in this way (and not as in (8)) is just to simplify a little bit the algebra and to have expressions similar to those in [8]. Since $\mathcal{A}_\varepsilon = 2A_\varepsilon + \varepsilon^2 A_\varepsilon^2$, bounds in Sobolev spaces for $\mathcal{A}_\varepsilon$ imply similar Sobolev bounds for A_ε and conversely. We shall now find Sobolev bounds for $\mathcal{A}_\varepsilon$ and φ_ε. It is easy to see that (TW_c) is equivalent to the following system for the phase φ and the modulus ρ (in the original variable x):

$$\begin{cases} c \dfrac{\partial}{\partial x_1}(\rho^2 - r_0^2) = 2\mathrm{div}(\rho^2 \nabla\phi), \\ \\ \Delta\rho - \rho|\nabla\phi|^2 + \rho F(\rho^2) = -c\rho \dfrac{\partial \phi}{\partial x_1}. \end{cases} \quad (17)$$

Multiplying the second equation by 2ρ, we write (17) in the form

$$\begin{cases} 2\mathrm{div}((\rho^2 - r_0^2)\nabla\phi) - c\dfrac{\partial}{\partial x_1}(\rho^2 - r_0^2) = -2r_0^2 \Delta\phi, \\ \\ \Delta(\rho^2 - r_0^2) - 2|\nabla U_c|^2 + 2\rho^2 F(\rho^2) + 2c(\rho^2 - r_0^2)\dfrac{\partial\phi}{\partial x_1} = -2cr_0^2 \dfrac{\partial \phi}{\partial x_1}. \end{cases} \quad (18)$$

Let $\eta = \rho^2 - r_0^2$. We apply the operator $-2c\dfrac{\partial}{\partial x_1}$ to the first equation in (18) and we take the Laplacian of the second one, then we add the resulting equalities to get

$$\left[\Delta^2 - \mathfrak{c}_s^2 \Delta + c^2 \frac{\partial^2}{\partial x_1^2} \right] \eta = \Delta \left(2|\nabla U_c|^2 - 2c\eta \frac{\partial\phi}{\partial x_1} - 2\rho^2 F(\rho^2) - \mathfrak{c}_s^2 \eta \right) + 2c \frac{\partial}{\partial x_1}(\mathrm{div}(\eta\nabla\phi)). \quad (19)$$

Since $\mathfrak{c}_s^2 = -2r_0^2 F'(r_0^2)$, using the Taylor expansion

$$2(s + r_0^2)F(s + r_0^2) + \mathfrak{c}_s^2 s = -\frac{\mathfrak{c}_s^2}{r_0^2} \left(1 - \frac{r_0^4 F''(r_0^2)}{\mathfrak{c}_s^2} \right) s^2 + r_0^2 \tilde{F}_3(s),$$

where $\tilde{F}_3(s) = \mathcal{O}(s^3)$ as $s \to 0$, we see that the right-hand side in (19) is at least quadratic in (η, ϕ). Then we perform a scaling and pass to the variable $z = (\varepsilon x_1, \varepsilon^2 x_\perp)$ (where $\varepsilon = \sqrt{\mathfrak{c}_s^2 - c^2}$), so that (19) becomes

$$\Big\{\partial_{z_1}^4 - \partial_{z_1}^2 - \mathfrak{c}_s^2 \Delta_{z_\perp} + 2\varepsilon^2 \partial_{z_1}^2 \Delta_{z_\perp} + \varepsilon^4 \Delta_{z_\perp}^2\Big\} \mathcal{A}_\varepsilon = \mathcal{R}_\varepsilon, \tag{20}$$

where $\mathcal{R}_\varepsilon$ contains terms at least quadratic in $(\mathcal{A}_\varepsilon, \varphi_\varepsilon)$:

$$\begin{aligned}
\mathcal{R}_\varepsilon = &\{\partial_{z_1}^2 + \varepsilon^2 \Delta_{z_\perp}\}\Big[2(1+\varepsilon^2\mathcal{A}_\varepsilon)\Big((\partial_{z_1}\varphi_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}\varphi_\varepsilon|^2\Big) + \varepsilon^2 \frac{(\partial_{z_1}\mathcal{A}_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}\mathcal{A}_\varepsilon|^2}{2(1+\varepsilon^2\mathcal{A}_\varepsilon)}\Big] \\
&- 2c\varepsilon^2 \Delta_{z_\perp}(\mathcal{A}_\varepsilon \partial_{z_1}\varphi_\varepsilon) + 2c\varepsilon^2 \sum_{j=2}^N \partial_{z_1}\partial_{z_j}(\mathcal{A}_\varepsilon \partial_{z_j}\varphi_\varepsilon) \\
&+ \{\partial_{z_1}^2 + \varepsilon^2\Delta_{z_\perp}\}\Big[\mathfrak{c}_s^2\Big(1 - \frac{r_0^4 F''(r_0^2)}{\mathfrak{c}_s^2}\Big)\mathcal{A}_\varepsilon^2 - \frac{1}{\varepsilon^4}\tilde{F}_3(r_0^2\varepsilon^2\mathcal{A}_\varepsilon)\Big].
\end{aligned}$$

In the two-dimensional case, uniform bounds (with respect to ε) in Sobolev spaces have been derived in [8] by using (20) and a bootstrap argument. This technique is based upon the fact that some kernels related to the linear part in (20), such as

$$\mathscr{F}^{-1}\Big(\frac{\xi_1^2}{\xi_1^4 + \xi_1^2 + \mathfrak{c}_s^2|\xi_\perp|^2 + 2\varepsilon^2\xi_1^2|\xi_\perp|^2 + \varepsilon^4|\xi_\perp|^4}\Big) \qquad \text{and} \qquad \mathscr{F}^{-1}\Big(\frac{\varepsilon^2|\xi_\perp|^2}{\xi_1^4 + \xi_1^2 + \mathfrak{c}_s^2|\xi_\perp|^2 + 2\varepsilon^2\xi_1^2|\xi_\perp|^2 + \varepsilon^4|\xi_\perp|^4}\Big)$$

are bounded in $L^p(\mathbb{R}^2)$ for p in some interval $[2, \bar{p})$, uniformly with respect to ε. However, this is no longer true in dimension $N = 3$: the above mentioned kernels are not in $L^2(\mathbb{R}^3)$ (but their Fourier transforms are uniformly bounded), and from the analysis in [23], the kernel

$$\mathscr{F}^{-1}\Big(\frac{\xi_1^2}{\xi_1^4 + \xi_1^2 + \mathfrak{c}_s^2|\xi_\perp|^2}\Big)$$

is presumably too singular near the origin to be in $L^p(\mathbb{R}^3)$ if $p \geq 5/3$. This lack of integrability of the kernels makes the analysis in the three dimensional case much more diffcult than in the case $N = 2$.

One of the main difficulties in the three dimensional case is to prove that for ε sufficiently small, $\mathcal{A}_\varepsilon$ is uniformly bounded in L^p for some $p > 2$. To do this we use a suitable decomposition of $\mathcal{A}_\varepsilon$ in the Fourier space (see the proof of Lemma 24 below). Then we improve the exponent p by using a bootstrap argument, combining the iterative argument in [8] (which uses the quadratic nature of $\mathcal{R}_\varepsilon$ in (20)) and the appropriate decomposition of $\mathcal{A}_\varepsilon$ in the Fourier space. This leads to some L^p bound with $p > 3 = N$. Once this bound is proved, the proof of the $W^{1,p}$ bounds follows the scheme in [8]. We get:

Proposition 17 *Under the assumptions of Theorem 6, there is $\varepsilon_0 > 0$ such that $\mathcal{A}_\varepsilon \in W^{4,p}(\mathbb{R}^N)$ and $\nabla\varphi_e \in W^{3,p}(\mathbb{R}^N)$ for all $\varepsilon \in (0, \varepsilon_0)$ and all $p \in (1, \infty)$. Moreover, for any $p \in (1, \infty)$ there exists $C_p > 0$ satisfying for all $\varepsilon \in (0, \varepsilon_0)$*

$$\|\mathcal{A}_\varepsilon\|_{L^p} + \|\nabla\mathcal{A}_\varepsilon\|_{L^p} + \|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon^2\|\nabla_{z_\perp}^2\mathcal{A}_\varepsilon\|_{L^p} \leq C_p \qquad \textit{and} \tag{21}$$

$$\begin{aligned}
&\|\partial_{z_1}\varphi_\varepsilon\|_{L^p} + \varepsilon\|\nabla_{z_\perp}\varphi_\varepsilon\|_{L^p} + \|\partial_{z_1}^2\varphi_\varepsilon\|_{L^p} + \varepsilon\|\nabla_{z_\perp}\partial_{z_1}\varphi_\varepsilon\|_{L^p} + \varepsilon^2\|\nabla_{z_\perp}^2\varphi_\varepsilon\|_{L^p} \\
&+\|\partial_{z_1}^3\varphi_\varepsilon\|_{L^p} + \varepsilon\|\nabla_{z_\perp}\partial_{z_1}^2\varphi_\varepsilon\|_{L^p} + \varepsilon^2\|\nabla_{z_\perp}^2\partial_{z_1}\varphi_\varepsilon\|_{L^p} \leq C_p.
\end{aligned} \tag{22}$$

The estimate (21) *is also valid with A_ε instead of $\mathcal{A}_\varepsilon$.*

Once these bounds are established, the estimates in Proposition 9 show that $(\mathfrak{c}_s^{-1}\partial_{z_1}\varphi_n)_{n\geq 0}$ is a minimizing sequence for the problem (6) if $N = 2$, respectively for the problem (7) if $N = 3$. Since Theorems 4 and 5 provide compactness properties for minimizing sequences, we get (pre)compactness of $(\mathfrak{c}_s^{-1}\partial_{z_1}\varphi_n)_{n\geq 0}$ in $\mathscr{Y}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, and then we complete the proof of Theorem 6 by standard interpolation in Sobolev spaces.

1.3 On the higher dimensional case

It is natural to ask what happens in the transsonic limit in dimension $N \geq 4$. Firstly, it should be noticed that even for the Gross-Pitaevskii nonlinearity the problem is critical if $N = 4$ and supercritical in higher dimensions, hence Theorem 3 does not apply directly.

The first crucial step is to investigate the behaviour of T_c as $c \to \mathfrak{c}_s$. In particular, in order to be able to use Proposition 14 to show that the solutions are vortexless in this limit, we would need to prove that $T_c \to 0$ as $c \to \mathfrak{c}_s$. We have not been able to prove (or disprove) this in dimension $N = 4$ and $N = 5$, except for the case $\Gamma = 0$. Quite surprisingly, for nonlinearities satisfying (A3) and (A4) (this is the case for both the Gross-Pitaevskii and the cubic-quintic nonlinearity), this is not true in dimension higher than 5, as shown by the following

Proposition 18 *Suppose that F satisfies (A3) and (A4) (and Γ is arbitrary). If $N \geq 6$, there exists $\delta > 0$ such that for any $0 \leq c \leq \mathfrak{c}_s$ and for any nonconstant solution $U \in \mathcal{E}$ of* (TW_c)*, we have*

$$E(U) + cQ(U) \geq \delta.$$

In particular,

$$\inf_{0<c<\mathfrak{c}_s} T_c > 0.$$

The same conclusion holds if $N \in \{4, 5\}$ provided that $\Gamma = 0$.

Therefore we do not know if the solutions constructed in Theorem 3 (for a subcritical nonlinearity) may vanish or not as $c \to \mathfrak{c}_s$ if $N \geq 6$. On the other hand we can show, in any space dimension $N \geq 4$, that we cannot scale the solutions in order to have compactness and convergence to a localized and nontrivial object in the transonic limit as soon as the quantity $E + cQ$ tends to zero.

Proposition 19 *Let $N \geq 4$ and suppose that F satisfies (A2), (A3) and (A4) (and Γ is arbitrary). Assume that there exists a sequence (U_n, c_n) such that $c_n \in (0, \mathfrak{c}_s]$, $U_n \in \mathcal{E}$ is a nonconstant solution of* (TW_{c_n}) *and $E_{c_n}(U_n) \to 0$ as $n \to \infty$. Then, for n large enough, there exist $\alpha_n, \beta_n, \lambda_n, \sigma_n \in \mathbb{R}$, $A_n \in H^1(\mathbb{R}^N)$ and $\varphi_n \in \dot{H}^1(\mathbb{R}^N)$ uniquely determined such that*

$$U_n(x) = r_0\Big(1 + \alpha_n A_n(z)\Big)\exp\Big(i\beta_n\varphi_n(z)\Big), \qquad \textit{where} \quad z_1 = \lambda_n x_1, \quad z_\perp = \sigma_n x_\perp,$$

$$\alpha_n \to 0 \qquad \textit{and} \qquad \|A_n\|_{L^\infty(\mathbb{R}^N)} = \|A_n\|_{L^2(\mathbb{R}^N)} = \|\partial_{z_1}\varphi_n\|_{L^2(\mathbb{R}^N)} = \|\nabla_{z_\perp}\varphi_n\|_{L^2(\mathbb{R}^N)} = 1.$$

Then we have $c_n \to \mathfrak{c}_s$ and

$$\|\partial_{z_1} A_n\|_{L^2(\mathbb{R}^N)} \to 0 \qquad \textit{as } n \to +\infty.$$

Consequently, even if one could show that $T_c \to 0$ as $c \to \mathfrak{c}_s$ in space dimension 4 or 5, we would not have a nontrivial limit (after rescaling) of the corresponding rarefaction pulses.

2 Three-dimensional ground states for (KP-I)

We recall the anisotropic Sobolev inequality (see [7], p. 323): for $N \geq 2$ and for any $2 \leq p < \frac{2(2N-1)}{2N-3}$, there exists $C = C(p, N)$ such that for all $\Theta \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ we have

$$\|\partial_{z_1}\Theta\|_{L^p(\mathbb{R}^N)} \leq C\|\partial_{z_1}\Theta\|_{L^2(\mathbb{R}^N)}^{1-\frac{(2N-1)(p-2)}{2p}} \|\partial_{z_1}^2\Theta\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-2)}{2p}} \|\nabla_{z_\perp}\Theta\|_{L^2(\mathbb{R}^N)}^{\frac{(N-1)(p-2)}{2p}}. \tag{23}$$

This shows that the energy $\mathscr{E}$ is well-defined on $\mathscr{Y}(\mathbb{R}^N)$ if $N = 2$ or $N = 3$. By (23) and the density of $\partial_{z_1} C_c^\infty(\mathbb{R}^3)$ in $\mathscr{Y}(\mathbb{R}^3)$ we get for any $w \in \mathscr{Y}(\mathbb{R}^3)$:

$$\|w\|_{L^3(\mathbb{R}^3)} \leq C\|w\|_{L^2(\mathbb{R}^3)}^{\frac{1}{6}}\|\partial_{z_1} w\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla_{z_\perp}\partial_{z_1}^{-1} w\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}}. \tag{24}$$

On the other hand, the following identities hold for any solution $\mathcal{W} \in \mathscr{Y}(\mathbb{R}^N)$ of (SW):

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{1}{\mathfrak{c}_s^2} (\partial_{z_1}\mathcal{W})^2 + |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 + \frac{\Gamma}{2}\mathcal{W}^3 + \frac{1}{\mathfrak{c}_s^2}\mathcal{W}^2 \, dz = 0 \\ \displaystyle\int_{\mathbb{R}^N} \frac{-1}{\mathfrak{c}_s^2} (\partial_{z_1}\mathcal{W})^2 + 3|\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 + \frac{\Gamma}{3}\mathcal{W}^3 + \frac{1}{\mathfrak{c}_s^2}\mathcal{W}^2 \, dz = 0 \\ \displaystyle\int_{\mathbb{R}^N} \frac{1}{\mathfrak{c}_s^2} (\partial_{z_1}\mathcal{W})^2 + |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 + \frac{\Gamma}{3}\mathcal{W}^3 + \frac{1}{\mathfrak{c}_s^2}\mathcal{W}^2 \, dz = \frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 \, dz. \end{cases} \tag{25}$$

The first identity is obtained by multiplying (SW) by $\partial_{z_1}^{-1}\mathcal{W}$ and integrating, whereas the two other equalities are the Pohozaev identities associated to the scalings in the z_1 and $z_\perp$ variables respectively. Formally, they are obtained by multiplying (SW) by $z_1\mathcal{W}$ and $z_\perp \cdot \nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}$ respectively and integrating by parts (see [20] for a complete justification). Combining the equalities in (25) we get

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{1}{\mathfrak{c}_s^2} (\partial_{z_1}\mathcal{W})^2 \, dz = \frac{N}{N-1}\int_{\mathbb{R}^N} |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 \, dz \\ \displaystyle\frac{\Gamma}{6}\int_{\mathbb{R}^N} \mathcal{W}^3 \, dz = -\frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 \, dz \\ \displaystyle\int_{\mathbb{R}^N} \frac{1}{\mathfrak{c}_s^2}\mathcal{W}^2 \, dz = \frac{7-2N}{N-1}\int_{\mathbb{R}^N} |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}|^2 \, dz. \end{cases} \tag{26}$$

Notice that for $N \geq 4$ we have $7 - 2N < 0$ and the last equality implies $\mathcal{W} = 0$.

We recall the following results about the ground states of (SW) and the compactness of minimizing sequences in $\mathscr{Y}(\mathbb{R}^3)$.

Lemma 20 ([20], [21]) *Let $N = 3$ and $\Gamma \neq 0$.*

(i) For $\lambda \in \mathbb{R}^*$, denote $I_\lambda = \inf\left\{ \|w\|^2_{\mathscr{Y}(\mathbb{R}^3)} \mid \int_{\mathbb{R}^3} w^3(z)\, dz = \lambda \right\}$. Then for any $\lambda \in \mathbb{R}^*$ we have $I_\lambda > 0$ and there is $w_\lambda \in \mathscr{Y}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} w_\lambda^3(z)\, dz = \lambda$ and $\|w_\lambda\|^2_{\mathscr{Y}(\mathbb{R}^3)} = I_\lambda$. Moreover, any sequence $(w_n)_{n\geq 1} \subset \mathscr{Y}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} w_n^3(z)\, dz \to \lambda$ and $\|w_n\|^2_{\mathscr{Y}(\mathbb{R}^3)} \to I_\lambda$ has a subsequence that converges in $\mathscr{Y}(\mathbb{R}^3)$ (up to translations) to a minimizer of I_λ.

(ii) There is $\lambda^* \in \mathbb{R}^*$ such that $w^* \in \mathscr{Y}(\mathbb{R}^3)$ is a ground state for (SW) *(that is, minimizes the action $\mathscr{S}$ among all solutions of* (SW)*) if and only if w^* is a minimizer of I_{λ^*}.*

The first part of Lemma 20 is a consequence of the proof of Theorem 3.2 p. 217 in [20] and the second part follows from Lemma 2.1 p. 1067 in [21].

Proof of Theorem 5. Given $w \in \mathscr{Y}(\mathbb{R}^3)$ and $\sigma > 0$, we denote $P(w) = \int_{\mathbb{R}^3} \frac{1}{\mathfrak{c}_s^2} w^2 + \frac{1}{\mathfrak{c}_s^2}|\partial_{z_1} w|^2 + \frac{\Gamma}{3} w^3 \, dz$ and $w_\sigma(z) = w(z_1, \frac{z_\perp}{\sigma})$. It is obvious that

$$\int_{\mathbb{R}^3} w_\sigma^p \, dz = \sigma^2 \int_{\mathbb{R}^3} w^p \, dz, \qquad \int_{\mathbb{R}^3} |\partial_{z_1}(w_\sigma)|^2 \, dz = \sigma^2 \int_{\mathbb{R}^3} |\partial_{z_1} w|^2 \, dz \qquad \text{and}$$

$$\int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1}(w_\sigma)|^2 \, dz = \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1}(w)|^2 w|^2 \, dz.$$

Let w^* be a ground state for (SW) (the existence of w^* is guaranteed by Lemma 20 above). Since w^* satisfies (25), we have $P(w^*) = 0$ and $\mathscr{S}(w^*) = \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1}(w^*)|^2 w|^2 \, dz$. Consider $w \in \mathscr{Y}(\mathbb{R}^3)$ such that $w \neq 0$ and $P(w) = 0$. Then $\frac{\Gamma}{3}\int_{\mathbb{R}^3} w^3 \, dz = -\frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3} w^2 + |\partial_{z_1} w|^2 \, dz < 0$ and it is easy to see that there is

$\sigma > 0$ such that $\displaystyle\int_{\mathbb{R}^3} w_\sigma^3\,dz = \int_{\mathbb{R}^3} (w^*)^3\,dz = \lambda^*$. From Lemma 20 it follows that $\|w_\sigma\|^2_{\mathscr{Y}(\mathbb{R}^3)} \geq \|w^*\|^2_{\mathscr{Y}(\mathbb{R}^3)}$, that is

$$\frac{\sigma^2}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3} w^2 + |\partial_{z_1} w|^2\,dz + \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1} w|^2\,dz \geq \frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3} (w^*)^2 + |\partial_{z_1} w^*|^2\,dz + \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1} w^*|^2\,dz.$$

Since $P(w) = 0$ and $P(w^*) = 0$ we have

$$\frac{\sigma^2}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3} w^2 + |\partial_{z_1} w|^2\,dz = -\sigma^2\frac{\Gamma}{3}\int_{\mathbb{R}^3} w^3\,dz = -\frac{\Gamma}{3}\int_{\mathbb{R}^3} (w^*)^3\,dz = \frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3} (w^*)^2 + |\partial_{z_1} w^*|^2\,dz$$

and the previous inequality gives $\displaystyle\int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1} w|^2\,dz \geq \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1} w^*|^2\,dz$, that is $\mathscr{S}(w) \geq \mathscr{S}(w^*)$. So far we have proved that the set $\mathcal{P} = \{w \in \mathscr{Y}(\mathbb{R}^3) \mid w \neq 0,\ P(w) = 0\}$ is not empty and any ground state w^* of (SW) minimizes the action $\mathscr{S}$ in this set. It is then clear that for any $\sigma > 0$, w^*_σ also belongs to $\mathcal{P}$ and mnimizes $\mathscr{S}$ on $\mathcal{P}$.

Conversely, let $w \in \mathcal{P}$ be such that $\mathscr{S}(w) = \mathscr{S}_*$. Let w^* be a ground state for (SW). It is clear that $\displaystyle\int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1} w|^2\,dz = \mathscr{S}_* = \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1} w^*|^2\,dz$. As above, there is a unique $\sigma > 0$ such that $\displaystyle\int_{\mathbb{R}^3} w_\sigma^3\,dz = \int_{\mathbb{R}^3} (w^*)^3\,dz = \lambda^*$ and then we have $\displaystyle\int_{\mathbb{R}^3} w_\sigma^2 + |\partial_{z_1} w_\sigma|^2\,dz = \int_{\mathbb{R}^3} (w^*)^2 + |\partial_{z_1} w^*|^2\,dz$. We find $\|w_\sigma\|^2_{\mathscr{Y}(\mathbb{R}^3)} = \|w^*\|^2_{\mathscr{Y}(\mathbb{R}^3)} = I_{\lambda^*}$, thus w_σ is a minimizer for I_{λ^*} and Lemma 20 (ii) implies that w_σ is a ground state for (SW).

Let $(\mathcal{W}_n)_{n\geq 1}$ be a sequence satisfying (i), (ii) and (iii). We have $P(\mathcal{W}_n) \to 0$ and

$$\frac{\Gamma}{3}\int_{\mathbb{R}^3} \mathcal{W}_n^3\,dz = P(\mathcal{W}_n) - \frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3} \mathcal{W}_n^2 + |\partial_{z_1}\mathcal{W}_n|^2\,dz \in \left[\frac{-2m_2}{\mathfrak{c}_s^2}, -\frac{m_1}{2\mathfrak{c}_s^2}\right] \qquad \text{for all } n \text{ sufficiently large.}$$

We infer that there are $n_0 \in \mathbb{N}$, $\underline{\sigma}, \bar{\sigma} > 0$ and a sequence $(\sigma_n)_{n\geq n_0} \subset [\underline{\sigma}, \bar{\sigma}]$ such that $\displaystyle\int_{\mathbb{R}^3} ((\mathcal{W}_n)_{\sigma_n})^3\,dz = \lambda^*$ for all $n \geq n_0$. Moreover,

$$\begin{aligned}
\|(\mathcal{W}_n)_{\sigma_n}\|^2_{\mathscr{Y}(\mathbb{R}^3)} &= \frac{\sigma_n^2}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3} \mathcal{W}_n^2 + |\partial_{z_1}\mathcal{W}_n|^2\,dz + \int_{\mathbb{R}^3} |\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}_n|^2\,dz \\
&= \sigma_n^2\left(P(\mathcal{W}_n) - \frac{\Gamma}{3}\int_{\mathbb{R}^3} \mathcal{W}_n^3\right) + (\mathscr{S}(\mathcal{W}_n) - P(\mathcal{W}_n)) \\
&= (\sigma_n^2 - 1)P(\mathcal{W}_n) + \mathscr{S}(\mathcal{W}_n) - \frac{\Gamma}{3}\int_{\mathbb{R}^3} (\mathcal{W}_n)^3_{\sigma_n}\,dz.
\end{aligned}$$

Passing to the limit in the above equality we get

$$\liminf_{n\to\infty} \|(\mathcal{W}_n)_{\sigma_n}\|^2_{\mathscr{Y}(\mathbb{R}^3)} = \liminf_{n\to\infty} \mathscr{S}(\mathcal{W}_n) - \frac{\Gamma}{3}\lambda^* \leq \mathscr{S}_* - \frac{\Gamma}{3}\lambda^* = \mathscr{S}(w^*) - \frac{\Gamma}{3}\int_{\mathbb{R}^3} (w^*)^3\,dz = \|w^*\|^2_{\mathscr{Y}(\mathbb{R}^3)} = I_{\lambda^*}.$$

Hence there is a subsequence of $((\mathcal{W}_n)_{\sigma_n})_{n\geq 1}$ which is a minimizing sequence for I_{λ^*}. Using Lemma 20 we infer that there exist a subsequence $(n_j)_{j\geq 1}$ such that $\sigma_{n_j} \to \sigma \in [\underline{\sigma}, \bar{\sigma}]$, a sequence $(z_j)_{j\geq 1} \subset \mathbb{R}^3$ and a minimizer $\mathcal{W}$ of I_{λ^*} (hence a ground state for (SW)) such that $(\mathcal{W}_{n_j})_{\sigma_{n_j}}(\cdot - z_j) \to \mathcal{W}$ in $\mathscr{Y}(\mathbb{R}^3)$. It is then straightforward that $\mathcal{W}_{n_j}(\cdot - z_j) \to \mathcal{W}_{\frac{1}{\sigma}}$ in $\mathscr{Y}(\mathbb{R}^3)$. □

We may give an alternate proof of Theorem 5 which does not rely directly on the analysis in [20], [21] by following the strategy of [36], which can be adapted to our problem up to some details.

3 Proof of Theorem 6

3.1 Proof of Proposition 9

For some given real valued functions A_ε and φ_ε, we consider the mapping

$$U_\varepsilon(x) = |U_\varepsilon|(x)\mathrm{e}^{i\phi(x)} = r_0\Big(1 + \varepsilon^2 A_\varepsilon(z)\Big)\mathrm{e}^{i\varepsilon\varphi_\varepsilon(z)}, \qquad \text{where} \quad z = (z_1, z_\perp) = (\varepsilon x_1, \varepsilon^2 x_\perp).$$

It is obvious that $U_\varepsilon \in \mathcal{E}$ provided that $A_\varepsilon \in H^1(\mathbb{R}^N)$ and $\nabla\varphi_\varepsilon \in L^2(\mathbb{R}^N)$. If ε is small and A_ε is uniformly bounded in $\mathbb{R}^N$, U_ε does not vanish and the momentum $Q(U_\varepsilon)$ is given by

$$Q(U_\varepsilon) = -\int_{\mathbb{R}^N} (|U_\varepsilon|^2 - r_0^2)\frac{\partial\phi}{\partial x_1}\ dx = -\varepsilon^{5-2N} r_0^2 \int_{\mathbb{R}^N} \Big(2A_\varepsilon + \varepsilon^2 A_\varepsilon^2\Big)\frac{\partial\varphi_\varepsilon}{\partial z_1}\ dz,$$

while the energy of U_ε is

$$\begin{aligned} E(U_\varepsilon) = & \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 + V(|U_\varepsilon|^2)\ dx \\ = & \ \varepsilon^{5-2N} r_0^2 \int_{\mathbb{R}^N} (\partial_{z_1}\varphi_\varepsilon)^2\Big(1+\varepsilon^2 A_\varepsilon\Big)^2 + \varepsilon^2|\nabla_{z_\perp}\varphi_\varepsilon|^2\Big(1+\varepsilon^2 A_\varepsilon\Big)^2 + \varepsilon^2(\partial_{z_1}A_\varepsilon)^2 + \varepsilon^4|\nabla_{z_\perp}A_\varepsilon|^2 \\ & + \mathfrak{c}_s^2 A_\varepsilon^2 + \varepsilon^2\mathfrak{c}_s^2\Big(1 - \frac{4r_0^4}{3\mathfrak{c}_s^2}F''(r_0^2)\Big)A_\varepsilon^3 + \frac{\mathfrak{c}_s^2}{\varepsilon^4}V_4\Big(\varepsilon^2 A_\varepsilon\Big)\ dz, \end{aligned}$$

where we have used the Taylor expansion

$$V\Big(r_0^2(1+\alpha)^2\Big) = r_0^2\Big\{\mathfrak{c}_s^2\alpha^2 + \mathfrak{c}_s^2\Big(1 - \frac{4r_0^4}{3\mathfrak{c}_s^2}F''(r_0^2)\Big)\alpha^3 + \mathfrak{c}_s^2 V_4(\alpha)\Big\} = r_0^2\mathfrak{c}_s^2\Big\{\alpha^2 + \Big(\frac{\Gamma}{3} - 1\Big)\alpha^3 + V_4(\alpha)\Big\} \qquad (27)$$

with $V_4(\alpha) = \mathcal{O}(\alpha^4)$ as $\alpha \to 0$. Consequently, with $\mathfrak{c}_s^2 = c^2(\varepsilon) + \varepsilon^2$ we get

$$\begin{aligned} E_{c(\varepsilon)}(U_\varepsilon) = & \ E(U_\varepsilon) + c(\varepsilon)Q(U_\varepsilon) \\ = & \ \varepsilon^{5-2N} r_0^2 \int_{\mathbb{R}^N} (\partial_{z_1}\varphi_\varepsilon)^2\Big(1+\varepsilon^2 A_\varepsilon\Big)^2 + \varepsilon^2|\nabla_{z_\perp}\varphi_\varepsilon|^2\Big(1+\varepsilon^2 A_\varepsilon\Big)^2 + \varepsilon^2(\partial_{z_1}A_\varepsilon)^2 + \varepsilon^4|\nabla_{z_\perp}A_\varepsilon|^2 \\ & + \mathfrak{c}_s^2 A_\varepsilon^2 + \varepsilon^2\mathfrak{c}_s^2\Big(1 - \frac{4r_0^4}{3\mathfrak{c}_s^2}F''(r_0^2)\Big)A_\varepsilon^3 + \frac{\mathfrak{c}_s^2}{\varepsilon^4}V_4\Big(\varepsilon^2 A_\varepsilon\Big) - c(\varepsilon)\Big(2A_\varepsilon + \varepsilon^2 A_\varepsilon^2\Big)\partial_{z_1}\varphi_\varepsilon\ dz \\ = & \ \varepsilon^{7-2N} r_0^2 \int_{\mathbb{R}^N} \frac{1}{\varepsilon^2}\Big(\partial_{z_1}\varphi_\varepsilon - c(\varepsilon)A_\varepsilon\Big)^2 + (\partial_{z_1}\varphi_\varepsilon)^2(2A_\varepsilon + \varepsilon^2 A_\varepsilon^2) + |\nabla_{z_\perp}\varphi_\varepsilon|^2(1+\varepsilon^2 A_\varepsilon)^2 + (\partial_{z_1}A_\varepsilon)^2 \\ & + \varepsilon^2|\nabla_{z_\perp}A_\varepsilon|^2 + A_\varepsilon^2 + \mathfrak{c}_s^2\Big(1 - \frac{4r_0^4}{3\mathfrak{c}_s^2}F''(r_0^2)\Big)A_\varepsilon^3 + \frac{\mathfrak{c}_s^2}{\varepsilon^6}V_4(\varepsilon^2 A_\varepsilon) - c(\varepsilon)A_\varepsilon^2\partial_{z_1}\varphi_\varepsilon\ dz. \end{aligned} \qquad (28)$$

Since the first term in the last integral is penalised by ε^{-2}, in order to get sharp estimates on $E_{c(\varepsilon)}$ one needs $\partial_{z_1}\varphi_\varepsilon \simeq c(\varepsilon)A_\varepsilon$.

