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Approximability of dynamical systems between trees of spheres

Matthieu Arfeux

July 31, 2014

Abstract

We study sequences of analytic conjugacy classes of rational maps which diverge in moduli space. In particular, we are interested in the notion of rescaling limits introduced by Jan Kiwi. In the continuity of [A1] we recall the notion of dynamical covers between trees of spheres for which a periodic sphere corresponds to a rescaling limit. We study necessary conditions for such a dynamical cover to be the limit of dynamically marked rational maps. With these conditions we classify them for the case of bicritical maps and we recover the second main result of Jan Kiwi regarding rescaling limits.

Contents

1 Introduction

In [K3], Jan Kiwi introduced the notion of rescaling limits as follows:

Definition. *For a sequence of rational maps $(f_n)_n$ of a given degree, a rescaling is a sequence of Moebius transformations $(M_n)_n$ such that there exist $k \in \mathbb{N}$ and a rational map g of degree ≥ 2 such that*

$$M_n \circ f_n^k \circ M_n^{-1} \to g$$

uniformly in compact subsets of $\mathbb{S}$ with finitely many points removed.
   If this k is minimum then it is called the rescaling period for $(f_n)_n$ at $(M_n)_n$ and g a rescaling limit for $(f_n)_n$.

On the set of rescaling limits we recalled the two equivalence relations of :

- equivalent rescalings and

- dynamically dependent rescalings.

Using the formalism of Berkovich spaces, he proved the two following theorems.

Theorem A. *[K3] For every sequence in Rat_d for $d \geq 2$ there are at most $2d-2$ dynamically independent rescalings classes with a non post-critically finite rescaling limit.*

Theorem B. *[K3] Every sequence in rat_2 admits at most 2 dynamically independent rescalings limits of period at least 2. Furthermore, in the case that a rescaling of period at least 2 exists, then exactly one of the following holds:*

1. *$(f_n)_n$ has exactly two dynamically independent rescalings, of periods $q' > q > 1$. The period q rescaling limit is a quadratic rational map with a multiple fixed point and a prefixed critical point. The period q' rescaling limit is a quadratic polynomial, modulo conjugacy.*

2. *$(f_n)_n$ has a rescaling whose corresponding limit is a quadratic rational map with a multiple fixed point and every other rescaling is dynamically dependent to it.*

In [A1] we proved the first theorem introducing the notion of dynamical systems between (stable) trees of spheres. More precisely we where interested in such dynamical systems approximable by dynamically marked rational maps. For these ones a cycle of spheres containing a critical sphere can be associated to a class of dynamically dependent rescalings.

In this paper we prove some necessary conditions for a dynamical systems between trees of spheres in order to be approximable by dynamically marked rational maps. Using these conditions we classify the trees which are limits of rational maps with exactly two critical points respectively to the behavior of cycle of spheres containing a critical sphere and we recover Theorem B.

Outline.

In section 2 we recall the main notions introduced in [A1] and we add new ones in 2.2. More precisely we define branches as a connected component of a tree minus an internal vertex. We define annuli as the non empty intersection of two branches. For v and v' two internal vertices we denote by $[v, v']$ the path between them and by $]v, v'[$ the annuli consisting on the intersection of branches on v and v' that has non empty intersection with $[v, v']$. We then prove some properties on branches and annuli of dynamical systems between trees of spheres.

In section 3, we establish some properties of the elements of $\partial \mathrm{Rat}_{\mathbf{F}, X}$, the set of dynamical systems of trees of spheres of portrait $\mathbf{F}$ which are limits of dynamical systems of marked spheres diverging in $\mathrm{Rat}_{\mathbf{F}, X}$. These properties are the following lemmas.

Lemma (Branches). *Let $(\mathcal{F}, \mathcal{T}^X) \in \partial \mathrm{Rat}_{\mathbf{F}, X}$, let v be a periodic internal vertex, let $a_0 \in \mathbb{S}_v$ and B be a branch on v such that for all $k \in \mathbb{N}$, the branch of T^Y attached to $f_v^k(a_0)$ maps inside the branch on T^Z attached to $f_v^{k+1}(a_0)$. Then*

- *B does not contain critical periodic vertex ;*

- *if B contains a periodic internal vertex then its cycle has degree 1 and a_0 is periodic.*

Lemma (Annuli). *Suppose that $(\mathcal{F}, \mathcal{T}^X) \in \partial \mathrm{Rat}_{\mathbf{F}, X}$ and that v and v' are distinct internal vertices of $\mathcal{T}^Y$ such that for $0 \le k \le k_0 - 1$ the annulus $]F^k(v), F^k(v')[^Y$ is defined and does not contain any critical leaf.*

- *If $\mathcal{F}$ has degree more than 1 on one of these $]F^k(v), F^k(v')[^Y$, we never have $[v_{k_0}, v'_{k_0}] \subseteq [v, v']$.*

- *If this is not the case and if $[v_{k_0}, v'_{k_0}] \subseteq [v, v']$ or $[v, v'] \subseteq [v_{k_0}, v'_{k_0}]$ we have*

    A) *either $v_{k_0} = v$ and $v'_{k_0} = v'$, then $i_v(v')$ and $i_{v'}(v)$ are fixed by $f_{v'}^{k_0}$ and the product of the associated multipliers is 1;*

    B) *or $v_{k_0} = v'$ and $v'_{k_0} = v$, then $f_v^{k_0}$ exchanges $i_v(v')$ and $i_{v'}(v)$ and the multiplier of the associated cycle is 1.*

From this lemma, on Figure 1, we provide an example of dynamical system between trees of spheres that is not a limit of dynamically marked rational maps.

In section 4, we classify the dynamical systems between trees of spheres with exactly two critical leaves (bi-critical) according the existence of rescalings.

Theorem 1 (Classification). *Let $\mathbf{F}$ be a portrait of degree d with $d + 1$ fixed points and exactly 2 critical points and let $(\mathcal{F}, \mathcal{T}^X) \in \partial \mathrm{Rat}_{\mathbf{F}, X}$. Suppose that there exists $(f_n, y_n, z_n) \xrightarrow{\lhd} \mathcal{F}$ in $\mathrm{Rat}_{\mathbf{F}, X}$ such that for all n, $x_n(X)$ contains all the fixed points of f_n. Then the map $\mathcal{F}$ has at most two critical cycles of spheres; they have degree d.*

Assume that there exits at least one rescaling limit. Then there is a vertex w_0 separating three fixed points which is fixed and such that f_{w_0} has finite order $k_0 > 1$. Denote by v_0 the critical vertex separating w_0 and the two critical leaves.

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1. *Either v_0 belongs to a critical cycle of period k_0 and*

   (a) *its associated cover has a parabolic fixed point;*

   (b) *if there is a second critical cycle then it has period $k_0' > k_0$, its associated cover has a critical fixed point with local degree d and the one associated to v_0 has a critical point that eventually maps to the parabolic fixed point.*

2. *Or v_0 is forgotten by F^k with $k < k_0$; in this case there is exactly one critical cycle; it has period $k_0' > k_0$ and its associated cover has a critical fixed point with local degree d.*

We then deduce from it a slight generalization of Theorem B and what appears in [A] to the bi-critical case :

Theorem 2. *Every sequence of bi-critical maps in rat_d admits at most 2 dynamically independent rescalings limits of period at least 2. Furthermore, in the case that a rescaling of period at least 2 exists, then exactly one of the following holds:*

1. *$(f_n)_n$ has exactly two dynamically independent rescalings, of periods $q' > q > 1$. The period q rescaling limit is a degree d rational map with a multiple fixed point and a prefixed critical point. The period q' rescaling limit is a degree d polynomial, modulo conjugacy.*

2. *$(f_n)_n$ has a rescaling whose corresponding limit is a degree d rational map with a multiple fixed point and every other rescaling is dynamically dependent to it.*

To conclude, in section 5 we discuss questions about rescalings on an explicit example and we do a comparison between our approach and J.Milnor's compactification in the special case of degree 2.

2 Preliminaries

2.1 Some definitions and notations

The reader that already read [A1] can directly go to section 2.2.

2.1.1 Cover between trees of spheres

Let X be a finite set with at least 3 elements. A (projective) tree of spheres $\mathcal{T}$ marked by X is the following data :

- a combinatorial tree T whose leaves are the elements of X and every internal vertex has at least valence 3,

- for each internal vertex v of T, an injection $i_v : E_v \to \mathcal{S}_v$ of the set of edges E_v adjacent to v into a topological sphere $\mathcal{S}_v$, and

- for every $v \in IV$ (internal vertex) of a projective structure on $\mathcal{S}_v$.

We use the notation $X_v := i_v(E_v)$ and define the map $a_v : X \to \mathcal{S}_v$ such that $a_v(x) := i_v(e)$ if x and e lie in the same connected component of $T - \{v\}$. We denote by $[v, v']$ the path between v and v' including these vertices and by $]v, v'[$ the path $[v, v']$ minus the two vertices v and v'.

A particular case is the notion of spheres marked by X defined below.

Definition 2.1 (Marked sphere). *A sphere marked (by X) is an injection*

$$x : X \to \mathbb{S}.$$

We identify trees with only one internal vertex with the marked spheres. In the same spirit we generalized the notion of rational maps marked by a portrait defined below :

Definition 2.2 (Marked rational maps). *A rational map marked by $\mathbf{F}$ is a triple (f, y, z) where*

- $f \in \mathrm{Rat}_d$

- $y : Y \to \mathbb{S}$ *and* $z : Z \to \mathbb{S}$ *are marked spheres,*

- $f \circ y = z \circ F$ *on* Y *and*

- $\deg_{y(a)} f = \deg(a)$ *for* $a \in Y$.

Where a portrait $\mathbf{F}$ of degree $d \geq 2$ is a pair $(F, \deg)$ such that

- $F : Y \to Z$ is a map between two finite sets Y and Z and

- $\deg : Y \to \mathbb{N} - \{0\}$ is a function that satisfies

$$\sum_{a \in Y} \big(\deg(a) - 1\big) = 2d - 2 \quad \text{and} \quad \sum_{a \in F^{-1}(b)} \deg(a) = d \quad \text{for all } b \in Z.$$

If (f, y, z) is marked by $\mathbf{F}$, we have the following commutative diagram :

$$
\begin{array}{ccc}
Y & \xrightarrow{\ y\ } & \mathbb{S} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ z\ } & \mathbb{S}
\end{array}
$$

Typically, $Z \subset \mathbb{S}$ is a finite set, $F : Y \to Z$ is the restriction of a rational map $F : \mathbb{S} \to \mathbb{S}$ to $Y := F^{-1}(Z)$ and $\deg(a)$ is the local degree of F at a. In this case, the Riemann-Hurwitz formula and the conditions on the function deg implies that Z contains the set V_F of the critical values of F so that $F : \mathbb{S} - Y \to \mathbb{S} - Z$ is a cover.

The generalization of marked rational maps is the notion of (holomorphic) cover between trees of spheres. A cover $\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z$ between two trees of spheres marked by Y and Z is the following data

- a map $F : T^Y \to T^Z$ mapping leaves to leaves, internal vertices to internal vertices, and edges to edges,

- for each internal vertex v of T^Y and $w := F(v)$ of T^Z, an holomorphic ramified cover $f_v : \mathcal{S}_v \to \mathcal{S}_w$ that satisfies the following properties:

  - the restriction $f_v : \mathcal{S}_v - Y_v \to \mathcal{S}_w - Z_w$ is a cover,
  - $f_v \circ i_v = i_w \circ F$,
  - if e is an edge between v and v', then the local degree of f_v at $i_v(e)$ is the same as the local degree of $f_{v'}$ at $i_{v'}(e)$.

We saw that a cover between trees of spheres $\mathcal{F}$ is surjective, has a global degree, denoted by $\deg(\mathcal{F})$.

