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BOUNDS FOR THE HILBERT TRANSFORM WITH MATRIX A_2 WEIGHTS

KELLY BICKEL, STEFANIE PETERMICHL⋆, AND BRETT D. WICK‡

Abstract. Let W denote a matrix A_2 weight. In this paper we implement a scalar argument using the square function to deduce related results for vector-valued functions on $L^2(\mathbb{R}, \mathbb{C}^d)$. These results are then used to study the boundedness of the Hilbert transform and Haar multipliers on $L^2(\mathbb{R}, \mathbb{C}^d)$. Our proof shortens the original argument by Treil and Volberg and improves the dependence on the A_2 characteristic. In particular, we prove that:

$$\|Tf\|_{L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2} \|f\|_{L^2(W)},$$

where T is either the Hilbert transform or a Haar multiplier.

1. Introduction

Write $L^2 \equiv L^2(\mathbb{R}, \mathbb{C}^d)$, namely those functions such that

$$\|f\|_{L^2}^2 \equiv \int_{\mathbb{R}} \|f(x)\|_{\mathbb{C}^d}^2 \, dx < \infty.$$

For a $d \times d$ positive self-adjoint matrix-valued function W we set $L^2(W) \equiv L^2(\mathbb{R}, W, \mathbb{C}^d)$ to be

$$\|f\|_{L^2(W)}^2 \equiv \int_{\mathbb{R}} \|W^{\frac{1}{2}}(x) f(x)\|_{\mathbb{C}^d}^2 \, dx = \int_{\mathbb{R}} \langle W(x) f(x), f(x) \rangle_{\mathbb{C}^d} \, dx < \infty.$$

We are in particular interested in those matrix weights W that satisfy the matrix A_2 Muckenhoupt condition:

$$\left[W\right]_{A_2} := \sup_I \left\| \langle W \rangle_I^{\frac{1}{2}} \langle W^{-1} \rangle_I^{\frac{1}{2}} \right\|^2 < \infty,$$

where $\|\cdot\|$ denotes the norm of the matrix acting on $\mathbb{C}^d$, the supremum is taken over all intervals and $\langle W \rangle_I \equiv \frac{1}{|I|} \int_I W$. Now, if $d = 1$ and $w \in A_2$, i.e. if w is a scalar-valued A_2 weight, then it is well-known that the Hilbert transform H maps $L^2(\mathbb{R}, w) \to L^2(\mathbb{R}, w)$. The question of sharp dependence was answered by Petermichl, who showed in [5] that

$$\|H\|_{L^2(\mathbb{R},w) \to L^2(\mathbb{R},w)} \lesssim [w]_{A_2}.$$

In [6], Petermichl and Pott provide a simple proof establishing the boundedness of H on $L^2(\mathbb{R}, w)$ with constant $[w]_{A_2}^{\frac{3}{2}}$. Key tools in the proof include the linear bound for the dyadic square function and the characterization of the Hilbert transform using dyadic shifts.

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Despite substantial complexities arising in the matrix valued case, Treil and Volberg showed in [7] that if $W \in A_2$, then the Hilbert transform $H : L^2(W) \to L^2(W)$ boundedly. Their paper does not track the dependence on $[W]_{A_2}$ and the question of the sharp constant remains open.

In this paper, we study the dependence of $\|H\|_{L^2(W)\to L^2(W)}$ on the A_2 characteristic $[W]_{A_2}$. Our arguments are strongly influenced by those in [6] and [7]. We first consider a matrix analogue of the dyadic square function. In Theorem 3.1, we obtain bounds on this square function-type object in terms of $[W]_{A_2}$. In Theorem 4.1, we establish that

$$\|H\|_{L^2(W)\to L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2}.$$

Although these constants do not appear to be sharp, they are better than what has previously appeared in the literature. Related results for the Haar multipliers T_σ are obtained in Theorem 5.2.

Further improvements of these estimates using the scalar proof strategy will likely require a matrix version of the weighted Carleson Embedding Theorem and sharp bounds on related testing conditions.

2. Basic Facts and Notation

Let $\mathcal{D}$ denote the standard dyadic grid. For $\alpha \in \mathbb{R}$ and $r > 0$, let $\mathcal{D}^{\alpha,r}$ denote the dyadic grid $\{\alpha + rI : I \in \mathcal{D}\}$ and let $\{h_I\}_{I\in\mathcal{D}^{\alpha,r}}$ denote the Haar functions adapted to $\mathcal{D}^{\alpha,r}$ and normalized in L^2. In much of what follows, we omit the α, r notation because the arguments hold for all such dyadic grids. Given $I \in \mathcal{D}$, let I_+ denote its right half and I_- denote its left half. Throughout this paper, $A \lesssim B$ indicates that $A \leq CB$, for some constant C that may depend on the dimension d.