Let $N = 3$. By Theorem 5, there exists a ground state $A \in \mathscr{Y}(\mathbb{R}^3)$ for (SW). It follows from Theorem 4.1 p. 227 in [21] that $A \in H^s(\mathbb{R}^3)$ for any $s \in \mathbb{N}$. Let $\varphi = \mathfrak{c}_s\partial_{z_1}^{-1}A$. We use (28) with $A_\varepsilon(z) = \frac{\lambda\mathfrak{c}_s}{c(\varepsilon)}A(\lambda z_1, z_\perp)$ and $\varphi_\varepsilon(z) = \varphi(\lambda z_1, z_\perp)$. For $\varepsilon > 0$ small and $\lambda \simeq 1$ (to be chosen later) we define

$$U_\varepsilon(x) = |U_\varepsilon|(x)\mathrm{e}^{i\phi_\varepsilon(x)} = r_0\Big(1 + \varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A(z)\Big)\mathrm{e}^{i\varepsilon\varphi(z)}, \qquad \text{where} \qquad z = (z_1, z_\perp) = (\varepsilon\lambda x_1, \varepsilon^2 x_\perp).$$

Notice that U_ε does not vanish if ε is sufficiently small. Since $\partial_{z_1}\varphi = \mathfrak{c}_s A$, we have $\partial_{z_1}\varphi_\varepsilon(z) = \lambda\partial_{z_1}\varphi(\lambda z_1, z_\perp) = \lambda\mathfrak{c}_s A(\lambda z_1, z_\perp) = c(\varepsilon)A_\varepsilon(z)$ and therefore

$$\begin{aligned} \lambda E_{c(\varepsilon)}(U_\varepsilon) = & \ \mathfrak{c}_s^2 r_0^2\varepsilon \int_{\mathbb{R}^3} \lambda^3\frac{\mathfrak{c}_s}{c(\varepsilon)}A^2\Big(2A + \varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A^2\Big) + \lambda^2|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2\Big(1 + \varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A\Big)^2 + \frac{\lambda^4}{c^2(\varepsilon)}(\partial_{z_1}A)^2 \\ & + \varepsilon^2\frac{\lambda^2}{c^2(\varepsilon)}|\nabla_{z_\perp}A|^2 + \frac{\lambda^2}{c^2(\varepsilon)}A^2 + \frac{\mathfrak{c}_s^3}{c^3(\varepsilon)}\lambda^3\Big(1 - \frac{4r_0^4}{3\mathfrak{c}_s^2}F''(r_0^2)\Big)A^3 + \frac{1}{\varepsilon^6}V_4\Big(\varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A\Big) \\ & - \lambda^3\frac{\mathfrak{c}_s}{c(\varepsilon)}A^3\ dz \\ = & \ \mathfrak{c}_s^2 r_0^2\varepsilon \int_{\mathbb{R}^3} \lambda^3\frac{\mathfrak{c}_s}{c(\varepsilon)}\Big(1 + \frac{\mathfrak{c}_s^2}{c^2(\varepsilon)}\Big[\frac{\Gamma}{3} - 1\Big]\Big)A^3 + \lambda^2|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2\Big(1 + \varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A\Big)^2 + \frac{\lambda^4}{c^2(\varepsilon)}(\partial_{z_1}A)^2 \\ & + \frac{\lambda^2}{c^2(\varepsilon)}A^2 + \varepsilon^2\frac{\lambda^2}{c^2(\varepsilon)}|\nabla_{z_\perp}A|^2 + \varepsilon^2\lambda^4\frac{\mathfrak{c}_s^2}{c^2(\varepsilon)}A^4 + \frac{1}{\varepsilon^6}V_4\Big(\varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A\Big)\ dz. \end{aligned}$$

On the other hand,

$$\lambda\int_{\mathbb{R}^3}|\nabla_\perp U_\varepsilon|^2\,dx = r_0^2\varepsilon\int_{\mathbb{R}^3}|\nabla_{z_\perp}\varphi|^2\Big(1+\varepsilon^2\lambda\frac{\mathfrak{c}_s}{c(\varepsilon)}A\Big)^2+\varepsilon^2\lambda^2\frac{\mathfrak{c}_s^2}{c^2(\varepsilon)}|\nabla_{z_\perp}A|^2\,dz$$
$$=\mathfrak{c}_s^2r_0^2\varepsilon\int_{\mathbb{R}^3}|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2\Big(1+\varepsilon^2\lambda\frac{\mathfrak{c}_s}{c(\varepsilon)}A\Big)^2+\varepsilon^2\frac{\lambda^2}{c^2(\varepsilon)}|\nabla_{z_\perp}A|^2\,dz.$$

Hence U_ε satisfies the constraint $P_{c(\varepsilon)}(U_\varepsilon)=0$ (or equivalently $E_{c(\varepsilon)}(U_\varepsilon)=\int_{\mathbb{R}^3}|\nabla_\perp U_\varepsilon|^2\,dx$) if and only if $G(\lambda,\varepsilon^2)=0$, where

$$G(\lambda,\varepsilon^2)=\int_{\mathbb{R}^3}\lambda^3\frac{\mathfrak{c}_s}{c(\varepsilon)}\Big(1+\frac{\mathfrak{c}_s^2}{c^2(\varepsilon)}\Big[\frac{\Gamma}{3}-1\Big]\Big)A^3+\lambda^2|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2\Big(1+\varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A\Big)^2+\frac{\lambda^4}{c^2(\varepsilon)}(\partial_{z_1}A)^2$$
$$+\frac{\lambda^2}{c^2(\varepsilon)}A^2+\varepsilon^2\frac{\lambda^2}{c^2(\varepsilon)}|\nabla_{z_\perp}A|^2+\varepsilon^2\lambda^4\frac{\mathfrak{c}_s^2}{c^2(\varepsilon)}A^4+\frac{1}{\varepsilon^6}V_4\Big(\varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}\lambda A\Big)\,dz$$
$$-\int_{\mathbb{R}^3}|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2\Big(1+\varepsilon^2\lambda\frac{\mathfrak{c}_s}{c(\varepsilon)}A\Big)^2+\varepsilon^2\frac{\lambda^2}{c^2(\varepsilon)}|\nabla_{z_\perp}A|^2\,dz.$$

Denote $\epsilon=\varepsilon^2$. Since A is a ground state for (SW), it satisfies the Pohozaev identities (25). The last of these identities is $\mathscr{S}(A)=\int_{\mathbb{R}^3}|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2\,dz$, or equivalently

$$G(\lambda=1,\epsilon=0)=0.$$

A straightforward computation using (26) gives

$$\frac{\partial G}{\partial\lambda}_{|(\lambda=1,\epsilon=0)}=\int_{\mathbb{R}^3}\Gamma A^3+2|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2+\frac{4}{\mathfrak{c}_s^2}(\partial_{z_1}A)^2+\frac{2}{\mathfrak{c}_s^2}A^2\,dz=3\int_{\mathbb{R}^3}|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2\neq0.$$

Then the implicit function theorem implies that there exists a function $\epsilon\longmapsto\lambda(\epsilon)=1+\mathcal{O}(\epsilon)=1+\mathcal{O}(\varepsilon^2)$ such that for all ϵ sufficiently small we have $G(\lambda(\epsilon),\epsilon)=0$, that is $U_{c(\varepsilon)}$ satisfies the Pohozaev identity $P_{c(\varepsilon)}(U_\varepsilon)=0$. Choosing $\lambda=\lambda(\varepsilon^2)$ and taking into account the last indetity in (25), we find

$$T_{c(\varepsilon)}\le E_{c(\varepsilon)}(U_\varepsilon)=\int_{\mathbb{R}^3}|\nabla_\perp U_\varepsilon|^2\,dx=\mathfrak{c}_s^2r_0^2\varepsilon\int_{\mathbb{R}^3}|\nabla_{z_\perp}\partial_{z_1}^{-1}A|^2+\mathcal{O}(\varepsilon^3)=\mathfrak{c}_s^2r_0^2\varepsilon\mathscr{S}_{\min}+\mathcal{O}(\varepsilon^3)$$

and the proof of (ii) is complete.

Next we turn our attention to the case $N=2$. Let $A=\mathfrak{c}_s^{-1}\partial_{z_1}\varphi\in\mathscr{Y}(\mathbb{R}^2)$ be a ground state of (SW). The existence of A is given by Theorem 4. By Theorem 4.1 p. 227 in [20] we have $A\in H^s(\mathbb{R}^2)$ for all $s\in\mathbb{N}$. For ε small, we define the map

$$U_\varepsilon(x)=|U_\varepsilon|(x)\mathsf{e}^{i\phi_\varepsilon(x)}=r_0\Big(1+\varepsilon^2\frac{\mathfrak{c}_s}{c(\varepsilon)}A(z)\Big)\mathsf{e}^{i\varepsilon\varphi(z)},\qquad\text{where}\qquad z=(z_1,z_2)=(\varepsilon x_1,\varepsilon^2x_2).$$

From the above computations and (26) we have

$$k_\varepsilon=\int_{\mathbb{R}^2}|\nabla U_\varepsilon|^2\,dx=r_0^2\varepsilon\int_{\mathbb{R}^2}(\partial_{z_1}\varphi_\varepsilon)^2\Big(1+\varepsilon^2A_\varepsilon\Big)^2+\varepsilon^2(\partial_{z_1}A_\varepsilon)^2+\varepsilon^2(\partial_{z_2}\varphi_\varepsilon)^2\Big(1+\varepsilon^2A_\varepsilon\Big)^2+\varepsilon^4(\partial_{z_2}A_\varepsilon)^2\,dz$$
$$=r_0^2\mathfrak{c}_s^2\varepsilon\int_{\mathbb{R}^2}A^2\Big(1+\frac{\varepsilon^2\mathfrak{c}_s}{c(\varepsilon)}A\Big)^2+\frac{\varepsilon^2}{c^2(\varepsilon)}(\partial_{z_1}A)^2+\varepsilon^2(\partial_{z_2}\partial_{z_1}^{-1}A)^2\Big(1+\frac{\varepsilon^2\mathfrak{c}_s}{c(\varepsilon)}A\Big)^2+\frac{\varepsilon^4}{c^2(\varepsilon)}(\partial_{z_2}A)^2\,dz$$
$$=r_0^2\mathfrak{c}_s^2\Big\{\varepsilon\int_{\mathbb{R}^2}A^2\,dz+\varepsilon^3\int_{\mathbb{R}^2}\Big(2A^3+\frac{(\partial_{z_1}A)^2}{\mathfrak{c}_s^2}+(\partial_{z_2}\partial_{z_1}^{-1}A)^2\Big)\,dz+\mathcal{O}(\varepsilon^5)\Big\}$$
$$=r_0^2\mathfrak{c}_s^2\Big\{\varepsilon\frac{3}{2}\mathfrak{c}_s^2\mathscr{S}(A)+\varepsilon^3\Big(2-\frac{12}{\Gamma}-\frac{1}{2}\Big)\mathscr{S}(A)+\mathcal{O}(\varepsilon^5)\Big\}$$

It is easy to see that $\varepsilon\mapsto k_\varepsilon$ is a smooth increasing diffeomorphism from an interval $[0,\bar\varepsilon]$ onto an interval $[0,\bar k=\bar k_{\bar\varepsilon}]$, and that $\varepsilon=\frac{k_\varepsilon}{r_0^2\mathfrak{c}_s^2\|A\|_{L^2}^2}+\mathcal{O}(k_\varepsilon^3)=\frac{k_\varepsilon}{\frac{3}{2}r_0^2\mathfrak{c}_s^4\mathscr{S}(A)}+\mathcal{O}(k_\varepsilon^3)$ as $\varepsilon\to0$. Moreover, denoting $U_\varepsilon^\sigma(x)=U_\varepsilon(x/\sigma)$ we have

$$\int_{\mathbb{R}^2}|\nabla U_\varepsilon^\sigma|^2\,dx=\int_{\mathbb{R}^2}|\nabla U_\varepsilon|^2\,dx$$

because $N=2$. Using the test function U_ε^σ, it follows that

$$I_{\min}(k_\varepsilon) \le I(U_\varepsilon^\sigma) \qquad \text{for any } \sigma > 0.$$

Since $Q(U_\varepsilon) < 0$, the mapping

$$\sigma \longmapsto I(U_\varepsilon^\sigma) = Q(U_\varepsilon^\sigma) + \int_{\mathbb{R}^2} V(|U_\varepsilon^\sigma|^2)\, dx = \sigma Q(U_\varepsilon) + \sigma^2 \int_{\mathbb{R}^2} V(|U_\varepsilon|^2)\, dx$$

achieves its minimum at $\sigma_0 = \dfrac{-Q(U_\varepsilon)}{2\displaystyle\int_{\mathbb{R}^2} V(|U_\varepsilon|^2)} > 0$, and the minimum value is $I(U_\varepsilon^{\sigma_0}) = \dfrac{-Q^2(U_\varepsilon)}{4\displaystyle\int_{\mathbb{R}^2} V(|U_\varepsilon|^2)\, dx}$.

Hence

$$I_{\min}(k_\varepsilon) \le I(U_\varepsilon^{\sigma_0}) = \frac{-Q^2(U_\varepsilon)}{4\displaystyle\int_{\mathbb{R}^2} V(|U_\varepsilon|^2)\, dx}.$$

Using (27) and (26) we find

$$\begin{aligned}\int_{\mathbb{R}^2} V(|U_\varepsilon|^2)\, dx &= \mathfrak{c}_s^2 r_0^2 \varepsilon \int_{\mathbb{R}^2} A^2 + \varepsilon^2\Big(\frac{\Gamma}{3}-1\Big)A^3 + \frac{1}{\varepsilon^4}V_4(\varepsilon^2 A)\, dz \\ &= \frac{3}{2}\mathfrak{c}_s^4 r_0^2 \mathscr{S}(A)\varepsilon - \mathfrak{c}_s^2 r_0^2\Big(\frac{\Gamma}{3}-1\Big)\frac{6}{\Gamma}\mathscr{S}(A)\varepsilon^3 + \mathcal{O}(\varepsilon^5)\end{aligned}$$

and

$$Q(U_\varepsilon) = -\varepsilon r_0^2 \mathfrak{c}_s \int_{\mathbb{R}^2} \Big(2A^2 + \varepsilon^2 A^3\Big)\, dz = -3 r_0^2 \mathfrak{c}_s^3 \mathscr{S}(A)\varepsilon + r_0^2 \mathfrak{c}_s \frac{6}{\Gamma}\mathscr{S}(A)\varepsilon^3.$$

Finally we obtain

$$\begin{aligned}
&I_{\min}(k_\varepsilon) + \frac{k_\varepsilon}{\mathfrak{c}_s^2} \le \frac{-Q^2(U_\varepsilon)}{4\displaystyle\int_{\mathbb{R}^2} V(|U_\varepsilon|^2)\, dx} + \frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}^2} |\nabla U_\varepsilon|^2\, dx \\
&= -\frac{\big(-3\mathfrak{c}_s^2 + \frac{6}{\Gamma}\varepsilon^2\big)^2 r_0^4 \mathfrak{c}_s^2 \mathscr{S}^2(A)\varepsilon^2}{4\Big[\frac{3}{2}\mathfrak{c}_s^2 - \Big(2 - \frac{6}{\Gamma}\Big)\varepsilon^2 + \mathcal{O}(\varepsilon^4)\Big] r_0^2 \mathfrak{c}_s^2 \mathscr{S}(A)\varepsilon} + \left[\frac{3}{2} r_0^2 \mathfrak{c}_s^2 \varepsilon + r_0^2\Big(\frac{3}{2} - \frac{12}{\Gamma}\Big)\varepsilon^3 + \mathcal{O}(\varepsilon^5)\right]\mathscr{S}(A) \\
&= -\frac{\big(3 r_0^2 \mathfrak{c}_s^2 \varepsilon^3 + \mathcal{O}(\varepsilon^5)\big)\mathscr{S}(A)}{2\left[3\mathfrak{c}_s^2 - \big(4 - \frac{12}{\Gamma}\big)\varepsilon^2 + \mathcal{O}(\varepsilon^4)\right]} = -\frac{1}{2} r_0^2 \mathscr{S}(A)\varepsilon^3 + \mathcal{O}(\varepsilon^5) \\
&= -\frac{1}{2} r_0^2 \mathscr{S}(A)\left[\frac{k_\varepsilon}{\frac{3}{2} r_0^2 \mathfrak{c}_s^4 \mathscr{S}(A)} + \mathcal{O}(k_\varepsilon^3)\right]^3 + \mathcal{O}\left(\left(\frac{k_\varepsilon}{\frac{3}{2} r_0^2 \mathfrak{c}_s^4 \mathscr{S}(A)} + \mathcal{O}(k_\varepsilon^3)\right)^5\right) = \frac{-4k_\varepsilon^3}{27 r_0^4 \mathfrak{c}_s{}^{12} \mathscr{S}_{\min}^2} + \mathcal{O}(k_\varepsilon^5).
\end{aligned}$$

Since $\varepsilon \longmapsto k_\varepsilon$ is a diffeomorphism from $[0, \bar{\varepsilon}]$ onto $[0, \bar{k}]$, Proposition 9 (i) is proven. □

3.2 Proof of Proposition 10

Given a function f defined on $\mathbb{R}^N$ and $a, b > 0$, we denote $f_{a,b}(x) = f(\frac{x_1}{a}, \frac{x_\perp}{b})$.

By Proposition 2.2 p. 1078 in [34], any solution of (TW$_c$) belongs to $W^{2,p}_{loc}(\mathbb{R}^N)$ for all $p \in [2, \infty)$, hence to $C^{1,\alpha}(\mathbb{R}^N)$ for all $\alpha \in (0,1)$.

(*i*) Let U be a minimizer of $E_c = E + cQ$ on $\mathscr{C}_c$ (where $\mathscr{C}_c$ is as in (5)) such that ψ solves (TW$_c$). Then U satisfies the Pohozaev identities (4).

If $Q(U) > 0$, let $\tilde{U}(x) = U(-x_1, x_\perp)$, so that $Q(\tilde{U}) = -Q(U) < 0$ and $P_c(\tilde{U}) = P_c(U) - 2cQ(U) = -2cQ(U) < 0$. Since for any function $\phi \in \mathcal{E}$ we have

$$P_c(\phi_{a,1}) = \frac{1}{a}\int_{\mathbb{R}^N} \Big|\frac{\partial \phi}{\partial x_1}\Big|^2 dx + a\frac{N-3}{N-1}\int_{\mathbb{R}^N} |\nabla_{x_\perp}\phi|^2\, dx + cQ(\phi) + a\int_{\mathbb{R}^N} V(|\phi|^2)\, dx, \tag{29}$$

we see that there is $a_0 \in (0,1)$ such that $P_c(\tilde{U}_{a_0,1}) = 0$. We infer that

$$T_c \le E_c(\tilde{U}_{a_0,1}) = \frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_{x_\perp}\tilde{U}_{a_0,1}|^2\,dx = a_0\frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_{x_\perp}U|^2\,dx = a_0E_c(U) = a_0T_c,$$

contradicting the fact that $T_c > 0$. Thus $Q(U) \le 0$.

Assume that $Q(U) = 0$. From the identities (4) with $Q(U) = 0$ we get

$$\int_{\mathbb{R}^N}\Big|\frac{\partial U}{\partial x_1}\Big|^2\,dx = -\frac{1}{N-2}\int_{\mathbb{R}^N} V(|U|^2)\,dx \qquad \text{and} \qquad \int_{\mathbb{R}^N}|\nabla_{x_\perp}U|^2\,dx = -\frac{N-1}{N-2}\int_{\mathbb{R}^N} V(|U|^2)\,dx. \quad (30)$$

Since $U \in \mathcal{E}$ and U is not constant, necessarily $\int_{\mathbb{R}^N} V(|U|^2)\,dx = -(N-2)\int_{\mathbb{R}^N}\Big|\frac{\partial U}{\partial x_1}\Big|^2\,dx < 0$ and this implies that the potential V must achieve negative values. Then it follows from Theorem 2.1 p. 100 in [15] that there is $\tilde{\psi}_0 \in \mathcal{E}$ such that $\int_{\mathbb{R}^N}|\nabla\tilde{\psi}_0|^2\,dx = \inf\Big\{\int_{\mathbb{R}^N}|\nabla\phi|^2\,dx \;\Big|\; \phi \in \mathcal{E},\ \int_{\mathbb{R}^N}V(|\phi|^2)\,dx = -1\Big\}$. Using Theorem 2.2 p. 102 in [15] we see that there is $\sigma > 0$ such that, denoting $\psi_0 = (\tilde{\psi}_0)_{\sigma,\sigma}$ and $-v_0 = \int_{\mathbb{R}^N} V(|\psi_0|^2)\,dx = -\sigma^N$, we have $\Delta\psi_0 + F(|\psi_0|^2)\psi_0 = 0$ in $\mathbb{R}^N$. Hence ψ_0 solves (TW$_0$) and

$$\int_{\mathbb{R}^N}|\nabla\psi_0|^2\,dx = \inf\Big\{\int_{\mathbb{R}^N}|\nabla\phi|^2\,dx \;\Big|\; \phi \in \mathcal{E},\ \int_{\mathbb{R}^N}V(|\phi|^2)\,dx = -v_0\Big\}.$$

Since all minimizers of this problem solve (TW$_0$) (after possibly rescaling), we know that they are C^1 in $\mathbb{R}^N$ and then Theorem 2 p. 314 in [35] imply that they are all radially symmetric (after translation). In particular, we have $Q(\psi_0) = 0$ and $\int_{\mathbb{R}^N}\Big|\frac{\partial\psi_0}{\partial x_j}\Big|^2\,dx = \frac{1}{N}\int_{\mathbb{R}^N}|\nabla\psi_0|^2\,dx$ for $j = 1,\dots,N$. By Lemma 2.4 p. 104 in [15] we know that ψ_0 satisfies the Pohozaev identity $\int_{\mathbb{R}^N}|\nabla\psi_0|^2\,dx = -\frac{N}{N-2}v_0$. It follows that $P_c(\psi_0) = 0$, hence $\psi_0 \in \mathscr{C}_c$ and we infer that $E_c(\psi_0) \ge T_c$, that is $\frac{2}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}\psi_0|^2\,dx \ge \frac{2}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U|^2\,dx$. Taking into account (30) and the radial symmetry of ψ_0, this gives $v_0 \ge -\int_{\mathbb{R}^N}V(|U|^2)\,dx$.

On the other hand, by scaling it is easy to see that ψ_0 is a minimizer of the functional $\phi \longmapsto \|\nabla\phi\|^2_{L^2(\mathbb{R}^N)}$ in the set $\mathcal{P} = \Big\{\phi \in \mathcal{E} \;\Big|\; \int_{\mathbb{R}^N}|\nabla\phi|^2\,dx = -\frac{N}{N-2}\int_{\mathbb{R}^N}V(|\phi|^2)\,dx\Big\}$. By (30) we have $U \in \mathcal{P}$, hence $\|\nabla U\|^2_{L^2(\mathbb{R}^N)} \ge \|\nabla\psi_0\|^2_{L^2(\mathbb{R}^N)}$ and consequently $-\int_{\mathbb{R}^N}V(|U|^2)\,dx \ge v_0$. Thus $\|\nabla U\|^2_{L^2(\mathbb{R}^N)} = \|\nabla\psi_0\|^2_{L^2(\mathbb{R}^N)}$, $\int_{\mathbb{R}^N}V(|U|^2)\,dx = \int_{\mathbb{R}^N}V(|\psi_0|^2)$ and U minimizes $\|\nabla\cdot\|^2_{L^2(\mathbb{R}^N)}$ in the set $\Big\{\phi \in \mathcal{E} \;\Big|\; \int_{\mathbb{R}^N}V(|\phi|^2)\,dx = -v_0\Big\}$. By Theorem 2.2 p. 103 in [15], U solves the equation $\Delta U + \lambda F(|U|^2)U = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ for some $\lambda > 0$ and using the Pohozaev identity associated to this equation we see that $\lambda = 1$, hence U solves (TW$_0$). Since U also solves (TW$_c$) for some $c > 0$ and $\frac{\partial U}{\partial x_1}$ is continuous, we must have $\frac{\partial U}{\partial x_1} = 0$ in $\mathbb{R}^N$. Together with the fact that $U \in \mathcal{E}$, this implies that U is constant, a contradiction. Therefore we cannot have $Q(U) = 0$ and we conclude that $Q(U) < 0$.

(ii) Fix $c_0 \in (0, \mathfrak{c}_s)$ and let $U_0 \in \mathcal{E}$ be a minimizer of E_{c_0} on $\mathscr{C}_{c_0}$, as given by Theorem 3. It follows from (29) that $P_c((U_0)_{a,1}) = \frac{1}{a}R_{c,U_0}(a)$, where

$$R_{c,U_0}(a) = \int_{\mathbb{R}^N}\Big|\frac{\partial U_0}{\partial x_1}\Big|^2\,dx + acQ(U_0) + a^2\left[\frac{N-3}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U_0|^2\,dx + \int_{\mathbb{R}^N}V(|u_0|^2)\,dx\right] \quad (31)$$

is a polynomial in a of degree at most 2. It is clear that $R_{c,U_0}(0) > 0$, $R_{c_0,U_0}(1) = P_{c_0}(U_0) = 0$ and for any $c > c_0$ we have $R_{c,U_0}(1) = P_{c_0}(U_0) + (c - c_0)Q(U_0) < 0$ because $Q(U_0) < 0$. Hence there is a unique $a(c) \in (0,1)$ such that $R_{c,U_0}(a(c)) = 0$, which means $P_c((U_0)_{a(c),1}) = 0$. We infer that

$$T_c \le E_c((U_0)_{a(c),1}) = \frac{2}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}(U_0)_{a(c),1}|^2\,dx = a(c)\frac{2}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U_0|^2\,dx = a(c)T_{c_0}. \quad (32)$$

Since $a(c) \in (0,1)$, we have proved that $T_c < T_{c_0}$ whenever $c_0 \in (0, \mathfrak{c}_s)$ and $c \in (c_0, \mathfrak{c}_s)$, thus $c \longmapsto T_c$ is decreasing. By a well-known result of Lebesgue, the function $c \longmapsto T_c$ has a derivative a.e.

(iii) Notice that (32) holds whenever c_0, U_{c_0} are as above and $a(c)$ is a positive root of R_{c,U_0}. Using the Pohozaev identities (4) we find

$$2\int_{\mathbb{R}^N}\Big|\frac{\partial U_0}{\partial x_1}\Big|^2\,dx = \frac{2}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U_0|^2\,dx - c_0Q(U_0) = T_{c_0} - c_0Q(U_0) \qquad \text{and then}$$

$$\frac{N-3}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U_0|^2\,dx + \int_{\mathbb{R}^N}V(|u_0|^2)\,dx = -c_0Q(U_0) - \int_{\mathbb{R}^N}\Big|\frac{\partial U_0}{\partial x_1}\Big|^2\,dx = -\frac{1}{2}c_0Q(U_0) - \frac{1}{2}T_{c_0}. \tag{33}$$

We now distinguish two cases: R_{c,U_0} has degree one or two.

Case (a): If $\frac{N-3}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U_0|^2\,dx + \int_{\mathbb{R}^N}V(|u_0|^2)\,dx = 0$, then R_{c,U_0} has degree one and we have $\int_{\mathbb{R}^N}\Big|\frac{\partial U_0}{\partial x_1}\Big|^2\,dx + c_0Q(U_0) = 0$ because $P_{c_0}(U_0) = 0$. Since R_{c,U_0} is an affine function, we find $a(c) = \frac{c_0}{c}$ for all $c > 0$, hence $a(c_0) = 1$. Moreover, the left-hand side in (33) is zero, thus we have $c_0Q(U_0) + T_{c_0} = 0$ and consequently $a'(c_0) = -\frac{1}{c_0} = \frac{Q(U_0)}{T_{c_0}}$.

