



# A general overview of cyclic transmultiplexers with cyclic modulation: Implementation and angular parametrization.

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# A general overview of cyclic transmultiplexers with cyclic modulation: Implementation and angular parametrization

Didier Pinchon and Pierre Siohan

June 23, 2014

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## Abstract

This preprint provides a general framework for cyclic transmultiplexers (TMUXs) with cyclic modulation. This TMUX also corresponds to a multicarrier modulation system of the Filtered MultiTone (FMT) type where the linear convolution is replaced by a cyclic one, hence the name Cyclic Block FMT (CB-FMT). In this preprint we present the Perfect Reconstruction (PR) conditions in the time and frequency domains. A duality theorem is proved showing that each PR solution in the frequency domain is connected to a dual PR solution in the time domain. Then, two decomposition theorems are established leading to modular implementations of the cyclic TMUX. For one of this implementation we provide an angular parametrization that only involves angles corresponding to independent parameters. Finally, a procedure to reconstruct the prototype function from all the elementary blocks of the modular implementation is described step-by step.

## 1 Introduction

Filter Bank MultiCarrier (FBMC) modulation has been gaining a lot of interest these last years. Indeed, in order to provide a flexible air interface for future 5G cellular communication systems, several MultiCarrier Modulation (MCM) schemes are actively studied in various projects [1, 2]. Differently from conventional Orthogonal Frequency Division Multiplex (OFDM), FBMC schemes can introduce more efficient filters that bring a significant advantage with respect to the  $\sin(x)/x$  OFDM spectrum. Recently the Generalized Frequency Division Multiplex (GFDM) proposal [3], in which the authors implement the filtering operation with a circular convolution instead of a linear one, has also opened new doors for FBMC-based systems. Indeed circular convolution variants have now been proposed for the two main FBMC schemes. It appears at first in a patent filled by Tonello [4] for a scheme named Cyclic Block Filtered MultiTone (CB-FMT) which corresponds to a cyclic transmultiplexer (TMUX) using a cyclic modulation. Since 2013, the same author, alone [5], [6], or with Girotto [7], [8], has presented several ideas and results that go beyond his initial idea [4]. As shown in different ways for FBMC systems using Offset Quadrature Amplitude Modulation (OQAM) [9, 10, 11], FBMC/OQAM schemes can also take advantage from circular convolution.

However, it is worth noting that the cyclic convolution before being used for MCM schemes has been proposed at first in 1999 by Vaidyanathan and Kirac who introduced in [12] the notion of cyclic filter banks. As for usual filter banks, the authors make use of the polyphase decomposition which then again is an efficient tool to introduce the fundamental notions of orthogonal filter banks and perfect reconstruction. These different concepts and properties are applied afterwards in [13] for Discrete Fourier Transform (DFT) modulated filter banks.

In all these previous publications, either concerning filter banks for sub-band coding or for transmission using TMUXs, the authors mention several advantages compared to classical structures using linear convolution. A first advantage of circular-convolution-based schemes is the possibility with such a block processing approach to avoid the transient edge effects. That means in the case of a TMUX, which is the case of interest in this preprint, there is no need for a packet transmission to introduce specific processing to mitigate the ramp-up ramp-down effects at the packet borders. Furthermore, similarly

to CP-OFDM, block processing makes possible the introduction of a CP, thus providing an increase robustness in front of a multipath transmission, together with a possibility to improve the Power Spectral Density compared to CP-OFDM.

In this preprint we only focus on perfect reconstruction (PR) cyclic TMUX transmitting complex symbols, e.g. QAM. Compared to the previous works by Tonello, we introduce a more general realization structure and a complete angular parametrization. A first important result consists in the introduction of a duality theorem stating that the prototype filter is PR if and only if its Fourier transform also holds the PR property for a set of symmetrical parameters. Then, a theorem of double decomposition is proved that allows us to provide a characterization of all the PR prototype filters. From what follows the possibility to define an angular parametrization of all these filters using independent angles. This independence is of a paramount practical interest when trying to optimize a given prototype filter for a given criterion.

Our preprint is organized as follows. In Section 2 we present the background concerning cyclic TMUX. Section 3 concerns the cyclic TMUX with cyclic modulation and also contains the proof of the double decomposition theorem with the corresponding realization schemes. Afterwards, Section 4 gives an angular parametric representation and the associated realization scheme.

## 2 Cyclic TMUX

**Notations 2.1.** *In this document the following conventions of notation will be respected :*

- $K, N, M, K_0, N_0, M_0, \Delta, \Delta_1, \dots$  indicate fixed strictly positive integer parameters,
- $a, b, n, m, k, l, \dots$  are summation indexes,
- $T$  is the time domain variable,
- $P$  is the frequency domain variable.

The following elementary classic notions are reminded.

For  $N > 1$ , the set of complex cyclic polynomials with length  $N$  and variable  $T$  is the quotient ring  $\mathbb{C}_N[T] = \mathbb{C}[T]/(T^N - 1)$ . So two polynomials  $p(T)$  and  $q(T)$  are equivalent if  $p(T) - q(T)$  is divisible by  $T^N - 1$  and the remainder of the euclidean division of  $p(T)$  by  $T^N - 1$  is chosen as the canonical representative of  $p(T)$ . The invertible elements of  $\mathbb{C}_N[T]$  are polynomials  $p(T)$  such that  $p(T)$  and  $T^N - 1$  are relatively prime.

In particular, for any  $k \in \mathbb{N}$ ,  $T^k$  is invertible and  $T^{-k} = T^r$  where  $-k = qN + r$  with  $0 \leq r < N$ .

A signal  $s = (s[n], 0 \leq n \leq N - 1)$  is said to be  $N$ -cyclic when the associated polynomial  $s(T) = \sum_{n=0}^{N-1} s[n]T^n$  belongs to  $\mathbb{C}_N[T]$  and this polynomial  $s(T)$  is identified with the signal  $s$ . By convention we shall admit that  $s[n + qN] \equiv s[n]$  for any  $q \in \mathbb{Z}$  as  $T^n \equiv T^{n+qN}$ .

**Remark 2.2.** *In notations it is avoided in the following to identify the  $N$ th root of unity  $\omega_N = e^{-2i\pi/N}$ , where  $i^2 = -1$ , with the variable or the polynomial  $T \in \mathbb{C}_N[T]$  because  $\omega_N$  is a root of a polynomial with degree strictly less than  $N$ , the cyclotomic polynomial  $C_N(T)$  (ref. Appendix in [14]).*

The (discrete) Fourier transform  $S = \mathcal{F}_N s$  of a  $N$ -cyclic signal  $s$  is the  $N$ -cyclic  $S$  signal defined by

$$S[k] = \sum_{n=0}^{N-1} s[n] \omega_N^{-nk}, \quad 0 \leq k \leq N-1. \quad (1)$$

The associated cyclic polynomial will be denoted  $S(P) \in \mathbb{C}_N[P]$ .

We denote by  $\mathbf{F}_N$  the  $N \times N$  matrix with elements  $[\mathbf{F}_N]_{r,c} = \omega_N^{-rc}$  and  $\mathbf{s}$  (resp.  $\mathbf{S}$ ) the vector with components  $s[n], 0 \leq n \leq N-1$  (resp.  $S[n], 0 \leq n \leq N-1$ ). Thus  $\mathbf{S} = \mathbf{F}_N \mathbf{s}$ .

The inverse Fourier transform  $\mathcal{F}_N^{-1}$  is the linear application associated to the matrix

$$\mathbf{F}_N^{-1} = \frac{1}{N} \overline{\mathbf{F}_N}. \quad (2)$$

Finally if  $x$  and  $y$  are two  $N$ -cyclic signals the circular convolution of  $x$  and  $y$ , denoted by  $x \star y$ , is the  $N$ -cyclic signal defined by

$$x \star y[k] = \sum_{i=0}^{N-1} x[i] y[k-i] = \sum_{i=0}^{N-1} y[i] x[k-i], \quad 0 \leq k \leq N-1, \quad (3)$$

and we get

$$\mathcal{F}_N(x \star y)[p] = \mathcal{F}_N(x)[p] \mathcal{F}_N(y)[p], \quad 0 \leq p \leq N-1. \quad (4)$$

When  $x$  and  $y$  are identified with polynomials in  $\mathbb{C}_N[T]$  then  $x \star y$  may be identified to the usual product of polynomials in  $\mathbb{C}_N[T]$ , i.e. to the product modulo  $T^N - 1$ .

**Definition 2.3.** Let  $\mathbf{A}(T)$  be a matrix with  $n$  rows and  $m$  columns with  $m \leq n$  and coefficients in  $\mathbb{C}_N[T]$ . We denote by  $\mathbf{A}^*(T)$  the  $m \times n$  matrix defined by

$$[\mathbf{A}^*(T)]_{c,r} = \overline{[\mathbf{A}(T^{-1})]_{r,c}}, \quad 0 \leq r \leq n-1, \quad 0 \leq c \leq m-1, \quad (5)$$

where  $\bar{z}$  is the complex conjugate number of  $z$ .

We say that  $\mathbf{A}(T)$  is a paraunitary matrix (with coefficients in  $\mathbb{C}_N[T]$ ) if

$$\mathbf{A}^*(T) \mathbf{A}(T) = \mathbf{I}_m. \quad (6)$$

This paragraph now resumes the notions introduced in [15] for cyclic filter banks by adapting them to the case of a cyclic transmultiplexer (TMUX).

Let us recall the following definitions of cyclic expansion and decimation for  $L$  and  $N$  two integers strictly bigger than 1 and  $M = LN$ .

**Definition 2.4.** The cyclic expander with factor  $N$  is the function  $E_N$  from  $\mathbb{C}_L[T]$  with values in  $\mathbb{C}_N[T]$  defined by  $E_N(x)(T) = x(T^N)$  for  $x \in \mathbb{C}_L[T]$ .

When  $X = \mathcal{F}_L x$  and  $Y = \mathcal{F}_M[E_N x]$ , it is easily checked that  $Y[k] = X[k] = X[k \bmod L]$ , that is

$$\mathcal{F}_M[E_N x](P) = \mathcal{F}_L[x](P) \cdot \sum_{n=0}^{N-1} P^n. \quad (7)$$

**Definition 2.5.** *The cyclic decimator with factor  $N$  is the function  $D_N$  from  $\mathbb{C}_M[T]$  with values in  $\mathbb{C}_L[T]$  defined by*

$$D_N y(T) = \sum_{l=0}^{L-1} y[lN]T^l \text{ for } y(T) = \sum_{n=0}^{M-1} y[n]T^n. \quad (8)$$

The functions  $Y = \mathcal{F}_M y \in \mathbb{C}_M[P]$  and  $X = \mathcal{F}_L x \in \mathbb{C}_L[P]$  are related by the equation

$$X[l] = \frac{1}{N} \sum_{n=0}^{N-1} Y[nL + l], \quad 0 \leq l \leq L-1. \quad (9)$$

Given two integers  $K, N$  and  $L$  such that  $K \leq N, L > 1$ , a cyclic transmultiplexer with parameters  $K, N, L$  is represented in Figure 1 where the expansion operator  $E_N$  is represented by  $\uparrow N$ , the decimation operator by  $\downarrow N$ . Input complex data symbols  $x_k, 0 \leq k \leq K-1$  and output complex data symbols  $\hat{x}_k, 0 \leq k \leq K-1$  are elements of  $\mathbb{C}_L[T]$ . The transmitter bank is denoted by  $\mathcal{E}(K, N, L, (g_k))$  and its filters  $g_k, 0 \leq k \leq K-1$  are members of  $\mathbb{C}_M[T]$  where  $M = NL$ . The receiver bank is denoted by  $\mathcal{R}(K, N, L, (h_k))$  and its filters  $h_k, 0 \leq k \leq K-1$  are also in  $\mathbb{C}_M[T]$ .