2.1.2 Dynamical systems between trees of spheres

Suppose in addition that $X \subseteq Y \cap Z$ and we show that we can associate a dynamical system to some covers between trees of spheres. More precisely we will say that $(\mathcal{F}, \mathcal{T}^X)$ is a dynamical system of trees of spheres if :

- $\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z$ is a cover between trees of spheres,

- $\mathcal{T}^X$ is a tree of spheres compatible with $\mathcal{T}^Y$ and $\mathcal{T}^Z$, ie :

  - $X \subseteq Y \cap Z$
  - each internal vertex v of T^X is an internal vertex common to T^Y and T^Z,
  - $\mathcal{S}_v^X = \mathcal{S}_v^Y = \mathcal{S}_v^Z$ and
  - $a_v^X = a_v^Y|_X = a_v^Z|_X$.

A very useful lemma about compatible trees is the following.

Lemma 2.3. *If T^X is compatible with T^Y and if an internal vertex $v \in IV^Y$ separates three vertices of V^X, then $v \in T^X$.*

With this definition we are able to compose covers along an orbit of vertices as soon as they are in T^X. When it is well defined we will denote by f_v^k the composition $f_{F^{k-1}(v)} \circ \ldots \circ f_{F(v)} \circ f_v$.

Dynamical covers between marked spheres can be naturally identified to dynamically marked rational maps:

Definition 2.4 (Dynamically marked rational map). *A rational map dynamically marked by* $(\mathbf{F}, X)$ *is a rational map* (f, y, z) *marked by* $\mathbf{F}$ *such that* $y|_X = z|_X$.

We denote by $\mathrm{Rat}_{\mathbf{F}, X}$ the set of rational maps dynamically marked by $(\mathbf{F}, X)$.

Let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system between trees of spheres. A period $p \geq 1$ cycle of spheres is a collection of spheres $(\mathcal{S}_{v_k})_{k \in \mathbb{Z}/p\mathbb{Z}}$ where the v_k are internal vertices of T^X that satisfies $F(v_k) = v_{k+1}$. It is critical if it contains a critical sphere, ie a sphere $\mathcal{S}_v$ such that $\deg(f_v)$ is greater or equal to two. If a sphere $\mathcal{S}_v$ on a critical cycle contains a critical point of its respective f_v that has infinite orbit, then the cycle is said non post-critically finite.

2.1.3 Convergence notions and approximability

Consider holomorphic covers between trees of spheres with a projective structure. We recall the three convergence notions introduced in [A1].

Definition 2.5 (Convergence of marked spheres). *A sequence* $\mathcal{A}_n$ *of marked spheres* $a_n : X \to \mathbb{S}_n$ *converges to a tree of spheres* $\mathcal{T}^X$ *if for all internal vertex* v *of* $\mathcal{T}^X$ *, there exists a (projective) isomorphism* $\phi_{n,v} : \mathbb{S}_n \to \mathbb{S}_v$ *such that* $\phi_{n,v} \circ a_n$ *converges to* a_v.

We use the notation $\mathcal{A}_n \to \mathcal{T}^X$ or $\mathcal{A}_n \xrightarrow[\phi_n]{} \mathcal{T}^X$.

Definition 2.6 (Non dynamical convergence). *Let* $\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z$ *be a cover between trees of spheres of portrait* $\mathbf{F}$. *A sequence* $\mathcal{F}_n := (f_n, a_n^Y, a_n^Z)$ *of marked spheres covers converges to* $\mathcal{F}$ *if their portrait is* $\mathbf{F}$ *and if for all pair of internal vertices* v *and* $w := F(v)$, *there exists sequences of isomorphisms* $\phi_{n,v}^Y : \mathbb{S}_n^Y \to \mathbb{S}_v$ *and* $\phi_{n,w}^Z : \mathbb{S}_n^Z \to \mathbb{S}_w$ *such that*

- $\phi_{n,v}^Y \circ a_n^Y : Y \to \mathbb{S}_v$ *converges to* $a_v^Y : Y \to \mathbb{S}_v$,

- $\phi_{n,w}^Z \circ a_n^Z : Z \to \mathbb{S}_w$ *converges to* $a_w^Z : Z \to \mathbb{S}_w$ *and*

- $\phi_{n,w}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1} : \mathbb{S}_v \to \mathbb{S}_w$ *converges locally uniformly outside* Y_v *to* $f_v : \mathbb{S}_v \to \mathbb{S}_w$.

We use the notation $\mathcal{F}_n \to \mathcal{F}$ or $\mathcal{F}_n \xrightarrow[(\phi_n^Y, \phi_n^Z)]{} \mathcal{F}$.

Definition 2.7 (Dynamical convergence). *Let* $(\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z, \mathcal{T}^X)$ *be a dynamical system of trees of spheres with portrait* $\mathbf{F}$. *A sequence* $(\mathcal{F}_n, a_n^Y, a_n^Z)_n$ *of dynamical systems of spheres marked by* $(\mathbf{F}, X)$ *converges to* $(\mathcal{F}, \mathcal{T}^X)$ *if*

$$\mathcal{F}_n \xrightarrow[\phi_n^Y, \phi_n^Z]{} \mathcal{F} \quad with \quad \phi_{n,v}^Y = \phi_{n,v}^Z$$

for all vertex $v \in IV^X$. *We say that* $(\mathcal{F}, \mathcal{T}^X)$ *is dynamically approximable by* $(\mathcal{F}_n)_n$.

We write $f_n \underset{\phi_n^Y, \phi_n^Z}{\overset{\lhd}{\longrightarrow}} \mathcal{F}$.

Recall some properties of these convergences.

Lemma 2.8. *Let v and v' be two distinct internal vertices of $\mathcal{T}^X$ (having each one at least three edges) and a sequence of marked spheres $(\mathcal{A}_n)_n$ such that $\mathcal{A}_n \underset{\phi_n}{\longrightarrow} \mathcal{T}^X$. Then the sequence of isomorphisms $(\phi_{n,v'} \circ \phi_{n,v}^{-1})_n$ converges locally uniformly outside $i_v(v')$ to the constant $i_{v'}(v)$.*

Lemma 2.9. *Let $\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z$ be a cover between trees of spheres with portrait $\mathbf{F}$ and of degree D. Let $v \in IV^Y$ with $\deg(v) = D$ and let $\mathcal{F}_n := (f_n, a_n^Y, a_n^Z)$ be a sequence of covers between trees of spheres that satisfies $\mathcal{F}_n \underset{\phi_n^Y, \phi_n^Z}{\longrightarrow} \mathcal{F}$. Then the sequence $\phi_{n,F(v)}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1} : \mathbb{S}_v \to \mathbb{S}_{F(v)}$ converges uniformly to $f_v : \mathbb{S}_v \to \mathbb{S}_{F(v)}$.*

Corollary 2.10. *Let $(\mathcal{F}, \mathcal{T}^X)$ be a dynamical system of trees of spheres of degree D, dynamically approximable by $\mathcal{F}_n := (f_n, a_n^Y, a_n^Z)$. Suppose that $v \in IV^X$ is a fixed vertex such that $\deg(v) = D$. Then the sequence $[f_n] \in \mathrm{rat}_D$ converges to the conjugacy classe $[f_v] \in \mathrm{rat}_D$.*

Lemma 2.11. *Let $(\mathcal{F}, \mathcal{T}^X)$ be dynamically approximable by $(\mathcal{F}_n)_n$. If $v \in IV(F^k)$ and if $w := F^k(v)$, then $(\phi_{n,w} \circ f_n^k \circ \phi_{n,v}^{-1})_n$ converges locally uniformly to f_v^k outside a finite set.*

To these results of [A1], we add the following corollary.

Corollary 2.12. *Let $(\mathcal{F}, \mathcal{T}^X)$ be dynamically approximable by $(\mathcal{F}_n)_n$. If $v \in IV(F^k)$ and if $F^k(v) \in B_w(e)$, then $(\phi_{n,w} \circ f_n^k \circ \phi_{n,v}^{-1})_n$ converges locally uniformly to the constant $i_w(e)$ outside a finite set.*

Proof. Indeed, we have

$$\phi_{n,w} \circ f_n^k \circ \phi_{n,v}^{-1} = (\phi_{n,w} \circ \phi_{n,F^k(v)}^{-1}) \circ (\phi_{n,F^k(v)} \circ f_n^k \circ \phi_{n,v}^{-1}).$$

According to lemma 2.11, the map on the right converges locally uniformly outside $Y(f^k)$ to a map whose image does not intersect $Z_{F^k(v)}$ and according to lemma 2.8, the map on the left converges locally uniformly to $i_w(e)$ outside $i_{F^k(v)}(w)$. $\qquad\square$

2.2 Branches and annuli

Given a tree T, the most natural open subgraphs to look at are the one defined in the following.

Definition 2.13 (Branch). *For v a vertex of a tree T and for $\star \in T - \{v\}$, a branch of $\star$ on v is the connected component of $T - \{v\}$ containing $\star$. It is denoted by $B_v(\star)$.*

Definition 2.14 (Annulus). *If v_1 and v_2 are two distinct internal vertices of T, the annulus $A :=]\!] v_1, v_2 [\![$ is the intersection of two branches $B_{v_1}(v_2)$ and $B_{v_2}(v_1)$. We define*

$$[\![v_1, v_2]\!] := \overline{A} := A \cup [v_1, v_2].$$

Note that $\overline{A} = A \cup \{v_1, v_2\}$. More generally, for every connected subset T' of the tree T, we denote by $\overline{T}'$ the smallest subtree of T containing T'. We proved in [A1] that given T'' an open, non empty and connected subset of T^Z and given T' a connected component of $F^{-1}(T'')$ there is a natural cover $\overline{\mathcal{F}} : \overline{\mathcal{T}}' \to \overline{\mathcal{T}}''$ defined by

- $\overline{F} := F : \overline{T}' \to \overline{T}''$ and

- $\overline{f}_v := f_v$ if $v \in V' - Y'$

which is a cover between trees of spheres. Recall the Riemann-Hurwitcz for trees proven in [A1]:

Proposition 2.15 (Riemann-Hurwitcz Formula). *Let $\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z$ be a cover between trees of spheres. Let T'' be a sub-tree of T^Z. Let T' be a connected component of $F^{-1}(T'')$. Then we have*

$$\chi_{T^Y}(T') = \deg(\mathcal{F}|_{\overline{\mathcal{T}}'}) \cdot \chi_{T^Z}(T'') - \sum_{y \in \mathrm{Crit} \mathcal{F} \cap T'} \mathrm{mult}\,(y).$$

From this formula we deduce the following useful corollaries.

Corollary 2.16. *If B is a branch of T^Y that does not contain any critical leaf, then its image $F(B)$ is a branch and $F : B \to F(B)$ is a bijection.*

Proof. Set $v \in V$ and e such that $B = B_v(e)$. Let T'' be the branch of $F(e)$ containing $F(v)$ and T' the component of $F^{-1}(T'')$ containing v. Then, T' is a sub-tree of B and it follows that $\mathrm{mult}\,T' = 0$. From the Riemann-Hurwitz formula,

$$\chi_{T^Y}(T') = \deg(F : T' \to T'') \cdot \chi_{T^Z}(T'') - \mathrm{mult}\,T' = \deg(F : T' \to T'') \cdot \chi_{T^Z}(T'').$$

It follows that $\deg(F : T' \to T'') = 1$ and $\chi_{T^Y}(T') = 1$. In particular, T' is a branch so $B = T'$. Moreover, $F(B) = T''$ and the degree of $F : B \to F(B)$ is equal to 1. $\qquad\square$

Using the same ideas we can find a formula for annuli.