Case (b): If $\frac{N-3}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U_0|^2\,dx + \int_{\mathbb{R}^N}V(|u_0|^2)\,dx \neq 0$, R_{c,U_0} has degree two, and the discriminant of this second-order polynomial is equal to

$$\Delta_{c,U_0} = (c^2 - c_0^2)Q^2(U_0) + T_{c_0}^2.$$

Consequently R_{c,U_0} has real roots as long as $(c^2 - c_0^2)Q^2(U_0) + T_{c_0}^2 \geq 0$. It is easy to see that if there are real roots, at least one of them is positive. Indeed, $R_{c,U_0}(0) > 0 > R'_{c,U_0}(0)$. If $\Delta_{c,U_0} \geq 0$, no matter of the sign of the leading order coefficient $\frac{N-3}{N-1}\int_{\mathbb{R}^N}|\nabla_{x_\perp}U_0|^2\,dx + \int_{\mathbb{R}^N}V(|u_0|^2)\,dx \neq 0$, the smallest positive root $a(c)$ of R_{c,U_0} is given by the formula

$$a(c) = \frac{-cQ(U_0) - \sqrt{(c^2 - c_0^2)Q^2(U_0) + T_{c_0}^2}}{-c_0Q(U_0) - T_{c_0}} = \frac{-c_0Q(U_0) + T_{c_0}}{-cQ(U_0) + \sqrt{(c^2 - c_0^2)Q^2(U_0) + T_{c_0}^2}}. \tag{34}$$

Therefore, the function $c \longmapsto a(c)$ is defined on the interval $[\tilde{c}_0, \infty)$ where $\tilde{c}_0 = \sqrt{c_0^2 - \frac{T_{c_0}^2}{Q^2(U_0)}} < c_0$, it is differentiable on $(\tilde{c}_0, \infty)$ and $a(c_0) = 1$. Moreover, a straightforward computation gives $a'(c_0) = \frac{Q(U_0)}{T_{c_0}}$. Note that in Case (a), the last expression in (34) is equal to $\frac{c_0}{c}$, which is then indeed $a(c)$.

By (32) we have $T_c \leq a(c)T_{c_0}$ and passing to the limit we get $\lim_{c\to c_0,\, c<c_0} T_c \leq \lim_{c\to c_0,\, c<c_0} a(c)T_{c_0} = T_{c_0}$. Since $c \longmapsto T_c$ is decreasing, $T_c > T_{c_0}$ for $c < c_0$ and we see that it is left contiuous at c_0. Moreover, we have

$$\frac{T_c - T_{c_0}}{c - c_0} \leq \frac{a(c) - a(c_0)}{c - c_0}T_{c_0} \quad \text{for } c > c_0, \qquad \text{respectively} \qquad \frac{T_c - T_{c_0}}{c - c_0} \geq \frac{a(c) - a(c_0)}{c - c_0}T_{c_0} \quad \text{for } c \in [\tilde{c}_0, c_0).$$

Passing to the limit in the above inequalities we obtain, since $a'(c_0) = \frac{Q(U_0)}{T_{c_0}}$ in Cases (a) and (b),

$$\limsup_{c\to c_0,\, c>c_0} \frac{T_c - T_{c_0}}{c - c_0} \leq a'(c_0)T_{c_0} = Q(U_0), \qquad \text{respectively} \qquad \liminf_{c\to c_0,\, c<c_0} \frac{T_c - T_{c_0}}{c - c_0} \geq a'(c_0)T_{c_0} = Q(U_0).$$

It is then clear that if $c \longmapsto T_c$ is differentiable at c_0, necessarily $\frac{dT_c}{dc}_{|c=c_0} = Q(U_0)$.

(iv) Fix $c_* \in (c_0, \mathfrak{c}_s)$. Passing to a subsequence we may assume that $c_0 < c_n < c_*$ for all n and $Q(U_n) \to -q_0 \leq 0$. Then $T_{c_0} > T_{c_n} > T_{c_*} > 0$ and $(c_0^2 - c_n^2)Q^2(U_n) + T_{c_n}^2 > (c_0^2 - c_n^2)Q^2(U_n) + T_{c_*}^2 > 0$ for all sufficiently large n. Hence for large n we may use (32) and (34) with (c_n, c_0) instead of (c_0, c) and we get

$$T_{c_0} \leq \frac{-c_nQ(U_n) + T_{c_n}}{-c_0Q(U_n) + \sqrt{(c_0^2 - c_n^2)Q^2(U_n) + T_{c_n}^2}}T_{c_n}.$$

Since T_{c_n} has a positive limit, passing to the limit as $n \to \infty$ in the above inequality and using the monotonicity of $c \longmapsto T_c$ we get $T_{c_0} \leq \liminf_{n\to\infty} T_{c_n} = \liminf_{c\to c_0,\, c>c_0} T_c$. This and the fact that T_c is decreasing and left continuous imply that T_c is continuous at c_0.

(v) Let $0 < c_1 < c_2 < \mathfrak{c}_s$ and U_1, U_2, $q_1 = Q(U_1) < 0$, $q_2 = Q(U_2) < 0$ be as in Proposition 10 (v). If $c_1^2 \leq c_2^2 - \frac{T_{c_2}^2}{q_2^2}$, the inequality in Proposition 10 (v) obviously holds. From now on we assume that $c_1^2 > c_2^2 - \frac{T_{c_2}^2}{q_2^2}$. The two discriminants $\Delta_{c_2,U_1} = (c_2^2 - c_1^2)q_1^2 + T_{c_1}^2$ and $\Delta_{c_1,U_2} = (c_1^2 - c_2^2)q_2^2 + T_{c_2}^2$ are positive: since $0 < c_1 < c_2$ for the first one, and by the assumption $c_1^2 > c_2^2 - \frac{T_{c_2}^2}{q_2^2}$ for the second one. Therefore, we may use (32) and (34) with the couples (c_1, c_2), respectively (c_2, c_1) instead of (c_0, c) to get

$$T_{c_2} \leq \frac{-c_1 q_1 + T_{c_1}}{-c_2 q_1 + \sqrt{(c_2^2 - c_1^2)q_1^2 + T_{c_1}^2}} T_{c_1}, \qquad \text{respectively} \qquad T_{c_1} \leq \frac{-c_2 q_2 + T_{c_2}}{-c_1 q_2 + \sqrt{(c_1^2 - c_2^2)q_2^2 + T_{c_2}^2}} T_{c_2}.$$

Since $T_{c_i} > 0$, we must have

$$\frac{-c_1 q_1 + T_{c_1}}{-c_2 q_1 + \sqrt{(c_2^2 - c_1^2)q_1^2 + T_{c_1}^2}} \cdot \frac{-c_2 q_2 + T_{c_2}}{-c_1 q_2 + \sqrt{(c_1^2 - c_2^2)q_2^2 + T_{c_2}^2}} \geq 1.$$

We set $y_1 = -\frac{T_{c_1}}{c_1 q_1} > 0$, and recast this inequality as

$$\frac{1 + y_1}{\frac{c_2}{c_1} + \sqrt{\frac{c_2^2}{c_1^2} - 1 + y_1^2}} \geq \frac{-c_1 q_2 + \sqrt{(c_1^2 - c_2^2)q_2^2 + T_{c_2}^2}}{-c_2 q_2 + T_{c_2}} = \frac{1 + \sqrt{1 - \frac{c_2^2}{c_1^2} + \frac{T_{c_2}^2}{c_1^2 q_2^2}}}{\frac{c_2}{c_1} - \frac{T_{c_2}}{c_1 q_2}}. \tag{35}$$

Denoting, for $y \in \mathbb{R}$, $g(y) = \dfrac{1 + y}{\frac{c_2}{c_1} + \sqrt{\frac{c_2^2}{c_1^2} - 1 + y^2}}$, (35) is exactly

$$g\Big(-\frac{T_{c_1}}{c_1 q_1}\Big) = g(y_1) \geq g\Big(\sqrt{1 - \frac{c_2^2}{c_1^2} + \frac{T_{c_2}^2}{c_1^2 q_2^2}}\Big).$$

If we show that g is increasing, then we obtain

$$-\frac{T_{c_1}}{c_1 q_1} \geq \sqrt{1 - \frac{c_2^2}{c_1^2} + \frac{T_{c_2}^2}{c_1^2 q_2^2}}, \qquad \text{or} \qquad \frac{T_{c_1}^2}{q_1^2} - c_1^2 \geq \frac{T_{c_2}^2}{q_2^2} - c_2^2,$$

which is the desired inequality. To check that g is increasing, we simply compute

$$g'(y) = \frac{\frac{c_2^2}{c_1^2} - 1 + \frac{c_2}{c_1}\sqrt{\frac{c_2^2}{c_1^2} - 1 + y^2} - y}{\Big(\frac{c_2}{c_1} + \sqrt{\frac{c_2^2}{c_1^2} - 1 + y^2}\Big)^2 \sqrt{\frac{c_2^2}{c_1^2} - 1 + y^2}},$$

which is positive since $\frac{c_2}{c_1} > 1$ and $\sqrt{\frac{c_2^2}{c_1^2} - 1 + y^2} > |y|$.

(vi) Since $c \longmapsto -T_c$ is increasing, by a well-known result of Lebesgue this map is differentiable a.e., the function $c \longmapsto \frac{dT_c}{dc}$ belongs to $L^1_{loc}(0, \mathfrak{c}_s)$ and for any $0 < c_1 < c_2 < \mathfrak{c}_s$ we have $\displaystyle\int_{c_1}^{c_2} -\frac{dT_c}{dc}\, dc \leq -T_{c_2} + T_{c_1}$.

We recall that $c(\varepsilon) = \sqrt{\mathfrak{c}_s^2 - \varepsilon^2}$ for all $\varepsilon \in (0, \mathfrak{c}_s)$. If $N = 3$, (A2) and (A4) hold and $\Gamma \neq 0$, by Proposition 9 (ii) there is $K > 0$ such that $T_{c(\varepsilon)} \leq K\varepsilon$ for all sufficiently small ε. Thus for $n \in \mathbb{N}$ large we have

$$\int_{c(2/n)}^{c(1/n)} -\frac{dT_c}{dc}\, dc \leq T_{c(2/n)} - T_{c(1/n)} \leq T_{c(2/n)} \leq \frac{2K}{n}.$$

Hence there exists $c_n \in (c(2/n), c(1/n))$ such that $c \mapsto T_c$ is differentiable at c_n and

$$-\frac{dT_c}{dc}_{|c=c_n} \leq \frac{1}{c(\frac{1}{n}) - c(\frac{2}{n})} \cdot \frac{2K}{n} \leq K'n.$$

Let $\varepsilon_n = \sqrt{\mathfrak{c}_s^2 - c_n^2}$, so that $c(\varepsilon_n) = c_n$. Since $c(2/n) \leq c_n \leq c(1/n)$, we have $\frac{1}{n} \leq \varepsilon_n \leq \frac{2}{n}$, so that $\varepsilon_n \to 0$ as $n \to \infty$. Let U_n be a minimizer of E_{c_n} on $\mathscr{C}_{c_n}$, scaled so that U_n solves (TW_{c_n}). From (i) and (iii) we get

$$|Q(U_n)| = -Q(U_n) = -\frac{dT_c}{dc}_{|c=c_n} \leq K'n \leq \frac{2K'}{\varepsilon_n}.$$

Since $E(U_n) + c_n Q(U_n) = T_{c_n} = \mathcal{O}(\varepsilon_n)$, it follows that

$$E(U_n) \leq -c_n Q(U_n) + T_{c_n} \leq \frac{K''}{\varepsilon_n}$$

and the proof is complete. □

3.3 Proof of Proposition 12

We postpone the proof of Proposition 11 and we prove Proposition 12.

Let $(\varepsilon_n)_{n\geq 1}$ be the sequence given by Proposition 10 (vi). For each n let $U_n \in \mathcal{E}$ be a minimizer of E_{c_n} on $\mathscr{C}_{c_n}$ which solves (TW_{c_n}). Passing to a subsequence if necessary and using Proposition 11, we may assume that $(\varepsilon_n)_{n\geq 1}$ is strictly decreasing, that $(\varepsilon_n, U_n)_{n\geq 1}$ satisfies the conclusion of Theorem 6 and

$$\frac{1}{2} r_0^2 \mathfrak{c}_s^4 \mathscr{S}_{\min} \frac{1}{\varepsilon_n} < E(U_n) < 2 r_0^2 \mathfrak{c}_s^4 \mathscr{S}_{\min} \frac{1}{\varepsilon_n}, \tag{36}$$

$$\frac{1}{2} r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min} \frac{1}{\varepsilon_n} < -Q(U_n) < 2 r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min} \frac{1}{\varepsilon_n} \qquad \text{for all } n. \tag{37}$$

We shall argue by contradiction. More precisely, we shall prove by contradiction that there exists $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$ and for any minimizer U of $E_{c(\varepsilon)}$ on $\mathscr{C}_{c(\varepsilon)}$ scaled so that U satisfies $(\mathrm{TW}_{c(\varepsilon)})$, we have

$$|Q(U)| \leq \frac{5 r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min}}{\varepsilon}.$$

In view of Proposition 9 (ii), we then infer that

$$E(U) = T_{c(\varepsilon)} - c(\varepsilon) Q(U) \leq \frac{K}{\varepsilon}$$

for some constant K depending only on r_0, $\mathfrak{c}_s$ and $\mathscr{S}_{\min}$, which is the desired result. We thus assume that there exist infinitely many n's such that there is $\tilde{\varepsilon}_n \in (\varepsilon_n, \varepsilon_{n-1})$ and there is a minimizer $\tilde{U}_n$ of $E_{c(\tilde{\varepsilon}_n)}$ on $\mathscr{C}_{c(\tilde{\varepsilon}_n)}$ which satisfies $(\mathrm{TW}_{c(\tilde{\varepsilon}_n)})$ and

$$|Q(\tilde{U}_n)| = -Q(\tilde{U}_n) > 5 r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min} \frac{1}{\tilde{\varepsilon}_n}. \tag{38}$$

Passing again to a subsequence of $(\varepsilon_n)_{n\geq 1}$, we may assume that (38) holds for all $n \geq 1$. Then for each $n \in \mathbb{N}^*$ we define

$$I_n = \Big\{ \varepsilon \in (\varepsilon_n, \varepsilon_{n-1}) \ \Big| \ \text{for all } \varepsilon' \in [\varepsilon_n, \varepsilon] \text{ and for any minimizer } U_{\varepsilon'} \text{ of } E_{c(\varepsilon')} \text{ on } \mathscr{C}_{c(\varepsilon')} \text{ which solves } (\mathrm{TW}_{c(\varepsilon')}) \text{ there holds } |Q(U_{\varepsilon'})| \leq 4 r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min} \cdot \tfrac{1}{\varepsilon'} \Big\}$$

and

$$\varepsilon_n^{\#} = \sup I_n.$$

By Proposition 10 (v), for $\varepsilon' \in (\varepsilon_n, \mathfrak{c}_s)$ and for any minimizer $U_{\varepsilon'}$ of $E_{c(\varepsilon')}$ on $\mathscr{C}_{c(\varepsilon')}$ which solves $(\mathrm{TW}_{c(\varepsilon')})$ we have

$$\frac{T^2_{c(\varepsilon')}}{Q^2(U_{\varepsilon'})} + (\varepsilon')^2 \geq \frac{T^2_{c(\varepsilon_n)}}{Q^2(U_n)} + \varepsilon_n^2,$$

which can be written as $\dfrac{Q^2(U_{\varepsilon'})}{T^2_{c(\varepsilon')}} \le \dfrac{Q^2(U_n)}{T^2_{c(\varepsilon_n)} + (\varepsilon_n^2 - (\varepsilon')^2)Q^2(U_n)}$ and this gives

$$(\varepsilon')^2 Q^2(U_{\varepsilon'}) \le \frac{(\varepsilon')^2 Q^2(U_n) T^2_{c(\varepsilon')}}{T^2_{c(\varepsilon_n)} + (\varepsilon_n^2 - (\varepsilon')^2)Q^2(U_n)}. \tag{39}$$

The mapping $\varepsilon \longmapsto T_{c(\varepsilon)}$ is right continuous (because $c \longmapsto T_c$ is left continuous) and using (37) we find

$$\lim_{\varepsilon' \to \varepsilon_n, \, \varepsilon' > \varepsilon_n} \frac{(\varepsilon')^2 Q^2(U_n) T^2_{c(\varepsilon')}}{T^2_{c(\varepsilon_n)} + (\varepsilon_n^2 - (\varepsilon')^2)Q^2(U_n)} = \varepsilon_n^2 Q^2(U_n) < (2r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min})^2.$$

Thus all $\varepsilon' \in (\varepsilon_n, \varepsilon_{n-1})$ sufficiently close to ε_n belong to I_n. In particular, I_n is not empty. On the other hand, (38) implies that any $\varepsilon' \in (\tilde{\varepsilon}_n, \varepsilon_{n-1})$ does not belong to I_n, hence $\varepsilon_n^\# = \sup I_n \in (\varepsilon_n, \tilde{\varepsilon}_n] \subset (\varepsilon_n, \varepsilon_{n-1})$.

Let $U_n^\#$ be a minimizer of $E_{c(\varepsilon_n^\#)}$ on $\mathscr{C}_{c(\varepsilon_n^\#)}$ which solves $(\mathrm{TW}_{c(\varepsilon_n^\#)})$. We claim that

$$|Q(U_n^\#)| = 4r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min} \frac{1}{\varepsilon_n^\#}. \tag{40}$$

Indeed, proceeding as in (39) we have for any $\varepsilon' \in (\varepsilon_n, \varepsilon_n^\#)$ and any minimizer $U_{\varepsilon'}$ of $E_{c(\varepsilon')}$ on $\mathscr{C}_{c(\varepsilon')}$ which satisfies $(\mathrm{TW}_{c(\varepsilon')})$

$$(\varepsilon_n^\#)^2 Q^2(U_n^\#) \le \frac{\left(\frac{\varepsilon_n^\#}{\varepsilon'}\right)^2 (\varepsilon')^2 Q^2(U_{\varepsilon'}) T^2_{c(\varepsilon_n^\#)}}{T^2_{c(\varepsilon')} + \left(1 - \left(\frac{\varepsilon_n^\#}{\varepsilon'}\right)^2\right)(\varepsilon')^2 Q^2(U_{\varepsilon'})}. \tag{41}$$

Notice that $(\varepsilon')^2 Q^2(U_{\varepsilon'}) \le (4r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min})^2$ because $\varepsilon' \in I_n$. In particular, $Q(U_{\varepsilon'})$ is bounded as $\varepsilon' \in (\varepsilon_n, \varepsilon_n^\#)$. Since $c(\varepsilon') \searrow c(\varepsilon_n^\#)$ as $\varepsilon' \nearrow \varepsilon_n^\#$, Proposition 10 (iv) implies that $c \longmapsto T_c$ is continuous at $c(\varepsilon_n^\#)$. Then passing to $\liminf$ as $\varepsilon' \nearrow \varepsilon_n^\#$ in (41) we get $(\varepsilon_n^\#)^2 Q^2(U_n^\#) \le (4r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min})^2$. We conclude that $\varepsilon_n^\# \in I_n$.

Next, for any $\varepsilon' \in (\varepsilon_n^\#, \mathfrak{c}_s)$ and any minimizer $U_{\varepsilon'}$ of $E_{c(\varepsilon')}$ on $\mathscr{C}_{c(\varepsilon')}$ that solves $(\mathrm{TW}_{c(\varepsilon')})$, inequality (39) holds with $\varepsilon_n^\#$ and $U_n^\#$ instead of ε_n and U_n, respectively. The limit of the right-hand side as $\varepsilon' \searrow \varepsilon_n^\#$ is $(\varepsilon_n^\#)^2 Q^2(U_n^\#)$. If $\varepsilon_n^\# |Q(U_n^\#| < 4r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min}$, as above we infer that there is $\delta_n > 0$ such that $[\varepsilon_n^\#, \varepsilon_n^\# + \delta_n] \subset I_n$, contradicting the fact that $\varepsilon_n^\# = \sup I_n$. The claim (40) is thus proved.

Now we turn our attention to the sequence $(\varepsilon_n^\#, U_n^\#)_{n \ge 1}$. It is clear that $\varepsilon_n^\# \to 0$ (because $\varepsilon_n^\# \in (\varepsilon_n, \varepsilon_{n-1})$). By Proposition 9 (ii) there is $K > 0$ such that

$$E(U_n^\#) + c(\varepsilon_n^\#) Q(U_n^\#) = E_{c(\varepsilon_n^\#)}(U_n^\#) = T_{c(\varepsilon_n^\#)} \le K \varepsilon_n^\#$$

and using (40) we find $|E(U_n^\#)| \le \frac{K'}{\varepsilon_n^\#}$ for some constant $K' > 0$ and for all n sufficiently large. Hence we may use Proposition 11 and we infer that there is a subsequence $(\varepsilon_{n_k}^\#, U_{n_k}^\#)_{k \ge 1}$ which satisfies the conclusion of Theorem 6. In particular, we have

$$\lim_{k \to \infty} \varepsilon_{n_k}^\# |Q(U_{n_k}^\#)| = r_0^2 \mathfrak{c}_s^3 \mathscr{S}_{\min}$$

and this contradicts the fact that $U_{n_k}^\#$ satisfies (40). Proposition 12 is thus proven. □

3.4 Proof of Proposition 14

(i) Since $U \in \mathcal{E}$, we have $|U| - r_0 \in H^1(\mathbb{R}^N)$ (see the Introduction of [17]) and then $\left|\dfrac{\partial}{\partial x_i}(|U| - r_0)\right| \le \left|\dfrac{\partial U}{\partial x_i}\right|$ a.e. in $\mathbb{R}^N$. It is well-known (see, for instance, [14] p. 164) that for any $\phi \in H^1(\mathbb{R}^N)$ there holds

$$\|\phi\|_{L^{2^*}(\mathbb{R}^N)} \le C_S \prod_{i=1}^N \left\| \frac{\partial \phi}{\partial x_i} \right\|_{L^2(\mathbb{R}^N)}^{\frac{1}{N}}.$$

We infer that

$$\| \, |U| - r_0 \|_{L^{2^*}(\mathbb{R}^N)} \le C_S \prod_{i=1}^N \left\| \frac{\partial U}{\partial x_i} \right\|_{L^2(\mathbb{R}^N)}^{\frac{1}{N}} \le C_S \left\| \frac{\partial U}{\partial x_1} \right\|_{L^2(\mathbb{R}^N)}^{\frac{1}{N}} \cdot \|\nabla_{x_\perp} U\|_{L^2(\mathbb{R}^N)}^{\frac{N}{N-1}}. \tag{42}$$

Assume first that (A2) holds. If $\left\| \frac{\partial U}{\partial x_1} \right\|_{L^2(\mathbb{R}^N)} \cdot \|\nabla_{x_\perp} U\|^{N-1}_{L^2(\mathbb{R}^N)} \leq 1$, from (42) we get $\| \, |U| - r_0 \|_{L^{2^*}(\mathbb{R}^N)} \leq C_S$. Let $\tilde{U}(x) = e^{-\frac{icx_1}{2}} U(x)$. Then $\tilde{U} \in H^1_{loc}(\mathbb{R}^N)$ and $\tilde{U}$ solves the equation

$$\Delta \tilde{U} + \left(\frac{c^2}{4} + F(|\tilde{U}|^2) \right) \tilde{U} = 0 \qquad \text{in } \mathbb{R}^N.$$

Since $\|\tilde{U}\|_{L^{2^*}(B(x,1))} \leq C$ for any $x \in \mathbb{R}^N$ and for some constant $C > 0$, using the above equation and a standard bootstrap argument (which works thanks to (A2)), we infer that $\|\tilde{U}\|_{W^{2,p}(B(x, \frac{1}{2^{n_0}}))} \leq \tilde{C}_p$ for some $n_0 \in \mathbb{N}$, $\tilde{C}_p > 0$ and for any $x \in \mathbb{R}^N$ and any $p \in [2, \infty)$. This clearly implies $\|U\|_{W^{2,p}(B(x, \frac{1}{2^{n_0}}))} \leq C_p$ for any $x \in \mathbb{R}^N$ and any $p \in [2, \infty)$. In particular, using the Sobolev embedding we see that there is $L > 0$ (independent on U) such that $\|\nabla U\|_{L^\infty(\mathbb{R}^N)} \leq L$.

Fix $\delta > 0$. If there is $x_0 \in \mathbb{R}^N$ such that $| \, |U(x_0)| - r_0| \geq \delta$, we infer that $\| \, |U(x)| - r_0| \geq \frac{\delta}{2}$ for any $x \in B(x_0, \frac{\delta}{2L})$ and consequently

$$\| \, |U| - r_0\|_{L^{2^*}(\mathbb{R}^N)} \geq \frac{\delta}{2} \left(\mathcal{L}^N \left(B(x_0 \frac{\delta}{2L}) \right) \right)^{\frac{1}{2^*}} = \frac{\delta}{2} \left(\frac{\delta}{2L} \right)^{\frac{N}{2^*}} \left(\mathcal{L}^N(B(0,1)) \right)^{\frac{1}{2^*}}. \tag{43}$$

Let $\mu(\delta) = \min \left(1, \frac{\delta}{2} \left(\frac{\delta}{2L} \right)^{\frac{N}{2^*}} \left(\mathcal{L}^N(B(0,1)) \right)^{\frac{1}{2^*}} \right)$. From (42) and (43) we infer that $| \, |U(x)| - r_0| < \delta$ for any solution $U \in \mathcal{E}$ of (TW$_c$) satisfying $\left\| \frac{\partial U}{\partial x_1} \right\|_{L^2(\mathbb{R}^N)} \cdot \|\nabla_{x_\perp} U\|^{N-1}_{L^2(\mathbb{R}^N)} \leq \mu(\delta)$.

If (A3) holds, it follows from the proof of Proposition 2.2 p. 1078-1080 in [34] thet there is $L > 0$, independent on U, such that $\|\nabla U\|_{L^\infty(\mathbb{R}^N)} \leq L$. The rest of the proof is as above.

(ii) By Proposition 2.2 p. 1078 in [34] we know that $U \in W^{2,p}_{loc}(\mathbb{R}^N)$ for any $p \in [2, \infty)$. In particular, $U \in C^1(\mathbb{R}^N)$. As in the proof of *(i)* we see that there is $L > 0$, independent on U, such that $||\nabla U\|_{L^\infty(\mathbb{R}^N)} \leq L$.

Fix $\delta > 0$ and assume that there is $x^0 = (x^0_1, \dots, x^0_N)$ such that $| \, |U(x^0)| - r_0| \geq \delta$. Then we have $| \, |U(x)| - r_0| \geq \frac{\delta}{2}$ for any $x \in B(x^0, \frac{\delta}{2L})$ and, in particular, $| \, |U(x_1, x^0_2, \dots, x^0_N)| - r_0| \geq \frac{\delta}{2}$ for any $x_1 \in [x^0_1 - \frac{\delta}{2L}, x^0_1 + \frac{\delta}{2L}]$. We infer that $| \, |U(x_1, x_\perp)| - r_0| \geq \frac{\delta}{4}$ for any $x_1 \in [x^0_1 - \frac{\delta}{2L}, x^0_1 + \frac{\delta}{2L}]$ and any $x_\perp \in B_{\mathbb{R}^{N-1}}(x^0_\perp, \frac{\delta}{4L})$. Consequently

$$\| \, |U(x_1, \cdot)| - r_0\|_{L^{\frac{2(N-1)}{N-3}}(\mathbb{R}^{N-1})} \geq \frac{\delta}{4} \left(\mathcal{L}^{N-1} \left(B_{\mathbb{R}^{N-1}} \left(x^0_\perp, \frac{\delta}{4L} \right) \right) \right)^{\frac{N-3}{2(N-1)}}$$

$$\geq \frac{\delta}{4} \left(\frac{\delta}{4L} \right)^{\frac{N-3}{2}} \left(\mathcal{L}^{N-1}(B_{\mathbb{R}^{N-1}}(0,1)) \right)^{\frac{N-3}{2(N-1)}} = K \delta^{\frac{N-1}{2}}$$

for all $x_1 \in [x^0_1 - \frac{\delta}{2L}, x^0_1 + \frac{\delta}{2L}]$. Using the Sobolev inequality in $\mathbb{R}^{N-1}$ we get for $x_1 \in \left[x^0_1 - \frac{\delta}{2L}, x^0_1 + \frac{\delta}{2L} \right]$,

$$\int_{\mathbb{R}^{N-1}} |\nabla_{x_\perp} U(x_1, x_\perp)|^2 \, dx_\perp \geq \frac{1}{\tilde{C}^2_S} \| \, |U(x_1, \cdot)| - r_0\|^2_{L^{\frac{2(N-1)}{N-3}}(\mathbb{R}^{N-1})} \geq \frac{K^2}{\tilde{C}^2_S} \delta^{N-1}.$$

Integrating the above inequality on $[x^0_1 - \frac{\delta}{2L}, x^0_1 + \frac{\delta}{2L}]$ we obtain $\|\nabla_{x_\perp} U\|^2_{L^2(\mathbb{R}^N)} \geq \frac{K^2}{L\tilde{C}^2_S} \delta^N = K_1 \delta^N$. We conclude that if $\|\nabla_{x_\perp} U\|^2_{L^2(\mathbb{R}^N)} < \min(1, K_1 \delta^N)$, then necessarily $| \, |U| - r_0| < \delta$ in $\mathbb{R}^N$. □

3.5 Proof of Proposition 16

It follows from Lemma 4.1 in [17] that there are $k_0 > 0$, $C_1, C_2 > 0$ such that for all $\psi \in \mathcal{E}$ with $\int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx \leq k_0$ we have

$$C_1 \int_{\mathbb{R}^2} (\chi^2(|\psi|) - r_0^2)^2 \, dx \leq \int_{\mathbb{R}^2} V(|\psi|^2) \, dx \leq C_2 \int_{\mathbb{R}^2} (\chi^2(|\psi|) - r_0^2)^2 \, dx. \tag{44}$$

We recall that in space dimension two, nontrivial solutions U_k to (TW$_c$) have been constructed in Theorem 2 by considering the minimization problem

$$\text{minimize } I(\psi) = Q(\psi) + \int_{\mathbb{R}^2} V(|\psi|^2) \, dx \quad \text{ in } \mathcal{E} \text{ under the constraint } \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx = k. \tag{$\mathcal{I}_k$}$$

If $\mathcal{U}_k$ is a minimizer for $(\mathcal{I}_k)$, there is $c_k > 0$ such that $U_k = (\mathcal{U}_k)_{c_k,c_k}$ solves (TW_{c_k}) and minimizes $E_{c_k} = E + c_k Q$ in the set $\left\{\psi \in \mathcal{E} \;\Big|\; \int_{\mathbb{R}^2} |\nabla \psi|^2\,dx = k\right\}$. Moreover, we have $c_k \to \mathfrak{c}_s$ as $k \to 0$. Lemma 13 implies that $|U_k| \to r_0$ uniformly on $\mathbb{R}^2$ as $k \to 0$; in particular, there is $k_1 > 0$ such that if $k \in (0, k_1)$, we have $|U_k| \geq \frac{r_0}{2}$ in $\mathbb{R}^2$. From the Pohozaev identities (4) we get $c_k Q(U_k) + 2\int_{\mathbb{R}^2} V(|U_k|^2)\,dx = 0$, and this gives

$$I_{\min}(k) = I(\mathcal{U}_k) = \frac{1}{c_k}Q(U_k) + \frac{1}{c_k^2}\int_{\mathbb{R}^2} V(|U_k|^2)\,dx = \frac{1}{2c_k}Q(U_k) = -\frac{1}{c_k^2}\int_{\mathbb{R}^2} V(|U_k|^2)\,dx. \tag{45}$$

By Lemma 5.2 in [17] there is $k_2 > 0$ such that $-\frac{2k}{\mathfrak{c}_s^2} \leq I_{\min}(k) \leq -\frac{k}{\mathfrak{c}_s^2}$ for all $k \in (0, k_2)$. Since $c_k \to \mathfrak{c}_s$ as $k \to 0$, the estimates (14) follow directly from (44) and (45).