Let $f \in L^2$. To define $\widehat{f}(I)$, let $e_1, \dots, e_d$ be an orthonormal basis in $\mathbb{C}^d$. Then,

$$\widehat{f}(I) = \sum_{j=1}^{d} \langle f, h_I e_j \rangle_{L^2} \, e_j,$$

where h_I is the standard Haar function defined by

$$h_I \equiv |I|^{-\frac{1}{2}} \left(\mathbf{1}_{I_+} - \mathbf{1}_{I_-} \right) \qquad \forall I \in \mathcal{D}.$$

Similarly, define $h_I^1 \equiv \mathbf{1}_I \frac{1}{|I|}$ for any $I \in \mathcal{D}$. Notice that for non-cancellative Haar functions we chose a different normalization. Now, let W be a matrix weight and for any interval I, set $W(I) \equiv \int_I W$. At a later point, we will require the use of disbalanced Haar functions adapted to W. In the matrix setting, these are considered by Treil and Volberg in [7]. To define them, fix $I \in \mathcal{D}$ and let $e_I^1, \dots, e_I^d$ be a set of orthonormal eigenvectors of $\langle W \rangle_I$. Define

$$w_I^k \equiv \left\| \langle W \rangle_I^{\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}^{-1} = \left\| \langle W \rangle_I^{-\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}.$$

Then, the vector-valued functions $\{w_I^k h_I e_I^k\}_{I\in\mathcal{D},1\le k\le d}$ are normalized in $L^2(W)$. Define the disbalanced Haar functions

$$g_I^{W,k} \equiv w_I^k h_I e_I^k + h_I^1 \tilde{e}_I^k,$$

where the vector $\tilde{e}_I^k = A(W,I)e_I^k$ and

$$A(W,I) = \frac{1}{2}|I|^{\frac{1}{2}} \langle W\rangle_I^{-1} \left(\langle W\rangle_{I_-} - \langle W\rangle_{I_+}\right) \langle W\rangle_I^{-\frac{1}{2}}.$$

Simple calculations, which appear in [7], show that

$$\left\langle g_I^{W,k}, g_J^{W,j}\right\rangle_W = 0 \qquad \forall\ J\neq I,\ 1\le j,k\le d, \tag{1}$$

and the functions satisfy $\|g_I^{W,k}\|_{L^2(W)}\le 5$. It is also clear that

$$h_I e_I^k = \left(w_I^k\right)^{-1} g_I^{W,k} - \left(w_I^k\right)^{-1} A(W,I) h_I^1 e_I^k, \quad \forall I\in\mathcal{D}, k=1,\dots,d. \tag{2}$$

3. Square Function Estimate

We first consider a generalization of the square function S_W to this matrix setting. Namely start with the simple Haar multiplier operator

$$T_\sigma f = \sum_{I\in\mathcal{D}} \sigma_I \widehat{f}(I) h_I.$$

where $\sigma_I\in\{-1;1\}$. Take expectation is in the natural probability space of sequences in $\{-1,1\}$ as follows:

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}} \langle W(x)T_\sigma f(x), T_\sigma f(x)\rangle_{\mathbb{C}^d}\,dx &= \int_{\mathbb{R}} \mathbb{E}\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}} \sigma_I\sigma_J h_I(x)h_J(x)\left\langle W(x)\widehat{f}(I), \widehat{f}(J)\right\rangle_{\mathbb{C}^d} dx\\ &= \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d}.\end{aligned}$$

Therefore

$$S_W : L^2(\mathbb{R},\mathbb{C}^d)\to L^2(\mathbb{R},\mathbb{R});\quad S_W f(x) := \sqrt{\mathbb{E}\,\langle W(x)T_\sigma f(x), T_\sigma f(x)\rangle_{\mathbb{C}^d}},$$

so that $\|S_W f\|_{L^2}^2 = \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d}$.

In the scalar situation, the square function is bounded on $L^2(\mathbb{R},w)$ with linear dependence on $[w]_{A_2}$. For matrix A_2 weights, we obtain a similar bound, which differs from the scalar bound by a logarithm:

Theorem 3.1. *Let W be a $d\times d$ matrix weight in A_2. Then*

$$\sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d} \lesssim [W]_{A_2}^2 \log{[W]_{A_2}}\ \|f\|_{L^2(W)}^2 \quad \forall f\in L^2(W).$$

To prove Theorem 3.1, we initially proceed as in Petermichl and Pott's proof of the scalar case in [6]. Some arguments generalize easily, but to finish the proof, we also require the following result of Treil and Volberg, which appears as Theorem 6.1 in [7]:

Theorem 3.2 (Treil and Volberg, [7]). *Let W be a $d \times d$ matrix weight in A_2. Then for all $f \in L^2$,*

$$\sum_{I \in \mathcal{D}} |I| \left\| \langle W \rangle_I^{-\frac{1}{2}} \left(\langle W \rangle_{I_-} - \langle W \rangle_{I_+} \right) \langle W \rangle_I^{-\frac{1}{2}} \right\|^2 \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\|^2 \lesssim [W]_{A_2} \log [W]_{A_2} \|f\|_{L^2}^2.$$

The constant $[W]_{A_2} \log [W]_{A_2}$ is not specified in Treil-Volberg's statement of the theorem. However, a careful reading of the proofs of their Lemma 3.1, Lemma 3.6, Theorem 4.1, and Theorem 6.1 reveal the above constant. A proof of the square function-type bound using only the arguments from [6] requires a matrix version of the weighted Carleson Embedding Theorem and testing conditions on a particular dyadic sum. We conjecture that such tools exist and given such tools, would have a proof of Conjecture 6.1.