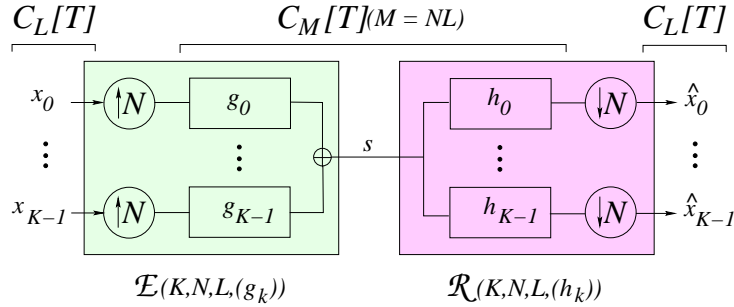


Figure 1: Cyclic transmultiplexer with parameters  $K, L$  and  $N$ .

For any  $0 \leq k \leq K-1$ ,  $g_k(T)$  may be written

$$g_k(T) = \sum_{n=0}^{N-1} E_{n,k}(T^N)T^n, \quad (10)$$

with

$$E_{n,k}(T) = \sum_{l=0}^{L-1} g_k[lN + n]T^l = D_N(T^{-n}g_k), \quad 0 \leq k \leq K-1, \quad 0 \leq n \leq N-1. \quad (11)$$

The functions  $E_{n,k} \in \mathbb{C}_L[T]$  are the type 1  $N$ -polyphase components of  $g_k$ .

In a similar way, we have

$$h_k(T) = \sum_{n=0}^{N-1} F_{k,n}(T^N)T^{-n}, \quad (12)$$

with

$$F_{k,n}(T) = \sum_{l=0}^{L-1} g[lN+n]T^l = D_N(T^n g_k), \quad 0 \leq k \leq K-1, \quad 0 \leq n \leq N-1. \quad (13)$$

The functions  $F_{k,n} \in \mathbb{C}_L[T]$  are the type 2  $N$ -polyphase components of  $h_k$ .

We may then write

$$\begin{aligned} s(T) &= \sum_{k=0}^{K-1} E_N(x_k(T))g_k(T) \\ &= \sum_{k=0}^{K-1} E_N(x_k(T)) \sum_{n=0}^{N-1} E_N(E_{n,k}(T))T^n \\ &= \sum_{n=0}^{N-1} \left( \sum_{k=0}^{K-1} E_N(E_{n,k}(T))E_N(x_k(T)) \right) T^n \\ &= \sum_{n=0}^{N-1} E_N \left( \sum_{k=0}^{K-1} E_{n,k}(T)x_k(T) \right) T^n. \end{aligned}$$

and, for  $0 \leq k \leq K-1$ ,

$$\begin{aligned} \hat{x}_k(T) &= D_N(h_k(T)s(T)) \\ &= D_N \left[ \sum_{n=0}^{N-1} s_k(T^N)T^n \cdot \sum_{n=0}^{N-1} F_{k,n}(T^N)T^{-n} \right] \\ &= \sum_{n=0}^{N-1} F_{k,n}(T^N)D_N(T^{-n}s(T)). \end{aligned}$$

These are the *noble identities* that allow in the diagram of Figure 1 to displace expanders  $E_N$  to the right and decimators  $D_N$  to the left. This is recalled in [15] and provides in Figure 2 an equivalent diagram of the cyclic TMUX. The only difference with the classic linear case is that polynomials are manipulated in  $\mathbb{C}_L[T]$  and  $\mathbb{C}_M[T]$  instead of polynomial computations in  $\mathbb{C}[z^{-1}]$ .

In Figure 2,  $\mathbf{E}((g_k)_{0 \leq k \leq K-1})$  the matrix with dimensions  $N \times K$  whose element of row  $n$  and column  $k$  is  $E_{n,k}(T)$  and  $\mathbf{R}((h_k)_{0 \leq k \leq K-1})$  is the matrix with dimensions  $K \times N$  whose element of row  $r$  and column  $r$  is  $F_{k,n}(T)$ .

**Definitions 2.6.** A cyclic TMUX with parameters  $K, L, N$  and filters  $(g_k)_{0 \leq k \leq K-1}$  and  $(h_k)_{0 \leq k \leq K-1}$  is said to have the perfect reconstruction property with a gain factor equal

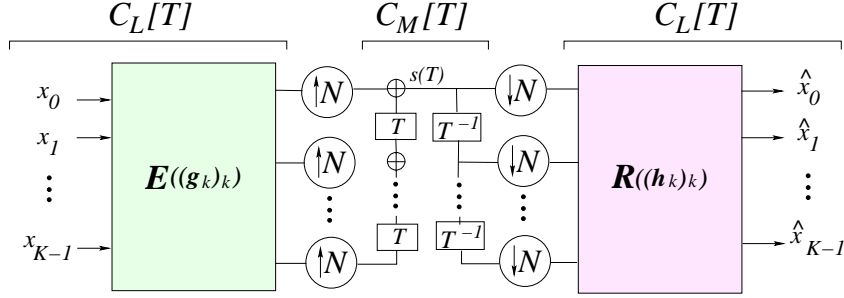


Figure 2: Equivalent diagram of a cyclic transmultiplexer with parameters  $K, L$  and  $N$ .

to  $\alpha > 0$  if  $\hat{x}_k = \alpha x_k$  for any input signals  $x_k \in \mathbb{C}_L[T]$ ,  $0 \leq k \leq K - 1$ . This condition is equivalent to the matrix equality

$$\mathbf{R}((h_k)_{0 \leq k \leq K-1}) \mathbf{E}((g_k)_{0 \leq k \leq K-1}) = \alpha \mathbf{I}_K, \quad (14)$$

where  $\mathbf{I}_K$  is the identity matrix of size  $K$ .

When the matrix  $\mathbf{E}((g_k)_{0 \leq k \leq K-1})$  is a paraunitary matrix, that is when

$$\mathbf{E}^*((g_k)_{0 \leq k \leq K-1}) \mathbf{E}((g_k)_{0 \leq k \leq K-1}) = \mathbf{I}_K, \quad (15)$$

and

$$\mathbf{R}((h_k)_{0 \leq k \leq K-1}) = \mathbf{E}^*((g_k)_{0 \leq k \leq K-1}), \quad (16)$$

then the TMUX is said to be orthogonal.

### 3 Cyclic TMUX with cyclic modulation

#### 3.1 Model definition

Let  $g \in \mathbb{C}_M[T]$ . If  $K$  is a divisor of  $M$ , and only in this case, the function  $n \in \mathbb{Z} \mapsto g(\omega_K^{-1}n)$ ,  $\omega_K = e^{-2i\pi/K}$ , is also  $M$ -cyclic we may set the following definition.

**Definition 3.1.** Let  $K, L, N$  with  $K \leq N$ ,  $M = NL$  such that  $K$  divides  $M$  and  $g \in \mathbb{C}_M[T]$ . The cyclic TMUX with cyclic modulation by the prototype function  $g$  is the cyclic TMUX with parameters  $K, L, N$  whose filters  $g_k$ ,  $0 \leq k \leq K - 1$  and  $h_k$ ,  $0 \leq k \leq K - 1$  are defined by

$$g_k(T) = g(\omega_K^{-k}T), h_k(T) = g_k^*(T) = \overline{g_k(T^{-1})}, \quad 0 \leq k \leq K - 1. \quad (17)$$

i.e. for  $0 \leq k \leq K - 1$

$$g_k[n] = \omega_K^{-nk} g[n], \quad (18)$$

$$h_k[n] = \omega_K^{-nk} g[-n]. \quad (19)$$



In the following matrix  $\mathbf{E}((g_k)_{0 \leq k \leq K-1})$  will be denoted by  $\mathbf{E}(g)$  and the matrix  $\mathbf{R}((h_k)_{0 \leq k \leq K-1})$  by  $\mathbf{R}(g)$ .

By construction,  $\mathbf{R}(g) = \mathbf{E}^*(g)$  thus TMUX is orthogonal with perfect reconstruction property if and only if the matrix  $\mathbf{E}(g)$  is a paraunitary matrix.

The parameters of a cyclic TMUX with cyclic modulation are thus the following :

- The number  $K$  of input sub-channels,
- The expansion/decimation factor  $N$  with  $K \leq N$ . The oversampling factor is  $r = N/M$ .
- The size  $L$  of blocks of symbols in each input sub-channel,
- The length  $M$  of cyclic function used for the modulation is such that  $M = NL$  but must also be a multiple of  $K$ , and  $Q$  such that  $KQ = M$ .

Let  $\Delta_1 = \text{gcd}(K, N)$  with  $K_0, N_0$  relatively primes such that  $K = \Delta_1 K_0$  and  $N = \Delta_1 N_0$ .

As  $M = NL = KQ$  and  $K \leq N$ , we get  $L \leq Q$ .

Let us set now  $\Delta_2 = \text{gcd}(L, Q)$  and let  $L_0$  and  $Q_0$  be relatively primes such that  $L = \Delta_2 L_0$  and  $Q = \Delta_2 Q_0$ . From  $NL = KQ$  we deduce that  $\Delta_1 N_0 \Delta_2 L_0 = \Delta_1 K_0 \Delta_2 Q_0$ . It follows that  $N_0 L_0 = K_0 Q_0$ . As  $N_0$  divides  $K_0 Q_0$  but is prime with respect to  $K_0$ ,  $N_0$  divides  $Q_0$  and as  $Q_0$  divides  $N_0 L_0$  and is prime with respect to  $L_0$ ,  $Q_0$  divides  $N_0$ . We deduce that  $Q_0 = N_0$  and thus  $L_0 = K_0$ .

The model parameters are thus completely described by the four strictly positive integer parameters  $K_0, N_0, \Delta_1, \Delta_2$  such that  $K_0 \leq N_0$  and  $\text{gcd}(K_0, N_0) = 1$ , and we get

$$K = \Delta_1 K_0, \quad N = \Delta_1 N_0, \quad L = \Delta_2 K_0, \quad Q = \Delta_2 N_0, \quad M = \Delta_1 \Delta_2 K_0 N_0. \quad (20)$$

From the TMUX description given by Figure 1, we get, using equations (18) and (19),

$$\begin{aligned} s[m] &= \sum_{k=1}^{K-1} = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} g_k[m - lN] x_k[l] \\ &= \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \omega_K^{-k(m-lN)} g[m - lN] x_k[l], \quad 0 \leq n, \end{aligned} \quad (21)$$

and then

$$\begin{aligned} \hat{x}_{k_1}[l_1] &= \sum_{m=0}^{M-1} s[m] h_{k_1}[l_1 N - m] \\ &= \sum_{m=0}^{M-1} s[m] \omega_K^{-k_1(m-l_1 N)} \overline{g[l_1 N - m]}. \end{aligned} \quad (22)$$

It follows, using (21) in (22)

$$\hat{x}_{k_1}[l_1] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x_k[l] \sum_{m=0}^{M-1} g[m - lN] \overline{g[m - l_1N]} \omega_K^{-k(m-lN) - k_1(l_1N - m)} \quad (23)$$

Making the substitutions  $k \rightarrow k_1 + k$  and  $l \rightarrow l_1 + l$  in sums, and then the substitution  $m \rightarrow m + l_1N + lN$ , we get

$$\hat{x}_{k_1}[l_1] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x_{k_1+k}[l_1 + l] \omega_K^{k_1 l N} \sum_{m=0}^{M-1} \omega_K^{-km} g[m] \overline{g[m + lN]}. \quad (24)$$

### 3.2 Perfect reconstruction relations

The TMUX has thus the perfect reconstruction property if and only if the relations, called the perfect reconstruction relations, are satisfied by the prototype function coefficients

$$\sum_{m=0}^{M-1} \omega_K^{-km} g[m] \overline{g[m + lN]} = \delta_{k,0} \delta_{l,0}, \quad 0 \leq k \leq K-1, \quad 0 \leq l \leq L-1, \quad (25)$$

where  $\delta_{m,n} = 1$  (resp. 0) when  $m = n$  (resp.  $m \neq n$ ).