Corollary 2.17. *If A is an annulus of T^Y that does not contain a critical leaf, then its image $F(A)$ is an annulus and A is a connected component of $F^{-1}(F(A))$.*
Moreover we have $F(\overline{A}) = \overline{F(A)}$. If in addition $A =]\!] v_1, v_2 [\![$ does not contain critical element, then $F : A \to F(A)$ and $\overline{F} : \overline{A} \to \overline{F(A)}$ are bijections.

Proof. Define $A = \rrbracket v_1, v_2 \llbracket$. If $A = \{\{v_1, v_2\}\}$ then the result follows directly from the definition of combinatorial trees maps. Suppose that it is not the case. Recall that $\overline{A}$ is the sub-tree of T^Y defined by adding v_1 and v_2 to the set of vertices of A.

Let e_1 and e_2 be two edges connecting v_1 and v_2 to the rest of $\overline{A}$. Let v be a vertex of $\overline{A}$, let T'' be the connected component of the graph $(V^Z, E^Z - \{F(e_1), F(e_2)\})$ containing $F(v)$ and T' be the component of $F^{-1}(T'')$ containing v. Then T' is a sub-tree of $\overline{A}$ and $\mathrm{mult}\,(T') = 0$. From the Riemann-Hurwitz formula

$$0 = \chi_{T^Y}(\overline{A}) \geq \chi_{T^Y}(T') = \deg(\mathcal{F} : \mathcal{T}' \to \mathcal{T}'') \cdot \chi_{T^Z}(T'').$$

The connected components of the graph $(V^Z, E^Z - \{F(e_1), F(e_2)\})$ has characteristic positive or equal to zero in T^Z. Then,

$$0 = \chi_{T^Y}(A') = \chi_{T^Y}(T') = \chi_{T^Z}(T'').$$

This proves that $\overline{A} = T'$ and $F(\overline{A}) = T''$.

Now suppose that A does not contain any critical vertex. Then the edges e_1 and e_2 have degree one and the map $\overline{\mathcal{F}} : \overline{\mathcal{T}}' \to \overline{\mathcal{T}}''$ has no critical leaves. So it has degree one. This proves that $F : \overline{A} \to F(\overline{A})$ is a bijection. In particular $F : A \to F(A)$ is a bijection.

Given that images of two adjacent vertices are adjacent vertices, the results about the adherence follow. $\square$

Corollary 2.18. *Let B be a branch on v in T^Y containing at most one critical leaf c. Then $F(B)$ is the branch on $F(v)$ attached at $a_{F(v)}(F(c))$.*

Proof. According to lemma 2.17, $F(\llbracket v, c \rrbracket) = \llbracket F(v), F(c) \rrbracket$ so in particular $F(B) = F(\rrbracket F(v), F(c) \llbracket) \cup \{F(c)\}$ which is a branch on $F(v)$. Otherwise the edge of B on v maps to the edge attached at $a_{F(v)}(F(c))$. $\square$

We remark that corollaries 2.16 and 2.17 can be proved by using the following lemma which is going to be very useful.

Lemma 2.19. *If $[v_1, v_2]$ is a path in T^Y having only vertices of degree one, then F is a bijection from it to $[F(v_1), F(v_2)]$.*

Proof. Let T' be the sub-tree such $[v_1, v_2]$; then $F(T')$ is a sub-tree of T^Y. Take one of its leaves. There exists $v \in T$ such that it is $F(v)$. Suppose that $v \neq v_1$ and $v \neq v_2$. Then the two edges of v in T' map to the unique edge of $F(v) \in F(T')$ and as an attaching point of an edge maps to the attaching point of the image of this edge, there is a point of $\mathcal{S}_{F(v)}$ that has two preimages. This contradicts the fact that vertices of T' have degree one.

So $F(T')$ is a subtree with at most two leaves. It follows that $F(T')$ is constituted of all the vertices of $[v_1, v_2]$. $\square$

3 Necessary conditions for approximability

In this section we look at the properties of a dynamical system between trees of spheres which are dynamically approximable by a sequence of dynamically marked rational maps.

3.1 Branches lemma

In this section, we are interested in some properties of branches of (stable) covers between trees of spheres which are approximable by a sequence of dynamical systems of spheres. We prove the following result :

Lemma 3.1 (Branches). *Let $(\mathcal{F}, \mathcal{T}^X) \in \partial \mathrm{Rat}_{\mathbf{F}, X}$, let v be a periodic internal vertex, let $a_0 \in \mathbb{S}_v$ and B be a branch on v such that for all $k \in \mathbb{N}$, the branch of T^Y attached to $f^k(a_0)$ maps inside the branch on T^Z attached to $f^{k+1}(a_0)$. Then*

- *B does not contain critical periodic vertex ;*

- *if B contains a periodic internal vertex then its cycle has degree 1 and a_0 is periodic.*

We first prove the general following lemma :

Lemma 3.2. *If a branch B on a vertex v maps to a branch, and if d is the degree of the attaching point of the edge of B on v, then the number of critical leaves in B, counting multiplicities, is $d - 1$.*

Proof. We apply the Riemann-Hurwitz formula to B then to $\overline{B}$ and the result follows directly. $\qquad \square$

We are now ready to prove the branches lemma.
Proof. [lemma 3.1] We denote by $B_k^\star$ the branch attached to $f^k(a_0)$ in $T^\star$. Let v' be a periodic vertex in B_0^X. Define $v_k := F^k(v)$ and $v'_k := F^k(v')$. As the iterates of v' are in the $B_k^\star$, a_0 orbit under f is periodic.

Suppose that a_0 is periodic with period k_0 and that the vertex v' has period k' multiple of k_0. Then v_0 and v'_0 lie in Z, we find z and $z' \in Z$ such that the path $[z, z']^Z$ passes through v_0 and v'_0 in this order. For $\star \in \{v_0, v'_0\} \subset T^Z$, we gives projective charts $\sigma_\star$ such that

$$\sigma_\star \circ a_\star(z) = \infty \quad \text{and} \quad \sigma_\star \circ a_\star(z') = 0.$$

Take ϕ_n^Y and ϕ_n^Z like in definition 2.7. After post-composing the isomorphisms $\phi_{n,\star}^Z$ by automorphisms of $\mathbb{S}_\star$ tending to identity when $n \to \infty$, we can suppose that

$$\phi_{n,\star}^Z \circ a_n(z) = a_\star(z) \quad \text{and} \quad \phi_{n,\star}^Z \circ a_n(z') = a_\star(z').$$

Define the projective charts on $\mathbb{S}_n$ by $\sigma_{n,\star} := \sigma_\star \circ \phi_{n,\star}^Z$.

The changes of coordinates $\sigma_{n,v'} \circ \sigma_{n,v}^{-1}$ fixe 0 and ∞. So they are similitudes centered on 0. We define λ_n by $\sigma_{n,v'} = \lambda_n \sigma_{n,v}$. The vertices v and v' have at least three edges, so lemma 2.8 assures that $\lambda_n \to \infty$.

Let $D \subset \mathbb{S}_v$ be a disk containing $a_v(z')$. Given that $f|_{\Sigma^x}$ is continuous, we can suppose that D is sufficiently small such that the set of its k' iterates under f contains at most a unique attaching point which is the iterate of a_0.

Define $D_n := (\phi_{n,v}^Z)^{-1}(D)$. Now we show that in the chart $\sigma_{n,v}$, we can take n large enough such that the map $f_n^{k'}$ has no poles on the disk D_n. For all $k > 0$, given that $F(B_{k-1}^Y) \subseteq B_k^Z$, then for n large enough the disk $D_{n,k}$ doesn't intersect edges attaching points of elements $z \in Z - B_k^Z$. We have $\phi_{n,v}^Z(\partial D_{n,k}) \to \partial D_k$, so from the maximum modulus principle we conclude that $\phi_{n,v}^Z(D_{n,k}) \to D_k$. By definition of D, $D_0 = D$ does not contain poles of $f^{k'}$ so D_n does not contain poles of $f_n^{k'}$.

In the charts $\sigma_{n,v}$ and $\sigma_{n,v'}$, we can develop the map $f_n^{k'} : D_n \to \mathbb{S}_n - \{a_n(z)\}$ as a power series in $a_n(z')$:

$$\sigma_{n,v} \circ f_n^{k'} = \sum_{j \in \mathbb{N}} c_{n,j} \cdot \sigma_{n,v}^j \quad \text{and} \quad \sigma_{n,v'} \circ f_n^{k'} = \sum_{j \in \mathbb{N}} c_{n,j}' \cdot \sigma_{n,v'}^j.$$

We have $\sigma_{n,v'} = \lambda_n \sigma_{n,v}$, so

$$c_{n,j}' = \lambda_n^{i-j} c_{n,j}.$$

Considering the development in Laurent series of $f^{k'} : \mathbb{S}_v \to \mathbb{S}_v$ in the neighborhood of $a_0 = a_v(z')$ in the chart σ_v and the one of $f^{k'} : \mathbb{S}_{v'} \to \mathbb{S}_{v'}$ in the neighborhood of $a_{v'}(z')$ in the chart $\sigma_{v'}$,

$$\sigma_v \circ f^{k'} = \sum_{j \in \mathbb{Z}} c_j \cdot \sigma_v^j \quad \text{and} \quad \sigma_{v'} \circ f^{k'} = \sum_{j \in \mathbb{Z}} c_j' \cdot \sigma_{v'}^j.$$

If $F^k(v) = v$ then lemma 2.11 assures that $\phi_{n,v} \circ f_n^k \circ \phi_{n,v}^{-1}$ converges locally uniformly to $\sigma_v \circ f^k \circ \sigma_v^{-1}$ on $D - \{a_v(z')\}$ so uniformly on D by the maximum principle, given that these maps have no poles in D. In the chart σ_v, we then have the convergence $c_{n,j} \xrightarrow[n \to \infty]{} c_j$. If $F^k(v) \neq v$, lemma 2.12 allows to conclude that $c_{n,j} \xrightarrow[n \to \infty]{} 0 = c_j$. Likewise, we have the convergence $c_{n,j}' \xrightarrow[n \to \infty]{} c_j'$. In particular, as $c_{n,j}' = 0$ for $j < 0$, we have $c_j' = 0$ and $c_j = 0$ (we recovered the fact that $f^{k'}$ fixes a_0).

If $F^k(v) \neq v$, all the coefficients c_j are zero, so coefficients of c_j' are zero : contradiction.

We conclude that $F^k(v) = v$. If we denote by d the local degree of $f^{k'}$ at a_0, then $c_j = 0$ for $j < d$ and $c_d \neq 0$. If $n \to \infty$ on $c_{n,j}' = \lambda_n^{1-j} c_{n,j}$, then $\lambda_n \to \infty$, and we have $c_1' = c_1$ and $c_j' = 0$ if $j > 1$. thus, if $d > 1$, the coefficients c_j' are again zero : contradiction. So the only possible case is $d = 1$. But d is the product of the d_k, where d_k is the degree of the attaching point of the edge of B_k^Y. Thus, if $d = 1$, then all the d_k are 1, then all the branches B_k^Y don't contain critical vertices according to lemma 3.2 applied with $d = 1$. Thus $f^{k'}$ has degree 1, then the map $f^{k'} : \mathbb{S}_{v'} \to \mathbb{S}_{v'}$ has degree 1 and this is absurd. $\square$

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3.2 Lemmas about annuli

In this part we continue the study of covers between trees of spheres which are approximable by a sequence of dynamical systems of marked spheres. We try to pass to the limit some usual properties on annuli. Recall that according to corollary 2.17, an annulus that does not contain critical leaves has a well defined degree. We prove the following lemma.