It remains to prove (15). By Proposition 9, there is $\mu_0 > 0$ such that for k sufficiently small we have $I_{\min}(k) \leq -\frac{k}{\mathfrak{c}_s^2} - \mu_0 k^3$. By scaling we have

$$\frac{1}{c_k^2}\Big(E_{c_k}(U_k) - \int_{\mathbb{R}^2} |\nabla U_k|^2\,dx\Big) = \frac{1}{c_k^2}\Big(c_k Q(U_k) + \int_{\mathbb{R}^2} V(|U_k|^2)\,dx\Big) = I(\mathcal{U}_k) = I_{\min}(k) \leq -\frac{k}{\mathfrak{c}_s^2} - \mu_0 k^3.$$

Since $\mathfrak{c}_s^2 - c_k^2 = \varepsilon_k^2$ and $\int_{\mathbb{R}^2} |\nabla U_k|^2\,dx = k$, we get

$$E_{c_k}(U_k) \leq k\Big(1 - \frac{c_k^2}{\mathfrak{c}_s^2}\Big) - \mu_0 c_k^2 k^3 = \frac{k\varepsilon_k^2}{\mathfrak{c}_s^2} - \mu_0 c_k^2 k^3. \tag{46}$$

The second Pohozaev identity (4) yields $E_{c_k}(U_k) = 2\int_{\mathbb{R}^2} |\partial_2 U_k|^2\,dx \geq 0$, thus $0 \leq k\Big(\frac{\varepsilon_k^2}{\mathfrak{c}_s^2} - \mu_0 c_k^2 k^2\Big)$ and this implies

$$\frac{\varepsilon_k^2}{\mathfrak{c}_s^2} \geq \mu_0 c^2 k^2.$$

Since $c \geq \mathfrak{c}_s/2$ for k small, the left-hand side inequality in (15) follows.

In order to prove the second inequality in (15), we need the next Lemma. In the case of the Gross-Pitaevskii nonlinearity, this result follows from Lemma 2.12 p. 597 in [8]. In the case of general nonlinearities, it was proved in [17].

Lemma 21 ([8, 17]) *Let $N \geq 2$. There is $\beta_* > 0$ such that any solution $U = \rho e^{i\phi} \in \mathcal{E}$ of (TW_c) verifying $r_0 - \beta_* \leq \rho \leq r_0 + \beta_*$ satisfies the identities*

$$E(U) + cQ(U) = \frac{2}{N}\int_{\mathbb{R}^N} |\nabla \rho|^2\,dx \qquad \textit{and} \tag{47}$$

$$2\int_{\mathbb{R}^N} \rho^2 |\nabla \phi|^2\,dx = c\int_{\mathbb{R}^N} (\rho^2 - r_0^2)\partial_1 \phi\,dx = -cQ(U). \tag{48}$$

Furthermore, there exist $a_1, a_2 > 0$ such that

$$a_1\|\rho^2 - r_0^2\|_{L^2(\mathbb{R}^N)} \leq \|\nabla U\|_{L^2(\mathbb{R}^N)} \leq a_2\|\rho^2 - r_0^2\|_{L^2(\mathbb{R}^N)}. \tag{49}$$

Proof. Identity (48) is Lemma 7.3 (i) in [17]. Formally, it follows by multiplying the first equation in (17) by ϕ and integrating by parts over $\mathbb{R}^N$; see [17] for a rigorous justification.

Combining the two Pohozaev identities in (4), we have

$$(N-2)\int_{\mathbb{R}^N} |\nabla U|^2\,dx + N\int_{\mathbb{R}^N} V(|U|^2)\,dx + c(N-1)Q(U) = 0.$$

Using that $|\nabla U|^2 = |\nabla \rho|^2 + \rho^2|\nabla \phi|^2$, we infer from (48)

$$\begin{aligned} N(E(U) + cQ(U)) = 2\int_{\mathbb{R}^N} |\nabla U|^2\,dx + cQ(U) &= 2\int_{\mathbb{R}^N} |\nabla \rho|^2\,dx + \Big(2\int_{\mathbb{R}^N} \rho^2|\nabla \phi|^2\,dx + cQ(U)\Big) \\ &= 2\int_{\mathbb{R}^N} |\nabla \rho|^2\,dx, \end{aligned}$$

and this establishes (47). The estimate (49) has been proven in [17] (see inequality (7.17) there). $\square$

We come back to the proof of Proposition 16. We write $U_k = \rho e^{i\phi}$ and we denote $\eta = \rho^2 - r_0^2$, so that ρ, ϕ and η satisfy (17)−(19) (with c_k instead of c). Taking the Fourier transform of (19) we get

$$\begin{aligned}\widehat{\eta}(\xi) \quad = \quad & \frac{|\xi|^2}{|\xi|^4 + \mathfrak{c}_s^2|\xi|^2 - c_k^2\xi_1^2}\mathscr{F}\left(-2|\nabla U_k|^2 + 2c_k\eta\frac{\partial\phi}{\partial x_1} + 2\rho^2 F(\rho^2) + \mathfrak{c}_s^2\eta\right) \\ & -2c_k\sum_{j=1}^{N}\frac{\xi_1\xi_j}{|\xi|^4 + \mathfrak{c}_s^2|\xi|^2 - c_k^2\xi_1^2}\mathscr{F}\left(\eta\frac{\partial\phi}{\partial x_j}\right).\end{aligned} \tag{50}$$

It is easy to see that $2\rho^2F(\rho^2) + \mathfrak{c}_s^2\eta = \mathcal{O}((\rho^2 - r_0^2)^2) = \mathcal{O}(\eta^2)$, hence

$$\|\mathscr{F}\left(2\rho^2F(\rho^2) + \mathfrak{c}_s^2\eta\right)\|_{L^\infty(\mathbb{R}^N)} \leq \|2\rho^2F(\rho^2) + \mathfrak{c}_s^2\eta\|_{L^1(\mathbb{R}^N)} \leq C\|\eta\|^2_{L^2(\mathbb{R}^N)}.$$

Since $r_0 - \beta_* < |U_k| < r_0 + \beta_*$ if k is sufficiently small and $|\nabla U_k|^2 = |\nabla\rho|^2 + \rho^2|\nabla\phi|^2$, using (49) we get

$$\left\|\mathscr{F}\left(\eta\frac{\partial\phi}{\partial x_j}\right)\right\|_{L^\infty(\mathbb{R}^N)} \leq \left\|\eta\frac{\partial\phi}{\partial x_j}\right\|_{L^1(\mathbb{R}^N)} \leq \|\eta\|_{L^2(\mathbb{R}^N)}\left\|\frac{\partial\phi}{\partial x_j}\right\|_{L^2(\mathbb{R}^N)} \leq C\|\eta\|^2_{L^2(\mathbb{R}^N)}$$

and $\|\mathscr{F}(|\nabla U_k|^2)\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla U_k\|^2_{L^2(\mathbb{R}^N)} \leq C\|\eta\|^2_{L^2(\mathbb{R}^N)}$. Coming back to (50) we discover

$$|\widehat{\eta}(\xi)| \leq C\|\eta\|^2_{L^2(\mathbb{R}^N)} \cdot \frac{|\xi|^2}{|\xi|^4 + \mathfrak{c}_s^2|\xi|^2 - c_k^2\xi_1^2}.$$

Using Plancherel's formula and the above estimate we find

$$\|\eta\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}|\widehat{\eta}(\xi)|^2\,d\xi \leq C\|\eta\|^4_{L^2(\mathbb{R}^N)}\int_{\mathbb{R}^N}\frac{|\xi|^4}{(|\xi|^4 + \mathfrak{c}_s^2|\xi|^2 - c_k^2\xi_1^2)^2}\,d\xi. \tag{51}$$

If $N = 2$, a straightforward computation using polar coordinates gives (see the proof of (2.59) p. 598 in [9]):

$$\int_{\mathbb{R}^2}\frac{|\xi|^4}{(|\xi|^4 + \mathfrak{c}_s^2|\xi|^2 - c_k^2\xi_1^2)^2}\,d\xi = \frac{\pi}{\mathfrak{c}_s\sqrt{\mathfrak{c}_s^2 - c_k^2}} = \frac{\pi}{\mathfrak{c}_s\varepsilon_k}.$$

From to (51) we get $\|\eta\|^2_{L^2(\mathbb{R}^2)} \leq \frac{C}{\varepsilon_k}\|\eta\|^4_{L^2(\mathbb{R}^2)}$ and taking into account (49) we infer that $\varepsilon_k \leq C\|\eta\|^2_{L^2(\mathbb{R}^2)} \leq \tilde{C}\|\nabla U_k\|^2_{L^2(\mathbb{R}^2)} = \tilde{C}k$. $\square$

Notice that at this stage, we have only upper bounds on the energy of travelling waves, and we will have to prevent convergence towards the trivial solution to (SW). This will be done with the help of the following result. It was proven in [9] in the case of the Gross-Pitaevskii nonlinearity (see Proposition 2.4 p. 595 there). We extend the proof to general nonlinearities.

Lemma 22 *Let $N \geq 2$ and assume that (A1) holds and F is twice differentiable at r_0^2. There is $C > 0$, depending only on N and on F, such that any travelling wave $U \in \mathcal{E}$ of* (NLS) *of speed $c \in [0, \mathfrak{c}_s]$ such that $\frac{r_0}{2} \leq |U| \leq \frac{3r_0}{2}$ satisfies*

$$\|\,|U| - r_0\|_{L^\infty(\mathbb{R}^N)} \geq C(\mathfrak{c}_s^2 - c^2) = C\varepsilon^2(U).$$

Proof. Let $U \in \mathcal{E}$ be a travelling wave such that $\frac{r_0}{2} \leq |U| \leq \frac{3r_0}{2}$ in $\mathbb{R}^N$. Then $U \in W^{2,p}_{loc}(\mathbb{R}^N)$, $\nabla U \in W^{1,p}(\mathbb{R}^N)$ for all $p \in [2,\infty)$ (see Proposition 2.2 p. 1078-1079 in [34]), and U admits a lifting $U = \rho e^{i\phi}$, where ρ and ϕ satisfy (17). Since $U \in \mathcal{E}$ we have $\rho^2 - r_0^2 \in H^1(\mathbb{R}^N)$ and then it is easy to see that $\frac{\rho^2 - r_0^2}{\rho} \in H^1(\mathbb{R}^N)$. Multiplying the second equation in (17) by $\frac{\rho^2 - r_0^2}{\rho}$ and integrating by parts we get

$$\int_{\mathbb{R}^N}\left(1 + \frac{r_0^2}{\rho^2}\right)|\nabla\rho|^2\,dx + \int_{\mathbb{R}^N}(\rho^2 - r_0^2)|\nabla\phi|^2 - (\rho^2 - r_0^2)F(\rho^2) - c(\rho^2 - r_0^2)\frac{\partial\phi}{\partial x_1}\,dx = 0. \tag{52}$$

Denote $\delta = \|\,|U| - r_0\|_{L^\infty(\mathbb{R}^N)} = \|\rho - r_0\|_{L^\infty(\mathbb{R}^N)}$. We have

$$\int_{\mathbb{R}^N}\left(1 + \frac{r_0^2}{\rho^2}\right)|\nabla\rho|^2\,dx \geq \left(1 + \frac{r_0^2}{(r_0 + \delta)^2}\right)\int_{\mathbb{R}^N}|\nabla\rho|^2\,dx \qquad \text{and} \tag{53}$$

$$\Big|\int_{\mathbb{R}^N}(\rho^2-r_0^2)|\nabla\phi|^2\,dx\Big|\le\int_{\mathbb{R}^N}\frac{|\rho^2-r_0^2|}{\rho^2}\rho^2|\nabla\phi|^2\,dx\le\frac{2r_0\delta+\delta^2}{(r_0-\delta)^2}\int_{\mathbb{R}^N}|\nabla U|^2\,dx. \tag{54}$$

There is $\tilde C>0$ such that $|F(s^2)-F'(r_0^2)(s^2-r_0^2)|\le\tilde C(s^2-r_0^2)^2$ for all $s\in[\frac{r_0}{2},\frac{3r_0}{2}]$. Remember that $-F'(r_0^2)=2a^2$ and $\mathfrak{c}_s=2ar_0$, thus

$$-(\rho^2-r_0^2)F(\rho^2)\ge-F'(r_0^2)(\rho^2-r_0^2)^2-\tilde C|\rho^2-r_0^2|^3\ge\Big(2a^2-\tilde C(2r_0\delta+\delta^2)\Big)(\rho^2-r_0^2)^2. \tag{55}$$

Using (48) and (3), then (52) and (53)-(55) we get

$$-2cQ(U)=2\int_{\mathbb{R}^N}\rho^2|\nabla\phi|^2\,dx+c\int_{\mathbb{R}^N}(\rho^2-r_0^2)\frac{\partial\phi}{\partial x_1}\,dx$$

$$=2\int_{\mathbb{R}^N}\rho^2|\nabla\phi|^2\,dx+\int_{\mathbb{R}^N}\left(1+\frac{r_0^2}{\rho^2}\right)|\nabla\rho|^2\,dx+\int_{\mathbb{R}^N}(\rho^2-r_0^2)|\nabla\phi|^2-(\rho^2-r_0^2)F(\rho^2)\,dx$$

$$\ge2\int_{\mathbb{R}^N}\rho^2|\nabla\phi|^2\,dx+\int_{\mathbb{R}^N}\left(1+\frac{r_0^2}{(r_0+\delta)^2}\right)|\nabla\rho|^2-\frac{2r_0\delta+\delta^2}{(r_0-\delta)^2}|\nabla U|^2+\Big(2a^2-\tilde C(2r_0\delta+\delta^2)\Big)(\rho^2-r_0^2)^2\,dx$$

and we infer that there exists $K>0$, depending only on F, such that

$$-2cQ(U)\ge2(1-K\delta)\int_{\mathbb{R}^N}|\nabla U|^2+a^2(\rho^2-r_0^2)^2\,dx. \tag{56}$$

On the other hand, using (3) we have

$$\begin{aligned}-Q(U)&=\frac{2ar_0}{\mathfrak{c}_s}\int_{\mathbb{R}^N}(\rho^2-r_0^2)\frac{\partial\phi}{\partial x_1}\,dx\le\frac{1}{\mathfrak{c}_s}\int_{\mathbb{R}^N}r_0^2\Big|\frac{\partial\phi}{\partial x_1}\Big|^2+a^2(\rho^2-r_0^2)^2\,dx\\&\le\frac{1}{\mathfrak{c}_s}\int_{\mathbb{R}^N}\frac{r_0^2}{(r_0-\delta)^2}\rho^2\Big|\frac{\partial\phi}{\partial x_1}\Big|^2+a^2(\rho^2-r_0^2)^2\,dx\le\frac{1}{\mathfrak{c}_s}\frac{r_0^2}{(r_0-\delta)^2}\int_{\mathbb{R}^N}|\nabla U|^2+a^2(\rho^2-r_0^2)^2\,dx.\end{aligned} \tag{57}$$

Since U is not constant we have $\int_{\mathbb{R}^N}|\nabla U|^2+a^2(\rho^2-r_0^2)^2\,dx>0$ and comparing (56) and (57) we get

$$\frac{c}{\mathfrak{c}_s}\frac{r_0^2}{(r_0-\delta)^2}\ge1-K\delta.$$

If $\delta>\frac{1}{2K}$ the conclusion of Lemma 22 holds because $\varepsilon(U)$ is bounded. Otherwise, the previous inequality is equivalent to $\frac{r_0^2}{(r_0-\delta)^2}\frac{1}{1-K\delta}\ge\frac{\mathfrak{c}_s}{\sqrt{\mathfrak{c}_s^2-\varepsilon^2(U)}}$. There are $K_1,\ K_2>0$ such that $\frac{r_0^2}{(r_0-\delta)^2}\frac{1}{1-K\delta}\le1+K_1\delta$ and $\frac{\mathfrak{c}_s}{\sqrt{\mathfrak{c}_s^2-\varepsilon^2}}\ge1+K_2\varepsilon^2$ for all $\delta\in[0,\frac{1}{2K}]$ and all $\varepsilon\in[0,\mathfrak{c}_s)$ and we infer that $1+K_1\delta\ge1+K_2\varepsilon^2(U)$, that is $\delta=\|\,|U|-r_0\|_{L^\infty(\mathbb{R}^N)}\ge\frac{K_2}{K_1}\varepsilon^2(U)$. □

3.6 Initial bounds for $\mathcal{A}_\varepsilon$

Let $U_c\in\mathcal{E}$ be a travelling wave to (NLS) of speed c provided by Theorems 1 or 2 if $N=2$, respectively by Theorem 3 if $N=3$, such that $\frac{r_0}{2}\le|U|\le\frac{3r_0}{2}$ in $\mathbb{R}^N$. As in (16), we write $U_c(x)=\rho(x)e^{i\phi(x)}=r_0\sqrt{1+\varepsilon^2\mathcal{A}_\varepsilon(z)}\ \mathrm{e}^{i\varepsilon\varphi_\varepsilon(z)}$, where $\varepsilon=\sqrt{\mathfrak{c}_s^2-c^2}$, $z_1=\varepsilon x_1$, $z_\perp=\varepsilon^2x_\perp$. According to Proposition 2.2 p. 1078-1079 in [34] we have

$$\|U_c\|_{C_b^1(\mathbb{R}^N)}\le C\qquad\text{and}\qquad\|\nabla U_c\|_{W^{1,p}(\mathbb{R}^N)}\le C_p\quad\text{for }p\in[2,\infty).$$

By scaling, we obtain the initial (rough) estimates

$$\|\mathcal{A}_\varepsilon\|_{L^\infty}\le\frac{C}{\varepsilon^2},\quad\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^\infty}\le\frac{C}{\varepsilon^3},\quad\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^\infty}\le\frac{C}{\varepsilon^4},\quad\|\partial_{z_1}\varphi_\varepsilon\|_{L^\infty}\le\frac{C}{\varepsilon^2},\quad\|\nabla_{z_\perp}\varphi_\varepsilon\|_{L^\infty}\le\frac{C}{\varepsilon^3} \tag{58}$$

and

$$\Big\|\frac{\partial^2\mathcal{A}_\varepsilon}{\partial z_1^2}\Big\|_{L^p}\le C_p\varepsilon^{-4+\frac{2N-1}{p}},\qquad\Big\|\frac{\partial^2\mathcal{A}_\varepsilon}{\partial z_1\partial z_j}\Big\|_{L^p}\le C_p\varepsilon^{-5+\frac{2N-1}{p}},\qquad\Big\|\frac{\partial^2\mathcal{A}_\varepsilon}{\partial z_j\partial z_k}\Big\|_{L^p}\le C_p\varepsilon^{-6+\frac{2N-1}{p}} \tag{59}$$

for any $p\in[2,\infty)$ and all $j,k\in\{2,\dots,N\}$. We have:

Lemma 23 *Assume that (A2) and (A4) are satisfied and* $\Gamma \neq 0$. *Let* U_c *be a solution to* (TW$_c$) *provided by Theorem 2 if* $N = 2$, *respectively by Theorem 3 if* $N = 3$ *and let* $\varepsilon = \sqrt{\mathfrak{c}_s^2 - c^2}$. *If* $N = 3$ *we assume moreover that* $E(U_c) \le \frac{K}{\varepsilon}$, *where* K *does not depend on* ε.

There exist $\varepsilon_0 > 0$ *and* $C > 0$ *(depending only on* F, N, K*) such that* U_c *admits a lifting as in* (16) *whenever* $\varepsilon \in (0, \varepsilon_0)$ *and the following estimate holds:*

$$\int_{\mathbb{R}^N} |\partial_{z_1}\varphi_\varepsilon|^2 + |\nabla_{z_\perp}\varphi_\varepsilon|^2 + \mathcal{A}_\varepsilon^2 + |\partial_{z_1}\mathcal{A}_\varepsilon|^2 + \varepsilon^2|\nabla_{z_\perp}\mathcal{A}_\varepsilon|^2 \, dz \le C.$$

Proof. If $N = 2$ it follows from Theorem 2 that $k = \displaystyle\int_{\mathbb{R}^2} |\nabla U_c|^2 \, dx$ is small if ε is small. Using Lemma 13 in the case $N = 2$, respectively Corollary 15 if $N = 3$, we infer that $|U_c|$ is arbitrarily close to r_0 if ε is sufficiently small and then it is clear that we have a lifting as in (16).

We will repeatedly use the fact that there is a constant C depending only on F such that

$$C|\partial_j U_c|^2 \ge |\partial_j(\rho^2)|^2 + |\partial_j\phi|^2 \qquad \text{for } 1 \le j \le N.$$

In view of the Taylor expansion of V near r_0^2, for ε sufficiently close to 0 (so that $|U_c|$ is sufficiently close to r_0) we have

$$V(|U_c|^2) \ge C(|U_c| - r_0)^2.$$

By scaling, we infer that for some $\delta_1 > 0$ depending only on F there holds

$$E(U_c) = \int_{\mathbb{R}^N} |\nabla U_c|^2 + V(|U_c|^2) \, dx \ge \delta_1 \varepsilon^{5-2N} \int_{\mathbb{R}^N} |\partial_{z_1}\varphi_\varepsilon|^2 + \mathcal{A}_\varepsilon^2 \, dz.$$

In the case $N = 2$ it follows from Proposition 16 that $E(U_c) \le C\varepsilon$ for some C independent of ε. In the case $N = 3$ we use the assumption $E(U_c) \le \frac{K}{\varepsilon}$. In both cases the previous inequality implies that

$$\int_{\mathbb{R}^N} |\partial_{z_1}\varphi_\varepsilon|^2 + \mathcal{A}_\varepsilon^2 \, dz \le C. \tag{60}$$

We have $E_c(U_c) = T_c = \mathcal{O}(\varepsilon)$ if $N = 3$ by Proposition 9 (ii), respectively $E_c(U_c) = \mathcal{O}(k\varepsilon^2) = \mathcal{O}(\varepsilon^3)$ by (46) and (15) in the case $N = 2$. From the Pohozaev identity $P_c(U_c) = 0$ (see (4)) we deduce

$$\frac{2r_0^2\varepsilon^{7-2N}}{N-1} \int_{\mathbb{R}^N} |\nabla_{z_\perp}\varphi_\varepsilon|^2 + \varepsilon^2|\nabla_{z_\perp}\mathcal{A}_\varepsilon|^2 \, dz \le C\frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_\perp U_c|^2 \, dx = CE_c(U_c) = \mathcal{O}(\varepsilon^{7-2N}).$$

Thus we get

$$\int_{\mathbb{R}^N} |\nabla_{z_\perp}\varphi_\varepsilon|^2 + \varepsilon^2|\nabla_{z_\perp}\mathcal{A}_\varepsilon|^2 \, dz \le C. \tag{61}$$

Furthermore, by scaling the identity (47) in Lemma 21 we obtain

$$r_0^2\varepsilon^{7-2N} \int_{\mathbb{R}^N} |\partial_{z_1}\mathcal{A}_\varepsilon|^2 \, dz \le C\int_{\mathbb{R}^N} |\partial_{x_1}\rho|^2 \, dx \le C\int_{\mathbb{R}^N} |\nabla\rho|^2 \, dx = C\frac{N}{2}E_c(U_c) = \mathcal{O}(\varepsilon^{7-2N}),$$

so that

$$\int_{\mathbb{R}^N} |\partial_{z_1}\mathcal{A}_\varepsilon|^2 \, dz \le C. \tag{62}$$

Gathering (60), (61) and (62) yields the desired inequality. □

Using the above estimates, we shall find L^q bounds for $\mathcal{A}_\varepsilon$. The proof is based on equation (20), that is

$$\Big\{\partial_{z_1}^4 - \partial_{z_1}^2 - \mathfrak{c}_s^2\Delta_{z_\perp} + 2\varepsilon^2\partial_{z_1}^2\Delta_{z_\perp} + \varepsilon^4\Delta_{z_\perp}^2\Big\}\mathcal{A}_\varepsilon = \mathcal{R}_\varepsilon, \tag{20}$$

where

$$\begin{aligned}\mathcal{R}_\varepsilon = {} & \{\partial_{z_1}^2 + \varepsilon^2\Delta_{z_\perp}\}\Big[2(1+\varepsilon^2\mathcal{A}_\varepsilon)\Big((\partial_{z_1}\varphi\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}\varphi_\varepsilon|^2\Big) + \varepsilon^2\frac{(\partial_{z_1}\mathcal{A}_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}\mathcal{A}_\varepsilon|^2}{2(1+\varepsilon^2\mathcal{A}_\varepsilon)}\Big] \\ & - 2c\varepsilon^2\Delta_{z_\perp}(\mathcal{A}_\varepsilon\partial_{z_1}\varphi_\varepsilon) + 2c\varepsilon^2\sum_{j=2}^N \partial_{z_1}\partial_{z_j}(\mathcal{A}_\varepsilon\partial_{z_j}\varphi_\varepsilon) \\ & + \{\partial_{z_1}^2 + \varepsilon^2\Delta_{z_\perp}\}\Big[\mathfrak{c}_s^2\Big(1 - \frac{r_0^4F''(r_0^2)}{\mathfrak{c}_s^2}\Big)\mathcal{A}_\varepsilon^2 - \frac{1}{\varepsilon^4}\tilde{F}_3(r_0^2\varepsilon^2\mathcal{A}_\varepsilon)\Big]\end{aligned}$$

and we recall that $\tilde{F}_3(\alpha) = \mathcal{O}(\alpha^3)$ as $\alpha \to 0$.

Let

$$D_\varepsilon(\xi) = \xi_1^4 + \xi_1^2 + \mathfrak{c}_s^2|\xi_\perp|^2 + 2\varepsilon^2\xi_1^2|\xi_\perp|^2 + \varepsilon^4|\xi_\perp|^4 = (\xi_1^2 + \varepsilon^2|\xi_\perp|^2)^2 + \xi_1^2 + \mathfrak{c}_s^2|\xi_\perp|^2.$$

We will consider the following kernels:

$$\mathcal{K}_\varepsilon^1(z) = \mathscr{F}^{-1}\Big(\frac{\xi_1^2}{D_\varepsilon(\xi)}\Big), \qquad \mathcal{K}_\varepsilon^\perp(z) = \mathscr{F}^{-1}\Big(\frac{|\xi_\perp|^2}{D_\varepsilon(\xi)}\Big) \qquad \text{and} \qquad \mathcal{K}_\varepsilon^{1,j}(z) = \mathscr{F}^{-1}\Big(\frac{\xi_1\xi_j}{D_\varepsilon(\xi)}\Big), \quad j = 2, \dots, N.$$

Then we may rewrite (20) as a convolution equation

$$\mathcal{A}_\varepsilon = \Big(\mathcal{K}_\varepsilon^1 + \varepsilon^2\mathcal{K}_\varepsilon^\perp\Big) * G_\varepsilon + 2c\varepsilon^2\mathcal{K}_\varepsilon^\perp * (\mathcal{A}_\varepsilon\partial_{z_1}\varphi_\varepsilon) - 2c(\varepsilon)\varepsilon^2\sum_{j=2}^N \mathcal{K}_\varepsilon^{1,j} * (\mathcal{A}_\varepsilon\partial_{z_j}\varphi_\varepsilon), \tag{63}$$

where

$$G_\varepsilon = (1 + \varepsilon^2\mathcal{A}_\varepsilon)\Big((\partial_{z_1}\varphi_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}\varphi_\varepsilon|^2\Big) + \varepsilon^2\frac{(\partial_{z_1}\mathcal{A}_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}\mathcal{A}_\varepsilon|^2}{4(1 + \varepsilon^2\mathcal{A}_\varepsilon)}$$
$$+ \frac{\mathfrak{c}_s^2}{4}(\Gamma - 2)\mathcal{A}_\varepsilon^2 - \frac{1}{\varepsilon^4}\tilde{F}_3(r_0^2\varepsilon^2\mathcal{A}_\varepsilon).$$

Lemma 24 *The following estimates hold for $N = 2$, 3 and ε small enough:*

- *(i) For all $2 \le p \le \infty$ we have $\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} \le C\varepsilon^{\frac{6}{p}-3}$.*
- *(ii) There exists $C > 0$ such that $\|\mathcal{A}_\varepsilon\|_{L^{3q}} \le C\varepsilon^{-\frac{2}{3}}\|\mathcal{A}_\varepsilon\|_{L^{2q}}^{\frac{2}{3}}$ for any $1 \le q \le \infty$.*
- *(iii) If $N = 3$, for any $2 \le p < 8/3$ there is $C_p > 0$ such that $\|\mathcal{A}_\varepsilon\|_{L^p(\mathbb{R}^3)} \le C_p$.*
- *(iv) If $N = 2$, for any $2 \le p < 4$ there is $C_p > 0$ such that $\|\mathcal{A}_\varepsilon\|_{L^p(\mathbb{R}^2)} \le C_p$.*

Proof. For (i), it suffices to notice that the estimate is true for $p = 2$ by Lemma 23 and for $p = \infty$ by (58), therefore it holds for any $2 \le p \le \infty$ by interpolation. For (ii) we just interpolate the exponent $3q$ between $2q$ and ∞ and we use (58):

$$\|\mathcal{A}_\varepsilon\|_{L^{3q}} \le \|\mathcal{A}_\varepsilon\|_{L^{2q}}^{\frac{2}{3}}\|\mathcal{A}_\varepsilon\|_{L^\infty}^{\frac{1}{3}} \le C\varepsilon^{-\frac{2}{3}}\|\mathcal{A}_\varepsilon\|_{L^{2q}}^{\frac{2}{3}}.$$

Next we prove (iii). As already mentioned, a uniform L^p bound (for $2 \le p \le 8/3$) on the kernels $\mathcal{K}_\varepsilon^1$, $\varepsilon^2\mathcal{K}_\varepsilon^\perp$ and $\varepsilon^2\mathcal{K}_\varepsilon^{1,j}$ is established in [8] by using a Sobolev estimate. Unfortunately this is no longer possible in dimension $N = 3$. We thus rely on a suitable decomposition of $\mathcal{A}_\varepsilon$ in the Fourier space. Some terms are controlled by using the energy bounds in Lemma 23, the others by using (63).