Setting  $m = qK + r$ ,  $0 \leq q \leq Q-1$ ,  $0 \leq r \leq K-1$ , we observe that  $\omega_K^{-km} = \omega_K^{-kr}$  and relations (25) may be written

$$\sum_{r=0}^{K-1} \omega_K^{-kr} \sum_{q=0}^{Q-1} g[r + qK] \overline{g[r + qK + lN]} = \delta_{k,0} \delta_{l,0}, \quad 0 \leq k \leq K-1, \quad 0 \leq l \leq L-1, \quad (26)$$

For given  $l$  with  $0 \leq l \leq L-1$ , let us denote  $\mathbf{V}_l$  and  $\mathbf{W}_l$  the dimension  $K$  vectors defined by

$$\mathbf{V}_l[k] = \sum_{q=0}^{Q-1} g[r + qK] \overline{g[r + qK + lN]}, \quad 0 \leq k \leq K-1, \quad (27)$$

$$\mathbf{W}_l[k] = \delta_{k,0} \delta_{l,0}. \quad (28)$$

Relations (26) can then be written in matrix form  $\mathbf{F}_K \mathbf{V}_l = \mathbf{W}_l$  and thus  $\mathbf{V}_l = \mathbf{F}_K^{-1} \mathbf{W}_l[k]$ , which provides the perfect reconstruction relations equivalent to (26)

$$\sum_{q=0}^{Q-1} g[r + qK] \overline{g[r + qK + lN]} = \frac{1}{K} \delta_{l,0}, \quad 0 \leq r \leq K-1, \quad 0 \leq l \leq L-1. \quad (29)$$

Let  $G = \mathcal{F}_M(g) \in \mathbb{C}_M[P]$  the Fourier transform of  $g$ . Observing that  $\omega_K = \omega_M^Q$ , relations (25) may be written

$$\sum_{m=0}^{M-1} \omega_M^{-kQm} g[m] \overline{g[m + lN]} = \delta_{k,0} \delta_{l,0}, \quad 0 \leq k \leq K-1, \quad 0 \leq l \leq L-1, \quad (30)$$

The function  $h_l \in \mathbb{C}_M[T]$  the coefficients of which are  $h_l[m] = \overline{g[m + lN]}$ ,  $0 \leq m \leq M-1$ , has the Fourier transform

$$\mathcal{F}_M h_l[m] = \sum_{m'=0}^{M-1} \omega_M^{-mm'} \overline{g[m + lN]} \quad (31)$$

$$= \sum_{m'=0}^{M-1} \omega_M^{m'm} g[m' + lN] \quad (32)$$

$$= \sum_{m_1=0}^{M-1} \omega_M^{m(m_1-lN)} g[m_2] \quad (33)$$

$$= \omega^{-mlN} \sum_{m_1=0}^{M-1} \omega_M^{mm_1} g[m_2] \quad (34)$$

$$= \omega^{mlN} \overline{G[-m]}. \quad (35)$$

As  $\mathcal{F}_M(g.h_l) = \mathcal{F}_M(g) \star \mathcal{F}_M(h_l)$ , relations (30) may be written

$$(\mathcal{F}_M(g) \star \mathcal{F}_M(h_l)) [kQ] = \delta_{k,0} \delta_{l,0}, \quad 0 \leq k \leq K-1, \quad 0 \leq l \leq L-1, \quad (36)$$

that is, as  $\omega_M^N = \omega_L$ ,

$$\sum_{m=0}^{M-1} \omega_L^{-ml} G[m] \overline{G[m + kQ]} = \delta_{k,0} \delta_{l,0}, \quad 0 \leq k \leq K-1, \quad 0 \leq l \leq L-1, \quad (37)$$

These are the perfect reconstruction equations relating  $G$  coefficients, equivalent to equations (25).

By setting  $m = nL + r$ ,  $0 \leq n \leq N-1$ ,  $0 \leq r \leq L-1$ , we get  $\omega_L^{-ml} = \omega_L^{-lr}$  and relations (37) become

$$\sum_{r=0}^{L-1} \omega_L^{-lr} \sum_{n=0}^{N-1} G[r + nL] \overline{G[r + nL + kQ]} = \delta_{k,0} \delta_{l,0}, \quad 0 \leq k \leq K-1, \quad 0 \leq l \leq L-1, \quad (38)$$

For a fixed  $k$  with  $0 \leq k \leq K-1$ , let us denote  $\mathbf{X}_k$  and  $\mathbf{Y}_k$  the dimension  $L$  vectors defined by

$$\mathbf{X}_k[l] = \sum_{n=0}^{N-1} G[r + nL] \overline{G[r + nL + kQ]}, \quad 0 \leq l \leq L-1, \quad (39)$$

$$\mathbf{Y}_l[l] = \delta_{k,0} \delta_{l,0}. \quad (40)$$

Relations (38) may then be written on a matrix form as  $\mathbf{F}_L \mathbf{X}_k = \mathbf{Y}_k$  and thus  $\mathbf{X}_k = \mathbf{F}_L^{-1} \mathbf{Y}_k$ , from which we deduce the perfect reconstruction relations equivalent to (38)

$$\sum_{n=0}^{N-1} G[r + nL] \overline{G[r + nL + kQ]} = \frac{1}{L} \delta_{k,0}, \quad 0 \leq k \leq K-1, \quad 0 \leq r \leq L-1. \quad (41)$$

Relations (41) are identical, up to a proportionality constant, to equations (17) and (18) in [8].

### 3.3 The duality theorem

We then note that equations (41) may be obtained from equations (29) by exchanging  $L$  and  $K$ ,  $Q$  and  $N$ ,  $g$  and  $G$ , that proves the following duality theorem.

**Theorem 3.2.** *The cyclic TMUX with parameters  $K_0, N_0, \Delta_1, \Delta_2$  and cyclic modulation  $g$  has the perfect reconstruction property if and only if the cyclic TMUX with parameters  $K_0, N_0, \Delta_2, \Delta_1$  and cyclic modulation  $G = \mathcal{F}_M g$  with  $M = K_0 N_0 \Delta_1 \Delta_2$  has the perfect reconstruction property.*

This theorem, that will be made more precise latter, has the following direct proof.

*Direct proof of theorem 3.2.*– Starting from the following expression of the output signal  $s(T) \in \mathbb{C}_M[T]$

$$s(T) = \sum_{k=0}^{M-1} \left( \sum_{l=0}^{L-1} x[k, l] T^{lN} \right) g(\omega_M^{-k} T), \quad (42)$$

we get

$$\mathcal{F}_M s[m] = \sum_{k=0}^{M-1} \mathcal{F}_M \left( \sum_{l=0}^{L-1} x[k, l] T^{lN} \right) [m] \cdot \mathcal{F}_M (g(\omega_M^{-k} T)) [m], \quad 0 \leq m \leq M-1. \quad (43)$$

It follows that

$$\begin{aligned} \mathcal{F}_M (g(\omega_M^{-k} T)) [m] &= \sum_{m_1=0}^{M-1} \omega_M^{-mm_1} g(\omega_M^{-k} T) [m_1] \\ &= \sum_{m_1=0}^{M-1} \omega_M^{-mm_1} \omega_K^{-m_1 k} g[m_1], \end{aligned} \quad (44)$$

and because

$$\mathcal{F}_M (T^a) [m] = \omega_M^{-ma}, \quad 0 \leq a \leq M-1,$$

$$\mathcal{F}_M \left( \sum_{l=0}^{L-1} x[k, l] T^{lN} \right) [m] = \sum_{l=0}^{L-1} x[k, l] \omega_M^{-mlN}. \quad (45)$$

It follows that

$$\begin{aligned} \mathcal{F}_M s[m] &= \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m_1=0}^{M-1} x[k, l] g[m_1] \omega_M^{-mlN - mm_1} \omega_K^{m_1 k} \\ &= \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m_1=0}^{M-1} x[k, l] g[m_1] \omega_L^{-ml} \omega_M^{-m_1(m+kQ)}, \end{aligned} \quad (46)$$

using  $\omega_K = \omega_M^Q$  and  $\omega_L = \omega_M^N$ .

We then deduce

$$\begin{aligned}
\mathcal{F}_{Ms}(P) &= \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} \sum_{m_1=0}^{M-1} x[k, l] \omega_L^{-ml} \omega_M^{-m_1(m+kQ)} g[m_1] P^m \\
&= \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x[k, l] \omega_L^{-ml} G[m+kQ] P^m,
\end{aligned} \tag{47}$$

because

$$\sum_{m_1=0}^{M-1} \omega_M^{-m_1(m+kQ)} g[m_1] = G[m+kQ].$$

Replacing  $m$  by  $m - kQ$ , we get

$$\begin{aligned}
\mathcal{F}_{Ms}(P) &= \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x[k, l] \omega_L^{klQ} P^{-kQ} \sum_{m_1=0}^{M-1} \omega_L^{-m_1 l} G[m_1] P^{m_1}, \\
&= \sum_{l=0}^{L-1} \left( \sum_{k=0}^{K-1} x[k, l] \omega_L^{klQ} P^{-kQ} \right) G(\omega_L^{-l} P), \\
&= \sum_{l=0}^{L-1} \left( \sum_{k=0}^{K-1} x[k, l] \omega_L^{klQ} P^{-k} \right) (P^Q). G(\omega_L^{-l} P).
\end{aligned} \tag{48}$$

We may then write

$$\sum_{k=0}^{K-1} x[k, l] \omega_M^{klNQ} P^{-k} = \sum_{k=0}^{K-1} X[l, k] P^k, \tag{49}$$

by setting

$$X[l, k] = x[(K-k) \bmod K, l] \omega_M^{[(K-k) \bmod K] l N Q}, \quad 0 \leq k \leq K-1, \quad 0 \leq l \leq L-1. \tag{50}$$

Let us denote  $\mathcal{A}_{K,N,L}$  the map for which the image of  $x = (x[k, l])_{0 \leq k \leq K-1, 0 \leq l \leq L-1}$  is  $X = (X[l, k])_{0 \leq l \leq L-1, 0 \leq k \leq K-1}$ . It is a linear invertible map that may be written as a permutation of the  $KL$  coordinates followed by the product of each coordinate by a constant. Formulas (48) and (50) are graphically represented in Figure 3.

**Notations 3.3.** Let  $g$  be a prototype function of a cyclic TMUX with cyclic modulation for parameters  $K_0, N_0, \Delta_1, \Delta_2$ . For  $0 \leq i \leq \Delta_1$ , we denote by  $p_i(T) \in \mathbb{C}_{K_0 N_0 \Delta_2}[T]$  the  $i$ -th  $\Delta_1$ -polyphase component of  $g$ .

For  $0 \leq j \leq \Delta_2$ , we denote by  $P_j(P) \in \mathbb{C}_{K_0 N_0 \Delta_1}[P]$  the  $j$ -th  $\Delta_2$ -polyphase component of  $G = \mathcal{F}_M g$  and  $p^{(j)} \in \mathbb{C}_{K_0 N_0 \Delta_1}[T]$  the function  $\mathcal{F}_{K_0 N_0 \Delta_1}^{-1}(P_j)$ . Functions  $p^{(j)}$  are called the  $\Delta_2$ -periodic components of  $g$ .