Lemma 3.3 (Annuli). *Suppose that $(\mathcal{F}, \mathcal{T}^X) \in \partial\mathrm{Rat}_{\mathbf{F},X}$ and that v and v' are distinct internal vertices of $\mathcal{T}^Y$ such that for $0 \leq k \leq k_0 - 1$ the annulus $]\!]F^k(v), F^k(v')[\![^Y$ is defined and does not contain any critical leaf.*

- *(Critical) If $\mathcal{F}$ has degree more than 1 on one of these $]\!]F^k(v), F^k(v')[\![^Y$, we never have $[v_{k_0}, v'_{k_0}] \subseteq [v, v']$.*

- *(Non critical) If this is not the case and if $[v_{k_0}, v'_{k_0}] \subseteq [v, v']$ or $[v, v'] \subseteq [v_{k_0}, v'_{k_0}]$ we have*

  A) *either $v_{k_0} = v$ and $v'_{k_0} = v'$, then $i_v(v')$ and $i_{v'}(v)$ are fixed by f^{k_0} and the product of the associated multipliers is 1;*

  B) *or $v_{k_0} = v'$ and $v'_{k_0} = v$, then f^{k_0} exchanges $i_v(v')$ and $i_{v'}(v)$ and the multiplier of the associated cycle is 1.*

Note that figure 1 show an example of a dynamical tree of sphere cover that does not satisfy the conclusion of the critical annulus lemma (3.3): the critical annulus between the full red and full black vertices has period 8.

Before proving this result we must note an interesting and open question:

Question 3.4. *Those two lemmas together with the branches lemma give some necessary conditions in order to be approximable by a sequence of dynamical systems of spheres. But are they sufficient conditions?*

The rest of this section is dedicated to the proof of this lemma.

We suppose that $(\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z, \mathcal{T}^X)$ is a trees of spheres dynamical system dynamically approximable by the sequence of dynamical systems between marked spheres (f_n, a_n^Y, a_n^Z).

Choice of coordinates. Define $w := v_{k_0}$ and $w' := v'_{k_0}$. Suppose that $[w, w']^Z \subseteq [v, v']^Z$ or that $[v, v']^Z \subseteq [w, w']^Z$. In the not critical case, we have to prove that $[w, w']^Z = [v, v']^Z$ and in the critical case that $[w, w']^Z \not\subseteq [v, v']^Z$.

Let $z, z' \in Z$ be such that the path $[z, z']^Z$ goes through v and v' in this order and through w and w', not necessarily in this order. For every internal vertex $\star$ on the path $[z, z']^Z$, we give projective charts $\sigma_\star$ such that

$$\sigma_\star \circ a_\star(z) = 0 \quad \text{and} \quad \sigma_\star \circ a_\star(z') = \infty.$$

Take ϕ_n^Y and ϕ_n^Z like in definition 2.7. Then, after post-composing the isomorphisms $\phi_{n,\star}^Z$ by automorphisms of $\mathbb{S}_\star$ tending to identity when $n \to \infty$, we can suppose that for every internal vertex $\star$ on the path $[z, z']$ we have

$$\phi_{n,\star}^Z \circ a_n(z) = a_\star(z) \quad \text{and} \quad \phi_{n,\star}^Z \circ a_n(z') = a_\star(z').$$
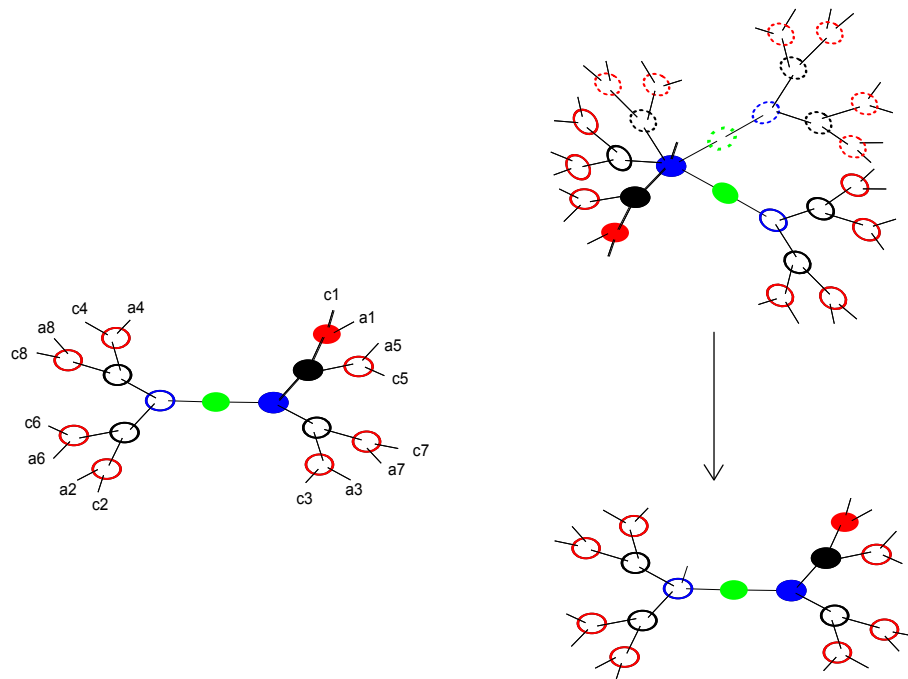
Figure 1: An example of dynamical system of degree 2 that does not satisfy the conclusion of theorem 4.1. To the left $\mathcal{T}^X$, to the right upstair $\mathcal{T}^Y$ and downstair $\mathcal{T}^Z$. The leaves c_i the period 8 cycle of the critical point. The vertices a_i are a cycle of same period. On $\mathcal{T}^X$, the full red (resp. full black, full blue) vertex is critical and of period 8 (resp. 4, 2) which orbit is the set of red (resp. black, blue) vertices. The full green vertex is fixed.

We then define projective charts on $\mathbb{S}_n$ by $\sigma_{n,\star} := \sigma_\star \circ \phi_{n,\star}^Z$.

The changes of coordinate maps fixe 0 and ∞ so they are similitudes centered on 0. We define λ_n, μ_n, ρ_n and ρ'_n by (cf figure 2):

$$\sigma_{n,v'} = \lambda_n \sigma_{n,v} \text{ and } \sigma_{n,w'} = \mu_n \sigma_{n,w}$$
$$\sigma_{n,v} = \rho_n \sigma_{n,w} \text{ and } \sigma_{n,v'} = \rho'_n \sigma_{n,w'}.$$

Note that as the vertices v, w, v' and w' have at least three edges, lemma 2.8 impose the behavior of λ_n, μ_n, ρ_n and ρ'_n according to the relative positions of the vertices.
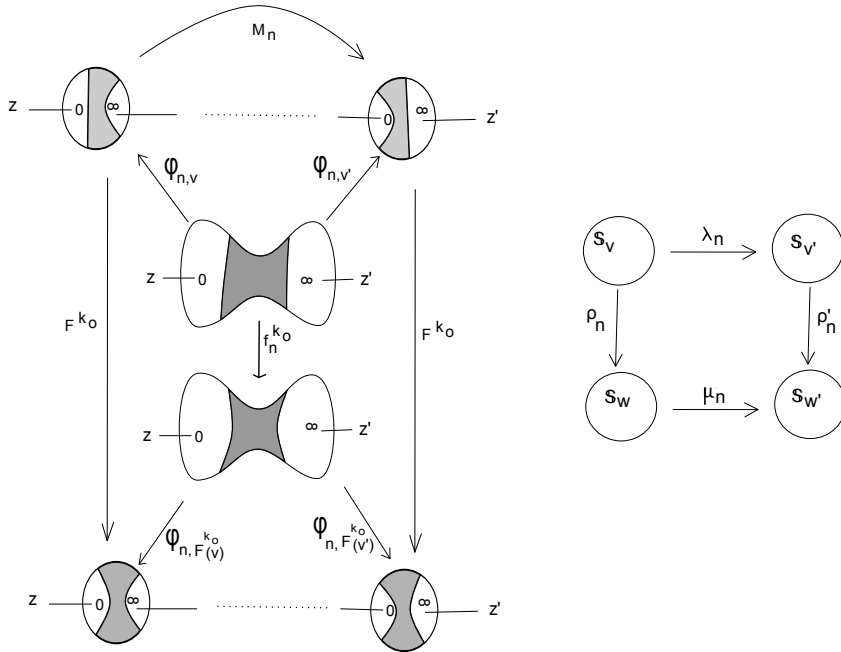

Figure 2: A simplified representation of notations in the proof of lemma 3.3.

Behavior of annuli. (cf figure 2) Recall that $Y(f^{k_0})$ is the set of points of Σ_{k_0} for which the image by an iterate of f^k with $k \in [1, k_0]$ is the attaching point of an edge in $\mathcal{T}^Z$. This is a finite set containing $i_v(v')$ and $i_{v'}(v)$.

Let $D \subset \mathbb{S}_v$ (respectively $D' \subset \mathbb{S}_{v'}$ be a disk containing $i_v(v')$ (respectively $i'_v(v)$) small enough such that its adherence does not contain any other point of $Y(f^{k_0})$ than $i_v(v')$ (respectively $i_{v'}(v)$) and that it does not contain $a_v(z)$ (resp. $a_{v'}(z')$). Define

$$A_n := (\phi_{n,v}^Y)^{-1}(D) \cap (\phi_{n,v'}^Y)^{-1}(D').$$

We are going to prove the following assertions :

1. for n large enough, A_n is an annulus contained in $\mathbb{S}_n - \{a_n(z), a_n(z')\}$;

2. $f_n^{k_0}(A_n)$ is contained in $\mathbb{S}_n - \{a_n(z), a_n(z')\}$;

3. Every compact of $D - \{i_v(v')\}$ is included in $\phi_{n,v}(A_n)$ for n large enough and $\phi_{n,w} \circ f_n^{k_0} \circ \phi_{n,v}^{-1}$ converges locally uniformly to f^{k_0} in $D - \{i_v(v')\}$;

4. Every compact sub-set of $D' - \{i_{v'}(v)\}$ is contained in $\phi_{n,v'}(A_n)$ for n large enough and $\phi_{n,w'} \circ f_n^{k_0} \circ \phi_{n,v'}^{-1}$ converges locally uniformly to f^{k_0} in $D' - \{i_{v'}(v)\}$.

Point 1. Define $M_n := \phi_{n,v'} \circ \phi_{n,v}^{-1}$. According to lemma 2.12, for n large enough, $M_n(\partial D)$ is contained in a neighborhood of $i_{v'}(v)$ that doesn't intersect $\partial D'$. Then, A_n is an annulus. when $n \to \infty$, $\phi_{n,v} \circ a_n(z)$ converges to $a_v(z)$ which is not in the adherence of D according to hypothesis. Consequently, for n large enough, D_n does not contain $a_n(z)$. Likewise, for n large enough, D_n' does not contain $a_n(z_n')$. Then, $A_n \subset \mathbb{S}_n - \{a_n(z), a_n'(z)\}$.

Point 2. Let $D_1 \subset \mathbb{S}_{v_1}$ be a disk containing $f(D)$ but no other attaching point of any edge of $\mathcal{T}^Z$ than $f(i_v(v'))$. Let $D_1' \subset \mathbb{S}_{v_1'}$ be a disk containing $f(D')$ but no other attaching point of any edge of $\mathcal{T}^Z$ than $f(i_{v'}(v))$. Just like in the point 1, for n large enough,

$$A_{1,n} := \phi_{n,v_1}(D_1) \cap \phi_{n,v_1'}^{-1}(D_1')$$

is an annulus.

Given that $\phi_{n,v_1} \circ f_n \circ \phi_{n,v}^{-1}$ converges uniformly to f in the neighborhood of ∂D and that $\phi_{n,v_1'} \circ f_n \circ \phi_{n,v'}^{-1}$ converges uniformly to f in the neighborhood of $\partial D'$, for n large enough we have $f_n(\partial A_n) \subset A_{1,n}$.