We consider a set of parameters α, β, $\gamma \in (1, 2)$ and $\nu > 5/2$ with $\alpha \ge \beta$ and $\alpha \ge \gamma$ (to be fixed later). For $\varepsilon \in (0, 1)$, let

$$E^I = \{\xi \in \mathbb{R}^N \mid |\xi_\perp| < 1\}, \quad E^{II} = \{\xi \in \mathbb{R}^N \mid |\xi_\perp| > \varepsilon^{-\alpha}\}, \quad E^{III} = \{\xi \in \mathbb{R}^N \mid \varepsilon^{-\beta} \le |\xi_\perp| \le \varepsilon^{-\alpha},\ |\xi_1| < 1\},$$

$$E^{IV} = \{\xi \in \mathbb{R}^N \mid \varepsilon^{-\gamma} \le |\xi_\perp| \le \varepsilon^{-\alpha},\ 1 \le |\xi_1|^\nu \le |\xi_\perp|\}, \qquad E^V = \{\xi \in \mathbb{R}^N \mid 1 \le |\xi_\perp| \le \varepsilon^{-\alpha},\ |\xi_1|^\nu > |\xi_\perp|\},$$

$$E^{VI} = \{\xi \in \mathbb{R}^N \mid 1 \le |\xi_\perp| < \varepsilon^{-\beta},\ |\xi_1| < 1\}, \qquad E^{VII} = \{\xi \in \mathbb{R}^N \mid 1 \le |\xi_\perp| < \varepsilon^{-\gamma},\ 1 \le |\xi_1|^\nu \le |\xi_\perp|\}.$$

It is easy to see that the sets $E^I, \dots, E^{VII}$ are disjoint and cover $\mathbb{R}^N$. For $J \in \{I, \dots, VII\}$ we denote $\mathcal{A}_\varepsilon^J = \mathscr{F}^{-1}(\widehat{\mathcal{A}}_\varepsilon\mathbf{1}_{E^J})$, so that $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^I + \dots + \mathcal{A}_\varepsilon^{VII}$, and we estimate each term separately.

For $\mathcal{A}_\varepsilon^I$ we use

$$\|\nabla_{z_\perp}\mathcal{A}_\varepsilon^I\|_{L^2} = \|\xi_\perp\widehat{\mathcal{A}}_\varepsilon\mathbf{1}_{\{|\xi_\perp|<1\}}\|_{L^2} \le \|\widehat{\mathcal{A}}_\varepsilon\mathbf{1}_{\{|\xi_\perp|\le 1\}}\|_{L^2} \le \|\widehat{\mathcal{A}}_{\mathbf{E}}\|_{L^2} = \|\mathcal{A}_{\mathbf{E}}\|_{L^2} \le C.$$

By Lemma 23, $\mathcal{A}_\varepsilon$ and $\partial_{z_1}\mathcal{A}_\varepsilon$ are uniformly bounded in L^2, thus we have

$$\|\mathcal{A}_\varepsilon^I\|_{L^2} + \|\partial_{z_1}\mathcal{A}_\varepsilon^I\|_{L^2} \le C.$$

Hence $\mathcal{A}_\varepsilon^I$ is uniformly bounded in H^1, and using the Sobolev embedding we deduce

$$\forall\, 2 \le p \le 6, \qquad \|\mathcal{A}_\varepsilon^I\|_{L^p} \le C. \tag{64}$$

We will use the Riesz-Thorin theorem to bound $\mathcal{A}_\varepsilon^{II}$: if $1 < q = \frac{p}{p-1} < 2$ is the conjugate exponent of $p \in (2, \infty)$, there holds

$$\|\mathcal{A}_\varepsilon^{II}\|_{L^p} \le C\|\widehat{\mathcal{A}}_\varepsilon^{II}\|_{L^q}.$$

Thus it suffices to bound $\|\widehat{\mathcal{A}}_\varepsilon^{II}\|_{L^q}$. Using the Hölder inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, we have

$$\begin{aligned}
\|\widehat{\mathcal{A}}_\varepsilon^{II}\|_{L^q}^q &= \int_{\mathbb{R}^3} \Big((|\xi_1| + \varepsilon|\xi_\perp|)|\widehat{\mathcal{A}}_\varepsilon|\Big)^q \times \frac{\mathbf{1}_{\{|\xi_\perp| > \varepsilon^{-\alpha}\}}}{(|\xi_1| + \varepsilon|\xi_\perp|)^q}\, d\xi \\
&\le \|(|\xi_1| + \varepsilon|\xi_\perp|)\widehat{\mathcal{A}}_\varepsilon\|_{L^2}^q \left(\int_{\mathbb{R}^3} \frac{\mathbf{1}_{\{|\xi_\perp| \ge \varepsilon^{-\alpha}\}}}{(|\xi_1| + \varepsilon|\xi_\perp|)^{\frac{2q}{2-q}}}\, d\xi\right)^{\frac{2-q}{q}} \\
&\le C_q(\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^2} + \varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^2})^q \left(\int_{\varepsilon^{-\alpha}}^{\infty} \frac{R\,dR}{(\varepsilon R)^{\frac{3q-2}{2-q}}}\right)^{\frac{2-q}{q}}.
\end{aligned}$$

(We have computed the integral in ξ_1 and we used cylindrical coordinates for the third line.) Provided that $\frac{3q-2}{2-q} > 2$ (or, equivalently, $q > 6/5$), the last integral in R is

$$C(q)\varepsilon^{-\frac{3q-2}{2-q}} \times \varepsilon^{\alpha\frac{5q-6}{2-q}} \le C_q$$

as soon as $\alpha \ge \frac{3q-2}{5q-6} = \frac{2+p}{6-p}$, that is $p \le 6 - \frac{8}{\alpha+1}$. Notice that $2 < 6 - \frac{8}{\alpha+1} < 6$ because $\alpha > 1$. By Lemma 23 we get

$$\forall\, 2 \le p \le 6 - \frac{8}{\alpha+1}, \qquad \|\mathcal{A}_\varepsilon^{II}\|_{L^p} \le C(\alpha). \tag{65}$$

Using similar arguments, we have

$$\begin{aligned}
\|\mathcal{A}_\varepsilon^{III}\|_{L^p}^q &\le C\|\widehat{\mathcal{A}}_\varepsilon^{III}\|_{L^q}^q \\
&= C\int_{\mathbb{R}^3} \Big(\varepsilon|\xi_\perp| \cdot |\widehat{\mathcal{A}}_\varepsilon|\Big)^q \times \frac{\mathbf{1}_{\{\varepsilon^{-\beta} \le |\xi_\perp| \le \varepsilon^{-\alpha},\, |\xi_1| < 1\}}}{(\varepsilon|\xi_\perp|)^q}\, d\xi \\
&\le C(\varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^2})^q \left(\int_{\mathbb{R}^3} \frac{\mathbf{1}_{\{\varepsilon^{-\beta} \le |\xi_\perp| \le \varepsilon^{-\alpha},\, |\xi_1| \le 1\}}}{(\varepsilon|\xi_\perp|)^{\frac{2q}{2-q}}}\, d\xi\right)^{\frac{2-q}{q}} \\
&\le C_q\left(\varepsilon^{-\frac{2q}{2-q}} \int_{\varepsilon^{-\beta}}^{\varepsilon^{-\alpha}} \frac{dR}{R^{\frac{4q-4}{2-q}+1}}\right)^{\frac{2-q}{q}} \le C_q
\end{aligned}$$

if $\beta\frac{4q-4}{2-q} - \frac{2q}{2-q} \ge 0$, that is $2\beta \ge \frac{q}{(q-1)} = p$. Consequently,

$$\forall\, 2 \le p \le 2\beta, \qquad \|\mathcal{A}_\varepsilon^{III}\|_{L^p} \le C(\beta). \tag{66}$$

Similarly we get a bound for $\mathcal{A}_\varepsilon^{IV}$:

$$\begin{aligned}
\|\mathcal{A}_\varepsilon^{IV}\|_{L^p}^q &\le C(\varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^2})^q \left(\int_{\mathbb{R}^3} \frac{\mathbf{1}_{\{\varepsilon^{-\gamma} \le |\xi_\perp| \le \varepsilon^{-\alpha},\, 1 \le |\xi_1|^\nu \le |\xi_\perp|\}}}{(\varepsilon|\xi_\perp|)^{\frac{2q}{2-q}}}\, d\xi\right)^{\frac{2-q}{q}} \\
&\le C_q\left(\varepsilon^{-\frac{2q}{2-q}} \int_{\varepsilon^{-\gamma}}^{\varepsilon^{-\alpha}} \frac{R^{\frac{1}{\nu}}\, dR}{R^{\frac{4q-4}{2-q}+1}}\right)^{\frac{2-q}{q}} \le C_q
\end{aligned}$$

provided that $\gamma\frac{4q-4}{2-q} - \frac{2q}{2-q} - \frac{\gamma}{\nu} \ge 0$, which is equivalent to $p \le \frac{2\gamma(2\nu+1)}{2\nu+\gamma}$ (notice that $\frac{2\gamma(2\nu+1)}{2\nu+\gamma} > 2$ because $\gamma > 1$). Therefore,

$$\forall\, 2 \le p \le \frac{2\gamma(2\nu+1)}{2\nu+\gamma}, \qquad \|\mathcal{A}_\varepsilon^{IV}\|_{L^p} \le C(\nu). \tag{67}$$

We use the fact that $\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^2}$ is bounded independently of ε (see part (i)) in order to estimate $\mathcal{A}_\varepsilon^V$:

$$\begin{aligned}\|\mathcal{A}_\varepsilon^V\|_{L^p}^q &\le C\|\widehat{\mathcal{A}}_\varepsilon^V\|_{L^q}^q \\ &= C\int_{\mathbb{R}^3}|\xi_1\widehat{\mathcal{A}}_\varepsilon|^q \times \frac{\mathbf{1}_{\{1\le|\xi_\perp|\le\varepsilon^{-\alpha},\,|\xi_\perp|<|\xi_1|^\nu\}}}{|\xi_1|^q}\,d\xi \\ &\le C\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^2}^q\left(\int_{\mathbb{R}^3}\frac{\mathbf{1}_{\{1\le|\xi_\perp|\le\varepsilon^{-\alpha},\,|\xi_\perp|\le|\xi_1|^\nu\}}}{|\xi_1|^{\frac{2q}{2-q}}}\,d\xi\right)^{\frac{2-q}{2}} \\ &\le C\left(\int_1^{\varepsilon^{-\alpha}}\frac{R\,dR}{R^{(\frac{2q}{2-q}-1)/\nu}}\right)^{\frac{2-q}{2}},\end{aligned}$$

by using cylindrical coordinates in the fourth line. We have $\frac{2q}{2-q}>1$ for $q\in[1,2)$ and the last integral is bounded independently of ε as soon as $\frac{1}{\nu}\left(\frac{2q}{2-q}-1\right)>2$, that is $p<\frac{4\nu+2}{2\nu-1}$. It is obvious that $\frac{4\nu+2}{2\nu-1}>2$ for $\nu>1/2$. As a consequence, we get

$$\forall\, 2\le p<\frac{4\nu+2}{2\nu-1},\qquad \|\mathcal{A}_\varepsilon^V\|_{L^p}\le C(p). \tag{68}$$

We use the convolution equation (63) to estimate $\mathcal{A}_\varepsilon^{VI}$ and $\mathcal{A}_\varepsilon^{VII}$. Applying the Fourier transform to (63) we obtain the pointwise bound

$$\begin{aligned}|\widehat{\mathcal{A}}_\varepsilon(\xi)| &= \left|\left(\widehat{\mathcal{K}}_\varepsilon^1+\varepsilon^2\widehat{\mathcal{K}}_\varepsilon^\perp\right)\widehat{G}_\varepsilon+2c(\varepsilon)\widehat{\mathcal{K}}_\varepsilon^\perp\mathscr{F}(\mathcal{A}_\varepsilon\partial_{z_1}\varphi_\varepsilon)-2c(\varepsilon)\varepsilon^2\sum_{j=2}^N\widehat{\mathcal{K}}_\varepsilon^{1,j}\mathscr{F}(\mathcal{A}_\varepsilon\partial_{z_j}\varphi_\varepsilon)\right| \\ &\le C\Big(|\widehat{\mathcal{K}}_\varepsilon^1|+\varepsilon^2|\widehat{\mathcal{K}}_\varepsilon^\perp|+\varepsilon^2\sum_{j=2}^N|\widehat{\mathcal{K}}_\varepsilon^{1,j}|\Big)\Big(\|\widehat{G}_\varepsilon\|_{L^\infty}+\|\mathscr{F}(\mathcal{A}_\varepsilon\partial_{z_1}\varphi_\varepsilon)\|_{L^\infty}+\sum_{j=2}^N\|\mathscr{F}(\mathcal{A}_\varepsilon\partial_{z_j}\varphi_\varepsilon)\|_{L^\infty}\Big).\end{aligned}$$

The estimates in Lemma 23 and the boundedness of $\mathscr{F}:L^1\to L^\infty$ imply that the second factor is bounded independently of ε. Therefore

$$|\widehat{\mathcal{A}}_\varepsilon(\xi)|\le C\Big(|\widehat{\mathcal{K}}_\varepsilon^1|+\varepsilon^2|\widehat{\mathcal{K}}_\varepsilon^\perp|+\varepsilon^2\sum_{j=2}^N|\widehat{\mathcal{K}}_\varepsilon^{1,j}|\Big)\le C\frac{\xi_1^2+\varepsilon^2|\xi_\perp|^2+\varepsilon^2|\xi_1|\cdot|\xi_\perp|}{D_\varepsilon(\xi)}\le C\frac{\xi_1^2+\varepsilon^2|\xi_\perp|^2}{D_\varepsilon(\xi)} \tag{69}$$

because $2\varepsilon^2|\xi_1|\cdot|\xi_\perp|\le\xi_1^2+\varepsilon^4|\xi_\perp|^2$. If $\xi\in E^{VI}$ we have $|\xi_1|\le 1$ and $1\le|\xi_\perp|\le\varepsilon^{-\beta}\le\varepsilon^{-2}$ (because $\beta<2$), hence there is some constant C depending only on $\mathfrak{c}_s$ such that

$$C|\xi_\perp|^2\ge D_\varepsilon(\xi)=\xi_1^4+\xi_1^2+\mathfrak{c}_s^2|\xi_\perp|^2+2\varepsilon^2\xi_1^2|\xi_\perp|^2+\varepsilon^4|\xi_\perp|^4\ge\frac{|\xi_\perp|^2}{C}.$$

Using the Riesz-Thorin theorem with exponents $2<p<\infty$ and $q=p/(p-1)\in(1,2)$ as well as (69) we find

$$\begin{aligned}\|\mathcal{A}_\varepsilon^{VI}\|_{L^p}^q &\le C\|\widehat{\mathcal{A}}_\varepsilon^{VI}\|_{L^q}^q \\ &\le C\int_{\mathbb{R}^3}\mathbf{1}_{\{1\le|\xi_\perp|\le\varepsilon^{-\beta},\,|\xi_1|\le1\}}\frac{(\xi_1^2+\varepsilon^2|\xi_\perp|^2)^q}{|\xi_\perp|^{2q}}\,d\xi \\ &\le C\int_{\mathbb{R}^3}\mathbf{1}_{\{1\le|\xi_\perp|\le\varepsilon^{-\beta},\,|\xi_1|\le1\}}\left(\frac{\xi_1^{2q}}{|\xi_\perp|^{2q}}+\varepsilon^{2q}\right)\,d\xi \\ &\le C\int_{|\xi_\perp|\ge1}\frac{d\xi_\perp}{|\xi_\perp|^{2q}}+C\varepsilon^{2q-2\beta}\le C_q\end{aligned}$$

provided that $q>1$ and $q\ge\beta$. We have $q\ge\beta$ if and only if $p\le\frac{\beta}{\beta-1}$. It is obvious that $\frac{\beta}{\beta-1}>2$ because $1<\beta<2$. Hence we obtain

$$\forall\, 2\le p\le\frac{\beta}{\beta-1},\qquad \|\mathcal{A}_\varepsilon^{VI}\|_{L^p}\le C(\beta). \tag{70}$$

In order to estimate $\mathcal{A}_\varepsilon^{VII}$ we notice that for $\xi \in E^{VII}$ we have $1 \le |\xi_\perp| \le \varepsilon^{-\gamma}$ and $1 \le |\xi_1|^\nu \le |\xi_\perp|$, thus $|\xi_1|^2 \le |\xi_\perp| \le \varepsilon^{-2}$ because $\nu \ge 5/2 > 2$ and $\gamma \le 2$. Hence there exists $C > 0$ depending only on $\mathfrak{c}_s$ such that

$$C|\xi_\perp|^2 \ge D_\varepsilon(\xi) = \xi_1^4 + \xi_1^2 + \mathfrak{c}_s^2|\xi_\perp|^2 + 2\varepsilon^2\xi_1^2|\xi_\perp|^2 + \varepsilon^4|\xi_\perp|^4 \ge \frac{|\xi_\perp|^2}{C}.$$

Using (69) we get

$$\|\mathcal{A}_\varepsilon^{VII}\|_{L^p}^q \le C\int_{\mathbb{R}^3} \mathbf{1}_{\{1\le|\xi_\perp|\le\varepsilon^{-\gamma},\, 1\le|\xi_1|^\nu\le|\xi_\perp|\}} \left(\frac{\xi_1^{2q}}{|\xi_\perp|^{2q}} + \varepsilon^{2q}\right) d\xi$$
$$\le C\int_{|\xi_\perp|\ge 1} \frac{|\xi_\perp|^{\frac{2q+1}{\nu}}}{|\xi_\perp|^{2q}}\, d\xi_\perp + C\varepsilon^{2q}\int_1^{\varepsilon^{-\gamma}} R^{1+\frac{1}{\nu}}\, dR \le C_q$$

provided that $2q - \frac{2q+1}{\nu} > 2$ and $2q - \gamma(2+\frac{1}{\nu}) \ge 0$. These inequalities are equivalent to $p < \frac{2\nu+1}{3}$ and $p \le \frac{\gamma(2\nu+1)}{\gamma(2\nu+1)-2\nu}$, respectively. Since $\nu > 5/2$, we have $\frac{2\nu+1}{3} > 2$ and $\frac{4\nu}{2\nu+1} > 5/3$ and $\frac{\nu}{\nu-1} < 5/3$. It is easy to see that $\frac{\gamma(2\nu+1)}{\gamma(2\nu+1)-2\nu} > 2$ if and only if $\gamma < \frac{4\nu}{2\nu+1}$, and that $\frac{\gamma(2\nu+1)}{\gamma(2\nu+1)-2\nu} > \frac{2\nu+1}{3}$ if and only if $\gamma < \frac{\nu}{\nu-1}$. Hence

$$\begin{cases} \forall\, 1 \le \gamma \le \dfrac{\nu}{\nu-1}, \quad \forall\, 2 \le p < \dfrac{2\nu+1}{3}, & \|\mathcal{A}_\varepsilon^{VII}\|_{L^p} \le C(p,\nu) \\[2ex] \forall\, \dfrac{\nu}{\nu-1} < \gamma \le \dfrac{5}{3}, \quad \forall\, 2 \le p \le \dfrac{\gamma(2\nu+1)}{\gamma(2\nu+1)-2\nu}, & \|\mathcal{A}_\varepsilon^{VII}\|_{L^p} \le C(\gamma,\nu). \end{cases} \tag{71}$$

We now choose the parameters α, β, γ and ν. In view of (66) and (70), we fix $\beta = 3/2$, so that $2\beta = \beta/(\beta-1) = 3$. We set $\alpha = 5/3 > 3/2 = \beta$. Then by (64), (65), (66) and (70) it follows that

$$\forall\, 2 \le p \le 3, \qquad \|\mathcal{A}_\varepsilon^{I}\|_{L^p} + \|\mathcal{A}_\varepsilon^{II}\|_{L^p} + \|\mathcal{A}_\varepsilon^{III}\|_{L^p} + \|\mathcal{A}_\varepsilon^{VI}\|_{L^p} \le C.$$

For the other terms, we notice that in the case $1 \le \gamma \le \frac{\nu}{\nu-1}$ we have

$$\frac{2\gamma(2\nu+1)}{2\nu+\gamma} \le \frac{4\nu+2}{2\nu-1},$$

with equality if $\gamma = \frac{\nu}{\nu-1}$. We also observe that

$$\frac{2\nu+1}{3} < \frac{4\nu+2}{2\nu-1} < \frac{8}{3} \quad \text{if} \quad \nu < \frac{7}{2}, \qquad \text{respectively} \qquad \frac{8}{3} < \frac{4\nu+2}{2\nu-1} < \frac{2\nu+1}{3} \quad \text{if} \quad \nu > \frac{7}{2}.$$

Then we fix $\nu = 7/2$ and $\gamma = \frac{\nu}{\nu-1} = 7/5 < 5/3$ and using (67), (68) and (71) we obtain

$$\forall\, 2 \le p < \frac{8}{3}, \qquad \|\mathcal{A}_\varepsilon^{IV}\|_{L^p} + \|\mathcal{A}_\varepsilon^{V}\|_{L^p} + \|\mathcal{A}_\varepsilon^{VII}\|_{L^p} \le C.$$

This concludes the proof of *(iii)*.

(iv) We use the same inequalities as in the three-dimensional case with $1 < \nu < 3$ and α, β, $\gamma \in (1,2)$ satisfying $\beta \le \alpha$ and $\gamma \le \alpha$. We get

$$\forall\, 2 \le p < \infty, \quad \|\mathcal{A}_\varepsilon^{I}\|_{L^p} \le C_p; \qquad \forall\, 2 \le p \le 4\alpha - 2, \quad \|\mathcal{A}_\varepsilon^{II}\|_{L^p} \le C_p;$$

$$\forall\, 2 \le p \le \frac{2\beta}{2-\beta}, \quad \|\mathcal{A}_\varepsilon^{III}\|_{L^p} \le C(\beta); \qquad \forall\, 2 \le p \le \frac{2\gamma(\nu+1)}{\gamma+\nu(2-\gamma)}, \quad \|\mathcal{A}_\varepsilon^{IV}\|_{L^p} \le C(\beta);$$

$$\forall\, 2 \le p < 2\frac{\nu+1}{\nu-1}, \quad \|\mathcal{A}_\varepsilon^{V}\|_{L^p} \le C_p; \qquad \forall\, 2 \le p < \infty, \quad \|\mathcal{A}_\varepsilon^{VI}\|_{L^p} \le C_p$$

and

$$\forall\, 1 \le \gamma \le \frac{\nu}{\nu-1}, \quad \forall\, 2 \le p < \frac{\nu+1}{3-\nu}, \qquad \|\mathcal{A}_\varepsilon^{VII}\|_{L^p} \le C_p.$$

Then we choose

$$\beta = \frac{4}{3}, \qquad \alpha = \frac{5}{3}, \qquad \nu = 3^-, \qquad \gamma = \frac{\nu}{\nu-1} = \frac{3}{2}^+,$$

so that $\alpha > \beta$ and $\alpha > \gamma$. We infer that

$$\forall\, 2 \le p < 4, \qquad \|\mathcal{A}_\varepsilon\|_{L^p} \le C_p.$$

This completes the proof in the case $N = 2$. $\square$

3.7 Proof of Proposition 17

We first recall the Fourier multiplier properties of the kernels $\mathcal{K}_\varepsilon^1$, $\mathcal{K}_\varepsilon^\perp$ and $\mathcal{K}_\varepsilon^{1,j}$. We skip the proof since it is the same as in section 5.2 in [8] and does not depend on the space dimension N.

Lemma 25 *Let $1 < q < \infty$. There exists $C_q > 0$ (depending also on $\mathfrak{c}_s$) such that for any $\varepsilon \in (0,1)$, any $2 \le j \le N$ and $h \in L^q$ we have*

$$\begin{aligned}
&\|\mathcal{K}_\varepsilon^1 \star h\|_{L^q}\\
&+ \|\partial_{z_1}\mathcal{K}_\varepsilon^1 \star h\|_{L^q} + \|\nabla_{z_\perp}\mathcal{K}_\varepsilon^1 \star h\|_{L^q}\\
&+ \|\partial_{z_1}^2\mathcal{K}_\varepsilon^1 \star h\|_{L^q} + \varepsilon\|\partial_{z_1}\nabla_{z_\perp}\mathcal{K}_\varepsilon^1 \star h\|_{L^q} + \varepsilon^2\|\nabla_{z_\perp}^2\mathcal{K}_\varepsilon^1 \star h\|_{L^q} \le C_q\|h\|_{L^q},
\end{aligned}$$

$$\begin{aligned}
&\|\mathcal{K}_\varepsilon^\perp \star h\|_{L^q}\\
&+ \varepsilon\|\partial_{z_1}\mathcal{K}_\varepsilon^\perp \star h\|_{L^q} + \varepsilon^2\|\nabla_{z_\perp}\mathcal{K}_\varepsilon^\perp \star h\|_{L^q}\\
&+ \varepsilon^2\|\partial_{z_1}^2\mathcal{K}_\varepsilon^\perp \star h\|_{L^q} + \varepsilon^3\|\partial_{z_1}\nabla_{z_\perp}\mathcal{K}_\varepsilon^\perp \star h\|_{L^q} + \varepsilon^4\|\nabla_{z_\perp}^2\mathcal{K}_\varepsilon^\perp \star h\|_{L^q} \le C_q\|h\|_{L^q}
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{K}_\varepsilon^{1,j} \star h\|_{L^q}\\
&+ \|\partial_{z_1}\mathcal{K}_\varepsilon^{1,j} \star h\|_{L^q} + \varepsilon\|\nabla_{z_\perp}\mathcal{K}_\varepsilon^{1,j} \star h\|_{L^q}\\
&+ \varepsilon\|\partial_{z_1}^2\mathcal{K}_\varepsilon^{1,j} \star h\|_{L^q} + \varepsilon^2\|\partial_{z_1}\nabla_{z_\perp}\mathcal{K}_\varepsilon^{1,j} \star h\|_{L^q} + \varepsilon^3\|\nabla_{z_\perp}^2\mathcal{K}_\varepsilon^{1,j} \star h\|_{L^q} \le C_q\|h\|_{L^q}.
\end{aligned}$$

The proof of (21) is then divided into 5 Steps.

Step 1. There is $\varepsilon_1 > 0$ and for any $1 < q < \infty$ there exists C_q (depending also on F) such that for all $\varepsilon \in (0, \varepsilon_1)$,

$$\begin{aligned}
\|\mathcal{A}_\varepsilon\|_{L^q} + &\|\nabla_z\mathcal{A}_\varepsilon\|_{L^q} + \|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon\|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon^2\|\nabla_{z_\perp}^2\mathcal{A}_\varepsilon\|_{L^q}\\
&\le C_q\Big(\|\mathcal{A}_\varepsilon\|_{L^{2q}}^2 + \varepsilon^2\Big[\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^{2q}} + \varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^{2q}}\Big]^2\Big).
\end{aligned}$$

The proof is very similar to that of Lemma 6.2 p. 268 in [8] and thus is only sketched. Indeed, if $U = \rho e^{i\phi}$ is a finite energy solution to (TW$_c$) such that $\frac{r_0}{2} \le \rho \le 2r_0$ then the first equation in (17) can be written as

$$2r_0^2\Delta\phi = c\frac{\partial}{\partial x_1}(\rho^2 - r_0^2) - 2\mathrm{div}\left((\rho^2 - r_0^2)\nabla\phi\right)$$

and this gives

$$2r_0^2\frac{\partial\phi}{\partial x_j} = cR_jR_1(\rho^2 - r_0^2) - 2\sum_{k=1}^N R_jR_k\left((\rho^2 - r_0^2)\frac{\partial\phi}{\partial x_k}\right),$$

where R_k is the Riesz transform (defined by $R_kf = \mathscr{F}^{-1}\left(\frac{i\xi_k}{|\xi|}\widehat{f}\right)$). It is well-known that the Riesz transform maps continuously $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. From the above we infer that for any $q \in (1,\infty)$ and any $j \in \{1,\dots,N\}$ we have

$$\Big\|\frac{\partial\phi}{\partial x_j}\Big\|_{L^q} \le C(q)\|\rho^2 - r_0^2\|_{L^q} + C(q)\sum_{k=1}^N\Big\|(\rho^2 - r_0^2)\frac{\partial\phi}{\partial x_j}\Big\|_{L^q} \le C(q)\|\rho^2 - r_0^2\|_{L^q} + C(q)\|\rho^2 - r_0^2\|_{L^\infty}\|\nabla\phi\|_{L^q}$$

and this implies

$$\|\nabla\phi\|_{L^q} \le C(q)\|\rho^2 - r_0^2\|_{L^q} + C(q)\|\rho^2 - r_0^2\|_{L^\infty}\|\nabla\phi\|_{L^q}.$$

If $\|\rho^2 - r_0^2\|_{L^\infty}$ is sufficiently small we get $\|\nabla\phi\|_{L^q} \le \tilde{C}(q)\|\rho^2 - r_0^2\|_{L^q} \le K(q)\|\rho - r_0\|_{L^q}$. By scaling, this estimate implies that for $1 < q < \infty$,

$$\|\partial_{z_1}\varphi_\varepsilon\|_{L^q} + \varepsilon\|\nabla_{z_\perp}\varphi_\varepsilon\|_{L^q} \le C_q\|\mathcal{A}_\varepsilon\|_{L^q}. \tag{72}$$

Hence, by Hölder's inequality and Lemma 24 (ii),

$$\|G_\varepsilon\|_{L^q} \le C_q\Big(\|\mathcal{A}_\varepsilon\|_{L^{2q}}^2 + \varepsilon^2\|\mathcal{A}_\varepsilon\|_{L^{3q}}^3 + \varepsilon^2\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^{2q}}^2 + \varepsilon^4\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^{2q}}^2\Big)$$
$$\le C_q\Big(\|\mathcal{A}_\varepsilon\|_{L^{2q}}^2 + \varepsilon^2\Big[\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^{2q}} + \varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^{2q}}\Big]^2\Big).$$

We take the derivatives up to order 2 of (63) and then the conclusion follows from Lemma 25.