A prototype function  $g$  may be expressed from its  $\Delta_1$ -polyphase components  $p_i$ ,  $0 \leq i \leq \Delta_1 - 1$  by the formula

$$g(T) = \sum_{i=0}^{\Delta_1-1} p_i(T^{\Delta_1}) T_i. \tag{51}$$

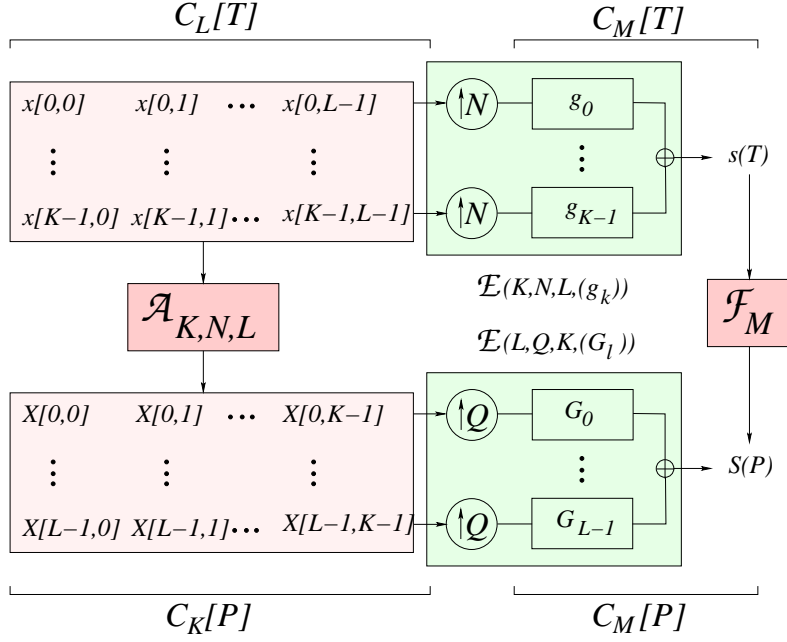


Figure 3: Illustration of Theorem 3.2 : correspondence of transmitter banks.

**Lemma 3.4.** *Let  $g$  be the prototype function of a cyclic TMUX with cyclic modulation for parameters  $K_0, N_0, \Delta_1, \Delta_2$ . The function  $g$  may be expressed in terms of its  $\Delta_2$ -periodic components  $p^{(j)}$ ,  $0 \leq j \leq \Delta_2 - 1$  by the formula*

$$g(T) = \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} p^{(j)}(\omega_M^j T) \left[ \sum_{s=0}^{\Delta_2-1} \omega_{\Delta_2}^{js} T^{sK_0 N_0 \Delta_1} \right]. \quad (52)$$

*Proof.*– Applying the inverse Fourier transform  $\mathcal{F}_M^{-1}$  to the members of equation

$$G(P) = \sum_{j=0}^{\Delta_2-1} P_j(P^{\Delta_2})P^j, \quad (53)$$

we deduce

$$g(T) = \sum_{j=0}^{\Delta_2-1} \mathcal{F}_M^{-1}[P_j(P^j P^{\Delta_2})](T). \quad (54)$$

Let now  $F(P) \in \mathbb{C}_M[P]$  and any  $a \in \mathbb{Z}$ . As

$$P^a F(p) = \sum_{m=0}^{M-1} F[m]P^{m+a} = \sum_{m=0}^{M-1} F[m-a]P^m, \quad (55)$$

it follows that

$$\begin{aligned} \mathcal{F}_M^{-1}(P^a F(p))[n] &= \frac{1}{M} \sum_{m=0}^{M-1} \omega_M^{mn} F[m-a] \\ &= \omega_M^{an} \mathcal{F}_M^{-1}(F)[n], \end{aligned}$$

and thus we get the following result

$$\mathcal{F}_M^{-1}(P^a F(p))(T) = \mathcal{F}_M^{-1}(F(p))(\omega^a T). \quad (56)$$

On the other hand we have  $P_j(P^{\Delta_2}) = E_{\Delta_2}(P_j(P))$  and from (7) by replacing variable  $T$  by variable  $P$  and after complex conjugation

$$\overline{\mathcal{F}_M(E_{\Delta_2}(P_j))}(T) = \overline{\mathcal{F}_{K_0 N_0 \Delta_1}(P_j)}(T) \cdot \sum_{s=0}^{\Delta_2-1} T^{s K_0 N_0 \Delta_1}. \quad (57)$$

Dividing by  $M$ , and observing that  $\omega_M^{K_0 N_0 \Delta_1} = \omega_{\Delta_2}$ , equations (2), (56), (54) and (57) prove equation (52).  $\square$

### 3.4 Decomposition theorems

We may then state the following decomposition theorem.

**Theorem 3.5.** *For a cyclic TMUX with cyclic modulation by the prototype function  $g \in \mathbb{C}_M[T]$  and parameters  $K_0, N_0, \Delta_1, \Delta_2$  with  $K_0 < N_0$  and  $\gcd(K_0, N_0) = 1$  the following three conditions are equivalent*

- (a) *The cyclic TMUX with prototype function  $g$  has the perfect reconstruction property,*
- (b) *For every  $0 \leq i \leq \Delta_1 - 1$ , the cyclic TMUX with parameters  $K_0, N_0, 1, \Delta_2$  and prototype function  $p_i$  has the perfect reconstruction property,*
- (c) *For every  $0 \leq j \leq \Delta_2 - 1$ , the cyclic TMUX with parameters  $K_0, N_0, \Delta_1, 1$  and prototype function  $p^{(j)}$  has the perfect reconstruction property.*

*Proof.*— Setting  $r = i + k_0 \Delta_1$ ,  $0 \leq i \leq \Delta_1 - 1$ ,  $0 \leq k_0 \leq K_0 - 1$  in equations (29), we get

$$\begin{aligned} & \sum_{q=0}^{Q-1} g[i + \Delta_1(k_0 + qK_0)] \overline{g[i + \Delta_1(k_0 + qK_0 + lN_0)]} = \\ & \frac{1}{\Delta_1 K_0} \delta_{l,0}, \quad 0 \leq k_0 \leq K_0 - 1, \quad 0 \leq l \leq L - 1, \end{aligned} \quad (58)$$

and as  $p_i[n] = g[i + \Delta_1 n]$ , one obtains for every  $0 \leq i \leq \Delta_1 - 1$ ,

$$\sum_{q=0}^{Q-1} p_i[k_0 + qK_0] \overline{p_i[k_0 + qK_0 + lN_0]} = \frac{1}{\Delta_1 K_0} \delta_{l,0}, \quad 0 \leq k_0 \leq K_0 - 1, \quad 0 \leq l \leq L - 1, \quad (59)$$

which are the perfect reconstruction relations for  $p_i$ . That proves the equivalence of (a) and (b).

In the same way setting  $r = j + \Delta_2 l_0$ ,  $0 \leq j \leq \Delta_2$ ,  $0 \leq l_0 \leq L_0 - 1$  in (41), we get

$$\begin{aligned} & \sum_{n=0}^{N-1} G[j + \Delta_2(l_0 + nL_0 + kQ_0)] \overline{G[j + \Delta_2(l_0 + nL_0 + kQ_0)]} = \\ & \frac{1}{\Delta_2 L_0} \delta_{k,0}, \quad 0 \leq k \leq K - 1, \quad 0 \leq l_0 \leq L_0 - 1, \end{aligned} \quad (60)$$

and because  $P_j[n] = G[j + n\Delta_2]$ , this proves that  $P_j$  has the perfect reconstruction property for every  $0 \leq j \leq \Delta_2 - 1$ .

As equations (29) and (41) are equivalent, one obtains that  $p^{(j)}$  has the perfect reconstruction property for every  $0 \leq j \leq \Delta_2 - 1$  and that the reverse is true.  $\square$

Let us introduce now the following functions.

**Notations 3.6.** For  $0 \leq i \leq \Delta_1 - 1$  and  $0 \leq j \leq \Delta_2 - 1$ , let us denote by  $[p_i]^{(j)}$  the  $j$ th  $\Delta_2$ -periodic component of the  $i$ th  $\Delta_1$ -polyphase component of  $g$  and by  $[p^{(j)}]_i$  the  $i$ th  $\Delta_1$ -polyphase component of the  $j$ th  $\Delta_2$ -periodic component of  $g$ .

**Lemma 3.7.** For  $g \in \mathbb{C}_M[T]$  and  $0 \leq i \leq \Delta_1 - 1$ ,  $0 \leq j \leq \Delta_2 - 1$ , the functions  $[p_i]^{(j)}$  and  $[p^{(j)}]_i$  in  $\mathbb{C}_{M_0}[T]$  with  $M_0 = K_0 N_0$  are related by the equation  $[p_i]^{(j)} = \omega_M^{ij} [p^{(j)}]_i$ .

*Proof.*– In (52) replace  $p^{(j)}$  by its expression in function of the  $[p^{(j)}]_i$ ,

$$p^{(j)}(T) = \sum_{i=0}^{\Delta_1-1} [p^{(j)}]_i (T^{\Delta_1}) T^i, \quad (61)$$

that gives

$$g(T) = \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} \sum_{i=0}^{\Delta_1-1} [p^{(j)}]_i (\omega_M^{j\Delta_1} T^{\Delta_1}) \left[ \sum_{s=0}^{\Delta_2-1} \omega_{\Delta_2}^{js} T^{sK_0 N_0 \Delta_1} \right] \omega_M^{ij} T^i. \quad (62)$$

Then let us consider expression (52) for the prototype function  $p_i(T) \in \mathbb{C}_{K_0 N_0 \Delta_2}[T]$  and parameters  $K_0, N_0, \Delta_1 = 1, \Delta_2$

$$p_i(T) = \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} [p_i]^{(j)} (\omega_{K_0 N_0 \Delta_2}^j T) \left[ \sum_{s=0}^{\Delta_2-1} \omega_{\Delta_2}^{js} T^{sK_0 N_0} \right]. \quad (63)$$

that we use in (51). This provides the equality

$$g(T) = \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} \sum_{i=0}^{\Delta_1-1} [p_i]^{(j)} (\omega_{K_0 N_0 \Delta_2}^j T^{\Delta_1}) \left[ \sum_{s=0}^{\Delta_2-1} \omega_{\Delta_2}^{js} T^{sK_0 N_0 \Delta_1} \right] T^i. \quad (64)$$

As  $\omega_{K_0 N_0 \Delta_2} = \omega_M^{\Delta_1}$ , comparison of (62) and (64) provides the result by identification.  $\square$

Applying theorem 3.5, condition (b) to the function  $g$ , then the same theorem, condition (c) to each function  $p_i$ ,  $0 \leq i \leq \Delta_1 - 1$ , we then immediately obtain the new following decomposition theorem.



**Theorem 3.8.** For a cyclic TMUX with cyclic modulation with parameters  $K_0, N_0, \Delta_1, \Delta_2$ ,  $K_0 < N_0$  and  $\gcd(K_0, N_0) = 1$ , and with the prototype function  $g \in \mathbb{C}_M[T]$  the following two conditions are equivalent

- (a) The cyclic TMUX with cyclic modulation  $g$  has the perfect reconstruction property,
- (b) For every  $0 \leq i \leq \Delta_1 - 1$  and every  $0 \leq j \leq \Delta_2 - 1$ , the cyclic TMUX with parameters  $K_0, N_0, 1, 1$  and cyclic modulation function  $[p_i]^{(j)}$  has the perfect reconstruction property.

Thanks to lemma 3.7, it is equivalent to consider the functions  $[p^{(j)}]_i$  in condition (b) of theorem 3.8.

## 3.5 Implementation schemes

### 3.5.1 The case $\Delta_1 > 1$

In a first step, the scheme of Figure 1 is transformed to take into account the decomposition of the prototype function  $g(T) \in \mathbb{C}_M[T]$  into its  $\Delta_1$ -polyphase components  $p_i(T)$ ,  $i = 0, \dots, \Delta_1 - 1$  with  $p_i(T) \in \mathbb{C}_{K_0 N_0 \Delta_2}[T]$ .

If  $s(T) \in \mathbb{C}_M(T)$  is the output signal of the transmitter bank, one get

$$s(T) = \sum_{k=0}^{K-1} g_k(T) x_k(T^N) = \sum_{k=0}^{K-1} g(\omega_K^{-k} T) x_k(T^N), \quad (65)$$

with  $x_k(T) = \sum_{l=0}^{L-1} x[k, l] T^l \in \mathbb{C}[T]$ ,  $k = 0, \dots, K - 1$ , using equation (17).