3.1. **Proof of Theorem 3.1.** The argument in [6] requires a lower bound on the square function. Our matrix analogue is Theorem 3.3 and the proof utilizes both arguments from [6] and Theorem 3.2.

Theorem 3.3. *Let W be a $d \times d$ matrix weight in A_2. Then*

$$\|f\|_{L^2(W)}^2 \lesssim [W]_{A_2} \log [W]_{A_2} \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \quad \forall f \in L^2(W).$$

Proof. As in [6], we can assume without loss of generality that W and W^{-1} are bounded. For more details, see Remark 3.4. Then $L^2(W)$ and L^2 are equal as sets. For ease of notation, define the constant

$$C_W \equiv [W]_{A_2} \log [W]_{A_2}.$$

Let $e_1, \dots, e_d$ be the standard orthonormal basis in $\mathbb{C}^d$. Define the discrete multiplication operator $D_W : L^2 \to L^2$ by

$$D_W : h_I e_k \mapsto \langle W \rangle_I h_I e_k \quad \forall I \in \mathcal{D}, k = 1, \dots, d,$$

and let M_W denote multiplication by W. Observe that

$$\langle D_W f, f \rangle_{L_2} = \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d}.$$

We can rewrite the desired inequality as:

$$\langle M_W f, f \rangle_{L^2} \lesssim C_W \langle D_W f, f \rangle_{L^2}, \qquad \forall f \in L^2. \tag{3}$$

As in [6], we convert this to an inverse inequality. Since W and W^{-1} are bounded, D_W and M_W are bounded and invertible with $M_W^{-1} = M_{W^{-1}}$ and D_W^{-1} defined by

$$D_W^{-1} : h_I e_k \mapsto \langle W\rangle_I^{-1} h_I e_k \quad \forall I \in \mathcal{D}, k = 1, \dots, d.$$

Since M_W and D_W and their inverses are positive, it is easy to show that (3) is equivalent to

$$\langle D_W^{-1} f, f\rangle_{L^2} \lesssim C_W \langle M_W^{-1} f, f\rangle_{L^2}, \qquad \forall f \in L^2. \tag{4}$$

So to prove Theorem 3.3, we need to establish:

$$\sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I^{-1} \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d} \lesssim C_W \|f\|^2_{L^2(W^{-1})} \quad \forall f \in L^2.$$

We will rewrite the sum using Haar functions adapted to W. First, for $I \in \mathcal{D}$, let $e_I^1, \dots, e_I^d$ be a set of orthonormal eigenvectors of $\langle W\rangle_I$. Recall that

$$w_I^k \equiv \left\| \langle W\rangle_I^{\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}^{-1} = \left\| \langle W\rangle_I^{-\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}.$$

Using these definitions, expand the sum as follows:

$$\begin{aligned}
\sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I^{-1} \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d} &= \sum_{I\in\mathcal{D}} \sum_{j,k=1}^{d} \left\langle \langle W\rangle_I^{-1} \langle f, h_I e_I^k\rangle_{L^2} e_I^k, \langle f, h_I e_I^j\rangle_{L^2} e_I^j \right\rangle_{\mathbb{C}^d} \\
&= \sum_{I\in\mathcal{D}} \sum_{j,k=1}^{d} \langle f, h_I e_I^k\rangle_{L^2} \overline{\langle f, h_I e_I^j\rangle_{L^2}} \left\langle \langle W\rangle_I^{-1} e_I^k, e_I^j \right\rangle_{\mathbb{C}^d} \\
&= \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left|\langle f, h_I e_I^k\rangle_{L^2}\right|^2 \left\langle \langle W\rangle_I^{-1} e_I^k, e_I^k \right\rangle_{\mathbb{C}^d} \\
&= \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left(w_I^k\right)^2 \left|\langle f, h_I e_I^k\rangle_{L^2}\right|^2.
\end{aligned}$$

Now, we can expand the $h_I e_I^k$ using the disbalanced Haar functions adapted to W as in (2). This transforms our sum as follows:

$$\begin{aligned}
\sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left(w_I^k\right)^2 \left|\langle f, h_I e_I^k\rangle_{L^2}\right|^2 &= \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left(w_I^k\right)^2 \left|\langle f, \left(w_I^k\right)^{-1} g_I^{W,k} - \left(w_I^k\right)^{-1} A(W,I) h_I^1 e_I^k\rangle_{L^2}\right|^2 \\
&\leq \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left|\langle f, g_I^{W,k}\rangle_{L^2}\right|^2 + 2 \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left|\langle f, g_I^{W,k}\rangle_{L^2} \langle f, A(W,I) h_I^1 e_I^k\rangle_{L^2}\right| \\
&\quad + \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left|\langle f, A(W,I) h_I^1 e_I^k\rangle_{L^2}\right|^2 \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