Step 2. Let $N = 3$. There is $\varepsilon_2 > 0$ and for any $1 < p < 3/2$ there exists C_p (also depending on F) such that for any $\varepsilon \in (0, \varepsilon_2)$ there holds

$$\|\mathcal{A}_\varepsilon\|_{L^p} + \|\nabla\mathcal{A}_\varepsilon\|_{L^p} + \|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon^2\|\nabla_{z_\perp}^2\mathcal{A}_\varepsilon\|_{L^p} \le C_p.$$

If $1 \le q \le 3/2$, we have by Lemma 24 (i)

$$\varepsilon\Big[\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^{2q}} + \varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^{2q}}\Big] \le C.$$

Thus for $1 < q \le 3/2$ we infer from Step 1 that

$$\|\mathcal{A}_\varepsilon\|_{L^q} + \|\nabla_z\mathcal{A}_\varepsilon\|_{L^q} + \|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon\|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon^2\|\nabla_{z_\perp}^2\mathcal{A}_\varepsilon\|_{L^q} \le C_q + C_q\|\mathcal{A}_\varepsilon\|_{L^{2q}}^2. \qquad (73)$$

If $1 < p < 4/3$, we use (73) combined with Lemma 24 (iii) with exponent $2p \in [2, 8/3)$ to get

$$\|\mathcal{A}_\varepsilon\|_{L^p} + \|\nabla_z\mathcal{A}_\varepsilon\|_{L^p} + \|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon^2\|\nabla_{z_\perp}^2\mathcal{A}_\varepsilon\|_{L^p} \le C_p. \qquad (74)$$

This proves Step 2 for $1 < p < 4/3$. In dimension $N = 3$, the Sobolev inequality does not enable us to improve the L^q integrability of $\mathcal{A}_\varepsilon$ to some $q > 8/3$. We thus rely on the decomposition of $\mathcal{A}_\varepsilon$ as $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^I + \mathcal{A}_\varepsilon^{II} + \mathcal{A}_\varepsilon^{III} + \mathcal{A}_\varepsilon^{IV} + \mathcal{A}_\varepsilon^{V} + \mathcal{A}_\varepsilon^{VI} + \mathcal{A}_\varepsilon^{VII}$, exactly as in Lemma 24. We choose $\alpha = 5/3$, $\beta = 3/2$. By the estimates in the proof of Lemma 24 (iii) we have then

$$\forall\, 2 \le p \le 3, \qquad \|\mathcal{A}_\varepsilon^I\|_{L^p} + \|\mathcal{A}_\varepsilon^{II}\|_{L^p} + \|\mathcal{A}_\varepsilon^{III}\|_{L^p} + \|\mathcal{A}_\varepsilon^{VI}\|_{L^p} \le C.$$

It remains to bound $\mathcal{A}_\varepsilon^{IV}$, $\mathcal{A}_\varepsilon^{V}$ and $\mathcal{A}_\varepsilon^{VII}$ in L^{3^-}. In view of (68), we choose $\nu = 5/2$, so that $\frac{4\nu+2}{2\nu-1} = 3$, and thus

$$\forall\, 2 \le p < 3, \qquad \|\mathcal{A}_\varepsilon^{V}\|_{L^p} \le C_p.$$

We cancel out $\mathcal{A}_\varepsilon^{IV}$ by taking $\gamma = 5/3 = \alpha$. Next we turn our attention to the "bad term" $\mathcal{A}_\varepsilon^{VII}$. By (74) we get

$$\forall\, 1 < p < \frac{4}{3}, \qquad \|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} \le C_p,$$

hence, by the Riesz-Thorin theorem,

$$\forall\, 4 < r < \infty, \qquad \|\xi_\perp\widehat{\mathcal{A}}_\varepsilon\|_{L^r} = \|\mathscr{F}(\nabla_{z_\perp}\mathcal{A}_\varepsilon)\|_{L^r} \le C_r.$$

Consequently, for $4 < r < \infty$, $2 < p < \infty$ and $q = p/(p-1) \in (1, 2)$, using once again the Riesz-Thorin theorem and the Hölder inequality with exponents $\frac{r}{q}$ and $\frac{r}{r-q}$ we get

$$\begin{aligned}
\|\mathcal{A}_\varepsilon^{VII}\|_{L^p}^q &\le C\|\widehat{\mathcal{A}}_\varepsilon^{VII}\|_{L^q}^q \\
&= C\int_{\mathbb{R}^3}(|\xi_\perp|\cdot|\widehat{\mathcal{A}}_\varepsilon|)^q \times \frac{\mathbf{1}_{\{1\le|\xi_\perp|\le\varepsilon^{-\gamma},\,1\le|\xi_1|^\nu\le|\xi_\perp|\}}}{|\xi_\perp|^q}\,d\xi \\
&\le C\|\xi_\perp\widehat{\mathcal{A}}_\varepsilon\|_{L^r}^q\Big(\int_{\mathbb{R}^3}\frac{\mathbf{1}_{\{1\le|\xi_\perp|\le\varepsilon^{-\gamma},\,1\le|\xi_1|^\nu\le|\xi_\perp|\}}}{|\xi_\perp|^{\frac{rq}{r-q}}}\,d\xi\Big)^{\frac{r-q}{r}} \\
&\le C_{r,q}\Big(\int_1^{\varepsilon^{-\gamma}}\frac{R^{1+\frac{1}{\nu}}}{R^{\frac{rq}{r-q}}}\,dR\Big)^{\frac{r-q}{r}} \le C_{r,q}
\end{aligned}$$

provided that $\frac{rq}{r-q} > 2 + \frac{1}{\nu} = 12/5$. Now let $2 \le p < 3$ be fixed, so that $3/2 < q \le 2$. Since $3/2 < q \le 2$ and $q \longmapsto \frac{4q}{4-q}$ is increasing on $(3/2, 2]$, we have $\frac{4q}{4-q} > 12/5$. Furthermore, we have $\frac{rq}{r-q} \to \frac{4q}{4-q} > 12/5$ as $r \to 4$. Hence we may choose $r > 4$ such that $\frac{rq}{r-q} > 2 + \frac{1}{\nu} = 12/5$. As a consequence, we have

$$\forall\, 2 \le p < 3, \qquad \|\mathcal{A}_\varepsilon^{VII}\|_{L^p} \le C_p.$$

Collecting the above estimates for $\mathcal{A}_\varepsilon^I$, ... , $\mathcal{A}_\varepsilon^{VII}$ we deduce

$$\forall\, 2 \le p < 3, \qquad \|\mathcal{A}_\varepsilon\|_{L^p} \le C_p.$$

Then we use once again (73) with exponent $p/2 \in (1, 3/2)$ to infer that Step 2 holds for $1 < p < 3/2$.

In order to be able to use Step 1 with some $q > 3/2$, we need to prove that $\mathcal{A}_\varepsilon$, $\varepsilon_n \partial_{z_1} \mathcal{A}_\varepsilon$ and $\varepsilon_n^2 \nabla_{z_\perp} \mathcal{A}_\varepsilon$ are uniformly bounded in L^p for some $p > 3$. This is what we will prove next.

Step 3. If $N = 3$, the following bounds hold:

$$\begin{cases} \forall\, 2 \le p < 15/4 = 3.75, & \|\mathcal{A}_\varepsilon\|_{L^p} \le C_p; \\ \forall\, 2 \le p < 18/5 = 3.6, & \varepsilon \|\partial_{z_1} \mathcal{A}_\varepsilon\|_{L^p} \le C_p; \\ \forall\, 2 \le p < 18/5 = 3.6, & \varepsilon^2 \|\nabla_{z_\perp} \mathcal{A}_\varepsilon\|_{L^p} \le C_p. \end{cases}$$

Fix $r \in (3, \infty)$, $p \in (2, \infty)$ and let $q = p/(p-1) \in (1, 2)$ be the conjugate exponent of p. By the Riesz-Thorin theorem and the Hölder inequality with exponents $\frac{r}{q}$ and $\frac{r}{r-q}$ we have

$$\begin{aligned} \|\mathcal{A}_\varepsilon\|_{L^p}^q &\le C \|\widehat{\mathcal{A}}_\varepsilon\|_{L^q}^q \\ &= C \int_{\mathbb{R}^3} \Big[(1 + |\xi_1|^2 + |\xi_\perp|) \cdot |\widehat{\mathcal{A}}_\varepsilon| \Big]^q \times \frac{d\xi}{(1 + |\xi_1|^2 + |\xi_\perp|)^q} \\ &\le C \Big(\|\widehat{\mathcal{A}}_\varepsilon\|_{L^r} + \|\xi_1^2 \widehat{\mathcal{A}}_\varepsilon\|_{L^r} + \|\xi_\perp \widehat{\mathcal{A}}_\varepsilon\|_{L^r} \Big)^q \Big(\int_{\mathbb{R}^3} \frac{d\xi}{(1 + |\xi_1|^2 + |\xi_\perp|)^{\frac{rq}{r-q}}} \Big)^{\frac{r-q}{r}}. \end{aligned} \tag{75}$$

We bound the first parenthesis using again the Riesz-Thorin theorem: since $r \in (3, \infty)$, its conjugate exponent $r/(r-1)$ belongs to $(1, 3/2)$ and then Step 2 holds for the exponent r instead of p, hence

$$\begin{aligned} \|\widehat{\mathcal{A}}_\varepsilon\|_{L^r} + \|\xi_1^2 \widehat{\mathcal{A}}_\varepsilon\|_{L^r} + \|\xi_\perp \widehat{\mathcal{A}}_\varepsilon\|_{L^r} &= \|\mathscr{F}(\mathcal{A}_\varepsilon)\|_{L^r} + \|\mathscr{F}(\partial_{z_1}^2 \mathcal{A}_\varepsilon)\|_{L^r} + \|\mathscr{F}(\nabla_{z_\perp} \mathcal{A}_\varepsilon)\|_{L^r} \\ &\le C \Big(\|\mathcal{A}_\varepsilon\|_{L^{\frac{r}{r-1}}} + \|\partial_{z_1}^2 \mathcal{A}_\varepsilon\|_{L^{\frac{r}{r-1}}} + \|\nabla_{z_\perp} \mathcal{A}_\varepsilon\|_{L^{\frac{r}{r-1}}} \Big) \le C_r. \end{aligned}$$

Next, we compute using cylindrical coordinates

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{d\xi}{(1 + |\xi_1|^2 + |\xi_\perp|)^{\frac{rq}{r-q}}} \\ &\le 4\pi \Big[\int_0^1 \int_0^{+\infty} \frac{R dR}{(1+R)^{\frac{rq}{r-q}}}\, d\xi_1 + \int_1^{+\infty} \int_0^{\xi_1^2} \frac{R dR}{\xi_1^{\frac{2rq}{r-q}}}\, d\xi_1 + \int_1^{+\infty} \int_{\xi_1^2}^{+\infty} \frac{R dR}{R^{\frac{rq}{r-q}}}\, d\xi_1 \Big] \\ &\le 4\pi \Big[\int_0^{+\infty} \frac{R dR}{(1+R)^{\frac{rq}{r-q}}} + \frac{1}{2} \int_1^{+\infty} \frac{\xi_1^4}{\xi_1^{\frac{2rq}{r-q}}}\, d\xi_1 + \frac{1}{\frac{rq}{r-q} - 2} \int_1^{+\infty} \frac{d\xi_1}{\xi_1^{2(\frac{rq}{r-q} - 2)}} \Big]. \end{aligned}$$

The integrals in the last line are finite provided that $\frac{rq}{r-q} > 2$ (for the first integral), $\frac{2rq}{r-q} > 5$ (for the second integral) and $2(\frac{rq}{r-q} - 2) > 1$ (for the third integral), hence their sum is finite if $\frac{rq}{r-q} > 5/2$. Note that $\frac{rq}{r-q} \to \frac{3q}{3-q}$ as $r \to 3$ and $\frac{3q}{3-q} > 5/2$ for $q \in (\frac{15}{11}, 3)$. If $2 < p < 15/4 = 3.75$ we have $15/11 < q < 2$ and we may choose $r > 3$ (and r close to 3) such that $\frac{rq}{r-q} > 5/2$. Then it follows from the two estimates above that

$$\forall\, 2 \le p < \frac{15}{4}, \qquad \|\mathcal{A}_\varepsilon\|_{L^p} \le C_p.$$

Now we turn our attention to the bound on $\varepsilon \partial_{z_1} \mathcal{A}_\varepsilon$. Let $r \in (1, \frac{3}{2})$, $q \in [2, \infty)$ and $s \in (r, q)$. We use the estimates in Step 2 for $\Big\| \frac{\partial^2 \mathcal{A}_\varepsilon}{\partial z_i \partial z_j} \Big\|_{L^r}$ and (59) with $N = 3$ for $\Big\| \frac{\partial^2 \mathcal{A}_\varepsilon}{\partial z_i \partial z_j} \Big\|_{L^q}$, then we interpolate to get

$$\Big\| \frac{\partial^2 \mathcal{A}_\varepsilon}{\partial z_1^2} \Big\|_{L^s} + \varepsilon \Big\| \frac{\partial^2 \mathcal{A}_\varepsilon}{\partial z_1 \partial z_j} \Big\|_{L^s} + \varepsilon^2 \Big\| \nabla_\perp^2 \mathcal{A}_\varepsilon \Big\|_{L^s} \le C_{r,q} \varepsilon^{\left(-4 + \frac{2N-1}{q}\right) \frac{1 - \frac{r}{s}}{1 - \frac{r}{q}}}. \tag{76}$$

If $s \in (r, 3)$, from the Sobolev inequality and the above estimate we obtain

$$\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^{\frac{3s}{3-s}}} \leq C_s \|\partial_{z_1}^2 \mathcal{A}_\varepsilon\|_{L^s}^{\frac{1}{3}} \|\partial_{z_1}\nabla_\perp \mathcal{A}_\varepsilon\|_{L^s}^{\frac{2}{3}} \leq C_{s,r,q}\varepsilon^{-\frac{2}{3}}\varepsilon^{\left(-4+\frac{5}{q}\right)\frac{1-\frac{r}{s}}{1-\frac{r}{q}}}. \tag{77}$$

We have $-\frac{2}{3} + \left(-4+\frac{5}{q}\right)\frac{1-\frac{r}{s}}{1-\frac{r}{q}} \to -\frac{14}{3} + \frac{4r}{s}$ as $q \to \infty$ uniformly with respect to $r \in [1, \frac{3}{2}]$ and $s \in [1, 3]$. If $1 < s < \frac{18}{11} \approx 1.636$ we have $-\frac{14}{3} + \frac{4r}{s} \to -\frac{14}{3} + \frac{6}{s} > -1$ as $r \to \frac{3}{2}$. For any fixed $s \in (1, \frac{18}{11})$ we may choose q sufficiently large and $r \in (1, \frac{3}{2})$ sufficiently close to $\frac{3}{2}$ such that $-\frac{2}{3} + \left(-4+\frac{5}{q}\right)\frac{1-\frac{r}{s}}{1-\frac{r}{q}} > -1$. Since $\frac{3s}{3-s} \nearrow \frac{18}{5}$ as $s \nearrow \frac{18}{11}$, from (77) we get

$$\forall\, p \in \left(1, \frac{18}{5}\right), \qquad \|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^p} \leq C_p \varepsilon^{-1}.$$

Let $r \in (1, \frac{3}{2})$, $q \in [3, \infty)$ and $s \in (r, 3)$. Using the Sobolev inequality and (76) we have

$$\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^{\frac{3s}{3-s}}} \leq C_p \|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^s}^{\frac{1}{3}} \|\nabla_{z_\perp}^2 \mathcal{A}_\varepsilon\|_{L^s}^{\frac{2}{3}} \leq C_{s,r,q}\varepsilon^{-\frac{5}{3}}\varepsilon^{\left(-4+\frac{5}{q}\right)\frac{1-\frac{r}{s}}{1-\frac{r}{q}}}.$$

Proceeding as above we infer that

$$\forall\, 1 < p < 18/5, \qquad \varepsilon^2 \|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} \leq C_p.$$

Step 4. Conclusion in the case $N = 3$.

Fix $1 < p < 9/5 = 1.8$. Since $2 < 2p < 18/5 < 15/4$, we may use Step 1 (with p instead of q) and Step 3 to deduce that

$$\begin{aligned}\|\mathcal{A}_\varepsilon\|_{L^p} &+ \|\nabla_z \mathcal{A}_\varepsilon\|_{L^p} + \|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon_n\|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon_n^2\|\nabla_{z_\perp}^2\mathcal{A}_\varepsilon\|_{L^p} \\ &\leq C_p\Big(\|\mathcal{A}_\varepsilon\|_{L^{2p}}^2 + \Big[\varepsilon_n\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^{2p}} + \varepsilon_n^2\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^{2p}}\Big]^2\Big) \leq C_p.\end{aligned} \tag{78}$$

Hence (21) holds for $p \in (1,\ 9/5)$. In particular, by the Sobolev imbeddding $W^{1,p} \hookrightarrow L^{\frac{3p}{3-p}}$ with $1 < p < 9/5$ we have

$$\forall\, 1 < q < 9/2 = 4.5, \qquad \|\mathcal{A}_\varepsilon\|_{L^q} \leq C_q.$$

On the other hand, for any $1 < p < 9/5$,

$$\varepsilon\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{W^{1,p}} = \varepsilon\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\nabla_{z_\perp}\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^p} \leq C_p \qquad \text{and} \qquad \varepsilon^2\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{W^{1,p}} \leq C_p,$$

hence by the Sobolev embdding,

$$\forall\, 1 < q < 9/2 = 4.5, \qquad \varepsilon\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^q} \leq C_q.$$

Thus we may apply Step 1 again to infer that (78) holds now for $1 < p < 9/4 = 2.25$. By the Sobolev embedding $W^{1,p} \hookrightarrow L^{\frac{3p}{3-p}}$, we deduce as before that

$$\forall\, 1 < q < 9, \qquad \|\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon^2\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^q} \leq C_q.$$

Applying Step 1, we discover that (78) holds for any $1 < p < 9/2$. Since $9/2 > 3$, the Sobolev embedding yields

$$\forall\, 1 < q \leq \infty, \qquad \|\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon^2\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} \leq C_p,$$

and the conclusion follows using again Step 1.

Step 5. Conclusion in the case $N = 2$. The proof of (21) in the two-dimensional case is much easier: for any $1 < p < \frac{3}{2}$, we have by Step 1 and Lemma 24 (i) and (iv)

$$\|\mathcal{A}_\varepsilon\|_{L^p} + \|\nabla_z \mathcal{A}_\varepsilon\|_{L^p} + \|\partial_{z_1}^2\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon\|\partial_{z_1}\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^p} + \varepsilon^2\|\nabla_{z_\perp}^2\mathcal{A}_\varepsilon\|_{L^p} \leq C_p.$$

Thus, by the Sobolev embedding $W^{1,p}(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{2-p}}(\mathbb{R}^2)$,

$$\forall\, 1 < q < 6, \qquad \|\mathcal{A}_\varepsilon\|_{L^q} \leq C_q \qquad \text{and} \qquad \varepsilon_n\Big[\|\partial_{z_1}\mathcal{A}_\varepsilon\|_{L^q} + \varepsilon_n\|\nabla_{z_\perp}\mathcal{A}_\varepsilon\|_{L^q}\Big] \leq C_q. \tag{79}$$

Applying Step 1 once again, we infer that (78) holds for any $p \in (1, 3)$. Since $3 > 2$, the Sobolev embedding implies that (79) holds for any $q \in (1, \infty]$. Repeating the argument we get the desired conclusion.

Since $A_\varepsilon = \varepsilon^{-2}(\sqrt{1 + \varepsilon^2 \mathcal{A}_\varepsilon} - 1)$, uniform bounds bounds on A_ε and its derivatives up to order 2 follow immediately from (21).

It remains to prove (22). The uniform bounds on $\partial_{z_1}\varphi_\varepsilon$ and $\varepsilon \nabla_{z_\perp}\varphi_\varepsilon$ follow from (72) and (21).
Let $U = \rho e^{i\phi}$ be a finite energy solution to (TW$_c$), from the first equation in (17) we have

$$2\rho^2 \Delta\phi = c\frac{\partial}{\partial x_1}(\rho^2 - r_0^2) - 2\nabla(\rho^2)\cdot\nabla\phi.$$

If $\rho \geq \frac{r_0}{2}$ and $c \in (0, \mathfrak{c}_s)$, using the properties of the Riesz transform we get for any $j, k \in \{1, \dots, N\}$ and any $q \in (1, \infty)$

$$\Big\| \frac{\partial^2 \phi}{\partial x_j \partial x_k} \Big\|_{L^q} = \|R_j R_k(\Delta\phi)\|_{L^q} \leq C\|\Delta\phi\|_{L^q} \leq C\Big\| \frac{\partial}{\partial x_1}(\rho^2 - r_0^2)\Big\|_{L^q} + C\|\nabla(\rho^2)\cdot\nabla\phi\|_{L^q}.$$

In the case $U = U_\varepsilon$, $\rho(x) = r_0\sqrt{1 + \varepsilon^2 \mathcal{A}_\varepsilon(z)}$, $\phi(x) = \varepsilon\varphi_\varepsilon(z)$, using (21) and (72) we get

$$\Big\| \frac{\partial^2 \phi}{\partial x_j \partial x_k} \Big\|_{L^q} \leq \varepsilon^{3-\frac{2N-1}{q}} \Big\|\frac{\partial \mathcal{A}_\varepsilon}{\partial z_1}\Big\|_{L^q} + C\varepsilon^{5-\frac{2N-1}{q}} \Big\|\frac{\partial \mathcal{A}_\varepsilon}{\partial z_1}\cdot\frac{\partial \varphi_\varepsilon}{\partial z_1}\Big\|_{L^q} + C\varepsilon^{7-\frac{2N-1}{q}} \sum_{j=2}^{N} \Big\|\frac{\partial \mathcal{A}_\varepsilon}{\partial z_j}\cdot\frac{\partial \varphi_\varepsilon}{\partial z_j}\Big\|_{L^q} \leq C_q \varepsilon^{3-\frac{2N-1}{q}}.$$

By scaling we find for $j, k \in \{2, \dots, N\}$,

$$\Big\|\frac{\partial^2 \varphi_\varepsilon}{\partial z_1^2}\Big\|_{L^q} + \varepsilon\Big\|\frac{\partial^2 \varphi_\varepsilon}{\partial z_1 \partial z_j}\Big\|_{L^q} + \varepsilon^2\Big\|\frac{\partial^2 \varphi_\varepsilon}{\partial z_j \partial z_k}\Big\|_{L^q} \leq C_q. \tag{80}$$

By assumption (A4) there is $\delta > 0$ such that F is C^2 on $(\,(r_0 - 2\delta)^2, (r_0 + 2\delta)^2)$. Let $U = \rho e^{i\phi}$ be a solution to (TW$_c$) such that $r_0 - \delta \leq \rho \leq r_0 + \delta$. Differentiating (TW$_c$) and using standard elliptic regularity theory it is not hard to see that $U \in W^{4,p}_{loc}(\mathbb{R}^N)$ and $\nabla U \in W^{3,p}(\mathbb{R}^N)$ for any $p \in (1, \infty)$ (see the proof Proposition 2.2 (ii) p. 1079 in [34]). We infer that $\nabla\rho$, $\nabla\phi \in W^{3,p}(\mathbb{R}^N)$ for $p \in (1, \infty)$. Differentiating the first equation in (17) with respect to x_1 we find

$$c\frac{\partial^2}{\partial x_1^2}\left(\rho^2 - r_0^2\right) = 2\nabla\left(\frac{\partial(\rho^2)}{\partial x_1}\right)\cdot\nabla\phi + 2\nabla(\rho^2)\cdot\nabla\left(\frac{\partial\phi}{\partial x_1}\right) + 2\frac{\partial(\rho^2)}{\partial x_1}\Delta\phi + 2\rho^2\Delta\left(\frac{\partial\phi}{\partial x_1}\right). \tag{81}$$

If $U = U_\varepsilon$, $\rho(x) = r_0\sqrt{1 + \varepsilon^2 \mathcal{A}_\varepsilon(z)}$ and $\phi(x) = \varepsilon\varphi_\varepsilon(x)$, we perform a scaling and then we use (21), (72) and (80) to get, for $1 < q < \infty$ and all ε sufficiently small,

$$\Big\|\frac{\partial^2}{\partial x_1^2}\left(\rho^2 - r_0^2\right)\Big\|_{L^q} = \varepsilon^{4+\frac{1-2N}{q}}\Big\|\frac{\partial^2 \mathcal{A}_\varepsilon}{\partial z_1^2}\Big\|_{L^q} \leq C_q\varepsilon^{4+\frac{1-2N}{q}},$$

$$\Big\|\frac{\partial^2(\rho^2)}{\partial x_1^2}\cdot\frac{\partial\phi}{\partial x_1}\Big\|_{L^q} \leq \Big\|\frac{\partial^2(\rho^2)}{\partial x_1^2}\Big\|_{L^{2q}}\Big\|\frac{\partial\phi}{\partial x_1}\Big\|_{L^{2q}} = \varepsilon^{6+\frac{1-2N}{q}}\Big\|\frac{\partial^2 \mathcal{A}_\varepsilon}{\partial z_1^2}\Big\|_{L^{2q}}\Big\|\frac{\partial\varphi_\varepsilon}{\partial z_1}\Big\|_{L^{2q}} \leq C_q\varepsilon^{6+\frac{1-2N}{q}},$$

$$\Big\|\frac{\partial^2(\rho^2)}{\partial x_1 \partial x_k}\cdot\frac{\partial\phi}{\partial x_k}\Big\|_{L^q} \leq \Big\|\frac{\partial^2(\rho^2)}{\partial x_1 \partial x_k}\Big\|_{L^{2q}}\Big\|\frac{\partial\phi}{\partial x_k}\Big\|_{L^{2q}} = \varepsilon^{8+\frac{1-2N}{q}}\Big\|\frac{\partial^2 \mathcal{A}_\varepsilon}{\partial z_1 \partial z_k}\Big\|_{L^{2q}}\Big\|\frac{\partial\varphi_\varepsilon}{\partial z_k}\Big\|_{L^{2q}} \leq C_q\varepsilon^{6+\frac{1-2N}{q}},$$

$$\Big\|\frac{\partial(\rho^2)}{\partial x_1}\cdot\frac{\partial^2\phi}{\partial x_1^2}\Big\|_{L^q} \leq \Big\|\frac{\partial(\rho^2)}{\partial x_1}\Big\|_{L^{2q}}\Big\|\frac{\partial^2\phi}{\partial x_1^2}\Big\|_{L^{2q}} = \varepsilon^{6+\frac{1-2N}{q}}\Big\|\frac{\partial \mathcal{A}_\varepsilon}{\partial z_1}\Big\|_{L^{2q}}\Big\|\frac{\partial^2\varphi_\varepsilon}{\partial z_1^2}\Big\|_{L^{2q}} \leq C_q\varepsilon^{6+\frac{1-2N}{q}},$$

$$\Big\|\frac{\partial(\rho^2)}{\partial x_k}\cdot\frac{\partial^2\phi}{\partial x_1 \partial x_k}\Big\|_{L^q} \leq \Big\|\frac{\partial(\rho^2)}{\partial x_k}\Big\|_{L^{2q}}\Big\|\frac{\partial^2\phi}{\partial x_1 \partial x_k}\Big\|_{L^{2q}} = \varepsilon^{8+\frac{1-2N}{q}}\Big\|\frac{\partial \mathcal{A}_\varepsilon}{\partial z_k}\Big\|_{L^{2q}}\Big\|\frac{\partial^2\varphi_\varepsilon}{\partial z_1 \partial z_k}\Big\|_{L^{2q}} \leq C_q\varepsilon^{7+\frac{1-2N}{q}},$$

$$\Big\|\frac{\partial(\rho^2)}{\partial x_1}\Big\|_{L^q} = \varepsilon^{3+\frac{1-2N}{q}}\Big\|\frac{\partial \mathcal{A}_\varepsilon}{\partial z_1}\Big\|_{L^q} \leq C_q\varepsilon^{3+\frac{1-2N}{q}},$$

$$\Big\|\frac{\partial^2\phi}{\partial x_1^2}\Big\|_{L^q} = \varepsilon^{3+\frac{1-2N}{q}}\Big\|\frac{\partial^2\varphi_\varepsilon}{\partial z_1^2}\Big\|_{L^q} \leq C_q\varepsilon^{3+\frac{1-2N}{q}} \qquad \text{and} \qquad \Big\|\frac{\partial^2\phi}{\partial x_k^2}\Big\|_{L^q} = \varepsilon^{5+\frac{1-2N}{q}}\Big\|\frac{\partial^2\varphi_\varepsilon}{\partial z_k^2}\Big\|_{L^q} \leq C_q\varepsilon^{3+\frac{1-2N}{q}}.$$

Hence $\|\Delta\phi\|_{L^q} \leq C_q \varepsilon^{3+\frac{1-2N}{q}}$ and then $\left\|\frac{\partial(\rho^2)}{\partial x_1} \cdot \Delta\phi\right\|_{L^q} \leq C_q \varepsilon^{6+\frac{1-2N}{q}}$. From (81) and the above estimates we infer that $\left\|\Delta\left(\frac{\partial\phi}{\partial x_1}\right)\right\|_{L^q} \leq C_q \varepsilon^{4+\frac{1-2N}{q}}$. As before, this implies $\left\|\frac{\partial^3\phi}{\partial x_1 \partial x_i \partial x_j}\right\|_{L^q} \leq C_q \varepsilon^{4+\frac{1-2N}{q}}$ for any $i, j \in \{1, \dots, N\}$. By scaling we find

$$\left\|\frac{\partial^3 \varphi_\varepsilon}{\partial z_1^3}\right\|_{L^q} + \varepsilon \left\|\nabla_{z_\perp} \frac{\partial^2 \varphi_\varepsilon}{\partial z_1^2}\right\|_{L^q} + \varepsilon^2 \left\|\nabla^2_{z_\perp} \frac{\partial \varphi_\varepsilon}{\partial z_1}\right\|_{L^q} \leq C_q.$$

Then (22) follows from the last estimate, (72) and (80). □

3.8 Proof of Proposition 11

Let $(U_n, \varepsilon_n)_{n\geq 1}$ be a sequence as in Proposition 11. We denote $c_n = \sqrt{\mathfrak{c}_s^2 - \varepsilon_n^2}$. By Corollary 15 we have $\| |U_n| - r_0 \|_{L^\infty(\mathbb{R}^3)} \to 0$ as $n \to \infty$, hence $|U_n| \geq \frac{r_0}{2}$ in $\mathbb{R}^3$ for all sufficiently large n, say $n \geq n_0$. For $n \geq n_0$ we have a lifting as in Theorem 6 or in (16), that is

$$U_n(x) = \rho_n(x) \mathrm{e}^{i\phi_n(x)} = r_0 \left(1 + \varepsilon_n^2 A_n(z)\right) \mathrm{e}^{i\varepsilon_n \varphi_n(z)} = r_0 \sqrt{1 + \varepsilon_n^2 \mathcal{A}_n(z)}\ \mathrm{e}^{i\varepsilon_n \varphi_n(z)},$$

where $z_1 = \varepsilon_n x_1,\ z_\perp = \varepsilon_n^2 x_\perp$. Let $\mathcal{W}_n = \partial_{z_1}\varphi_n / \mathfrak{c}_s$. Our aim is to show that $(\mathcal{W}_n)_{n\geq n_0}$ is a minimizing sequence for $\mathscr{S}_*$ in the sense of Theorem 5. To that purpose we expand the functional $E_{c_n}(U_n)$ in terms of the (KP-I) action of $\mathcal{W}_n = \partial_{z_1}\varphi_n / \mathfrak{c}_s$. Recall that by (28) we have

$$\begin{aligned} E_{c_n}(u_n) = \varepsilon_n r_0^2 \int_{\mathbb{R}^3} \frac{1}{\varepsilon_n^2} \Big(\partial_{z_1}\varphi_n - c_n A_n\Big)^2 + (\partial_{z_1}\varphi_n)^2 (2A_n + \varepsilon_n^2 A_n^2) + |\nabla_{z_\perp}\varphi_n|^2 (1 + \varepsilon_n^2 A_n)^2 \\ + (\partial_{z_1} A_n)^2 + \varepsilon_n^2 |\nabla_{z_\perp} A_n|^2 + A_n^2 + \mathfrak{c}_s^2 \Big(\frac{\Gamma}{3} - 1\Big) A_n^3 + \frac{\mathfrak{c}_s^2}{\varepsilon_n^6} V_4(\varepsilon_n^2 A_n) \\ - c_n A_n^2 \partial_{z_1}\varphi_n \, dz. \end{aligned}$$

By Proposition 17, $(A_n)_{n\geq n_0}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ for all $p \in (1, \infty)$, hence it is bounded in $L^\infty(\mathbb{R}^3)$. Since $F(r_0^2(1+\varepsilon^2 A_\varepsilon)) = F(r_0^2) - \mathfrak{c}_s^2 \varepsilon^2 A_\varepsilon + \mathcal{O}(\varepsilon^4 A_\varepsilon^2) = -c^2(\varepsilon)\varepsilon^2 A_\varepsilon - \varepsilon^4 A_\varepsilon + \mathcal{O}(\varepsilon^4 A_\varepsilon)$, from the second equation in (9), Lemma 23 and Proposition 17 we get

$$\|\partial_{z_1}\varphi_n - c_n A_n\|_{L^2} = \mathcal{O}(\varepsilon_n^2). \tag{82}$$

In particular, we have $\int_{\mathbb{R}^3} \frac{1}{\varepsilon_n^2} \Big(\partial_{z_1}\varphi_n - c_n A_n\Big)^2 dz = \mathcal{O}(\varepsilon_n^2)$ as $n \to \infty$.