Like for the point 1, for n large enough, A_n doesn't intersect $a_n(Y -]\!]v, v'[\![)$ so, $f_n(A_n)$ doesn't intersect $a_n(Z -]\!]v_1, v_1'[\![)$. Likewise, for n large enough, $A_{1,n}$ doesn't intersect $a_n(Z -]\!]v_1, v_1'[\![)$.

In conclusion, $f_n(\partial A_n) \subset A_{1,n}$ and $f_n(A_n)$ doesn't intersect at least one point in each component of $A_{1,n}$ complementary. It follows from the maximum modulus principle that $f_n(A_n) \subset A_{1,n}$.

For n large enough, $D - a_v(v')$ (resp. $D' - a_{v'}(v)$) doesn't intersect the set $a_v(Y(f^{k_0}) -]\!]v, v'[\![)$ (resp. the set $a_{v'}(Y(f^{k_0}) -]\!]v, v'[\![)$), so $D_1 - a_{F(v)}(F(v'))$ (resp. $D_1' - a_{F(v')}(F(v))$) doesn't intersect $a_{F(v)}(Y(f^{k_0-1}) -]\!]F(v), F(v')[\![)$ (resp $a_{F(v')}(Y(f^{k_0-1}) -]\!]F(v), F(v')[\![)$). Thus we can do the same if we replace D and D' by D_1 and D_1', the vertices v and v' by v_1 and v_1' and iterate this $k_0 - 1$ times. This proves that $f_n^{k_0}(A_n)$ doesn't intersect $a_n(Z -]\!]v_{k_0}, v_{k_0}'[\![)$, in particular $a_n(z)$ and $a_n(z')$.

Points 3 and 4. These assertions follow from lemma 2.11.

Developing in Laurent series and convergence.

In the charts from $\sigma_{n,v}$ to $\sigma_{n,w}$, the map $f_n^{k_0} : A_n \to \mathbb{S}_n - \{a_n(z), a_n(z')\}$ has a Laurent series development:

$$\sigma_{n,w} \circ f_n^{k_0} = \sum_{j \in \mathbb{Z}} c_{n,j} \cdot \sigma_{n,v}^j.$$

In the charts from $\sigma_{n,v'}$ to $\sigma_{n,w'}$, this Laurent series development become

$$\sigma_{n,w'} \circ f_n^{k_0} = \sum_{j \in \mathbb{Z}} c'_{n,j} \cdot \sigma_{n,v'}^j.$$

As we have

$$\sigma_{n,v'} = \lambda_n \sigma_{n,v} \quad \text{and} \quad \sigma_{n,w'} = \mu_n \sigma_{n,w},$$

it follows that

$$c'_{n,j} = \frac{\mu_n}{\lambda_n^j} c_{n,j}.$$

Now consider the Laurent series development of $f^{k_0} : \mathbb{S}_v \to \mathbb{S}_w$ in the neighborhood of $a_v z$ in the charts form σ_v to σ_w and $f^{k_0} : \mathbb{S}_{v'} \to \mathbb{S}_{w'}$ development in the neighborhood of $a_{v'}(z')$ in the charts from $\sigma_{v'}$ to $\sigma_{w'}$

$$\sigma_w \circ f^{k_0} = \sum_{j \in \mathbb{Z}} c_j \cdot \sigma_v^j \quad , \quad \sigma_{w'} \circ f^{k_0} = \sum_{j \in \mathbb{Z}} c'_j \cdot \sigma_{v'}^j.$$

Convergence. Here we prove that

$$c_{n,j} \xrightarrow[n \to \infty]{} c_j \text{ and } c'_{n,j} \xrightarrow[n \to \infty]{} c'_j.$$

As $\mathcal{F}_n \to \mathcal{F}$ and for all $0 \le k < k_0$, $\phi_{n,v_k}^{-1}(\partial D_{k,n})$ doesn't intersect Z_v, we have

$$\sigma_{v_{k+1}} \circ \left(\phi_{n,v_{k+1}} \circ f_n \circ \phi_{n,v_k}^{-1} \right) \circ \sigma_{v_k}^{-1} \to \sigma_{v_{k+1}} \circ f_{v_k} \circ \sigma_{v_k}^{-1}$$

uniformly on $\sigma_{v_k}^{-1}(\phi_{n,v_k}^{-1}(\partial D_{k,n}))$. So by composition, we have

$$\sigma_w \circ \left(\phi_{n,w} \circ f_n^{k_0} \circ \phi_{n,v}^{-1} \right) \circ \sigma_v^{-1} \to \sigma_w \circ f^{k_0} \circ \sigma_v^{-1}$$

uniformly on $\sigma_v^{-1}(\partial D)$.

Otherwise, we have

$$(\sigma_w \circ \phi_{n,w}) \circ f_n^{k_0} \circ (\phi_{n,v}^{-1} \circ \sigma_v^{-1}) = \sigma_{n,w} \circ f_n^{k_0} \circ \sigma_{n,v}^{-1}.$$

The uniform convergence implies the uniform convergence of the coefficients of Laurent series so we have $c_{n,j} \xrightarrow[n \to \infty]{} c_j$. The proof is the same for $c'_{n,j} \xrightarrow[n \to \infty]{} c'_j$.

Conclusions and multipliers.

Case A. Suppose that the path connecting z to z' in T^Z goes through w and w' in this order. On one hand, we have $\sigma_{n,w} = \rho_n \sigma_{n,v}$ with $\rho_n \in \mathbb{C} - \{0\}$ and according to lemma 2.8:

- $\rho_n \to 0$ if and only if the path connecting z to z' goes through v before going through w and

- $\rho_n \to \infty$ if and only if the path connecting z to z' goes through w before going through v.

Likewise, we have $\sigma_{n,w'} = \rho'_n \sigma_{n,v'}$ with $\rho'_n \in \mathbb{C} - \{0\}$ and

- $\rho'_n \to 0$ if and only if the path de x à x' goes through v' before going through w' and

- $\rho'_n \to \infty$ if and only if the path connecting x to x' goes through w' before going through v'.

Note that
$$\rho_n \mu_n \sigma_{n,v} = \mu_n \sigma_{n,w} = \sigma_{n,w'} = \rho'_n \sigma_{n,v'} = \rho'_n \lambda_n \sigma_{n,v}$$

so
$$\rho'_n = \frac{\mu_n}{\lambda_n} \rho_n.$$

On the other hand, the development of $f^{k_0} : \mathbb{S}_v \to \mathbb{S}_w$ in the neighborhood of z in the charts from σ_v to σ_w is the one of a function defined in the neighborhood of infinity and mapping infinity to infinity with local degree d_0. Consequently, $c_j = 0$ if $j \geq d_0 + 1$ and $c_{d_0} \neq 0$. Likewise, the development of $f^{k_0} : \mathbb{S}_{v'} \to \mathbb{S}_{w'}$ in the neighborhood of z' in the charts from $\sigma_{v'}$ to $\sigma_{w'}$ is the one of a function defined in the neighborhood of 0 mapping 0 to 0 with local degree d_0. Consequently, $c'_j = 0$ if $j \leq d_0 - 1$ and $c'_{d_0} \neq 0$.

Under hypothesis of lemma 3.3, as $d_0 = 1$

$$c'_{n,1} = \frac{\mu_n}{\lambda_n^1} c_{n,1}, \text{ so } \frac{\mu_n}{\lambda_n} \xrightarrow{n \to \infty} \frac{c'_1}{c_1} \in \mathbb{C} - \{0\}.$$

It follows that $\rho_n \to 0$ if and only if $\rho'_n \to 0$. Likewise, $\rho_n \to \infty$ if and only if $\rho'_n \to \infty$. Given that $[w,w']^Z \subseteq [v,v']^Z$ or $[v,v']^Z \subseteq [w,w']^Z$, the only possibility is that ρ_n and ρ'_n tend neither to 0 or to infinity, which implies that $v = w$ and $v' = w'$. Then we can chose $\sigma_v = \sigma_w$ and $\sigma_{v'} = \sigma_{w'}$, which implies that $\lambda_n = \mu_n$ so $c_1 = c'_1$. The multiplier of f^{k_0} at z is $1/c_1$ and the multiplier of f^{k_0} at z' is c'_1. The multiplier product is 1 as required.

Under hypothesis of lemma 3.3, we have $d_0 > 1$ so

$$c'_{n,d_0} = \frac{\mu_n}{\lambda_n^{d_0}} c_{n,d_0} \text{ so } \frac{\rho'_n}{\rho_n} \approx \lambda_n^{d_0-1} \left(\frac{c'_{d_0}}{c_{d_0}} \right).$$

Thus if we suppose that $w \in [v,v']$, ie $\rho_n \to \mathbb{C}^\star \cup \{\infty\}$, we can deduce that $\rho'_n \to \infty$, ie $w' \notin [v,v']$. The case $w' \in [v,v']$ is symetric.

Case B. Suppose that the path connecting z to z' in T^Z goes through w' and w in this order. On one hand, we have $\sigma_{n,w} = \rho_n \sigma_{n,v'}$ with $\rho_n \in \mathbb{C} - \{0\}$. Likewise, we have $\sigma_{n,w'} = \rho'_n \sigma_{n,v}$ with $\rho'_n \in \mathbb{C} - \{0\}$. Note that

$$\rho'_n \sigma_{n,v} = \sigma_{n,w'} = \mu_n \sigma_{n,w} = \mu_n \rho_n \sigma_{n,v'} = \mu_n \rho_n \lambda_n \sigma_{n,v}$$

so
$$\rho'_n = \mu_n \rho_n \lambda_n.$$

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On the other hand, the development of $f^{k_0} : \mathbb{S}_v \to \mathbb{S}_w$ in the neighborhood of z in the charts from σ_v to σ_w is the one of a function defined in the neighborhood of infinity and that maps infinity to 0 with local degree d_0. Consequently, $c_j = 0$ if $j \geq d_0 - 1$ and $c_{-d_0} \neq 0$. Likewise, the development of $f^{k_0} : \mathbb{S}_{v'} \to \mathbb{S}_{w'}$ in the neighborhood of z' in the charts from $\sigma_{v'}$ to $\sigma_{w'}$ is the one of a function defined in the neighborhood of 0 and that maps 0 on infinity with local degree d_0. Consequently, $c'_j = 0$ if $j \leq -(d_0 + 1)$ and $c'_{-d_0} \neq 0$.

Under hypothesis of lemma 3.3, as $d_0 = 1$ we have

$$c'_{-1} \underset{n \to \infty}{\longleftarrow} c'_{n,-1} = \mu_n \lambda_n^{-1} c_{n,-1} \underset{n \to \infty}{\longrightarrow} c_{-1},$$

so

$$\mu_n \lambda_n \underset{n \to \infty}{\longrightarrow} \frac{c'_{-1}}{c_{-1}} \in \mathbb{C} - \{0\}.$$

Then $\rho_n \to 0$ if and only if $\rho'_n \to 0$. Likewise, $\rho_n \to \infty$ if and only if $\rho'_n \to \infty$. given that $[w, w']^Z \subseteq [v, v']^Z$ or $[v, v']^Z \subseteq [w, w']^Z$, the only possibility is that ρ_n and ρ'_n tend neither to 0 or infinity, which according to lemma 2.8 implies that $v = w'$ and $v' = w$. Then we can chose $\sigma_v = \sigma_{w'}$ and $\sigma_{v'} = \sigma_w$, which implies that $\lambda_n \mu_n = 1$ so the cycle multiplier, ie c'_{-1}/c_{-1} is 1 as required.