It is clear that

$$S_1 = \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left| \langle f, g_I^{W,k} \rangle_{L^2} \right|^2 = \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left| \langle W^{-1} f, g_I^{W,k} \rangle_{L^2(W)} \right|^2 \lesssim \|f\|^2_{L^2(W^{-1})},$$

since the $g_I^{W,k}$ satisfy (1) and are uniformly bounded in $L^2(W)$. Since $S_2 \lesssim S_1^{\frac{1}{2}} S_3^{\frac{1}{2}}$, the main term to understand is S_3. It can be written as

$$(5) \qquad S_3 = \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left\langle f, \frac{1}{2}|I|^{\frac{1}{2}} \langle W\rangle_I^{-1} \left(\langle W\rangle_{I_+} - \langle W\rangle_{I_-}\right) \langle W\rangle_I^{-\frac{1}{2}} h_I^1 e_I^k \right\rangle_{L^2}^2$$

$$= \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left\langle f, \langle W\rangle_I^{-1} \widehat{W}(I) \langle W\rangle_I^{-\frac{1}{2}} h_I^1 e_I^k \right\rangle_{L^2}^2$$

$$= \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left\langle \langle f\rangle_I, \langle W\rangle_I^{-1} \widehat{W}(I) \langle W\rangle_I^{-\frac{1}{2}} e_I^k \right\rangle_{\mathbb{C}^d}^2$$

$$(6) \qquad = \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left\langle \langle W\rangle_I^{-\frac{1}{2}} \langle f\rangle_I, \langle W\rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W\rangle_I^{-\frac{1}{2}} e_I^k \right\rangle_{\mathbb{C}^d}^2.$$

Now, we can bound S_3 as follows:

$$S_3 \leq \sum_{I\in\mathcal{D}} \sum_{k=1}^{d} \left\| \langle W\rangle_I^{-\frac{1}{2}} \langle f\rangle_I \right\|_{\mathbb{C}^d}^2 \left\| \langle W\rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W\rangle_I^{-\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}^2$$

$$\lesssim \sum_{I\in\mathcal{D}} \left\| \langle W\rangle_I^{-\frac{1}{2}} \langle f\rangle_I \right\|_{\mathbb{C}^d}^2 \left\| \langle W\rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W\rangle_I^{-\frac{1}{2}} \right\|^2$$

$$\lesssim [W]_{A_2} \log [W]_{A_2} \|f\|^2_{L^2(W^{-1})},$$

where we used Theorem 3.2 applied to $g = W^{-\frac{1}{2}} f$. This also implies a similar bound for S_2, and combining our estimates for S_1, S_2, S_3 completes the proof of Theorem 3.3. □

Using Theorem 3.3, we can easily prove Theorem 3.1:

Proof. Again, assume without loss of generality that W and W^{-1} are bounded and define the constant B_W by

$$B_W = [W]^2_{A_2} \log [W]_{A_2}.$$

Using our previous notation, Theorem 3.1 is equivalent to the inequality

$$\langle D_W f, f\rangle_{L^2} \lesssim B_W \langle M_W f, f\rangle_{L^2}, \quad \forall f \in L^2.$$

We require the following operator inequality

$$D_W \leq [W]_{A_2} \left(D_{W^{-1}}\right)^{-1}.$$

The A_2 condition implies that for every $I \in \mathcal{D}$ and vector $e_I \in \mathbb{C}^d$,

$$\left\langle \langle W \rangle_I^{\frac{1}{2}} \langle W^{-1} \rangle_I^{\frac{1}{2}} e_I, \langle W \rangle_I^{\frac{1}{2}} \langle W^{-1} \rangle_I^{\frac{1}{2}} e_I \right\rangle_{\mathbb{C}^d} \leq [W]_{A_2} \|e_I\|_{\mathbb{C}^d}^2.$$

Fixing $g \in L^2$ and setting $e_I = \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I)$, we can conclude

$$\left\langle \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I), \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d} \leq [W]_{A_2} \left\langle \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I), \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d}.$$

Then

$$\begin{aligned} \langle D_W g, g \rangle_{L^2} &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I), \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d} \\ &\leq [W]_{A_2} \sum_{I \in \mathcal{D}} \left\langle \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I), \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d} \\ &= [W]_{A_2} \langle (D_{W^{-1}})^{-1} g, g \rangle_{L^2}. \end{aligned}$$