By Proposition 17, $\partial_{z_1}\varphi_n \in W^{2,p}(\mathbb{R}^N)$ for $p \in (1, \infty)$. Integrating by parts we have

$$\int_{\mathbb{R}^N} (\partial_{z_1} A_n)^2 - \frac{(\partial_{z_1}^2 \varphi_n)^2}{c_n^2} \, dz = -\int_{\mathbb{R}^N} \left(A_n - \frac{\partial_{z_1}\varphi_n}{c_n}\right)\left(\partial_{z_1}^2 A_n + \frac{\partial_{z_1}^3 \varphi_n}{c_n}\right) dz$$

From the above identity, the Cauchy-Schwarz inequality, (82) and Proposition 17 we get

$$\Big|\int_{\mathbb{R}^N} (\partial_{z_1} A_n)^2 - \frac{(\partial_{z_1}^2 \varphi_n)^2}{\mathfrak{c}_s^2} \, dz\Big| \leq \left(\frac{1}{c_n^2} - \frac{1}{\mathfrak{c}_s^2}\right) \int_{\mathbb{R}^N} (\partial_{z_1}^2 \varphi_n)^2 \, dz + \left\|A_n - \frac{\partial_{z_1}\varphi_n}{c_n}\right\|_{L^2} \left\|\partial_{z_1}^2 A_n + \frac{\partial_{z_1}^3 \varphi_n}{c_n}\right\|_{L^2} = \mathcal{O}(\varepsilon_n^2).$$

Similarly, using (82), Hölder's inequality and Proposition 17 we find

$$\begin{aligned} &\Big|\int_{\mathbb{R}^3} A_n^2 - \frac{(\partial_{z_1}\varphi_n)^2}{\mathfrak{c}_s^2} \, dz\Big| + \Big|\int_{\mathbb{R}^3} A_n^3 - \frac{(\partial_{z_1}\varphi_n)^3}{\mathfrak{c}_s^3} \, dz\Big| \\ &+ \Big|\int_{\mathbb{R}^3} A_n^2 \partial_{z_1}\varphi_n - \frac{(\partial_{z_1}\varphi_n)^3}{\mathfrak{c}_s^2} \, dz\Big| + \Big|\int_{\mathbb{R}^3} A_n (\partial_{z_1}\varphi_n)^2 - \frac{(\partial_{z_1}\varphi_n)^3}{\mathfrak{c}_s} \, dz\Big| = \mathcal{O}(\varepsilon_n^2). \end{aligned}$$

Since $(A_n)_{n\geq n_0}$ is bounded in $L^\infty(\mathbb{R}^3)$, using Lemma 23 we find

$$\int_{\mathbb{R}^3} |\nabla_{z_\perp}\varphi_n|^2 (1 + \varepsilon_n^2 A_n)^2 \, dz = \int_{\mathbb{R}^3} |\nabla_{z_\perp}\varphi_n|^2 \, dz + \mathcal{O}(\varepsilon_n^2) = \mathfrak{c}_s^2 \int_{\mathbb{R}^3} |\nabla_{z_\perp} \partial_{z_1}^{-1} \mathcal{W}_n|^2 \, dz + \mathcal{O}(\varepsilon_n^2).$$

Recall that $V_4(\alpha) = \mathcal{O}(\alpha^4)$ as $\alpha \to 0$, hence Proposition 17 implies that

$$\int_{\mathbb{R}^3} \varepsilon_n^2 A_n^2 (\partial_{z_1}\varphi_n)^2 + \varepsilon_n^2 |\nabla_{z_\perp} A_n|^2 + \frac{\mathfrak{c}_s^2}{\varepsilon_n^6} V_4(\varepsilon_n^2 A_n)\, dz = \mathcal{O}(\varepsilon_n^2).$$

Inserting the above estimates into (28) we obtain

$$\frac{E_{c(\varepsilon_n)}(U_n)}{\mathfrak{c}_s^2 r_0^2 \varepsilon_n} = \int_{\mathbb{R}^3} \left|\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}_n\right|^2 + \frac{1}{\mathfrak{c}_s^2}(\partial_{z_1}\mathcal{W}_n)^2 + \frac{\Gamma}{3}\mathcal{W}_n^3 + \frac{1}{\mathfrak{c}_s^2}\mathcal{W}_n^2\, dz + \mathcal{O}(\varepsilon_n^2) = \mathscr{S}(\mathcal{W}_n) + \mathcal{O}(\varepsilon_n^2). \tag{83}$$

From the above estimate and the upper bound on $E_{c_n}(U_n) = T_{c_n}$ given by Proposition 9 (ii) we infer that

$$\mathscr{S}(\mathcal{W}_n) = \frac{E_{c(\varepsilon_n)}(U_n)}{\mathfrak{c}_s^2 r_0^2 \varepsilon_n} + \mathcal{O}(\varepsilon_n^2) = \frac{T_{c_n}}{\mathfrak{c}_s^2 r_0^2 \varepsilon_n} + \mathcal{O}(\varepsilon_n^2) \le \mathscr{S}_{\min} + \mathcal{O}(\varepsilon_n^2) = \mathscr{S}_* + \mathcal{O}(\varepsilon_n^2).$$

Similarly we have

$$\int_{\mathbb{R}^3} |\nabla_{x_\perp} U_n|^2\, dx = r_0^2\varepsilon_n \int_{\mathbb{R}^3} (1+\varepsilon_n^2 A_n)^2 |\nabla_{x_\perp}\varphi_n|^2 + \varepsilon_n^2 |\nabla_{x_\perp} A_n|^2\, dz = r_0^2 \mathfrak{c}_s^2 \varepsilon_n \int_{\mathbb{R}^3} \left|\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W}_n\right|^2 dz + \mathcal{O}(\varepsilon_n^3).$$

Since U_n satisfies the Pohozaev identity $E_{c_n}(U_n) = \int_{\mathbb{R}^3} |\nabla_{z_\perp} U_n|^2\, dz$, comparing the above equation to the expression of $E_{c_n}(U_n)$ in (83) we find

$$\int_{\mathbb{R}^3} \frac{1}{\mathfrak{c}_s^2}(\partial_{z_1}\mathcal{W}_n)^2 + \frac{\Gamma}{3}\mathcal{W}_n^3 + \frac{1}{\mathfrak{c}_s^2}\mathcal{W}_n^2\, dz = \mathcal{O}(\varepsilon_n^2).$$

In order to apply Theorem 5, we have to check that there is $m_1 > 0$ such that for all n sufficiently large there holds

$$\int_{\mathbb{R}^3} \mathcal{W}_n^2 + (\partial_{z_1}\mathcal{W}_n)^2\, dz \ge m_1.$$

By Lemma 22, there are $k > 0$ depending only on F and $n_1 \ge n_0$ such that

$$\forall n \ge n_1, \quad \|A_n\|_{L^\infty} \ge k.$$

Since A_n tends to 0 at infinity, after a translation we may assume that

$$|A_n(0)| = \|A_n\|_{L^\infty} \ge k.$$

By Proposition 17 we know that for all $p \in (1,\infty)$ there is $C_p > 0$ such that $\|A_n\|_{W^{1,p}} \le C_p$ for any $n \ge n_0$. Then Morrey's inequality (see e.g. Theorem IX.12 p. 166 in [14]) implies that for any $\alpha \in (0,1)$ there is $C_\alpha > 0$ such that for all $n \ge n_0$ and all $x, y \in \mathbb{R}^3$ we have $|A_n(x) - A_n(y)| \le C_\alpha |x-y|^\alpha$. We infer that $|A_n| \ge k/2$ in $B_r(0)$ for some $r > 0$ independent of n, hence there is $m_1 > 0$ such that

$$\|A_n\|_{L^2} \ge \|A_n\|_{L^2(B_r(0))} \ge 2m_1.$$

From (82) it follows that $\|\mathcal{W}_n - A_n\|_{L^2} \to 0$ as $n \to \infty$, hence

$$\|\mathcal{W}_n\|_{L^2} \ge \|\mathcal{W}_n\|_{L^2(B_r(0))} \ge m_1 \qquad \text{for all } n \text{ sufficiently large.}$$

Then Theorem 5 implies that there exist $\mathcal{W} \in \mathscr{Y}(\mathbb{R}^3)$, a subsequence of $(\mathcal{W}_n)_{n\ge n_0}$ (still denoted $(\mathcal{W}_n)_{n\ge n_0}$), and a sequence $(z^n)_{n\ge n_0} \subset \mathbb{R}^3$ such that

$$\mathcal{W}_n(\cdot - z^n) \to \mathcal{W} \qquad \text{in} \quad \mathscr{Y}(\mathbb{R}^3).$$

Moreover, there is $\sigma > 0$ such that $z \longmapsto \mathcal{W}(z, \frac{1}{\sigma} z_\perp)$ is a ground state (with speed $1/(2\mathfrak{c}_s^2)$) of (KP-I). We will prove that $\sigma = 1$.

Let $x^n = \left(\frac{z_1^n}{\varepsilon_n}, \frac{z_\perp^n}{\varepsilon_n^2}\right)$. We denote $\tilde{\mathcal{W}}_n = \mathcal{W}_n(\cdot - z^n)$, $\tilde{A}_n = A_n(\cdot - z^n)$, $\tilde{\varphi}_n = \varphi_n(\cdot - z^n)$, $\tilde{U}_n = U_n(\cdot - x^n)$. It is obvious that $\tilde{U}_n$ satisfies (TW$_{c_n}$) and all the previous estimates hold with $\tilde{A}_n$, $\tilde{\varphi}_n$ and $\tilde{U}_n$ instead of A_n, φ_n and U_n, respectively.

Since $\tilde{\mathcal{W}}_n = \frac{1}{\mathfrak{c}_s}\partial_{z_1}\tilde{\varphi}_n$ and $\tilde{\mathcal{W}}_n \to \mathcal{W}$ in $\mathscr{Y}(\mathbb{R}^3)$, we have

$$\partial_{z_1}\tilde{\varphi}_n \to \mathfrak{c}_s\mathcal{W}, \qquad \partial_{z_1}^2\tilde{\varphi}_n \to \mathfrak{c}_s\partial_{z_1}\mathcal{W} \qquad \text{and} \qquad \nabla_{z_\perp}\tilde{\varphi}_n \to \mathfrak{c}_s\nabla_{z_\perp}\partial_{z_1}^{-1}\mathcal{W} \qquad \text{in } L^2(\mathbb{R}^3). \tag{84}$$

Integrating by parts, then using the Cauchy-Schwarz inequality, Proposition 17 and (82) we find

$$\int_{\mathbb{R}^3}\left|\partial_{z_1}^2\tilde{\varphi}_n - c_n\partial_{z_1}\tilde{A}_n\right|^2 dz = -\int_{\mathbb{R}^3}(\partial_{z_1}\tilde{\varphi}_n - c_n\tilde{A}_n)(\partial_{z_1}^3\tilde{\varphi}_n - c_n\partial_{z_1}^2\tilde{A}_n)\,dz$$

$$\leq \|\partial_{z_1}\tilde{\varphi}_n - c_n\tilde{A}_n\|_{L^2}\|\partial_{z_1}^3\tilde{\varphi}_n - c_n\partial_{z_1}^2\tilde{A}_n\|_{L^2} = \mathcal{O}(\varepsilon_n^2),$$

hence $\|\partial_{z_1}^2\tilde{\varphi}_n - c_n\partial_{z_1}\tilde{A}_n\|_{L^2} = \mathcal{O}(\varepsilon_n) \to 0$. Since $c_n \to \mathfrak{c}_s$, from (82) and (84) we get

$$\tilde{A}_n \to \mathcal{W} \qquad \text{and} \qquad \partial_{z_1}\tilde{A}_n \to \partial_{z_1}\mathcal{W} \qquad \text{in } L^2(\mathbb{R}^3) \quad \text{as } n\to\infty. \tag{85}$$

It is obvious that $\tilde{A}_n$, $\tilde{\varphi}_n$ and ε_n satisfy (11). Let $\psi \in C_c^\infty(\mathbb{R}^3)$. We multiply (11) by ψ, integrate by parts, then pass to the limit as $n\to\infty$. We use Proposition 17, (84) and (85) and after a straightforward computation we discover that $\mathcal{W}$ satisfies the equation (SW) in $\mathcal{D}'(\mathbb{R}^3)$. This implies that necessarily $\sigma = 1$ and $\mathcal{W}$ is a ground state of speed $1/(2\mathfrak{c}_s^2)$ to (KP-I). In particular, $\mathcal{W}$ satisfies the Pohozaev identities (25) and (26).

Since $\tilde{\mathcal{W}}_n \to \mathcal{W}$ in $\mathscr{Y}(\mathbb{R}^3)$, we have $\mathscr{S}(\mathcal{W}_n) = \mathscr{S}(\tilde{\mathcal{W}}_n) \to \mathscr{S}(\mathcal{W})$ and (83) implies

$$\frac{E_{c(\varepsilon_n)}(U_n)}{\mathfrak{c}_s^2 r_0^2\varepsilon_n} = \mathscr{S}(\mathcal{W}_n) + \mathcal{O}(\varepsilon_n^2) = \mathscr{S}(\mathcal{W}) + o(1) = \mathscr{S}_{\min} + o(1),$$

that is (13) holds. Using the expression for the momentum in (3), then (84), (85), Proposition 17 and the Pohozaev identities (25) and (26) we get

$$-\frac{\varepsilon_n}{r_0^2\mathfrak{c}_s^3}Q(U_n) = \frac{\varepsilon_n}{r_0^2\mathfrak{c}_s^3}\int_{\mathbb{R}^3}(\rho_n^2 - r_0^2)\frac{\partial\phi_n}{\partial x_1}\,dx = \frac{1}{\mathfrak{c}_s^3}\int_{\mathbb{R}^3}(2A_n(z)+\varepsilon_n^2A_n^2(z))\frac{\partial\varphi_n}{\partial z_1}(z)\,dz \longrightarrow \frac{2}{\mathfrak{c}_s^2}\int_{\mathbb{R}^3}\mathcal{W}^2(z)\,dz = \mathscr{S}(\mathcal{W}).$$

Hence $-\mathfrak{c}_sQ(U_n) \sim r_0^2\mathfrak{c}_s^4\mathscr{S}_{\min}\varepsilon^{-1}$ as $n\to\infty$. Together with (13) this implies that $(U_n)_{n\geq n_0}$ satisfies (12).

By Proposition 17 we know that $(\tilde{A}_n)_{n\geq n_0}$, $(\partial_{z_1}\tilde{A}_n)_{n\geq n_0}$, $(\partial_{z_1}\tilde{\varphi}_n)_{n\geq n_0}$ and $(\partial_{z_1}^2\tilde{\varphi}_n)_{n\geq n_0}$ are bounded in $L^p(\mathbb{R}^3)$ for $1<p<\infty$. From (84), (85) and standard interpolation in L^p spaces we find as $n\to\infty$

$$\tilde{A}_n \to \mathcal{W}, \qquad \partial_{z_1}\tilde{A}_n \to \partial_{z_1}\mathcal{W}, \qquad \partial_{z_1}\tilde{\varphi}_n \to \mathfrak{c}_s\mathcal{W} \quad \text{and} \quad \partial_{z_1}^2\tilde{\varphi}_n \to \mathfrak{c}_s\partial_{z_1}\mathcal{W} \quad \text{in } L^p \tag{86}$$

for any $p\in(1,\infty)$.

Proceeding as in [8] (see Lemma 4.6 p. 262 and Proposition 6.1 p. 266 there) one can prove that for any multiindex $\alpha\in\mathbb{N}^N$ with $|\alpha|\leq 2$, the sequences $(\partial^\alpha\tilde{A}_n)_{n\geq n_0}$, $(\partial^\alpha\partial_{z_1}\tilde{A}_n)_{n\geq n_0}$, $(\partial^\alpha\partial_{z_1}\tilde{\varphi}_n)_{n\geq n_0}$ and $(\partial^\alpha\partial_{z_1}^2\tilde{\varphi}_n)_{n\geq n_0}$ are bounded in $L^p(\mathbb{R}^3)$ for $1<p<\infty$. Then by interpolation we see that (86) holds in $W^{1,p}(\mathbb{R}^3)$ for all $p\in(1,\infty)$. □

3.9 Proof of Theorem 6 completed in the case $N = 2$

Assume that $N=2$. Let (U_n, c_n) be a sequence of travelling waves to (NLS) satisfying assumption (b) in Theorem 6 such that $c_n \to \mathfrak{c}_s$ as $n\to\infty$. Let $\varepsilon_n = \sqrt{\mathfrak{c}_s^2 - c_n^2}$. By Theorem 1 we have $\int_{\mathbb{R}^2}|\nabla U_n|^2\,dx \to 0$ as $n\to\infty$ and then Lemma 13 implies that $\|\,|U_n| - r_0\|_{L^\infty} \to 0$; in particular, for n sufficiently large we have a lifting $U_n(x) = \rho_n(x)\mathrm{e}^{i\phi_n(x)} = r_0\Big(1+\varepsilon_n^2A_n(z)\Big)\mathrm{e}^{i\varepsilon_n\varphi_n(z)}$ as in (8) and the conclusion of Proposition 17 holds for A_n and φ_n. As in the proof of Proposition 11 we obtain

$$\|\partial_{z_1}\varphi_n - c_nA_n\|_{L^2} = \mathcal{O}(\varepsilon_n^2) \qquad \text{and} \qquad \|\partial_{z_1}^2\varphi_n - c_n\partial_{z_1}A_n\|_{L^2} = \mathcal{O}(\varepsilon_n) \qquad \text{as } n\to\infty. \tag{87}$$

Let $k_n = \int_{\mathbb{R}^2}|\nabla U_n(x)|^2\,dx$. We denote $\mathcal{W}_n = \mathfrak{c}_s^{-1}\partial_{z_1}\varphi_n$. By (87) we have $\|\mathcal{W}_n - A_n\|_{L^2} = \mathcal{O}(\varepsilon_n^2)$. As in the proof of Proposition 11 we find $\Big|\int_{\mathbb{R}^2}(\partial_{z_1}A_n)^2 - (\partial_{z_1}\mathcal{W}_n)^2\,dz\Big| = \Big|\int_{\mathbb{R}^2}(\partial_{z_1}A_n)^2 - \frac{(\partial_{z_1}^2\varphi_n)^2}{\mathfrak{c}_s^2}\,dz\Big| = \mathcal{O}(\varepsilon_n^2)$.

Using (87) and Proposition 17 we get

$$
\begin{aligned}
k_n = \int_{\mathbb{R}^2} |\nabla U_n|^2 \, dx &= \varepsilon_n r_0^2 \int_{\mathbb{R}^2} (\partial_{z_1}\varphi_n)^2 (1+\varepsilon_n^2 A_n)^2 + \varepsilon_n^2 (\partial_{z_1} A_n)^2 + \varepsilon_n^2 (\partial_{z_2}\varphi_n)^2 (1+\varepsilon_n^2 A_n)^2 + \varepsilon_n^4 (\partial_{z_2} A_n)^2 \, dz \\
&= \varepsilon_n r_0^2 \int_{\mathbb{R}^2} (\partial_{z_1}\varphi_n)^2 \, dz + \varepsilon_n^3 r_0^2 \int_{\mathbb{R}^2} \Big(2A_n (\partial_{z_1}\varphi_n)^2 + (\partial_{z_1} A_n)^2 + (\partial_{z_2}\varphi_n)^2 \Big) \, dz + \mathcal{O}(\varepsilon_n^5) \\
&= \varepsilon_n r_0^2 \mathfrak{c}_s^2 \int_{\mathbb{R}^2} \mathcal{W}_n^2 \, dz + \varepsilon_n^3 r_0^2 \mathfrak{c}_s^2 \int_{\mathbb{R}^2} \Big(2\mathcal{W}_n^3 + \frac{1}{\mathfrak{c}_s^2} (\partial_{z_1} \mathcal{W}_n)^2 + (\partial_{z_2} \partial_{z_1}^{-1} \mathcal{W}_n)^2 \Big) \, dz + \mathcal{O}(\varepsilon_n^5). \qquad (88)
\end{aligned}
$$

Inverting this expansion we find the following expression of ε_n in terms of k_n:

$$
\varepsilon_n = \frac{k_n}{r_0^2 \mathfrak{c}_s^2 \|\mathcal{W}_n\|_{L^2}^2} - \frac{k_n^3}{r_0^6 \mathfrak{c}_s^6 \|\mathcal{W}_n\|_{L^2}^8} \int_{\mathbb{R}^2} \Big(2\mathcal{W}_n^3 + \frac{1}{\mathfrak{c}_s^2} (\partial_{z_1} \mathcal{W}_n)^2 + (\partial_{z_2} \partial_{z_1}^{-1} \mathcal{W}_n)^2 \Big) \, dz + \mathcal{O}(k_n^5). \qquad (89)
$$

Recall that the mapping $U_n(c_n \cdot)$ is a minimizer of the functional $I(\psi) = Q(\psi) + \int_{\mathbb{R}^2} V(|\psi|^2) \, dx$ under the constraint $\int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx = k_n$. Using this information, Proposition 9 (i), the fact that $c_n^2 = \mathfrak{c}_s^2 - \varepsilon_n^2$ and (89) we get

$$
\begin{aligned}
c_n Q(U_n) + \int_{\mathbb{R}^2} V(|U_n|^2) \, dx &= c_n^2 I(U_n(c_n \cdot)) = c_n^2 I_{\min}(k_n) \\
&\leq c_n^2 \left(-\frac{k_n}{\mathfrak{c}_s^2} - \frac{4k_n^3}{27 r_0^4 \mathfrak{c}_s^{12} \mathscr{S}_{\min}^2} + \mathcal{O}(k_n^5) \right) \\
&= -k_n + \frac{k_n^3}{r_0^4 \mathfrak{c}_s^6 \|\mathcal{W}_n\|_{L^2}^4} - \frac{4k_n^3}{27 r_0^4 \mathfrak{c}_s^{10} \mathscr{S}_{\min}^2} + \mathcal{O}(k_n^5). \qquad (90)
\end{aligned}
$$

Moreover, using the Taylor expansion (27), we find

$$
\int_{\mathbb{R}^2} V(|U_n|^2) \, dx = r_0^2 \mathfrak{c}_s^2 \varepsilon_n \int_{\mathbb{R}^2} \Big(A_n^2 + \varepsilon_n^2 \Big[\frac{\Gamma}{3} - 1 \Big] A_n^3 + \frac{V_4(\varepsilon_n^2 A_n)}{\varepsilon_n^4} \Big) \, dz
$$

and by (3) we have

$$
Q(U_n) = -\varepsilon_n r_0^2 \int_{\mathbb{R}^2} \Big(2A_n + \varepsilon_n^2 A_n^2 \Big) \frac{\partial \varphi_n}{\partial z_1} \, dz.
$$

Taking into account (87) and the equality $c_n^2 = \mathfrak{c}_s^2 - \varepsilon_n^2$, then using expansion of ε_n in terms of k_n (89) we get

$$
\begin{aligned}
c_n Q(U_n) + & \int_{\mathbb{R}^2} V(|U_n|^2) \, dx \\
&= r_0^2 \mathfrak{c}_s^2 \left(\varepsilon_n \int_{\mathbb{R}^2} \Big(-2A_n \mathcal{W}_n + A_n^2 \Big) \, dz + \varepsilon_n^3 \int_{\mathbb{R}^2} \Big(-A_n^2 \mathcal{W}_n + \Big[\frac{\Gamma}{3} - 1 \Big] A_n^3 + \frac{1}{\mathfrak{c}_s^2} A_n \mathcal{W}_n \Big) \, dz + \mathcal{O}(\varepsilon_n^5) \right) \\
&= r_0^2 \mathfrak{c}_s^2 \left(\varepsilon_n \|\mathcal{W}_n - A_n\|_{L^2}^2 - \varepsilon_n \int_{\mathbb{R}^2} \mathcal{W}_n^2 \, dz + \varepsilon_n^3 \int_{\mathbb{R}^2} \Big[\frac{\Gamma}{3} - 2 \Big] \mathcal{W}_n^3 + \frac{\mathcal{W}_n^2}{\mathfrak{c}_s^2} \, dz + \mathcal{O}(\varepsilon_n^5) \right) \\
&= r_0^2 \mathfrak{c}_s^2 \left(-\varepsilon_n \int_{\mathbb{R}^2} \mathcal{W}_n^2 \, dz + \varepsilon_n^3 \int_{\mathbb{R}^2} \Big[\frac{\Gamma}{3} - 2 \Big] \mathcal{W}_n^3 + \frac{\mathcal{W}_n^2}{\mathfrak{c}_s^2} \, dz + \mathcal{O}(\varepsilon_n^5) \right) \\
&= -k_n + \frac{k_n^3}{r_0^4 \mathfrak{c}_s^4 \|\mathcal{W}_n\|_{L^2}^6} \mathscr{S}(\mathcal{W}_n) + \mathcal{O}(k_n^5). \qquad (91)
\end{aligned}
$$

Inserting (91) into (90) we discover

$$
\frac{k_n^3}{r_0^4 \mathfrak{c}_s^4 \|\mathcal{W}_n\|_{L^2}^6} \mathscr{S}(\mathcal{W}_n) + \mathcal{O}(k_n^5) \leq \frac{k_n^3}{r_0^4 \mathfrak{c}_s^6 \|\mathcal{W}_n\|_{L^2}^4} - \frac{4k_n^3}{27 r_0^4 \mathfrak{c}_s^{10} \mathscr{S}_{\min}^2} + \mathcal{O}(k_n^5),
$$

that is

$$
\mathscr{S}(\mathcal{W}_n) \leq \frac{1}{\mathfrak{c}_s^2} \|\mathcal{W}_n\|_{L^2}^2 - \frac{4}{27 \mathfrak{c}_s^6 \mathscr{S}_{\min}^2} \|\mathcal{W}_n\|_{L^2}^6 + \mathcal{O}(k_n^2)
$$

or equivalently

$$\mathscr{E}(\mathcal{W}_n) = \mathscr{S}(\mathcal{W}_n) - \frac{1}{\mathfrak{c}_s^2}\int_{\mathbb{R}^2} \mathcal{W}_n^2 \, dz \le -\frac{1}{2\mathscr{S}_{\min}^2}\Big(\frac{2}{3}\Big)^3 \cdot \Big(\frac{1}{\mathfrak{c}_s^2}\|\mathcal{W}_n\|_{L^2}^2\Big)^3 + \mathcal{O}(k_n^2). \tag{92}$$

As in the proof of Proposition 11, it follows from Lemma 22 and Proposition 17 that there are some positive constants m_1, m_2 such that

$$m_1 \le \|\mathcal{W}_n\|_{L^2}^2 \le m_2 \qquad \text{for all sufficiently large } n.$$

Denote $\lambda_n = \frac{\|\mathcal{W}_n\|_{L^2}^2}{\mathfrak{c}_s^2}$. Passing to a subsequence if necessary we may assume that $\lambda_n \to \lambda$, where $\lambda \in (0, +\infty)$. Let

$$\mathcal{W}_n^{\#}(z) = \frac{\mu^2}{\lambda_n^2}\mathcal{W}_n\Big(\frac{\mu}{\lambda_n}z_1, \frac{\mu^2}{\lambda_n^2}z_2\Big),$$

where μ is as in Theorem 4. Then $\mathcal{W}_n^{\#}$ satisfies

$$\int_{\mathbb{R}^2} \frac{1}{\mathfrak{c}_s^2}(\mathcal{W}_n^{\#})^2 \, dz = \frac{\mu}{\lambda_n}\int_{\mathbb{R}^2}\frac{1}{\mathfrak{c}_s^2}\mathcal{W}_n^2 \, dz = \mu \qquad \text{and} \qquad \mathscr{E}(\mathcal{W}_n^{\#}) = \frac{\mu^3}{\lambda_n^3}\mathscr{E}(\mathcal{W}_n).$$

Plugging this into (92) and recalling that $\mu = \frac{3}{2}\mathscr{S}_{\min}$, we infer that

$$\mathscr{E}(\mathcal{W}_n^{\#}) = \frac{\mu^3}{\lambda_n^3}\mathscr{E}(\mathcal{W}_n) \le -\frac{1}{2\mathscr{S}_{\min}^2}\Big(\frac{2\mu}{3}\Big)^3 + \mathcal{O}(k_n^2) = -\frac{1}{2}\mathscr{S}_{\min} + \mathcal{O}(k_n^2).$$

Therefore $(\mathcal{W}_n^{\#})_{n\ge n_0}$ is a minimizing sequence for (6).