The  $\Delta_1$ -polyphase decomposition of  $g(T)$

$$g(T) = \sum_{i=0}^{\Delta_1-1} p_i(T^{\Delta_1}) T^i, \quad (66)$$

with  $p_i(T) \in \mathbb{C}_{K_0 N_0 \Delta_2}[T]$  gives

$$g(\omega_K^{-k} T) = \sum_{i=0}^{\Delta_1-1} p_i(\omega_K^{-k \Delta_1} T^{\Delta_1}) \omega_K^{-ki} T^i, \quad (67)$$

and thus plugging into (65)

$$s(T) = \sum_{i=0}^{\Delta_1-1} \sum_{k=0}^{K-1} p_i(\omega_K^{-k \Delta_1} T^{\Delta_1}) \omega_K^{-ki} T^i x_k(T^N). \quad (68)$$

With  $k = r K_0 + k_0$ ,  $0 \leq k_0 \leq K_0 - 1$ ,  $0 \leq r \leq \Delta_1 - 1$ , (68) becomes

$$\begin{aligned} s(T) &= \sum_{i=0}^{\Delta_1-1} T^i \sum_{k_0=0}^{K_0-1} p_i(\omega_{K_0}^{-k_0} T^{\Delta_1}) \omega_{K_0}^{k_0 i} \sum_{r=0}^{\Delta_1-1} \omega_{\Delta_1}^{-ri} x_{r K_0 + r}(T^N), \\ &= \sum_{i=0}^{\Delta_1-1} T^i \left( p_i(\omega_{K_0}^{-k_0} T^{\Delta_1}) \omega_{K_0}^{k_0 i} \sum_{r=0}^{\Delta_1-1} \omega_{\Delta_1}^{-ri} x_{r K_0 + r}(T^{N_0}) \right) (T^{\Delta_1}). \end{aligned} \quad (69)$$

Let us denote by  $\mathbf{\Pi}$  the  $K \times K$  permutation matrix defined by  $[\mathbf{\Pi}]_{r,c} = 1$  when  $r = q\Delta_1 + d$  and  $c = dK_0 + q$ , and 0 elsewhere, and by  $\mathbf{D}$  the  $K \times K$  diagonal matrix such that  $[\mathbf{D}]_{r,r} = \omega_K^{-dk_0}$  if  $r = dK_0 + k_0$ . Equation (80) allows therefore to obtain the following equivalent scheme in Figure 4 for a cyclic TMUX with cyclic modulation when  $\Delta_1 > 1$ .

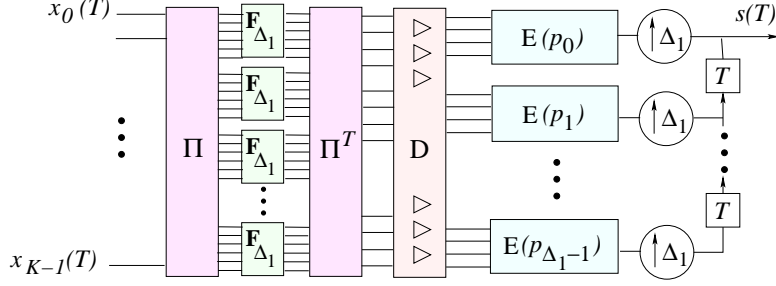


Figure 4: Equivalent form of the transmitter bank when  $\Delta_1 > 1$ .

### 3.5.2 The case $\Delta_1 = 1$ and arbitrary $\Delta_2$

In this paragraph let us denote by  $K_0$  and  $N_0$  two strictly positive integers such that  $K_0 < N_0$  and  $\gcd(K_0, N_0) = 1$ .

As in [16], we consider three functions  $a(r, c)$ ,  $p(r, c)$  and  $q(r, c)$  defined on  $\{(r, c), 0 \leq r \leq N_0 - 1 \text{ and } 0 \leq c \leq K_0 - 1\}$  with the properties

$$0 \leq a(r, c) \leq K_0 N_0 - 1, \quad 0 \leq p(r, c) \leq K_0 - 1, \quad 0 \leq q(r, c) \leq N_0 - 1, \quad (70)$$

$$a(r, c) = r + p(r, c)N_0 = c + q(r, c)K_0. \quad (71)$$

For  $0 \leq a \leq K_0 N_0 - 1$ ,  $r = a \bmod N_0$  and  $c = a \bmod K_0$ , we get  $a = a(r, c)$ . In another hand if  $\bar{K}_0$  and  $\bar{N}_0$  are the two integers such that  $0 \leq \bar{K}_0 \leq N_0 - 1$ ,  $0 \leq \bar{N}_0 \leq K_0 - 1$  and  $\bar{K}_0 K_0 + \bar{N}_0 N_0 = K_0 N_0 + 1$ , then, for every  $r$  and  $c$  such that  $0 \leq r \leq N_0 - 1$  and  $0 \leq c \leq K_0 - 1$ ,

$$p(r, c) = [(c - r)\bar{N}_0] \bmod K_0, \quad (72)$$

$$q(r, c) = [(r - c)\bar{K}_0] \bmod N_0. \quad (73)$$

Let us consider a cyclic TMUX with cyclic modulation function  $g$  and parameters  $(K_0, N_0, \Delta_1 = 1, \Delta_2)$ . It follows that  $K = K_0$ ,  $N = N_0$ ,  $L = \Delta_2 K_0$ ,  $Q = \Delta_2 N_0$ ,  $M = \Delta_2 K_0 N_0$ . Let us then denote by  $W_a(T) \in \mathbb{C}_{\Delta_2}[T]$ ,  $0 \leq a \leq K_0 N_0 - 1$ , the  $K_0 N_0$ -polyphase components of prototype function  $g$  when  $\Delta_1 = 1$  ;

$$g(T) = \sum_{a=0}^{K_0 N_0 - 1} W_a(T^{K_0 N_0}) T^a. \quad (74)$$

As

$$s(T) = \sum_{k=0}^{K_0 - 1} g_k(T) x_k(T^{N_0}), \quad (75)$$

where  $x_k(T) \in \mathbb{C}_L[T] = \sum_{l=0}^{L-1} x[k, l]T^l$ ,  $0 \leq k \leq K_0 - 1$ , because

$$g_k(T) = g(\omega_{K_0}^{-k}T) = \sum_{a=0}^{K_0 N_0 - 1} W_a(T^{K_0 N_0}) \omega_{K_0}^{-ka} T^a, \quad (76)$$

it follows that

$$s(T) = \sum_{k=0}^{K_0-1} \sum_{a=0}^{K_0 N_0 - 1} W_a(T^{K_0 N_0}) \omega_{K_0}^{-ka} T^a x_k(T^{N_0}). \quad (77)$$

Taking  $a = a(r, c)$  with  $0 \leq r \leq N_0 - 1$ ,  $0 \leq c \leq K_0 - 1$  in (68) and using relations (71), it comes

$$\begin{aligned} s(T) &= \sum_{k=0}^{K_0-1} \sum_{c=0}^{K_0-1} \sum_{r=0}^{N_0-1} W_{a(r,c)}(T^{K_0 N_0}) \omega_{K_0}^{-k(c+q(r,c)K_0)} T^{r+p(r,c)N_0} x_k(T^{N_0}) \\ &= \sum_{r=0}^{N_0-1} T^r \left[ \sum_{c=0}^{K_0-1} T^{p(r,c)N_0} W_{a(r,c)}(T^{K_0 N_0}) \sum_{k=0}^{K_0-1} \omega_{K_0}^{-kc} x_k(T^{N_0}) \right]. \end{aligned} \quad (78)$$

Introducing vector  $\mathbf{X}(T)$  with components  $x_k(T)$ ,  $0 \leq k \leq K_0 - 1$  and matrix  $\mathbf{W}(g)$  with dimensions  $K_0 \times N_0$  whose elements are

$$[\mathbf{W}(g)]_{r,c} = T^{p(r,c)} W_{a(r,c)}(T^{K_0}), \quad (79)$$

we get

$$s(T) = \sum_{r=0}^{N_0-1} [\mathbf{W}(g) \mathbf{F}_{K_0} \mathbf{X}]_r (T^{N_0}) T^r. \quad (80)$$

Formula (80) allows to obtain in Figure 5 an implementation of the transmitter bank when  $\Delta_1 = 1$ .

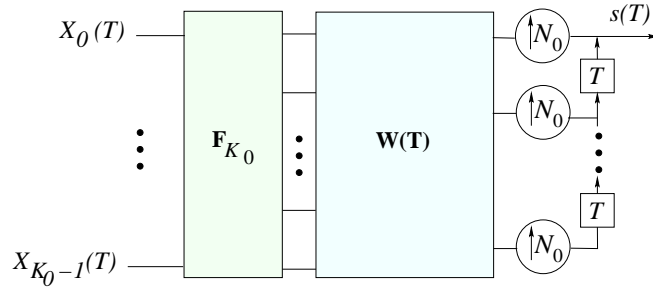


Figure 5: Equivalent form of the transmitter bank when  $\Delta_1 = 1$ .

Considering the schemes given in Figures 4 and 5 and using the well-known factorization of Fourier matrix  $\mathbf{F}_K$  for  $K = \Delta_1 K_0$  ([17])

$$\mathbf{F}_K = \mathbf{\Pi} \mathbf{F}_{K_0}^{(\Delta_1)} \mathbf{D} \mathbf{\Pi}^T \mathbf{F}_{\Delta_1}^{(K_0)} \mathbf{\Pi} \quad (81)$$

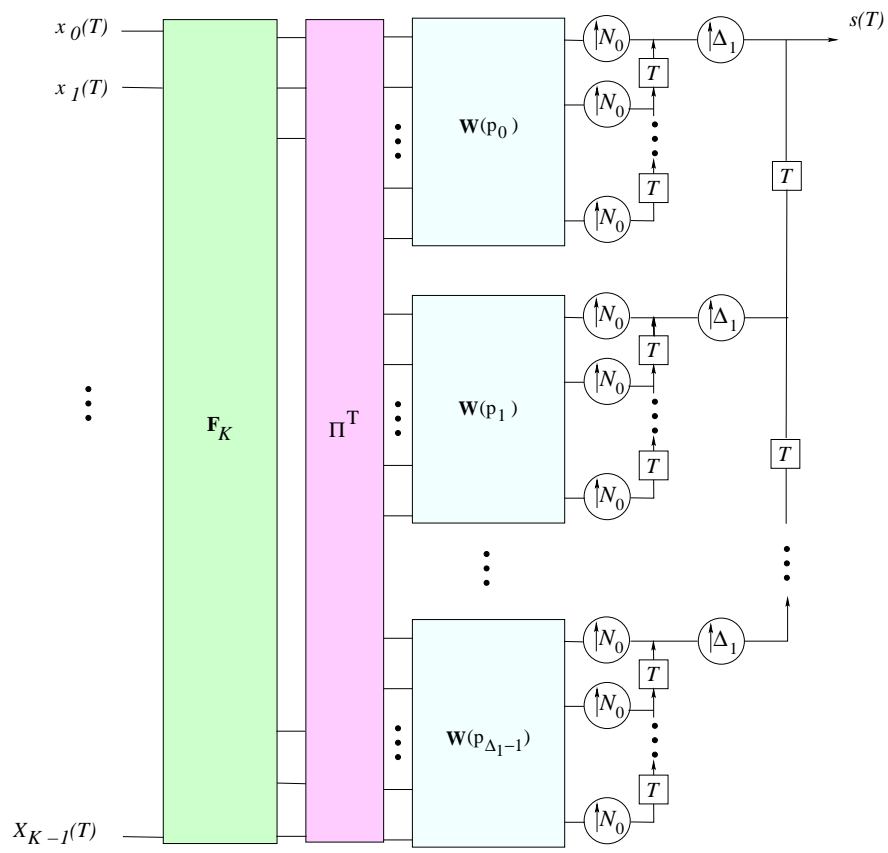


Figure 6: Equivalent form of the transmitter bank when  $\Delta_1 > 1$ .

where  $\mathbf{F}_{K_0}^{(\Delta_1)}$  is the block diagonal  $K \times K$  matrix with  $\Delta_1$  diagonal blocks equal to  $\mathbf{F}_{K_0}$  and  $\mathbf{F}_{\Delta_1}^{(K_0)}$  is the block diagonal  $K \times K$  matrix with  $K_0$  diagonal blocks equal to  $\mathbf{F}_{\Delta_1}$ , one may obtain a new scheme for the transmitter bank given in Figure 6.