Combining that estimate with (4) from Theorem 3.3 applied to W^{-1} gives:

$$\langle D_W g, g \rangle_{L^2} \lesssim [W]_{A_2} \langle (D_{W^{-1}})^{-1} g, g \rangle_{L^2} \lesssim [W]_{A_2} C_W \langle M_{W^{-1}}^{-1} g, g \rangle_{L^2} = B_W \|g\|_{L^2(W)} \qquad \forall g \in L^2,$$

which completes the proof. □

Remark 3.4 (Reducing to Bounded Weights)**.** The proof of Theorems 3.1 and 3.3 only handles weights W with both W and W^{-1} bounded. To reduce to this case, fix $W \in A_2$ and write

$$W(x) = \sum_{j=1}^{d} \lambda_j(x) P_{E_j}(x) \quad \text{ for } x \in \mathbb{R},$$

where the $\lambda_j(x)$ are eigenvalues of $W(x)$, the $E_j(x)$ are the associated orthogonal eigenspaces, and the $P_{E_j(x)}$ are the orthogonal projections onto the $E_j(x)$. Define

$$\begin{aligned} E_1^n(x) &\equiv \text{Eigenspaces of } W(x) \text{ corresponding to eigenvalues } \lambda_j(x) \leq \tfrac{1}{n}; \\ E_2^n(x) &\equiv \text{Eigenspaces of } W(x) \text{ corresponding to eigenvalues } \tfrac{1}{n} < \lambda_j(x) < n; \\ E_3^n(x) &\equiv \text{Eigenspaces of } W(x) \text{ corresponding to eigenvalues } \lambda_j(x) \geq n. \end{aligned}$$

Using these spaces, truncate $W(x)$ as follows:

$$W_n(x) = \tfrac{1}{n} P_{E_1^n(x)} + P_{E_2^n(x)} W(x) P_{E_2^n(x)} + n P_{E_3^n(x)}.$$

It is easy to see that $W_n, W_n^{-1} \leq n I_{d \times d}$. Each W_n is also an A_2 weight with

$$[W_n]_{A_2} \equiv \sup_I \left\| \langle W_n \rangle_I^{\frac{1}{2}} \langle W_n^{-1} \rangle_I^{\frac{1}{2}} \right\|^2 \lesssim [W]_{A_2}, \tag{7}$$

where the constant depends on the dimension d. This is not hard to show, but relies on the following two facts about positive self-adjoint matrices:

Fact 1: If $A_1, A_2 > 0$, then $\left\| A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \right\|^2 \approx \mathrm{Tr}(A_1 A_2)$.

Fact 2: If $A_1, A_2, B_1, B_2 \geq 0$ and each $A_j \leq B_j$, then $\mathrm{Tr}(A_1 A_2) \leq \mathrm{Tr}(B_1 B_2)$.

Here, the implied constants again depend on d. Fact 1 allows us to equate $\left\| \langle W_n \rangle_I^{\frac{1}{2}} \langle W_n^{-1} \rangle_I^{\frac{1}{2}} \right\|^2 \approx \mathrm{Tr}(\langle W_n \rangle_I \langle W_n^{-1} \rangle_I)$. Then, using Fact 2 and the matrix inequalities

$$\langle W_n \rangle_I \leq n I_{d\times d}$$
$$\left\langle P_{E_2^n(x)} W(x) P_{E_2^n(x)} + n P_{E_3^n(x)} \right\rangle_I \leq \langle W \rangle_I$$

for W_n and similar ones for W_n^{-1}, one can easily deduce (7). Then, Theorem 3.3 gives:

$$\|f\|^2_{L^2(W_n)} \lesssim [W_n]_{A_2} \log [W_n]_{A_2} \sum_{I\in\mathcal{D}} \left\langle \langle W_n \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \quad \forall f \in L^2(W_n). \tag{8}$$

Using basic convergence theorems, one can show that both

$$\lim_{n\to\infty} \|f\|^2_{L^2(W_n)} = \|f\|^2_{L^2(W)} \quad \text{and} \quad \lim_{n\to\infty} \sum_{I\in\mathcal{D}} \left\langle \langle W_n \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} = \sum_{I\in\mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d},$$

for $f \in L^2 \cap L^2(W)$. Combining this with (7) and (8) gives Theorem 3.3 for general W. Theorem 3.1 follows similarly.

4. The Hilbert Transform

The bounds given in Theorems 3.1 and 3.3 imply similar bounds for the Hilbert transform on $L^2(W)$. First, fix $\alpha \in \mathbb{R}$ and $r > 0$. The densely-defined shift operator $\text{Ш}^{\alpha,r}$ on $L^2(\mathbb{R})$ is given by

$$\text{Ш}^{\alpha,r} f \equiv \frac{1}{\sqrt{2}} \sum_{I\in\mathcal{D}^{\alpha,r}} \widehat{f}(I) \left(h_{I_-} - h_{I_+} \right),$$