Under hypothesis of lemma 3.3, as $d_0 > 1$ we have

$$\mu_n \lambda_n = \lambda^{d_0 - 1} \left(\frac{c'_{n,-d_0}}{c_{-d_0}} \right) \text{ so } \frac{\rho'_n}{\rho_n} \approx \lambda_n^{d_0 - 1} \left(\frac{c'_{d_0}}{c_{d_0}} \right).$$

Thus supposing that $w \in [v, v']$, ie $\rho_n \to \mathbb{C}^\star \cup \{\infty\}$, as $\lambda_n^{d_0 - 1} \to \infty$ then we have $\rho'_n \to \infty$, ie $w' \notin [v, v']$. The case $w' \in [v, v']$ is symetric.

4 Bi-critical case

We now want to study rescaling-limits in the case of rational maps of degree d with exactly two critical points (including for example the case of degree two rational maps). In this case the critical points have exactly multiplicity $d - 1$. Such maps are said bicritical.

In this subsection we prove the following theorem and conclude with the proof of Theorem B:

Theorem 4.1 (Classification). *Let $\mathbf{F}$ be a portrait of degree d with $d + 1$ fixed points and exactly 2 critical points and let $(\mathcal{F}, \mathcal{T}^X) \in \partial \mathrm{Rat}_{\mathbf{F}, X}$. Suppose that there exists $(f_n, y_n, z_n) \overset{\lhd}{\longrightarrow} \mathcal{F}$ in $\mathrm{Rat}_{\mathbf{F}, X}$ such that for all n, $x_n(X)$ contains all the fixed points of f_n. Then the map $\mathcal{F}$ has at most two critical cycles of spheres; they have degree d.*

Assume that there exits at least one rescaling limit. Then there is a vertex w_0 separating three fixed points which is fixed and such that f_{w_0} has finite order $k_0 > 1$. Denote by v_0 the critical vertex separating w_0 and the two critical leaves.

1. Either v_0 belongs to a critical cycle of period k_0 and

(a) its associated cover has a parabolic fixed point;

(b) if there is a second critical cycle then it has period $k_0' > k_0$, its associated cover has a critical fixed point with local degree d and the one associated to v_0 has a critical point that eventually maps to the parabolic fixed point.

2. *Or v_0 is forgotten by F^k with $k < k_0$; in this case there is exactly one critical cycle; it has period $k_0' > k_0$ and its associated cover has a critical fixed point with local degree d.*

Our interest for these maps comes from the following lemma. That makes this case easier to understand.

Lemma 4.2. *Let $\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z$ be a cover between trees of spheres. Every critical vertex lies in a path connecting two critical leaves. Each vertex on this path is critical.*

Proof.   Let v be a critical vertex of $\mathcal{F}$. Then f_v has at least two distinct critical points. There are at least two distinct edges attached to v. So v is on a path of critical vertices.

Let $[v_1, v_2]$ be such a path with a maximal number of vertices. From this maximality property, we see that there is only one critical edge (edge with degree strictly greater than one) attached to v_1. If v_1 is not a leaf then f_{v_1} has just one critical point and that is not possible. So v_1 is a leaf. As well, v_2 is a leaf. $\qquad\square$

Notation.   We will denote by c and c' the two critical leaves. Then according to lemma 4.2 the critical vertices are the vertices of $C_d := [c, c']$. We will use the notation $c_k := F^k(c)$ and $c_k' := F^k(c')$ when they are defined.

The fixed vertex w_0 in Theorem 4.1.

We will call a principal branch every branch attached to a critical vertex in $\mathcal{T}^Y$ with vertices of degree one and containing a fixed leaf.

If v_0 is critical then every branch containing a fixed leave maps bijectively to its image?

First prove that there are two fixed leaves lying in the same principal branch and that the vertex separating them and C_d is fixed.

Case1. At least two fixed leaves are not critical.

Case1a. Two non critical leaves α and β lie in the same principal branch. We will see that it is the only non absurd case.

Denote by w_0 the vertex separating α, β and a critical leave. Then w_0 is not critical and both of the critical leaves lie in the same branch on it. Let v_0 be the vertex separating w_0 and the two critical leaves. Then $A := [w_0, v_0] \cup C_d$ maps bijectively to its image which is inside a branch B on $F(w_0)$. The iterates of w_0 lie in $]\alpha, \beta[$. Suppose that w_0 is not fixed and for example that w_0, $F(w_0)$ and β lie on $]\alpha, \beta[$ in this order. Then the only pre-image of $a_{F(w_0)}(\alpha)$ is $a_{w_0}(\alpha)$. It follows that $F(B_{w_0}(\beta)) \subset B_{w_0}(\beta)$ and that $B \subset B_{w_0}(\beta)$ so B cannot contain any critical periodic vertex. Thus $F(w_0) = w_0$.

As $w_0 \in [\alpha, \beta]$, it follows that f_{w_0} is conjugated to a rotation. If it has not finite order, then the iterates of B are branches attached at the iterates of $f_{w_o}(a_{w_0}(v_0))$ which are all distinct and contains the iterates of C_d which is absurd. So w_0 has finite order.

Case1b. All the critical leaves lie in different principal branches. Let α and β be two critical leaves. Then there exists a critical vertex w_0 in $]\alpha, \beta[$, the closest to α. The vertex w_0 is not fixed. We can prove that there exists a fixed leave γ such that $B_{F(w_0)}(\gamma) \subset B_{w_0}(\gamma)$. Indeed, suppose that there is not such a γ. For every fixed leave δ, the point $a_{w_0}(\delta)$ is a preimage of $a_{F(w_0)}(v_0)$. Either one of the fixed leave c is critical so the $a_{w_0}(c)$ the $a_{w_0}(\alpha)(\neq a_{w_0}(c))$ would be $d+1$ preimage of $a_{F(w_0)}(v_0)$ counting with multiplicities. Or all the fixed leaves lie in different branches on w_0; thus as f_{w_0} is the uniform limit of $\phi_{n,F(w_0)}^{-1} \circ f_n \circ \phi_{n,w_0}$ in the neighborhood of the $a_{w_0}(\delta)$, the the $a_{w_0}(\delta)$ would be again exactly $d+1$ preimage of $a_{F(w_0)}(v_0)$ counting with multiplicities. In both cases we conclude a contradiction.

Define $B_0 := B_{w_0}(\gamma)$, we have $F(B_0) \subset B_0$. Then $F(C_d) \subset B_0$ because if not c_0 is a critical leave then

$$f(a_{w_0}(c_0)) = a_{F(w_0)}(w_0) = F(a_{w_0}(\alpha))$$

so $a_{F(w_0)}(w_0)$ would have at least $d+1$ preimages counted with multiplicities which is absurd. It follows that every periodic vertex lies in B_0.

Suppose that v is a periodic critical internal vertex. If $F(w_0) \notin C_d$ then $v \notin B_0$ but $F(v) \in B_0$. But $F(B_0) \subset B_0$ so v cannot be periodic which is absurd. Thus $F(w_0) \in C_d$ and it follows that $B_v(F(v))$ maps to a branch on $F(v)$ so maps inside itself. But $F(v) \neq v$ so v cannot be periodic which is absurd.

Case2. There is just one non critical fixed leave α. Then the vertex separating α and the two critical leaves is fixed and has maximal degree so, according to lemma 2.10, the approximating sequence converges in rat_d. Absurd.

Case3. All fixed leaves are critical. Then C_d maps bijectively to itself. Denote by v the periodic critical internal vertex. As the approximating sequence does not converge in rat_d then v is not fixed. From the critical annulus lemma (3.3) $F^2(v) \notin [v, F(v)]$. As C_d maps bijectively to itself the order of the vertices $v, F(v)$ and $F^2(v)$ has to be preserved so v cannot be periodic. This is absurd.

Conclusion. We are in Case1a and this conclude the proof of this point.

Notation. The vertex w_0 has a finite order that we denote by k_0. The vertex w_0 is not critical, so it doesn't lie in C_d. We denote by $v_0 \in T^X$ the vertex separating c, c' and w_0.

Remark 4.3. Suppose that v' is a critical periodic internal vertex. Denote by B_0^Y the branch on w_0 containing the two critical points (so v' too) and for $1 \leq k \leq k_0$, denote by $B_k^\star \in IV^\star$ the branch attached to $a_{w_0}^\star(F^k(v'))$. We have $F : B_k^Y \to B_{k+1}^Z$ is a bijection.

Point 1a of Theorem 4.1. Here we suppose that v_0 is not forgotten by F_{k_0}. The annulus $A_0 :=]w_0, v_0[$ does not contain vertices of degree d so from
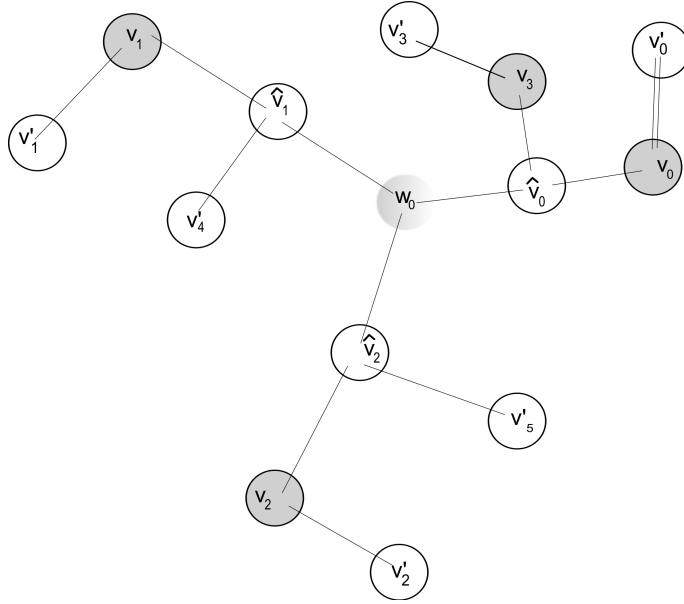
Figure 3: Example of the absurd case of the proof of point 1 with $k_0 = 3$

corollary 2.17 we know that F is injective on $\overline{A}_0$. Moreover, elements of C_d have maximum degree, so F is injective on $A_0 \cup C_d$. According to remark 4.3, F is bijective from B_k^Y to B_{k+1}^Z for $1 \leq k < k_0$, thus, the vertex $v_k := F^k(v_0) \in B_k^Z$ so $v_{k_0} \in B_0$.

We have three cases: $v_{k_0} \in [w_0, v_0], v_0 \in [w_0, v_{k_0}]$ or $v_{k_0} \in A_0 - [w_0, v_0]$. In he two first cases, as we know that $F^k(A_0 \cup C_d)$ for $1 \leq k < k_0$ is included in B_k which contains only vertices of degree 1, then the non critical annulus lemma (3.3) applied to $F^k(A_0)$ assures that $v_{k_0} = v_0$. So the vertex v_0 is periodic with period k_0. It is the only vertex of degree d in its cycle. As $a_{w_0}(v_0)$ is a fixed point of f^{k_0} with multiplier 1, according to the same lemma the rational map $f^{k_0} : \mathbb{S}_{v_0} \to \mathbb{S}_{v_0}$ has a parabolic fixed point at $i_{v_0}(w_0)$. According to corollary 2.10 we have $k_0 > 1$.

To finish the proof we show that this third case is absurd. Indeed, in this case, v_{k_0}, v_0 and w_0 are not on a common path. So there is a vertex $\hat{v}_0 \in [w_0, v_0]$ that separates v_{k_0}, v_0 and w_0. Otherwise, every critical cycle of spheres has to correspond to a cycle of vertices that intersects C_d because by definition a critical cycle of spheres doesn't have degree 1. Let v'_0 be a vertex in this intersection. The vertices v'_0 and v_0 lie on a same branch of $\hat{v}_0$ disjoint from the one containing v_{k_0}. As $A_0 \cup C_d$ and its iterates map bijectively to their image, we know that v_{k_0} and v'_{k_0} lie on a same branch of $\hat{v}_0$ so $\hat{v}_0$ separates the vertices w_0, v'_0 and v'_{k_0}. These four vertices lie in A_0 and are not forgotten by F^{k_0}, so according to lemma 2.3 the k_0 iterates of $\hat{v}_0$ are well defined.