By Theorem 4 we infer that there exist a subsequence of $(\mathcal{W}_n^{\#})_{n\ge n_0}$, still denoted $(\mathcal{W}_n^{\#})_{n\ge n_0}$, a sequence $(z^n)_{n\ge n_0} = (z_1^n, z_2^n)_{n\ge n_0} \subset \mathbb{R}^2$ and a ground state $\mathcal{W}$ (with speed $1/(2\mathfrak{c}_s^2)$) of (KP-I) such that $\mathcal{W}_n^{\#}(\cdot - z^n) \longrightarrow \mathcal{W}$ strongly in $\mathscr{Y}(\mathbb{R}^2)$as $n \to \infty$.

Let $x^n = \Big(\frac{\mu}{\varepsilon_n\lambda_n}z_1^n, \frac{\mu^2}{\varepsilon_n^2\lambda_n^2}z_2^n\Big)$ and $\tilde{U}_n = U(\cdot - x^n)$, $\tilde{A}_n(z) = A_n\Big(z_1 - \frac{\mu}{\lambda_n}z_1^n, z_2 - \frac{\mu^2}{\lambda_n^2}z_2^n\Big)$, $\tilde{\varphi}_n(z) = \varphi_n\Big(z_1 - \frac{\mu}{\lambda_n}z_1^n, z_2 - \frac{\mu^2}{\lambda_n^2}z_2^n\Big)$, $\tilde{\mathcal{W}}_n(z) = \mathcal{W}_n\Big(z_1 - \frac{\mu}{\lambda_n}z_1^n, z_2 - \frac{\mu^2}{\lambda_n^2}z_2^n\Big)$. We denote $\tilde{\mathcal{W}}(z) = \frac{\lambda^2}{\mu^2}\mathcal{W}(\frac{\lambda}{\mu}z_1, \frac{\lambda^2}{\mu^2}z_2)$. It is obvious that $\tilde{U}_n(x) = r_0\Big(1 + \varepsilon_n^2\tilde{A}_n(z)\Big)\mathrm{e}^{i\varepsilon_n\tilde{\varphi}_n(z)}$ is a solution to (TW_{c_n}) with the same properties as U_n and the functions $\tilde{A}_n$, $\tilde{\varphi}_n$, $\tilde{\mathcal{W}}_n$ satisfy the same estimates as A_n, φ_n and $\mathcal{W}_n$, respectively. Moreover, we have $\tilde{\mathcal{W}}_n = \frac{1}{\mathfrak{c}_s}\partial_{z_1}\tilde{\varphi}_n$ and $\tilde{\mathcal{W}}_n \longrightarrow \tilde{\mathcal{W}}$ strongly in $\mathscr{Y}(\mathbb{R}^2)$ as $n \to \infty$.

It is clear that $\tilde{A}_n$, $\tilde{\varphi}_n$ and ε_n satisfy (11). For any fixed $\psi \in C_c^\infty(\mathbb{R}^3)$ we mutiply (11) by ψ, integrate by parts, then pass to the limit as $n \to \infty$. Proceeding as in the proof of Proposition 11 we find that $\tilde{\mathcal{W}}$ satisfies equation (SW) in $\mathcal{D}'(\mathbb{R}^2)$. We know that $\mathcal{W}$ also solves (SW) and comparing the equations for $\mathcal{W}$ and $\tilde{\mathcal{W}}$ we infer that $\Big(\frac{\lambda^3}{\mu^3} - \frac{\lambda^5}{\mu^5}\Big)\partial_{z_1}\mathcal{W} = 0$ in $\mathbb{R}^2$. Since $\partial_{z_1}\mathcal{W} \ne 0$, $\lambda > 0$ and $\mu > 0$, we have necessarily $\lambda = \mu$, that is $\tilde{\mathcal{W}} = \mathcal{W}$.

In particular, we have $\mathscr{S}(\mathcal{W}_n) = \mathscr{S}(\tilde{\mathcal{W}}_n) \longrightarrow \mathscr{S}(\mathcal{W}) = \mathscr{S}_{\min}$ as $n \to \infty$. Since $\int_{\mathbb{R}^2}|\nabla U_n|^2\,dx = k_n$, using (91) and (88) we get

$$E(U_n) + c_nQ(U_n) = \frac{k_n^3}{r_0^4\mathfrak{c}_s^4\|\mathcal{W}_n\|_{L^2}^6}\mathscr{S}(\mathcal{W}_n) + \mathcal{O}(k_n^5) \sim \varepsilon_n^3 r_0^2\mathfrak{c}_s^2\mathscr{S}_{\min} \qquad \text{as } n \to \infty.$$

Hence (13) holds. As in the proof of Proposition 11 we have

$$Q(U_n) = -\int_{\mathbb{R}^2}(\rho_n^2 - r_0^2)\frac{\partial\phi}{\partial x_1} = -r_0^2\varepsilon_n\int_{\mathbb{R}^2}(2A_n(z) + \varepsilon_n^2A_n^2(z))\frac{\partial\varphi_n}{\partial z_1}(z)\,dz$$

$$\sim -2r_0^2\mathfrak{c}_s\varepsilon_n\int_{\mathbb{R}^2}\mathcal{W}^2(z)\,dz = -3r_0^2\mathfrak{c}_s^3\mathscr{S}(\mathcal{W})\varepsilon_n.$$

The above computation and (13) imply (12).

Finally, the convergence in (86) as well as the similar property in $W^{1,p}(\mathbb{R}^2)$ are proven exactly as in the three dimensional case. □

4 The higher dimensional case

4.1 Proof of Proposition 18

We argue by contradiction. Suppose that the assumptions of Proposition 18 hold and there is a sequence $(U_n)_{n\geq 1} \subset \mathcal{E}$ of nonconstant solutions to (TW_{c_n}) such that $E_{c_n}(U_n) \to 0$ as $n \to +\infty$. By Proposition 14 (ii) we have $|U_n| \to r_0 > 0$ uniformly in $\mathbb{R}^N$. Hence for n sufficiently large we have the lifting $U_n(x) = \rho_n(x)\mathrm{e}^{i\phi_n(x)}$. We write

$$\mathcal{B}_n = \frac{|U_n|}{r_0} - 1, \qquad \text{so that} \qquad \rho_n = r_0(1+\mathcal{B}_n) \qquad \text{and} \qquad \mathcal{B}_n \to 0 \quad \text{as } n \to \infty.$$

Recall that U_n satisfies the Pohozaev identities (4). The identity $P_{c_n}(U_n) = 0$ can be written as

$$\int_{\mathbb{R}^N} \Big|\frac{\partial U_n}{\partial x_1}\Big|^2 + \frac{N-3}{N-1}|\nabla_{x_\perp} U_n|^2\,dx + c_n Q(U_n) + \int_{\mathbb{R}^N} V(|U_n|^2)\,dx = 0.$$

Using the formula (3) for $Q(U_n)$ and the Taylor expansion (27) for $V(r_0^2(1+\mathcal{B}_n)^2)$ we get

$$r_0^2\int_{\mathbb{R}^N}\Big|\frac{\partial \mathcal{B}_n}{\partial x_1}\Big|^2 + (1+\mathcal{B}_n)^2\Big|\frac{\partial \phi_n}{\partial x_1}\Big|^2 + \frac{N-3}{N-1}|\nabla_{x_\perp}\mathcal{B}_n|^2 + \frac{N-3}{N-1}(1+\mathcal{B}_n)^2|\nabla_{x_\perp}\phi_n|^2$$
$$- c_n(2\mathcal{B}_n + \mathcal{B}_n^2)\frac{\partial \phi_n}{\partial x_1} + \mathfrak{c}_s^2\left(\mathcal{B}_n^2 + \Big(\frac{\Gamma}{3}-1\Big)\mathcal{B}_n^3 + V_4(\mathcal{B}_n)\right)\,dx = 0,$$

where $V_4(\alpha) = \mathcal{O}(\alpha^4)$ as $\alpha \to 0$. After rearranging terms, the above equality yields

$$\begin{aligned}
&\int_{\mathbb{R}^N}(\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2 + (\partial_{x_1}\mathcal{B}_n)^2 + \frac{N-3}{N-1}|\nabla_{x_\perp}\phi_n|^2(1+\mathcal{B}_n)^2 + \frac{N-3}{N-1}|\nabla_{x_\perp}\mathcal{B}_n|^2 + \varepsilon_n^2\mathcal{B}_n^2\,dx\\
&= -\int_{\mathbb{R}^6}(\partial_{x_1}\phi_n)^2(2\mathcal{B}_n + \mathcal{B}_n^2) + \mathfrak{c}_s^2\Big(\frac{\Gamma}{3}-1\Big)\mathcal{B}_n^3 + \mathfrak{c}_s^2 V_4(\mathcal{B}_n) - c_n\mathcal{B}_n^2\partial_{x_1}\phi_n\,dx\\
&= -\Big[\frac{\Gamma}{3}\mathfrak{c}_s^2 - \varepsilon_n^2\Big]\int_{\mathbb{R}^N}\mathcal{B}_n^3\,dz - \mathfrak{c}_s^2\int_{\mathbb{R}^N}V_4(\mathcal{B}_n)\,dx - \int_{\mathbb{R}^N}(\partial_{x_1}\phi_n)^2\mathcal{B}_n^2\,dx\\
&\quad + \int_{\mathbb{R}^N}\mathcal{B}_n\Big((\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2 - 3c_n\mathcal{B}_n(\partial_{x_1}\phi_n - c_n\mathcal{B}_n)\Big)\,dx
\end{aligned}$$

and this can be written as

$$\begin{aligned}
&\int_{\mathbb{R}^N}(\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2 + (\partial_{x_1}\mathcal{B}_n)^2 + \frac{N-3}{N-1}|\nabla_{x_\perp}\phi_n|^2(1+\mathcal{B}_n)^2 + \frac{N-3}{N-1}|\nabla_{x_\perp}\mathcal{B}_n|^2 + \varepsilon_n^2(1-\mathcal{B}_n)\mathcal{B}_n^2\,dx\\
&= -\frac{\Gamma}{3}\mathfrak{c}_s^2\int_{\mathbb{R}^N}\mathcal{B}_n^3\,dz - \mathfrak{c}_s^2\int_{\mathbb{R}^N}V_4(\mathcal{B}_n)\,dx - \int_{\mathbb{R}^N}(\partial_{x_1}\phi_n)^2\mathcal{B}_n^2\,dx \qquad (93)\\
&\quad + \int_{\mathbb{R}^N}\mathcal{B}_n\Big((\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2 - 3c_n\mathcal{B}_n(\partial_{x_1}\phi_n - c_n\mathcal{B}_n)\Big)\,dx.
\end{aligned}$$

For n sufficiently large we have $\frac{1}{2}\mathcal{B}_n \leq (1-\mathcal{B}_n)\mathcal{B}_n^2 \leq \frac{3}{2}\mathcal{B}_n^2$ and then all the terms in the left-hand side of (93) are nonnegative. We will find an upper bound for the right-hand side of (93). First we notice that the third integral there is nonnegative. Since $\mathcal{B}_n \to 0$ in L^∞ and $V_4(\alpha) = \mathcal{O}(\alpha^4)$ as $\alpha \to 0$, we have

$$\Big|\mathfrak{c}_s^2\int_{\mathbb{R}^N}V_4(\mathcal{B}_n)\,dx\Big| \leq C\|\mathcal{B}_n\|_{L^4}^4 \leq C\|\mathcal{B}_n\|_{L^\infty}\|\mathcal{B}_n\|_{L^3}^3. \qquad (94)$$

Using the fact that $\|\mathcal{B}_n\|_{L^\infty} \leq 1/4$ for n large enough and the inequality $2ab \leq a^2 + b^2$, we get

$$\int_{\mathbb{R}^N}\mathcal{B}_n\Big((\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2 - 3c_n\mathcal{B}_n(\partial_{x_1}\phi_n - c_n\mathcal{B}_n)\Big)\,dx \leq \frac{1}{2}\int_{\mathbb{R}^N}(\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2\,dx + 9\mathfrak{c}_s^2\int_{\mathbb{R}^N}\mathcal{B}_n^4\,dx. \qquad (95)$$

It is easy to see that $\mathcal{B}_n \in H^1(\mathbb{R}^N)$ (see the Introduction of [17]). We recall the critical Sobolev embedding: for any $h \in H^1(\mathbb{R}^N)$ (with $N \geq 3$) there holds

$$\|h\|_{L^{\frac{2N}{N-2}}} \leq C\|\partial_{x_1}h\|_{L^2}^{\frac{1}{N}}\|\nabla_{x_\perp}h\|_{L^2}^{\frac{N-1}{N}}. \qquad (96)$$

Assume first that $N \geq 6$. Then $2^* = \frac{2N}{N-2} \leq 3$. Using the Sobolev embedding (96) and the fact that $\|\mathcal{B}_n\|_{L^\infty}$ is bounded we get

$$\|\mathcal{B}_n\|_{L^3}^3 \leq \|\mathcal{B}_n\|_{L^\infty}^{3-2^*} \|\mathcal{B}_n\|_{L^{2^*}}^{2^*} \leq C \|\partial_{x_1}\mathcal{B}_n\|_{L^2}^{\frac{2^*}{N}} \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2^*(N-1)}{N}}. \tag{97}$$

Using the inequalities $\|\mathcal{B}_n\|_{L^4}^4 \leq \|\mathcal{B}_n\|_{L^\infty}\|\mathcal{B}_n\|_{L^3}^3$ and $1 + \mathcal{B}_n \geq 1/2$ for n large, we deduce from (93) that

$$\int_{\mathbb{R}^N} (\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2 + (\partial_{x_1}\mathcal{B}_n)^2 + |\nabla_{x_\perp}\phi_n|^2 + |\nabla_{x_\perp}\mathcal{B}_n|^2 + \varepsilon_n^2\mathcal{B}_n^2 \, dx \leq C\|\mathcal{B}_n\|_{L^3}^3. \tag{98}$$

From (98) and (97) we obtain

$$\|\nabla_{x_\perp}\phi_n\|_{L^2}^2 + \|\partial_{x_1}\mathcal{B}_n\|_{L^2}^2 + \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^2 \leq C\|\mathcal{B}_n\|_{L^3}^3 \leq C\|\partial_{x_1}\mathcal{B}_n\|_{L^2}^{\frac{2}{N-2}} \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2N-2}{N-2}}. \tag{99}$$

Assume now that ($N = 4$ or $N = 5$) and $\Gamma \neq 0$. From (93), (94) and (95) we get

$$\int_{\mathbb{R}^N} (\partial_{x_1}\phi_n - c_n\mathcal{B}_n)^2 + (\partial_{x_1}\mathcal{B}_n)^2 + |\nabla_{x_\perp}\phi_n|^2 + |\nabla_{x_\perp}\mathcal{B}_n|^2 + \varepsilon_n^2\mathcal{B}_n^2 \, dx \leq C\|\mathcal{B}_n\|_{L^4}^4. \tag{100}$$

We have $2^* = 4$ if $N = 4$ and $2^* = \frac{10}{3} < 4$ if $N = 5$. By the Sobolev embedding we have

$$\|\mathcal{B}_n\|_{L^4}^4 \leq \|\mathcal{B}_n\|_{L^\infty}^{4-2^*} \|\mathcal{B}_n\|_{L^{2^*}}^{2^*} \leq C\|\mathcal{B}_n\|_{L^{2^*}}^{2^*} \leq C\|\partial_{x_1}\mathcal{B}_n\|_{L^2}^{\frac{2^*}{N}} \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2^*(N-1)}{N}}. \tag{101}$$

The two inequalities above give

$$\|\nabla_{x_\perp}\phi_n\|_{L^2}^2 + \|\partial_{x_1}\mathcal{B}_n\|_{L^2}^2 + \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^2 \leq C\|\mathcal{B}_n\|_{L^4}^4 \leq C\|\partial_{x_1}\mathcal{B}_n\|_{L^2}^{\frac{2}{N-2}} \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2N-2}{N-2}}. \tag{102}$$

From either (99) or (102) we obtain

$$\|\partial_{x_1}\mathcal{B}_n\|_{L^2}^2 \leq C\|\partial_{x_1}\mathcal{B}_n\|_{L^2}^{\frac{2}{N-2}} \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2N-2}{N-2}},$$

which gives $\|\partial_{x_1}\mathcal{B}_n\|_{L^2}^{\frac{2N-6}{N-2}} \leq C\|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2N-2}{N-2}}$, or equivalently

$$\|\partial_{x_1}\mathcal{B}_n\|_{L^2} \leq C\|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{N-1}{N-3}}. \tag{103}$$

Now we plug (103) into (98) or (100) to discover

$$\|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^2 \leq C\|\partial_{x_1}\mathcal{B}_n\|_{L^2}^{\frac{2}{N-2}} \|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2N-2}{N-2}} \leq C\|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2}^{\frac{2(N-1)}{N-3}}.$$

Since $\frac{2(N-1)}{N-3} > 2$ we infer that there is a constant $m > 0$ such that $\|\nabla_{x_\perp}\mathcal{B}_n\|_{L^2} \geq m$ for all sufficiently large n. On the other hand U_n satisfies the Pohozaev identity $P_{c_n}(U_n) = 0$, hence for large n we have

$$E_{c_n}(U_n) = \frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_{x_\perp}U_n|^2 \, dx \geq \frac{2}{N-1} r_0^2 \int_{\mathbb{R}^N} |\nabla_{x_\perp}\mathcal{B}_n|^2 \, dx \geq \frac{1}{N-1} r_0^2 m^2.$$

This contradicts the assumption that $E_{c_n}(U_n) \to 0$ as $n \to \infty$. The proof of Proposition 18 is complete. ☐

Remark 26 We do not know whether T_c tends to zero or not as $c \to \mathfrak{c}_s$ if $N = 4$ or $N = 5$ and $\Gamma \neq 0$.

4.2 Proof of Proposition 19

Let $N \geq 4$ and let $(U_n, c_n)_{n \geq 1}$ be a sequence of nonconstant, finite energy solutions solution of (TW$_{c_n}$) such that $E_{c_n}(U_n) \to 0$. By Proposition 14 (ii) we have $|U_n| \to r_0 > 0$ uniformly in $\mathbb{R}^N$, hence for n sufficiently large we may write

$$U_n(x) = \rho_n(x)\mathrm{e}^{i\phi_n(x)} = r_0\Big(1 + \alpha_n A_n(z)\Big)\exp\Big(i\beta_n\varphi_n(z)\Big) \qquad \text{where } z_1 = \lambda_n x_1, \quad z_\perp = \sigma_n x_\perp,$$

and $\alpha_n = \frac{1}{r_0}\|\rho_n - r_0\|_{L^\infty} \to 0$. Using the Pohozaev identity $P_{c_n}(U_n) = 0$ and (47) we have

$$\frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_{x_\perp} U_n(x)|^2\,dx = E(U_n) + c_n Q(U_n) = \frac{2}{N}\int_{\mathbb{R}^N} |\nabla\rho_n|^2\,dx.$$

Since $U_n \in \mathcal{E}$ and U_n is not constant, we have $\int_{\mathbb{R}^N} |\nabla_{x_\perp} U_n(x)|^2\,dx > 0$ and the above identity implies that ρ_n is not constant. The equality $E(U_n) + c_n Q(U_n) = \frac{2}{N}\int_{\mathbb{R}^N} |\nabla\rho_n|^2\,dx$ can be written as

$$\left(1 - \frac{2}{N}\right)\int_{\mathbb{R}^N} |\nabla\rho_n|^2\,dx + \int_{\mathbb{R}^N} \rho_n^2|\nabla\phi_n|^2\,dx + c_n Q(U_n) + \int_{\mathbb{R}^N} V(\rho_n^2)\,dx = 0.$$

Since $\rho_n \to r_0$ uniformly in $\mathbb{R}^N$ as $n \to \infty$, for n large we have $V(\rho_n^2) \geq 0$ and from the last identity we infer that $0 > c_n Q(U_n) = \int_{\mathbb{R}^N} (r_0^2 - \rho_n^2)\frac{\partial\phi}{\partial x_1}\,dx$, which implies $\|\partial_{x_1}\phi_n\|_{L^2} > 0$. We must have $\|\nabla_{x_\perp}\phi_n\|_{L^2} > 0$ (otherwise ϕ would depend only on x_1, contradicting the fact that $\int_{\mathbb{R}^N} |\nabla\phi_n|^2\,dx$ is finite).

The choice of α_n implies $\|A_n\|_{L^\infty} = 1$. Since A_n, $\partial_{z_1}\phi_n$ and $\nabla_{z_\perp}\phi_n$ are nonzero, by scaling it is easy to see that

$$\|A_n\|_{L^2} = \|\partial_{z_1}\varphi_n\|_{L^2} = \|\nabla_{z_\perp}\varphi_n\|_{L^2} = 1 \tag{104}$$

if and only if

$$\lambda_n\sigma_n^{N-1} = \frac{\|\,|U_n| - r_0\|_{L^\infty}^2}{\|\,|U_n| - r_0\|_{L^2}^2}, \qquad \lambda_n\beta_n = \|\partial_{x_1}\phi_n\|_{L^2}\frac{\|\,|U_n| - r_0\|_{L^\infty}}{\|\,|U_n| - r_0\|_{L^2}}, \qquad \beta_n\sigma_n = \|\nabla_{x_\perp}\phi_n\|_{L^2}\frac{\|\,|U_n| - r_0\|_{L^\infty}}{\|\,|U_n| - r_0\|_{L^2}}.$$

Since $N \geq 3$, the above equalities allow to compute λ_n, β_n and σ_n. Hence the scaling parameters $(\alpha_n, \beta_n, \lambda_n, \sigma_n)$ are uniquely determined if (104) holds and $\|A_n\|_{L^\infty} = 1$.

The Pohozaev identity $P_{c_n}(U_n) = 0$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^N} \lambda_n^2\beta_n^2(\partial_{z_1}\varphi_n)^2\Big(1 + \alpha_n A_n\Big)^2 + \alpha_n^2\lambda_n^2(\partial_{z_1}A_n)^2 \\
&+ \frac{N-3}{N-1}\beta_n^2\sigma_n^2|\nabla_{z_\perp}\varphi_n|^2\Big(1 + \alpha_n A_n\Big)^2 + \frac{N-3}{N-1}\alpha_n^2\sigma_n^2|\nabla_{z_\perp}A_n|^2 + \frac{1}{r_0^2}V\Big(r_0^2(1 + \alpha_n A_n)^2\Big)\,dz \\
&\qquad = 2c_n\int_{\mathbb{R}^N} 2\lambda_n\alpha_n\beta_n A_n\partial_{z_1}\varphi_n + \lambda_n\alpha_n^2\beta_n A_n^2\partial_{z_1}\varphi_n\,dz.
\end{aligned} \tag{105}$$

By (104), the right-hand side of (105) is $\mathcal{O}(\lambda_n\alpha_n\beta_n)$. Since $\alpha_n \to 0$ and $\|A_n\|_{L^\infty} = 1$ for n large enough we have $1 + \alpha_n A_n \geq 1/2$, and by (27) we get $V(r_0^2(1 + \alpha_n A_n)^2) \geq \frac{1}{2}r_0^2\mathfrak{c}_s^2\alpha_n^2 A_n^2$. If $N \geq 3$ all the terms in the left-hand side of (105) are non-negative and we infer that

$$\int_{\mathbb{R}^N} \lambda_n^2\beta_n^2(\partial_{z_1}\varphi_n)^2 + \alpha_n^2 A_n^2\,dz = \mathcal{O}(\lambda_n\alpha_n\beta_n).$$

From the normalization (104) it follows that

$$\lambda_n^2\beta_n^2 = \mathcal{O}(\lambda_n\alpha_n\beta_n), \qquad \text{and} \qquad \alpha_n^2 = \mathcal{O}(\lambda_n\alpha_n\beta_n),$$

which yields

$$C_1 \leq \frac{\lambda_n\beta_n}{\alpha_n} \leq C_2 \qquad \text{for some } C_1,\, C_2 > 0. \tag{106}$$

Let $\theta_n = \frac{\lambda_n\beta_n}{\alpha_n}$. We use the Taylor expansion (27) for the potential V, multiply (105) by $\frac{1}{\alpha_n^2}$ and write the resulting equality in the form

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(\theta_n\partial_{z_1}\varphi_n - c_n A_n\Big)^2 + \lambda_n^2(\partial_{z_1}A_n)^2 + \frac{N-3}{N-1}\frac{\theta_n^2\sigma_n^2}{\lambda_n^2}|\nabla_{z_\perp}\varphi_n|^2\Big(1 + \alpha_n A_n\Big)^2 + \frac{N-3}{N-1}\sigma_n^2|\nabla_{z_\perp}A_n|^2 \\
&\qquad\qquad + (\mathfrak{c}_s^2 - c_n^2)A_n^2\,dz \\
&= -\int_{\mathbb{R}^N} \theta_n^2\alpha_n(\partial_{z_1}\varphi_n)^2\Big(2A_n + \alpha_n A_n^2\Big) + \mathfrak{c}_s^2\alpha_n\Big(\frac{\Gamma}{3} - 1\Big)A_n^3 + \mathfrak{c}_s^2\frac{V_4(\alpha_n A_n)}{\alpha_n^2} - 2c_n\theta_n\alpha_n A_n^2\partial_{z_1}\varphi_n\,dz.
\end{aligned}$$

By (104) and (106) the right-hand side of the above equality is $\mathcal{O}(\alpha_n)$. If $N \geq 3$ all the terms in the left-hand side are nonnegative. In particular, we get $(\mathfrak{c}_s^2 - c_n^2) \int_{\mathbb{R}^N} A_n^2 \, dz = \mathfrak{c}_s^2 - c_n^2 = \mathcal{O}(\alpha_n)$, so that $c_n \to \mathfrak{c}_s$. Assuming that $N \geq 4$, we also infer that

$$\int_{\mathbb{R}^N} \lambda_n^2 (\partial_{z_1} A_n)^2 + \frac{\sigma_n^2}{\lambda_n^2} |\nabla_{z_\perp} \varphi_n|^2 \, dz = \mathcal{O}(\alpha_n).$$

Together with (104) and (106), this implies

$$\frac{\sigma_n^2}{\lambda_n^2} = \mathcal{O}(\alpha_n) \qquad \text{and} \qquad \int_{\mathbb{R}^N} (\partial_{z_1} A_n)^2 \, dz = \mathcal{O}\Big(\frac{\alpha_n}{\lambda_n^2}\Big). \tag{107}$$

The Pohozaev identity $P_{c_n}(U_n) = 0$ and (104) imply that for each n such that $1 + \alpha_n A_n \geq \frac{1}{2}$ we have

$$\begin{aligned} E_{c_n}(U_n) &= \frac{2}{N-1} \int_{\mathbb{R}^N} |\nabla_\perp U_n|^2 \, dx \\ &= \frac{2 r_0^2}{(N-1)\lambda_n \sigma_n^{N-1}} \int_{\mathbb{R}^N} \beta_n^2 \sigma_n^2 |\nabla_{z_\perp} \varphi_n|^2 \Big(1 + \alpha_n A_n\Big)^2 + \alpha_n^2 \sigma_n^2 |\nabla_{z_\perp} A_n|^2 \, dz \\ &\geq \frac{r_0^2 \alpha_n^2 \theta_n^2}{2(N-1)\lambda_n^3 \sigma_n^{N-3}} \int_{\mathbb{R}^N} |\nabla_{z_\perp} \varphi_n|^2 \, dz \geq \frac{\alpha_n^2}{C \lambda_n^3 \sigma_n^{N-3}}. \end{aligned} \tag{108}$$

However, in view of (107) we have

$$\frac{\alpha_n^2}{\lambda_n^3 \sigma_n^{N-3}} = \frac{\alpha_n^2}{\lambda_n^N (\sigma_n/\lambda_n)^{N-3}} \geq \Big(\frac{\alpha_n}{\lambda_n^2}\Big)^{N/2} \frac{\alpha_n^2}{C \alpha_n^{N/2} \alpha_n^{(N-3)/2}} = \Big(\frac{\alpha_n}{\lambda_n^2}\Big)^{N/2} \frac{1}{C \alpha_n^{(2N-7)/2}}. \tag{109}$$

Notice that $\alpha_n^{(2N-7)/2} \to 0$ as $\alpha_n \to 0$ because $N \geq 4$. The fact that $E_{c_n}(U_n) \longrightarrow 0$, (108) and (109) imply that $\frac{\alpha_n}{\lambda_n^2} \to 0$ as $n \to +\infty$. Then using (107) we find

$$\int_{\mathbb{R}^N} (\partial_{z_1} A_n)^2 \, dz = \mathcal{O}\Big(\frac{\alpha_n}{\lambda_n^2}\Big) \to 0$$

and the proof is complete. □

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