For  $\Delta_1 = 1$ , relation (52) may be written

$$g(T) = \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} p^{(j)}(\omega_M^j T) \left[ \sum_{s=0}^{\Delta_2-1} \omega_{\Delta_2}^{js} T^{sK_0 N_0} \right]. \quad (82)$$

and because the matrix  $\mathbf{W}(g)$  linearly depends on  $g$ , denoting by  $\mathbf{W}_j$  the matrix  $\mathbf{W}(g^{(j)})$  and by  $q_{\Delta_2}(T)$  the polynomial  $\sum_{s=0}^{\Delta_2-1} T^{sK_0}$ , we get

$$[\mathbf{W}_g]_{r,c}(T) = \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} (q_{\Delta_2}(T) \mathbf{W}(g^{(j)})) (\omega_M^j T). \quad (83)$$

Putting back this relation in (80), we get

$$\begin{aligned} s(T) &= \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} \sum_{r=0}^{N_0-1} [q_{\Delta_2}(\omega_M^j T) \mathbf{W}_j(\omega_M^j T) \mathbf{F}_{K_0} \mathbf{X}]_r (T^{N_0}) T^r, \\ &= \frac{1}{\Delta_2} \sum_{r=0}^{N_0-1} \left[ \sum_{j=0}^{\Delta_2-1} \left\{ \sum_{r=0}^{N_0-1} [q_{\Delta_2}(T) \mathbf{W}_j(T) \mathbf{F}_{K_0} \mathbf{X}(\omega_M^{-j} T)]_r (T^{N_0}) \right\} (\omega_M^j T) \right] T^r. \end{aligned} \quad (84)$$

The scheme of Figure 7 shows the implementation of matrix  $\mathbf{W}(g)$  that corresponds to formula (84) :  $\mathbf{B} = \mathbf{W}(g) \mathbf{A}$  where  $\mathbf{A} = (a_0, a_1, \dots, a_{K_0-1})$  and  $\mathbf{B} = (b_0, b_1, \dots, b_{N_0-1})^T$ .

In this scheme the symbol  $\otimes_{\omega_M^j}$  stands for the linear transformation that maps a signal  $s(T)$  to the signal  $s(\omega_M^j T)$ . The matrix  $\mathbf{\Pi}_2$  is a permutation matrix with dimensions  $(\Delta_2 N_0) \times (\Delta_2 N_0)$  which allows to group the  $\Delta_2$   $k$ th outputs of the  $\Delta_2$  inputs blocks for  $k$  varying from 0 to  $N_0 - 1$ .

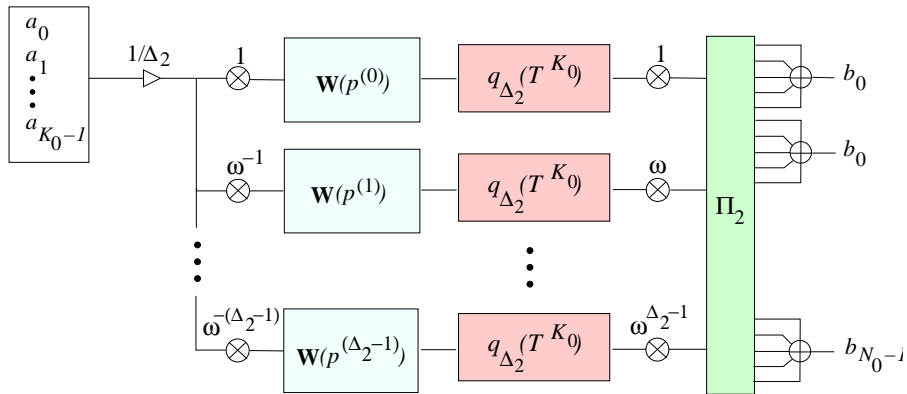


Figure 7: Equivalent form of matrix  $\mathbf{W}(g)$  when  $\Delta_1 = 1$  and  $\Delta_2 > 1$ .

## 4 Angular parametrization

In this paragraph we first obtain an angular parametric representation for a cyclic TMUX with cyclic modulation and parameters  $K_0, N_0, \Delta_1, \Delta_2$  with  $\Delta_1 = \Delta_2 = 1$ . Then with such a representation for each function  $[g_i]^{(j)}$  of theorem 3.8 we precisely show how to reconstruct the prototype function  $g$  of a cyclic TMUX with cyclic modulation and general parameters  $K_0, N_0, \Delta_1, \Delta_2$  for any  $\Delta_1$  and  $\Delta_2$ . This procedure leads to its angular parametrization.

### 4.1 The case $\Delta_1 = \Delta_2 = 1$

Relation (79) allows to state the following theorem.

**Theorem 4.1.** *Let  $g \in \mathbb{C}_{M_0}$  be the prototype function of a cyclic TMUX with cyclic modulation with parameters  $K_0, N_0, \Delta_1, \Delta_2$  and  $\Delta_1 = \Delta_2 = 1$  and  $K_0 < N_0, \gcd(K_0, N_0) = 1$ . This implies  $L = K_0, Q = N_0$  and  $M = M_0 = K_0 N_0$ . The  $N_0 \times K_0$  matrix  $\mathbf{E}(g)$  satisfies the following equality*

$$\mathbf{E}(g) = \mathbf{W}(g) \mathbf{F}_{K_0}, \quad (85)$$

where  $\mathbf{W}(g)$  is the  $N_0 \times K_0$  matrix defined by (79) :

$$[\mathbf{W}(g)]_{r,c} = g[a(r,c)]T^{p(r,c)}, \quad 0 \leq r \leq N_0 - 1, \quad 0 \leq c \leq K_0 - 1. \quad (86)$$

Matrix  $\mathbf{W}(g)$  may be factorized as

$$\mathbf{W}(g) = \mathbf{D}_1 \mathbf{U}(g) \mathbf{D}_2, \quad (87)$$

where  $\mathbf{D}_1$  is the diagonal matrix of size  $N_0$  whose element in row  $r$ ,  $0 \leq r \leq N_0 - 1$  is  $T^{-rN_0}$ ,  $\mathbf{D}_2$  is the diagonal matrix of size  $K_0$  whose element in column  $c$ ,  $0 \leq c \leq K_0 - 1$  is  $T^{cN_0}$  and  $\mathbf{U}(g)$  is the  $N_0 \times K_0$  matrix whose element of row  $r$  and column  $c$  is given by

$$[\mathbf{U}(g)]_{r,c} = g[a(r,c)]. \quad (88)$$

**Proof.**– The row  $r$  and column  $c$  element in matrix  $\mathbf{E}(g)$  is  $E_{r,c}(T) \in \mathbb{C}_{K_0}[T]$ , the  $r$ th  $N_0$ -polyphase component of  $g_c(T)$ ,

$$E_{r,c}(T) = \sum_{l=0}^{K_0-1} g_c[r + lN_0]T^l, \quad (89)$$

$$= \sum_{l=0}^{K_0-1} g[r + lN_0]\omega_{K_0}^{-c(r+lN_0)}T^l, \quad (90)$$

and thus

$$[\mathbf{E}(g)\mathbf{F}_{K_0}^{-1}]_{r,c} = \frac{1}{K_0} \sum_{k=0}^{K_0-1} E_{r,k}(T)\omega_{K_0}^{kc} \quad (91)$$

$$= \frac{1}{K_0} \sum_{k=0}^{K_0-1} \sum_{l=0}^{K_0-1} \omega_{K_0}^{kc} g[r + lN_0] \omega_{K_0}^{-k(r+lN_0)} T^l \quad (92)$$

$$= \frac{1}{K_0} g[r + lN_0] T^l \sum_{k=0}^{K_0-1} \omega_{K_0}^{k[c-r-lN_0]}. \quad (93)$$

The sum over  $k$  is null if  $[c - r - lN_0] \bmod K_0 \neq 0$  and is equal to  $K_0$  if  $[c - r - lN_0] \bmod K_0 = 0$ , that is  $r + lN_0 = c + \alpha K_0$  where  $\alpha$  is an integer. Because  $0 \leq r + lN_0 \leq N_0 K_0 - 1$ , this implies that  $l = p(r, c)$  and  $r + lN_0 = a(r, c)$ . We deduce that

$$\mathbf{E}(g)\mathbf{F}_{K_0}^{-1} = \mathbf{W}(g),$$

that proves (85) and (86).

Because in  $\mathbb{C}_{K_0}$ ,  $T^n = 1$  if  $n \bmod K_0 = 0$ , we deduce from (72) that

$$T^{p(r,c)} = T^{(c-r)\bar{N}_0} = T^{-r\bar{N}_0} T^{c\bar{N}_0}.$$

□

Relation

$$\mathbf{E}(g) = \mathbf{D}_1 \mathbf{U}(g) \mathbf{D}_2 \mathbf{F}_{K_0}, \quad (94)$$

that results from (85) and (86), and because matrices  $\mathbf{D}_1$ ,  $\mathbf{D}_2$  and  $\mathbf{F}_{K_0}$  are unitary matrices, shows  $\mathbf{E}(g)$  is unitary if and only if  $\mathbf{U}(g)$  is unitary.

An angular parametrization of all unitary matrices  $\mathbf{U}$  with dimension  $N_0 \times K_0$  thus corresponds to an angular parametrization of all prototype functions  $g$  such that the cyclic TMUX with parameters  $K_0, N_0, \Delta_1 = 1, \Delta_2 = 1$  and cyclic modulation by  $g$  has the perfect reconstruction property.

The decomposition theorem 3.8 then allows, from an independent parametrization of each function  $[g_i]^{(j)}$ ,  $0 \leq i \leq \Delta_1 - 1$ ,  $0 \leq j \leq \Delta_2 - 1$ , to get an angular parametrization of all cyclic TMUX with cyclic modulation for parameters  $K_0, N_0, \Delta_1, \Delta_2$  with the perfect reconstruction property. An algorithmic construction is given in paragraph 4.3.

## 4.2 Angular parametrization of unitary matrices

**Definitions 4.2.** Let  $N$  and  $K$  be two positive integers with  $1 \leq K \leq N$  and denote by  $U(N, M)$  the set of all matrices  $\mathbf{A}$  with dimensions  $N \times K$  and complex coefficients such that  $\mathbf{A}^* \mathbf{A} = \mathbf{I}_K$ . Such matrices are called unitary matrices. When  $M = N$ , the set  $U(N, K)$  has a group structure for the matrix product : this is the unitary group denoted by  $U(N)$ .