As $A'_0 :=]w_0, \hat{v}_0[\subset A_0$, its k_0 iterates map bijectivement to their images which

are the $]w_0, \hat{v}_k[$ which all contain vertices of degree one exept maybe $]w_0, \hat{v}_{k_0}[$. But w_0, $\hat{v}_0$ and v_0 are aligned in this order, so for $0 \leq k \leq k_0$, it is the same for w_0, $\hat{v}_k$ and v_k then both $\hat{v}_0$ and $\hat{v}_{k_0}$ lie on the path $[w_0, v_{k_0}]$. So we have the inclusion $]w_0, \hat{v}_0[\subset]w_0, v_{k_0}[$ or the inverse inclusion. Thus according to the non critical annulus lemma (3.3) we have $\hat{v}_0 = \hat{v}_{k_0}$. Then the situation is similar to the one on figure 3.

As we did in previous cases for v_0, we can prove that the cover associated to the cycle containing $\hat{v}_0$ has a fixed point with multiplier 1 using the non critical annulus lemma. As $\hat{v}_0$ has degree 1, this cover in a projective chart is the identity or a translation. If it is a translation then the branch $B_{\hat{v}_0}(v'_{k_0})$ would be of infinite orbit which contradicts the existence of v'_{k_0} lying on it and which is periodic. If it is the identity then $F(B_{\hat{v}_0}(v'_{k_0})) \subset B_{\hat{v}_0}(v'_{k_0})$, this contradicts the fact that v'_0 lies in the orbit of v'_{k_0}. So it is again absurd.

Notation. As on figure 4, we define $D_i^\star := T^\star - D_{v_i}^\star(v_0)$. Denote by $\alpha := a_{v_0}(w_0)$ the parabolic fixed point and $\beta_1, \ldots, \beta_{d-1}$ its preimages by f^{k_0}. If there is a branch in T^Y attached to β_i on v_0, we denote it by B'_i. If it is not the case, we set $B'_i = \emptyset$.

Remark 4.4. As $D_k^\star \subset B_k^\star$, according to remark 4.3 we have $F : D_k^Y \to D_{k+1}^Z$ is a bijection for $1 \leq k < k_0$. Because α is not critical, the B'_i contain only elements of degree 1 so from corollary 2.16, we know that $F(B'_i)$ is the branch on v_1 attached to $f(\alpha)$. This means $F(B'_0) = B_{v_0}(w_0)$. We deduce that $B'_i \neq \emptyset$. Moreover, as $F : T^Y \to T^Z$ is surjective, we have $F(D_0 - B'_i) = D_1$.

Remark 4.5. As $k_0 > 1$ we can deduce from the last remark that there are no critical fixed leaves.

Corollary 4.6. *There exists a critical leaf c_0 such that $B_{v_0}(c_0)$ does not contain any critical spheres cycle.*

Proof. The orbit of a critical point z_0 lies in the basin of the parabolic fixed point $f(\alpha)$ of $f^{k_0} : \mathbb{S}_{v_0} \to \mathbb{S}_{v_0}$ (cf [M2] for example). The branch B of T^Y on v_0 corresponding to this critical point contains a critical leaf c_0.

If B contains a periodic vertex, then its iterates are defined as soon as they are branches. Iterates of $a_{v_0}(c_0)$ doesn't intersect α, because this one is prefixed so all the iterates of B are branches. Indeed, either they lie in $D_0 - \bigcup B'_i$ and we can apply corollary 2.18, or they lie in the D_i with $i > 0$ and then we have corollary 2.16. From this we deduce by lemma 3.1 that B does not contain critical spheres cycle. $\square$

Notation. From now, we will denote by c' this critical leaf and c the other.

Point 1b of Theorem 4.1.

Every critical spheres cycle has at least degree d so it has a vertex v'_0 in C_d. We define $v'_k := F^k(v'_0)$ and k'_0 the period of this cycle.

Define $D'_k := T^Y - B_{v'_k}(w_0)$. Let prove that $F(D'_k) = D'_{k+1}$ and that $f(a_{v'_k}(w_0)) = a_{v'_{k+1}}(w_0)$. According to remark 4.4, it is true for all k when
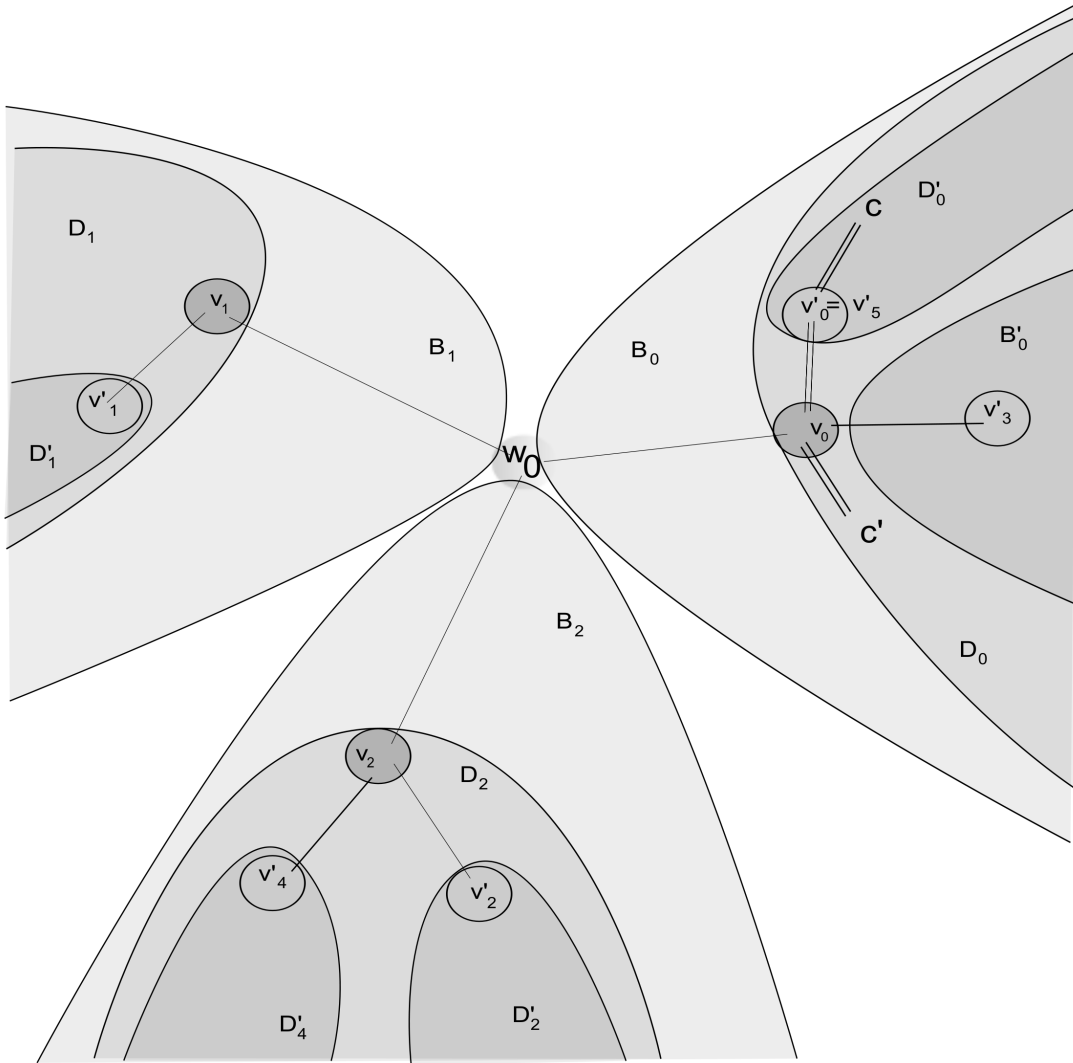
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Figure 4: Simplified representation of a tree $\mathcal{T}^X$ for an example of cover $\mathcal{F} : \mathcal{T}^Y \to \mathcal{T}^Z$ limit of degree 2 rational maps having a two critical spheres cycle. One of these has period 3 and the other period 5 with the notations introduced in the proof of Theorem 4.1.

$v'_k \notin B'_0$ and we always have $f(a_{v'_k}(w_0)) = a_{v'_{k+1}}(w_0)$. Suppose that there exists i such that $v'_k \in B'_i$. As we have $D'_k = B_{v_0}(v'_k) -]\!]v_0, v'_k[\![$ and F is bijective on $B_{v_0}(v'_k)$, we deduce that

$$F(D'_k) = B_{v_1}(v'_{k+1}) -]\!]v_1, v'_{k+1}[\![= D'_{k+1}.$$

Moreover F is a bijection between the edges of v'_k and the one of v'_{k+1}. We deduce that $f(a_{v'_k}(w_0)) = a_{v'_{k+1}}(w_0)$.

From $F(D'_k) = D'_{k+1}$ we conclude by lemma 3.1 that D'_0 does not contain any critical periodic internal vertex so after supposing that v'_0 is the degree d sphere of the cycle the closest to w_0, we deduce that v'_0 is the only critical vertex of the cycle. Thus the associated cover has degree 1. As $f(a_{v'_k}(w_0)) = a_{v'_{k+1}}(w_0)$ and $a_{v'_0}(w_0)$ is critical, this cover is conjugated to a degree d polynomial.

Now prove that $k'_0 > k_0$. As f_{v_0} has degree d, a critical point of degree d cannot be a preimage of the parabolic fixed point. So $v'_0 \in D_0 - \bigcup B'_i$. Thus according to remark 4.4 the v'_k lie in the D_k for $0 \le k \le k_0$ so $k'_0 \ge k_0$. If there is equality then we have $F(]\!]v_0, v'_0[\![) =]\!]F^k(v_0), F^k(v'_0)[\![=]\!]v_0, F^k(v'_0)[\![$ which contradicts lemma 3.3.

Suppose that one of the iterates of v'_0 lies in some B'_i. Let v'_i be the first iterate of v'_0 in B'_i. According to the above, for $0 \le k < k + k_0 \le i$ we have $F^{k_0} :]\!]v_0, v'_k[\![\to]\!]v_0, v'_{k+k_0}[\![$ is a bijection. Thus

$$f^{k_0+i} \circ a_{v_0}(v'_0) = f^{k_0} \circ f^i \circ a_{v_0}(c_0) = f^{k_0} \circ a_{v_0}(v_i) = f^{k_0}(\beta_i) = \alpha.$$

So a preimage of the parabolic fixed point is a critical point.

Let prove by the absurd that one of the iterates of v'_0 lies in some B'_{i_0}, which will finish the proof. If it is not the case, we can apply lemma 3.1 to the branch $B := B_{v_0}(c)$ because the iterates of B lie in the D_i for $i \ne 1$ or in $D_0 - \bigcup B'_i$ according to remark 4.4 and because these iterates are disjoints to $B_{v_0}(c')$.

End of point 1 (Number of rescalings).

We prove that in the case 1, there are at most two critical cycles of spheres. We have $C_d = [c', v_0] \cup [v_0, v'_0] \cup [v'_0, c]$. But $[c', v_0] - \{v_0\}$ does not contain periodic vertices. According to corollary 4.6, it is the same for $[v_0, v'_0] - \{v_0, v'_0\}$ by definition of v'_0 and $[v'_0, c]$ because $[v'_0, c] \subset D'_0$ so C_d contains only two periodic vertices.

Point 2 of Theorem 4.1.

First recall the following lemma from [A1].