where I_- is the left half of I and I_+ is the right half of I. In [4], S. Petermichl showed that the Hilbert transform H on $L^2(\mathbb{R})$ is basically an average of these dyadic shifts. Specifically, there is a constant c and $L^\infty(\mathbb{R})$ function b such that $H = cT + M_b$, where T is in the weak operator closure of the convex hull of the set $\{\text{Ш}^{\alpha,r}\}_{\alpha,r}$ in $\mathcal{L}(L^2(\mathbb{R}))$ and M_b is multiplication by b. The Hilbert transform on $L^2(\mathbb{R}, \mathbb{C}^d)$, also denoted H, is the scalar Hilbert transform applied component-wise. The dyadic shift operators $\text{Ш}^{\alpha,r}$ on $L^2(\mathbb{R}, \mathbb{C}^d)$ are similarly defined by

$$\text{Ш}^{\alpha,r} f \equiv \frac{1}{\sqrt{2}} \sum_{I\in\mathcal{D}^{\alpha,r}} \widehat{f}(I) \left(h_{I_-} - h_{I_+} \right),$$

which is the same as applying the scalar $\text{Ш}^{\alpha,r}$ shifts component-wise. Using the scalar-result, the Hilbert transform H on $L^2(\mathbb{R}, \mathbb{C}^d)$ satisfies $H = c\widetilde{T} + M_b$ where $\widetilde{T}$ is T applied component-wise and so, is in the weak operator closure of the convex hull of the set $\{Ш^{\alpha,r}\}_{\alpha,r}$ in $\mathcal{L}(L^2(\mathbb{R},\mathbb{C}^d))$.

In [7], Treil and Volberg show that for A_2 weights W, the Hilbert transform is bounded on $L^2(W)$, but do not track the dependence on the A_2 characteristic $[W]_{A_2}$. In contrast, using our square function estimates, we are able to establish the following:

Theorem 4.1. *Let W be a $d \times d$ matrix weight in A_2. Then*

$$\|Hf\|_{L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2} \|f\|_{L^2(W)} \qquad \forall\ f \in L^2(W).$$

Proof. As before, we omit the α, r notation. Observe that the square-function object in Theorems 3.1 and 3.3 does not "see" dyadic shifts. Specifically, let $\tilde{I}$ denote the parent of I in the dyadic grid. Then

$$\begin{aligned}\sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{Ш f}(I), \widehat{Ш f}(I) \right\rangle_{\mathbb{C}^d} &= \frac{1}{2}\sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{f}(\tilde{I}), \widehat{f}(\tilde{I}) \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I\in\mathcal{D}} \left\langle \frac{1}{2}\left(\langle W\rangle_{I_-} + \langle W\rangle_{I_+}\right) \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d}.\end{aligned}$$

Now, using Theorems 3.1 and 3.3, we have

$$\begin{aligned}\|Шf\|^2_{L^2(W)} &\lesssim [W]_{A_2} \log [W]_{A_2} \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{Ш f}(I), \widehat{Ш f}(I) \right\rangle_{\mathbb{C}^d} \\ &= [W]_{A_2} \log [W]_{A_2} \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &\lesssim [W]^3_{A_2} (\log [W]_{A_2})^2 \|f\|^2_{L^2(W)}.\end{aligned}$$

The formula for H in terms of dyadic shifts implies that

$$\|Hf\|^2_{L^2(W)} \lesssim \sup_{\alpha,r} \|Ш^{\alpha,r} f\|^2_{L^2(W)} + \|b\|^2_\infty \|f\|^2_{L^2(W)} \lesssim [W]^3_{A_2} (\log [W]_{A_2})^2 \|f\|^2_{L^2(W)},$$

as desired. □

5. Haar Multipliers

The arguments above extend easily to Haar multipliers. To begin, let $\sigma = \{\sigma_I\}_{I\in\mathcal{D}}$ be a sequence of matrices and define the Haar multiplier T_σ by

$$T_\sigma f \equiv \sum_{I\in\mathcal{D}} \sigma_I \widehat{f}(I) h_I.$$

To obtain boundedness on $L^2(W)$, it is crucial that the matrices σ_I interact well with W. To be precise, fix a weight $W \in A_2$ and define

$$\|\sigma\|_\infty \equiv \inf\left\{C : \langle W\rangle_I^{-\frac{1}{2}}\, \sigma_I^*\, \langle W\rangle_I\, \sigma_I\, \langle W\rangle_I^{-\frac{1}{2}} \leq C^2 I_{d\times d} \quad \forall I \in \mathcal{D}\right\}.$$

Equivalently, we could define $\|\sigma\|_\infty = \sup_{I\in\mathcal{D}} \left\|\langle W\rangle_I^{\frac{1}{2}}\, \sigma_I\, \langle W\rangle_I^{-\frac{1}{2}}\right\|$. Then, a variant of the following result is established by Isralowitz, Kwon and Pott in [2]:

Theorem 5.1. *Let $W \in A_2$ and $\sigma = \{\sigma_I\}_{I\in\mathcal{D}}$ a sequence of matrices. Then the Haar multiplier T_σ is bounded on $L^2(W)$ if and only if $\|\sigma\|_\infty < \infty$.*

Here, we have translated their result to the notation of this paper. It should also be noted that their paper handles the entire range $1 < p < \infty$. Now, we provide a new and simpler proof of this boundedness result for $p = 2$. Using our previous arguments, we are also able to track the dependence on $[W]_{A_2}$.