A complex Givens rotation matrix with real angular parameters  $\theta$  and  $\phi$  is the matrix  $\mathbf{R}(\theta, \phi) \in U(2)$  defined here by [18]

$$\mathbf{R}(\theta, \phi) = \begin{bmatrix} e^{i\phi} \cos \theta & -e^{i\phi} \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad (95)$$

$$= \begin{bmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \quad (96)$$

**Lemma 4.3.** *If  $a, b \in \mathbb{C}$  then there exist  $\theta, \phi \in \mathbb{R}$  and  $c \in \mathbb{C}$  such that*

$$\mathbf{R}^*(\theta, \phi) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}. \quad (97)$$

*Proof.*— Let  $r_a \geq 0, \alpha \in \mathbb{R}$  be such that  $a = r_a e^{i\alpha}$  and  $r_b \geq 0, \beta \in \mathbb{R}$  such that  $b = r_b e^{i\beta}$ . Now let us define  $r \geq 0, \theta \in \mathbb{R}, \phi \in \mathbb{R}$  by  $r_a + ir_b = re^{i\theta}$  and  $\phi = \alpha - \beta$ . It is now straightforward to check that

$$\mathbf{R}^*(\theta, \phi) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} e^{-i\phi} \cos \theta & \sin \theta \\ -e^{-i\phi} \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} re^{i\alpha} \cos \theta \\ re^{i\beta} \sin \theta \end{bmatrix} = \begin{bmatrix} re^{i\beta} \\ 0 \end{bmatrix}.$$

and so (97) is verified with  $c = re^{i\beta}$ . □

**Remark 4.4.** *Several other definitions of complex Givens rotation matrices have been considered in the literature, for example in [19], all of them satisfying the property of lemma 4.3 (see for example [18], paragraph 5.1.3 page 243–246). The main sought objective is the realization of devices, with real time efficiency, for the QR factorisation of matrices or the computation of their inverse.*

**Notations 4.5.** *Let  $N \geq 2$  and  $0 \leq r_1, r_2 \leq N - 1$  two row indexes with  $r_1 \neq r_2$ . We denote by  $\mathbf{R}_{r_1, r_2}(\theta, \phi)$  the Givens rotation matrix of size  $N \times N$  whose non null coefficients are*

$$\begin{aligned} [\mathbf{R}_{r_1, r_2}(\theta, \phi)]_{r_1, r_1} &= e^{i\phi} \cos \theta, \\ [\mathbf{R}_{r_1, r_2}(\theta, \phi)]_{r_1, r_2} &= -e^{i\phi} \sin \theta, \\ [\mathbf{R}_{r_1, r_2}(\theta, \phi)]_{r_2, r_1} &= \sin \theta, \\ [\mathbf{R}_{r_1, r_2}(\theta, \phi)]_{r_2, r_2} &= \cos \theta, \\ [\mathbf{R}_{r_1, r_2}(\theta, \phi)]_{k, k} &= 1, \quad 0 \leq k \leq N - 1, \quad k \neq r_1, \quad k \neq r_2. \end{aligned}$$

$\mathbf{R}_{r_1, r_2}(\theta, \phi)$  is simply denoted by  $\mathbf{R}_{r_1, r_2}(\theta)$  when  $\phi = 0 \pmod{2\pi}$ .

On the other hand, for  $0 \leq r \leq N - 1$  and a non null function  $f$ ,  $\mathbf{J}_r(f)$  denotes the diagonal matrix of size  $N$  for which all diagonal elements are equal to 1 except on row  $r$  where the element is equal to  $f$ .

So, from (96), one has

$$\mathbf{R}_{r_1, r_2}(\theta, \phi) = \mathbf{J}_{r_1}(e^{i\phi}) \mathbf{R}_{r_1, r_2}(\theta). \quad (98)$$

Now, let us denote by  $\mathbf{B}_{r_1, r_2}^{(n)}(\theta, \phi)$  the matrix

$$\mathbf{B}_{r_1, r_2}^{(n)}(\theta, \phi) = \mathbf{J}_{r_1}(T^n e^{i\phi}) \mathbf{R}_{r_1, r_2}(\theta). \quad (99)$$



$\mathbf{B}_{r_1, r_2}^{(1)}(\theta, \phi)$  will be simply denoted by  $\mathbf{B}_{r_1, r_2}(\theta, \phi)$ .

Finally we denote by  $\mathbf{I}_{N, K}$  with  $1 \leq K \leq N$  the matrix of size  $N \times K$  such that  $[\mathbf{I}_{N, K}]_{r, r} = 1$ ,  $0 \leq r \leq K - 1$ , all the other elements being 0.

Using the QR decomposition method, such as described in [18] for example, we may prove the following theorem.

**Theorem 4.6.** *Let  $N, K$  be positive integers with  $1 \leq K \leq N$  and  $\mathbf{A} \in U(N, K)$  an unitary matrix. There exist  $K(2N - K)$  real angular parameters  $\theta_{c, r}, \phi_{c, r}, 0 \leq c \leq K - 1, c + 1 \leq r \leq N - 1$  and  $\psi_c, 0 \leq c \leq K - 1$  such that*

$$\mathbf{A} = \prod_{c=0}^{K-1} \prod_{r=c+1}^{N-1} \mathbf{R}_{c, r}(\theta_{c, r}, \phi_{c, r}) \cdot \prod_{c=0}^{K-1} \mathbf{J}_c(\psi_c) \mathbf{I}_{N, K}. \quad (100)$$

*Proof.*— Using lemma 4.3, we may choose complex Givens rotations  $\mathbf{R}_{1, r}(\theta_{1, r}, \phi_{1, r})$  for  $r$  varying from  $N - 1$  to 2 to cancel element  $\mathbf{A}_{r, 1}$ . So we get

$$\mathbf{A} = \prod_{r=1}^{N-1} \mathbf{R}_{0, r}(\theta_{0, r}, \phi_{0, r}) \left[ \begin{array}{c|ccc} b_{0,0} & b_{0,1} & \cdots & b_{0, K-1} \\ \hline 0 & & & \\ \vdots & & \mathbf{A}_1 & \\ 0 & & & \end{array} \right]. \quad (101)$$

As Givens rotation matrices are unitary, the left member matrix in (101) too. One deduces that  $|b_{0,0}|^2 = 1$ , and so  $b_{0,0}$  has the form  $b_{0,0} = e^{i\psi_0}$  with  $\psi_0 \in \mathbb{R}$ . As column vectors of an unitary matrix are mutually orthogonal, we get  $b_{0,c} = 0$ ,  $1 \leq c \leq K - 1$  and the matrix  $\mathbf{A}_1$  with size  $(N - 1) \times (K - 1)$  is unitary. This process is then iterated for matrix  $\mathbf{A}_1$  then on the obtained matrix  $\mathbf{A}_2$  with size  $(N - 2) \times (K - 2)$ , and so on until a matrix  $\mathbf{A}_{K-1}$  with dimensions  $(N - K + 1) \times 1$ .  $\square$

**Remark 4.7.** *When  $K = N$ , we retrieve the fact that a matrix in  $U(N)$  may be expressed with  $N^2$  real angular parameters and the known result that the dimension  $U(N)$  as a real algebraic manifold is  $N^2$ .*

We shall now be interested especially in the case  $K = K_0 \geq 2$ ,  $N = N_0 = K_0 + 1$ . The number of angular parameters is thus equal to  $K_0(K_0 + 2)$ .

The factorization theorem 4.6 for unitary matrices associated to relation (94) allows to obtain a factorization of matrix  $\mathbf{W}(g)$  when  $\Delta_1 = \Delta_2 = 1$ . However the following theorem gives a simplified expression of it with less products by variable  $T$ .

**Theorem 4.8.** *Let  $g$  be the prototype function of a cyclic TMUX with cyclic modulation and parameters  $K_0, N_0, \Delta_1, \Delta_2$  and  $\Delta_1 = \Delta_2 = 1, K_0 < N_0, \gcd(K_0, N_0) = 1$  that satisfies the perfect reconstruction conditions. There exist  $K_0(2N_0 - K_0)$  real angular parameters  $\theta_{c, r}, \phi_{c, r}, 0 \leq c \leq K_0 - 1, c + 1 \leq r \leq N_0 - 1$  and  $\psi_c, 0 \leq c \leq K_0 - 1$  such that*

$$\mathbf{W}(g) = \prod_{c=0}^{K_0-1} \prod_{r=c+1}^{N_0-1} \mathbf{B}_{c, r}^{(N_0)}(\theta_{c, r}, \phi_{c, r}) \cdot \prod_{c=0}^{K_0-1} \mathbf{J}_c(e^{i\psi_c} T^{(c+1)N_0-1}) \mathbf{I}_{N_0, K_0}. \quad (102)$$

**Proof.**– For  $0 \leq c \leq K_0$  and  $c \leq r \leq N_0 - 1$ , let us define the diagonal matrix  $\mathbf{A}_{c,r}$  of size  $N_0$  with elements in  $\mathbb{C}_{K_0}[T]$  by

$$[\mathbf{A}_{c,r}]_{r',r'} = \begin{cases} 1, & 0 \leq r' \leq c-1, \\ T^{(N_0-1-r)\bar{N}_0}, & r' = c, \\ T^{(N_0-1-r')\bar{N}_0}, & c+1 \leq r' \leq N_0-1. \end{cases} \quad (103)$$

In particular one may note that, for  $0 \leq c \leq K_0 - 1$ ,

$$\mathbf{A}_{c,N_0-1} = \mathbf{A}_{c+1,c+1}, \quad (104)$$

and that the matrix  $\mathbf{D}_1$  is related to  $\mathbf{A}_{0,0}$  by

$$\mathbf{D}_1 = T^{-(N_0-1)\bar{N}_0} \mathbf{A}_{0,0} = T^{\bar{N}_0-1} \mathbf{A}_{0,0}, \quad (105)$$

because  $N_0\bar{N}_0 = 1 \pmod{K_0}$ .

Now for fixed  $c$  and  $r$  such that  $0 \leq c \leq K_0 - 1$  and  $c+1 \leq r \leq N_0 - 1$ , we may write  $\mathbf{A}_{c,r-1} = \mathbf{J}_c(T^{\bar{N}_0})\mathbf{A}_{c,r}$  and because  $\mathbf{A}_{c,r}$  has the same diagonal element  $T^{(N_0-1-r)\bar{N}_0}$  on rows  $c$  and  $r$  it commutes with  $\mathbf{R}_{c,r}(\theta_{c,r}, \phi_{c,r})$ , and we get

$$\begin{aligned} \mathbf{A}_{c,r-1}\mathbf{R}_{c,r}(\theta_{c,r}, \phi_{c,r}) &= \mathbf{J}_c(T^{\bar{N}_0})\mathbf{R}_{c,r}(\theta_{c,r}, \phi_{c,r})\mathbf{A}_{c,r} \\ &= \mathbf{B}_{c,r}^{(\bar{N}_0)}(\theta_{c,r}, \phi_{c,r})\mathbf{A}_{c,r}. \end{aligned} \quad (106)$$

Using (106) for  $r$  varying from  $c+1$  to  $N_0 - 1$  and (104), we get, for any fixed  $c$  with  $0 \leq c \leq K_0 - 1$

$$\mathbf{A}_{c,c} \prod_{r=c+1}^{N_0-1} \mathbf{R}_{c,r}(\theta_{c,r}, \phi_{c,r}) = \left( \prod_{r=c+1}^{N_0-1} \mathbf{B}_{c,r}^{(\bar{N}_0)}(\theta_{c,r}, \phi_{c,r}) \right) \mathbf{A}_{c+1,c+1}. \quad (107)$$

Using (107) for  $c$  varying from 0 to  $K_0 - 1$  gives

$$\mathbf{A}_{0,0} \prod_{c=0}^{K_0-1} \prod_{r=c+1}^{N_0-1} \mathbf{R}_{c,r}(\theta_{c,r}, \phi_{c,r}) = \left( \prod_{c=0}^{K_0-1} \prod_{r=c+1}^{N_0-1} \mathbf{B}_{c,r}^{(\bar{N}_0)}(\theta_{c,r}, \phi_{c,r}) \right) \mathbf{A}_{K_0,K_0}. \quad (108)$$

From equation (87), Theorem 4.6, and equation (105), we deduce that

$$\mathbf{W}(g) = \prod_{c=0}^{K_0-1} \prod_{r=c+1}^{N_0-1} \mathbf{B}_{c,r}^{(\bar{N}_0)}(\theta_{c,r}, \phi_{c,r}) \cdot \left( T^{\bar{N}_0-1} \mathbf{A}_{K_0,K_0} \prod_{c=0}^{K_0-1} \mathbf{J}_c(e^{i\psi_c}) \right) \mathbf{I}_{N_0,K_0} \mathbf{D}_2. \quad (109)$$

The matrix inside parenthesis in (109) is a diagonal matrix of size  $N_0$  whose diagonal element on row  $c$ ,  $0 \leq c \leq K_0 - 1$ , is  $T^{\bar{N}_0-1} e^{i\psi_c}$  while the  $\mathbf{D}_2$  diagonal element on row  $c$  is  $T^{c\bar{N}_0}$ , which achieves the proof.  $\square$

When  $N_0 = K_0 + 1$  with  $K_0 > 1$ ,  $\bar{N}_0 = 1$  and the following corollary holds.