Theorem 5.2. *Let W be a $d \times d$ matrix weight in A_2 and let $\sigma = \{\sigma_I\}_{I\in\mathcal{D}}$ be a sequence of matrices. Then T_σ is bounded on $L^2(W)$ if and only if $\|\sigma\|_\infty < \infty$. Moreover,*

$$\|T_\sigma f\|_{L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2} \|\sigma\|_\infty \|f\|_{L^2(W)}.$$

Proof. Necessity is almost immediate. Fix $I \in \mathcal{D}$ and $e \in \mathbb{C}^d$ and simply set $f \equiv \langle W\rangle_I^{-\frac{1}{2}} h_I e$. Then simple computations prove that $T_\sigma f = \sigma_I \langle W\rangle_I^{-\frac{1}{2}} h_I e$ and the following norm equalities:

$$\|f\|^2_{L^2(W)} = \|\langle W\rangle_I^{-\frac{1}{2}} h_I e\|^2_{L^2(W)} = \|e\|^2_{\mathbb{C}^d}$$

$$\|T_\sigma f\|^2_{L^2(W)} = \left\|\sigma_I \langle W\rangle_I^{-\frac{1}{2}} h_I e\right\|^2_{L^2(W)} = \left\langle \langle W\rangle_I^{-\frac{1}{2}}\, \sigma_I^*\, \langle W\rangle_I\, \sigma_I\, \langle W\rangle_I^{-\frac{1}{2}}\, e, e\right\rangle_{\mathbb{C}^d}.$$

Assuming T_σ is bounded on $L^2(W)$, we can then conclude:

$$\left\langle \langle W\rangle_I^{-\frac{1}{2}}\, \sigma_I^*\, \langle W\rangle_I\, \sigma_I\, \langle W\rangle_I^{-\frac{1}{2}}\, e, e\right\rangle_{\mathbb{C}^d} = \|T_\sigma f\|^2_{L^2(W)} \leq \|T_\sigma\|^2_{L^2(W)\to L^2(W)} \|e\|^2_{\mathbb{C}^d}.$$

Since $e \in \mathbb{C}^d$ and $I \in \mathcal{D}$ was arbitrary we have that $\|\sigma\|_\infty < \infty$.

The proof of sufficiency, with the desired constant, is largely a repetition of computations from earlier in the paper. Again, observe that the square function in Theorems 3.1 and 3.3 does not "see" martingale transforms. Specifically,

$$\begin{aligned}
\sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{T_\sigma f}(I), \widehat{T_\sigma f}(I)\right\rangle_{\mathbb{C}^d} &= \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \sigma_I \widehat{f}(I), \sigma_I \widehat{f}(I)\right\rangle_{\mathbb{C}^d} \\
&= \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \sigma_I \langle W\rangle_I^{-\frac{1}{2}} \langle W\rangle_I^{\frac{1}{2}} \widehat{f}(I), \sigma_I \langle W\rangle_I^{-\frac{1}{2}} \langle W\rangle_I^{\frac{1}{2}} \widehat{f}(I)\right\rangle_{\mathbb{C}^d} \\
&\leq \|\sigma\|^2_\infty \sum_{I\in\mathcal{D}} \left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d}.
\end{aligned}$$

Simple applications of Theorems 3.1 and 3.3 then yield

$$\begin{aligned}\|T_\sigma f\|^2_{L^2(W)} &\lesssim [W]_{A_2}\log[W]_{A_2}\sum_{I\in\mathcal{D}}\left\langle \langle W\rangle_I \widehat{T_\sigma f}(I), \widehat{T_\sigma f}(I)\right\rangle_{\mathbb{C}^d}\\ &\leq [W]_{A_2}\log[W]_{A_2}\|\sigma\|^2_\infty\sum_{I\in\mathcal{D}}\left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d}\\ &\lesssim [W]^3_{A_2}(\log[W]_{A_2})^2\|\sigma\|^2_\infty\|f\|^2_{L^2(W)},\end{aligned}$$

which gives the desired bound. □

6. Open Questions

6.1. **Square Function Estimates.** If w is a scalar-valued A_2 weight, then Theorem 3.1 is true with $[w]^2_{A_2}$ replacing $[w]^2_{A_2}\log[w]_{A_2}$. This motivates the following conjecture:

Conjecture 6.1. *Let W be a $d\times d$ matrix weight in A_2. Then*

$$\sum_{I\in\mathcal{D}}\left\langle \langle W\rangle_I \widehat{f}(I), \widehat{f}(I)\right\rangle_{\mathbb{C}^d} \lesssim [W]^2_{A_2}\|f\|^2_{L^2(W)} \quad \forall f\in L^2(W).$$