**Corollary 4.9.** *Let  $g$  be the prototype function of a cyclic TMUX with parameters  $K_0, N_0 = K_0 + 1, \Delta_1 = \Delta_2 = 1$  and cyclic modulation by  $g$  satisfying the perfect reconstruction property.*

*There exist  $K_0(K_0 + 2)$  real angular parameters  $\theta_{c,r}, \phi_{c,r}, 0 \leq c \leq K_0 - 1, c + 1 \leq r \leq N_0 - 1$  and  $\psi_c, 0 \leq c \leq K_0 - 1$  such that*

$$\mathbf{W}(g) = \prod_{c=0}^{K_0-1} \prod_{r=c+1}^{N_0-1} \mathbf{B}_{c,r}(\theta_{c,r}, \phi_{c,r}) \cdot \prod_{c=0}^{K_0-1} \mathbf{J}_c(e^{i\psi_c} T^c) \mathbf{I}_{N_0, K_0}. \quad (110)$$

An example of implementation of matrix  $\mathbf{W}(g)$  is given in Figure 8 for  $K_0 = 3$  and  $N_0 = 5$  ( $\bar{N}_0 = 2$ ).

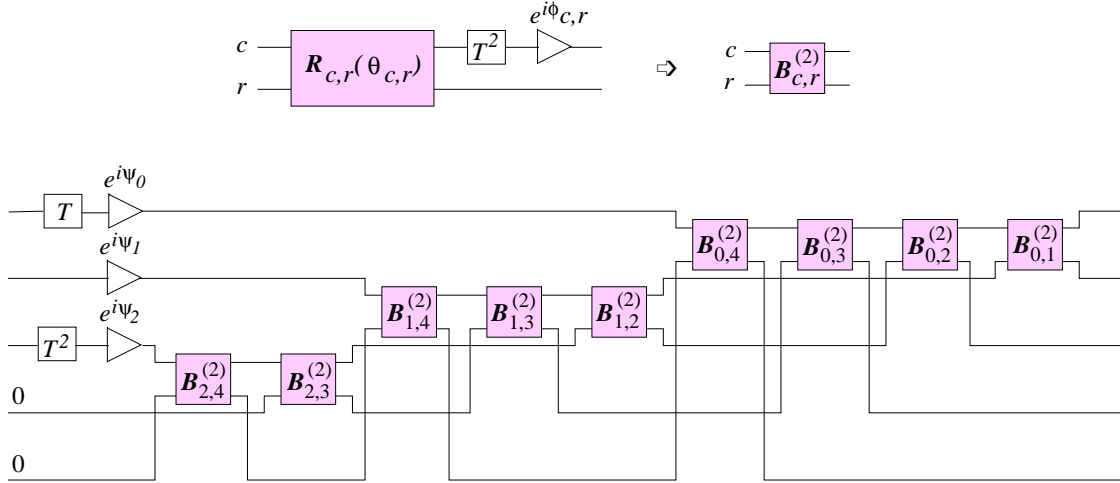


Figure 8: Implementation scheme of  $\mathbf{W}(g)$  for  $K_0 = 3, N_0 = 5$  using Theorem 4.8.

### 4.3 Prototype function reconstruction

Let  $g(T) \in \mathbb{C}_M[T]$  be the prototype function of a cyclic TMUX with cyclic modulation and parameters  $K_0, N_0, \Delta_1, \Delta_2$  with  $M = \Delta_1 \Delta_2 K_0 N_0$  and satisfying the perfect reconstruction property. From Theorem 3.8, each component  $[g_i]^{(j)}(T) \in \mathbb{C}_{K_0 N_0}[T]$  is the prototype function of a cyclic TMUX with cyclic modulation and parameters  $K_0, N_0, 1, 1$  with perfect reconstruction property.

We may then construct such a prototype function  $g$  using the following three steps algorithm :

1. For each pair  $(i, j)$  with  $0 \leq i \leq \Delta_1 - 1$  and  $0 \leq j \leq \Delta_2 - 1$ , an unitary matrix  $\mathbf{U}_{i,j}$  with size  $N_0 \times K_0$  is constructed using  $K_0(2N_0 - K_0)$  real angular parameters  $\theta_{i,j,c,r}, \alpha_{i,j,c,r}, 0 \leq c \leq K_0 - 1, c + 1 \leq r \leq N_0 - 1$  and  $\beta_{i,j,c}, 0 \leq c \leq K_0 - 1$ . This matrix is identified to a matrix  $\mathbf{U}([p_i]^{(j)}(T))$  following the formula

$$[p_i]^{(j)}(T) = \sum_{r=0}^{N_0-1} \sum_{c=0}^{K_0-1} [\mathbf{U}_{i,j}]_{r,c} T^{a(r,c)}, \quad (111)$$

which allows to construct  $[p_i]^{(j)}(T) \in \mathbb{C}_{K_0 N_0}[T]$ .

2. For each  $0 \leq i \leq \Delta_1 - 1$ , components  $p_i(T) \in \mathbb{C}_{K_0 N_0 \Delta_2}[T]$  are built using equation (52)

$$p_i(T) = \frac{1}{\Delta_2} \sum_{j=0}^{\Delta_2-1} [p_i]^{(j)}(\omega_M^{j\Delta_1} T) \left[ \sum_{s=0}^{\Delta_2-1} \omega_{\Delta_2}^{js} T^{sK_0 N_0} \right]. \quad (112)$$

3. Finally  $g(T) \in \mathbb{C}_M[T]$  is obtained thanks to the formula (51)

$$g(T) = \sum_{i=0}^{\Delta_1-1} p_i(T^{\Delta_1}) T^i. \quad (113)$$

It is then easy to verify by calculus that such an obtained function  $g$  has well the perfect reconstruction property for the given set of parameters  $K_0, N_0, \Delta_1, \Delta_2$ . To check this property, one may choose at random the  $K_0(2N_0 - K_0)\Delta_1\Delta_2$  angular parameters with rational sine and cosine, which allows to get exact expressions for the coefficients of  $g$  in the algebraic extension field  $\mathbb{Q}(i, \omega_M)$  ( $i^2 = -1$ ).

## 5 Conclusion

As shown with some recent publications, transmultiplexers based on circular convolution offer a number of advantages. In this preprint, focusing on the case of perfect reconstruction cyclic TMUX with cyclic modulation, we have opened new perspectives concerning the design of CB-FMT systems. Indeed, our theoretical analysis provides a detailed description of this TMUX based on two-level decomposition theorems. Application of this theory is made easy with the introduction of a complete angular parametrization of these cyclic TMUX. Contrary to [8], the angular parametrization leads to an independent set of angles. This finding thus provides a significant potential advantage when the aim is to design prototype functions for a given criterion.

It is also worth noting the higher simplicity of the resulting parametrization compared to the one necessary to exhaustively represent conventional modulated TMUX or filter banks [16].

As a natural follow up of the present study, the next step will consist in defining appropriate design criteria for cyclic prototype functions in relation with certain transmission scenarios. In this respect, the duality theorem proved in our preprint already shows that a frequency selectivity criterion, whatever its precise definition, is equivalent to a selectivity criterion in the time domain. In a computational cost perspective, which becomes of paramount importance when the number of angular parameters is very high, it has to be examined if the compact representation method introduced in [20] can be adapted to the cyclic TMUX case.

Obviously, if the PR property can provide some interesting advantages, as e.g. the derivation of an efficient realization structure, from a communication point of view PR property no longer holds in realistic situations. Then, the resilience of PR cyclic TMUX to channel distortion and system impairments needs to be compared to the one provided by non PR TMUX.

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## Nomenclature

- $[p^{(j)}]_i$   $i$ th  $\Delta_1$ -polyphase component of the  $j$ th  $\Delta_2$ -periodic component of  $g$ , page 15
- $[p_i]^{(j)}$   $j$ th  $\Delta_2$ -periodic component of the  $i$ th  $\Delta_1$ -polyphase component of  $g$ , page 15
- $\alpha(r, c), p(r, c)$  and  $q(r, c)$  Arithmetic functions, page 17
- $\Pi$   $K \times K$  permutation matrix, page 17
- $\mathbf{B}_{r_1, r_2}(\theta, \phi)$  stands for  $\mathbf{B}_{r_1, r_2}^{(1)}(\theta, \phi)$ , page 24
- $\mathbf{B}_{r_1, r_2}^{(n)}(\theta, \phi)$  Matrix of size  $N_0 \times N_0$  depending on two angles, an integer  $n$ , and variable  $T$ , page 23
- $\mathbf{D}$   $K \times K$  diagonal matrix such that  $[\mathbf{D}]_{r, r} = \omega_K^{-dk_0}$  if  $r = dK_0 + k_0$ , page 17
- $\mathbf{D}_1$  Diagonal matrix of size  $N_0$  whose element in row  $r$ ,  $0 \leq r \leq N_0 - 1$  is  $T^{-r\bar{N}_0}$ , page 21
- $\mathbf{D}_2$  Diagonal matrix of size  $K_0$  whose element in column  $c$ ,  $0 \leq c \leq K_0 - 1$  is  $T^{c\bar{N}_0}$ , page 21
- $\mathbf{E}(g)$   $N \times K$  transmitter matrix for the cyclic TMUX with a cyclic modulation by  $g$ , page 8
- $\mathbf{F}_N$   $N \times N$  matrix for the Fourier transform  $\mathcal{F}_N$ , page 4
- $\mathbf{I}_K$  Identity matrix with size  $K$ , page 7
- $\mathbf{J}_r(f)$  Diagonal matrix of size  $N \times N$ , page 23
- $\mathbf{R}(\theta, \phi)$   $2 \times 2$  complex Givens rotation matrix, page 23
- $\mathbf{R}(g)$   $K \times N$  receiver matrix for the cyclic TMUX with a cyclic modulation by  $g$ , page 8
- $\mathbf{R}_{r_1, r_2}(\theta, \phi)$   $N \times N$  Givens rotation matrix modifying rows  $r_1$  and  $r_2$ , page 23
- $\mathbf{U}(g)$  Constant  $N_0 \times K_0$  matrix defined from prototype function  $g$  when  $\Delta_1 = \Delta_2 = 1$ , page 21
- $\mathbf{W}(g)$   $K_0 \times N_0$  matrix built from elements  $W_a(T)$ ,  $0 \leq a \leq K_0N_0 - 1$  ( $\Delta_1 = 1$ ), page 18
- $\mathbf{W}_j$  Another notation for matrix  $\mathbf{W}(g^{(j)})$  ( $\Delta_1 = 1$ ), page 20
- $P$  Frequency domain variable, page 3
- $T$  Time domain variable, page 3
- $a, b, n, m, k, l, \dots$  Summation indexes, page 3
- $D_N$  Cyclic decimator with factor  $N$ , page 5
- $E_N$  Cyclic expander with factor  $N$ , page 4

$E_{n,k} \in \mathbb{C}_L[T]$	Type 1 $N$ -polyphase component of a filter $g_k \in \mathbb{C}_{LN}[T]$ , page 6
$F_{k,n} \in \mathbb{C}_L[T]$	Type 2 $N$ -polyphase component of a filter $h_k \in \mathbb{C}_{LN}[T]$ , page 6
$K, N, M, K_0, N_0, M_0, \Delta, \Delta_1, \dots$	Strictly positive integer parameters, page 3
$p^{(j)} \in \mathbb{C}_{K_0 N_0 \Delta_1}[T]$	$\Delta_2$ -periodic component of a prototype filter $g$ , page 12
$p_i(T) \in \mathbb{C}_{K_0 N_0 \Delta_2}[T]$	$\Delta_1$ -polyphase component of a prototype filter $g$ , page 12
$S = \mathcal{F}_{Ns}$	Order $N$ Fourier transform of a $N$ -cyclic signal, page 4
$W_a(T)$	$K_0 N_0$ -polyphase components of prototype function $g$ when $\Delta_1 = 1$ , page 17

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