To prove Conjecture 6.1, we would need to control the term S_3 from (5) in a more optimal way. Our current method of using Theorem 3.2 introduces the troublesome $\log[W]_{A_2}$ term. An alternate method of controlling S_3 would use a matrix version of the weighted Carleson Embedding Theorem. One would first control S_3 by

$$\begin{aligned}S_3 &= \sum_{I\in\mathcal{D}}\sum_{k=1}^{d}\left\langle \langle W\rangle_I^{-\frac12}\widehat{W}(I)\langle W\rangle_I^{-1}\langle f\rangle_I, e_I^k\right\rangle^2_{\mathbb{C}^d}\\ &\lesssim \sum_{I\in\mathcal{D}}\left\| \langle W\rangle_I^{-\frac12}\widehat{W}(I)\langle W\rangle_I^{-1}\langle f\rangle_I\right\|^2_{\mathbb{C}^d}\\ &= \sum_{I\in\mathcal{D}}\left\langle \langle W\rangle_I^{-1}\widehat{W}(I)\langle W\rangle_I^{-1}\widehat{W}(I)\langle W\rangle_I^{-1}\langle f\rangle_I, \langle f\rangle_I\right\rangle_{\mathbb{C}^d}.\end{aligned}$$

To control this estimate for S_3, we need two things:

6.1.1. *A matrix version of the weighted Carleson Embedding Theorem.* Suppose that A_I is a sequence of non-negative operators on $\mathbb{C}^d$. We would like to prove that

$$\sum_{I\in\mathcal{D}}\langle A_I\langle f\rangle_I, \langle f\rangle_I\rangle_{\mathbb{C}^d} \lesssim C\|f\|^2_{L^2(W^{-1})}$$

if and only if (in the sense of positive operators)

$$\frac{1}{|J|}\sum_{I\subset J}\langle W\rangle_I A_I\langle W\rangle_I \leq C\langle W\rangle_J \qquad \forall\, J\in\mathcal{D}.$$

By fixing $e\in\mathbb{C}^d$ and $J\in\mathcal{D}$ and setting $f(x)=\mathbf{1}_J W(x)e$, one can show that the embedding implies that the testing condition holds. We need to prove the converse. In the unweighted case, the scalar-valued Carleson Embedding Theorem can be used to obtain the unweighted version of this matrix embedding theorem. The analogous arguments do not immediately work for weights. However, similar results have been established by Nazarov, Pisier, Treil, and Volberg in [3] in the unweighted setting. Additionally, a recent result of Isralowitz, Kwon, and Pott, see [2], provides a proof of an estimate as above, but does not explicitly track the constants (and a reading of their proof shows that the method used will introduce some additional characteristic of W into the estimates).

6.1.2. *Bounds on a related dyadic sum.* The A_I that appear in S_3 are the operators

$$\langle W\rangle_I^{-1}\widehat{W}(I)\langle W\rangle_I^{-1}\widehat{W}(I)\langle W\rangle_I^{-1}.$$

Given the matrix weighted Carleson Embedding Theorem conjectured above, we will need the appropriate sum estimate to apply it to S_3. Indeed, we require

$$\frac{1}{|J|}\sum_{I\subset J}\widehat{W}(I)\langle W\rangle_I^{-1}\widehat{W}(I)\lesssim [W]_{A_2}\left\langle W\right\rangle_J, \quad \forall J\in\mathcal{D}.$$

In the scalar case, this is proved by Wittwer in [8] using estimates from Buckley [1]. Currently, it is not clear how to obtain the matrix analogue of this estimate.

6.2. **The Hilbert Transform and Haar Multipliers.** If w is a scalar-valued A_2 weight, then Theorems 4.1 and 5.2 are true with $[w]_{A_2}$ replacing $[w]_{A_2}^{\frac{3}{2}}\log[w]_{A_2}$. This motivates the following conjecture:

Conjecture 6.2. *Let W be a $d\times d$ matrix weight in A_2 and let $\{\sigma_I\}$ be a sequence of matrices satisfying $\|\sigma\|_\infty<\infty$. Then*

$$\|H\|_{L^2(W)\to L^2(W)}\lesssim [W]_{A_2};$$
$$\|T_\sigma\|_{L^2(W)\to L^2(W)}\lesssim [W]_{A_2}\left\|\sigma\right\|_\infty.$$

Given our current tools, those estimates seem out of reach. However, if we could prove Conjecture 6.1 by establishing a bound of $[W]_{A_2}$ in Theorem 3.3, then the arguments from the proof of Theorems 4.1 and 5.2 would immediately imply that

$$\|H\|_{L^2(W)\to L^2(W)}\lesssim [W]_{A_2}^{\frac{3}{2}};$$
$$\|T_\sigma\|_{L^2(W)\to L^2(W)}\lesssim [W]_{A_2}^{\frac{3}{2}}\left\|\sigma\right\|_\infty.$$

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