Lemma 4.7. *Suppose that $f_n \overset{\lhd}{\longrightarrow} \mathcal{F}$ and $z \in Z \setminus X$ then after passing to a subsequence there exists extensions $(f_n, y_n, z_n)_n \lhd (\tilde{f}_n, \tilde{y}_n, \tilde{z}_n)_n$ with $z \in \tilde{X}$ and $\forall n \in \mathbb{N}, \tilde{x}_n(z) = z_n(z)$ and $\tilde{\mathcal{F}}$ such that $\tilde{f}_n \overset{\lhd}{\longrightarrow} \tilde{\mathcal{F}}$ and*

- $\mathcal{T}^X \lhd \mathcal{T}^{\tilde{X}}$, $\mathcal{T}^Y \lhd \mathcal{T}^{\tilde{Y}}$, and $\mathcal{T}^X \lhd \mathcal{T}^{\tilde{Z}}$,

- $\forall v \in IV^Y, F(v) \in T^X \implies \tilde{f}_v = f_v$.

Denote by v' a periodic critical internal vertex. According to this lemma, after considering a subsequence, we can find such a $(\tilde{\mathcal{F}}, \mathcal{T}^{\tilde{X}})$ such that $\tilde{X}$ contains the k_0 first iterates of c' (where c' is the critical leaf such that $v' \notin]c', v_0[$). Then we can show for k from 1 to k_0 that the vertex $\tilde{F}^k(v_0)$ separate the vertices w_0, $\tilde{F}^k(c')$ and $\tilde{F}^k(v')$. It follows from lemma 2.3 that v_0 is not forgotten by $\tilde{F}^{k_0}$. Then we conclude by applying point 1 and using the fact that in this case there can be just two different critical cycles of spheres. It follows again that in this case there cannot be more than two critical cycles of spheres.

Remark 4.8. We just proved that with the hypothesis of Theorem 4.1, if there exists a critical cycle of spheres with a polynomial associated cover then, after passing to a subsequence, we are in case 1.

Proof of Theorem 2.

First recall the following theorem from [A1].

Theorem 4.9. *Given a sequence $(f_n)_n$ in Rat_d for $(d \geq 2)$ with $p \in \mathbb{N}^*$ classes $M_1, \ldots, M_p$ of rescalings. Then, passing to a subsequence, there exists a portrait $\mathbf{F}$, a sequence $(f_n, y_n, z_n)_n \in \mathrm{Rat}_{\mathbf{F},X}$ and a dynamical system between trees of spheres $(\mathcal{F}, \mathcal{T}^X)$ such that*

- $f_n \xrightarrow[\phi_n^Y, \phi_n^Z]{\lhd} \mathcal{F}$ *and*

- $\forall i \in [1, p], \exists v_i \in \mathcal{T}^Y, M_i \sim (\phi_{n, v_i}^Y)_n$.

Take a sequence of bi-critical maps in Rat_d and suppose that it admits $p \geq 2$ dynamically independent rescalings of period at least 2. Applying this theorem, passing to a subsequence we obtain such a $(\mathcal{F}, \mathcal{T}^X)$. Then according to lemma 4.7 we can suppose after passing to a subsequence that the portrait $\mathbf{F}$ satisfies the hypothesis of Theorem 4.1. Then the conclusions follow immediately.

5 The case of degree 2

5.1 Example

In this part we propose to understand on a concrete example how to compute rescaling limits in the case of degree 2. Theorem 4.1 allows to know how to find them but not how to prove that they exist.

In [M1] J. Milnor notices that we have the following surprising relation : the set of rational maps of degree 2 having a period 2 cycle with multiplier -3 (denoted by $Per_2(-3)$) has always a period 3 cycle with multiplier 1 (so is included in $Per_3(1)$).

During a MRC program session in June 2013 organized in Snowbird, Laura De Marco and Jan-Li Lin tried to understand this decomposition and studied the family $Per_2(-3)$. We have a parametrization (not injective) of this family given by

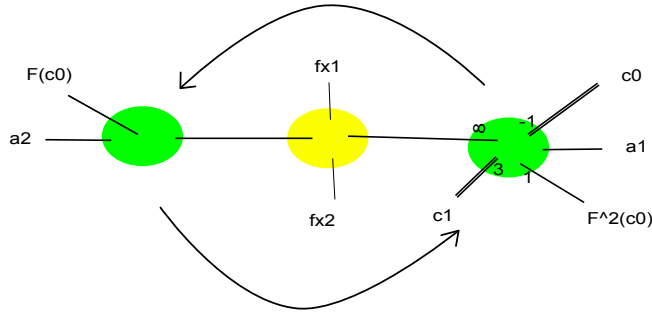$$f_a := \frac{(1 + 3a)(-a + z)}{(1 - a)(3az + z^2)}.$$

Figure 5: The yellow is fixed and the associated cover is $-Id$. The green spheres have period 2 and are adjacent to a period 2 cycle $(a1 \rightarrow a2 \rightarrow a1)$ with multiplier -3. On the green sphere on the right, the rescaling limit has a parabolic fixed point at infinity.

Here, the 2-cycle with multiplier -3 is $\{0, \infty\}$. When $a \rightarrow 1$, the family f_a diverges and $[f_a]$ diverges in rat_2.

In this case we can find two rescaling limits and the knowledge of the limiting dynamics gives us the good rescalings that we have to look in order to compute them.

As we have a persistent period 2 cycle that converges when $a \rightarrow 1$, we know that for some normalization the second iterate of f_a converges to a quadratic rational map with a parabolic fixed point and that it separates critical points. As critical points of f_a are $-a$ and $3a$ and converge to different limits, we are in this normalization. After computations we verify that

$$f_a^2 \rightarrow f_1^2 := z(3+z)/(z-1) \text{ when } a \rightarrow 1 \quad \text{(cf figure 5).}$$

Infinity is a parabolic fixed point for f_1^2. We can also verify that the fixed sphere is in the branch attached at infinity (two of the three fixed points of f_a converge to infinity). The third fixed point is constant equal to 1 which is the second preimage of infinity for f_1^2. Otherwise we note that -1 is a critical point of f_1^2 and that $f_1^2(-1) = 1$ so -1 is prefixed (so the other critical point 3 lies in the parabolic basin of infinity). So there is no contradictions for the existence of a second resealing limit. J.Milnor's remark suggest to see if there is not a rescaling limit of period 3 marked by a period 3 cycle. We would be in the configuration of figure 6.

We know that such a sphere would be marked by the critical point tending to 1 and an element of this orbit. Thus we are looking for a point of period three f_a that tends to -1 and after calculus we see that there exists exactly one like it that we will denote by $p3_a$.

We chose the Moebius transformation M_a such that

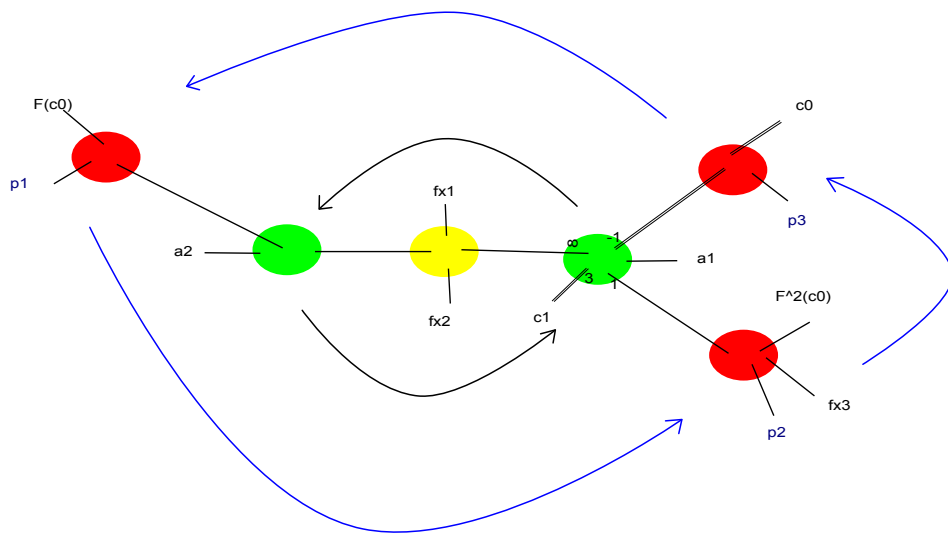$$M_a(3a) = \infty, M_a(-a) = 0 \text{ and } M_a(p3_a) = 1,$$

Figure 6: The yellow sphere is fixed and the cover associated is $-Id$. The green spheres have period 2 and are adjacent to a period 2 cycle ($a1 \rightarrow a2 \rightarrow a1$) with multiplier -3. On the green sphere on the right, the rescaling limit has a parabolic fixed point at infinity, $c1$ is in its direct basin and $c0$ maps after two iterates to infinity. The red spheres are a cycle of period 3 and are adjacent to a period three cycle with multiplier 1 ($p1 \rightarrow p2 \rightarrow p3 \rightarrow p1$). The rescaling limit associated to the up and right sphere is a quadratic polynomial.

ie :

$$M_a(z) = \frac{z+a}{z-3a} \cdot \frac{p3_a - 3a}{p3_a + a} \ .$$

Thus if such a rescaling limit exists, then the branch containing one of the critical points would be at infinity and fixed and the other one would be at 0, so we would obtain a quadratic polynomial of the form $z^2 + C$ with $c \in \mathbb{C}$. After computation we find:

$$\lim_{a \to 1} M_a \circ f_a^3 \circ M_a^{-1}(z) = z^2 + 1/4.$$

Remark 5.1. We can see that in this case, if we didn't mark the cycle a_1, a_2 then we are in the case 2 of Theorem 1. In figure 5 we are in the case 1a of Theorem 1 and in figure 6, in the case 1b of Theorem 1.

5.2 Comparison with J.Milnor's compactification

J. Milnor's point of view.

In [M1], J. Milnor provides a parametrization of rat_2 by looking at two of the symmetric functions of the multipliers at the fixed points. Such a way, rat_2 can be viewed as a subset of $\mathbb{C}^2$ and be compactified as a subset of $\mathbb{C}P^2$.

Every rational map f of degree 2 that has three distinct fixed points a, b, c, is after a conjugacy by a Moebius transformation, of the form

$$f(z) = z\frac{z+\alpha}{\beta z + 1} \text{ with } \alpha\beta \neq 1,$$

where α, β, γ are the respective multipliers of $a = 0, b = \infty$ and c.

Thus if a sequence of degree 2 rational maps $(f_n)_n$ diverges, one of the multipliers diverges. Suppose for example that $\gamma_n \to \infty$.

The index formula assures that

$$\frac{1}{1-\alpha_n} + \frac{1}{1-\beta_n} + \frac{1}{1-\gamma_n} = 1 \text{ so } \alpha_n\beta_n \to 1.$$

Suppose that $\beta_n \to \beta_\infty \in \mathbb{C}^\star$. Then $f_n \to (\beta_\infty \cdot Id)$ locally uniformly outside a point which is the limit of the critical points of the f_n. J. Milnor proved that the intersection points between the boundary of rat_2 and the curves corresponding to rational maps with a cycle of given period and multiplier are the points where two of the multipliers are conjugated roots of unity.

Our point of view on rat_2.

Consider a rational map f of degree 2 with 3 distinct and non super-attractive fixed points α, β, γ. We define $X = \{\alpha, \beta, \gamma\}$. By the Riemann-Hurwitz formula we know that f has exactly two critical points, that we will denote by c and c', so two critical values $v := f(c)$ and $v' : +f(c')$. We set $Z := X \cup \{v, v'\}$ and $Y := f^{-1}(Z)$. We define $\mathbf{F} := (f|_Y, deg_f|_Y)$ the corresponding portrait.

Consider the set of rational maps of degree 2 with 3 distinct and non super-attractive fixed points. We can define three injections injections x, y and z
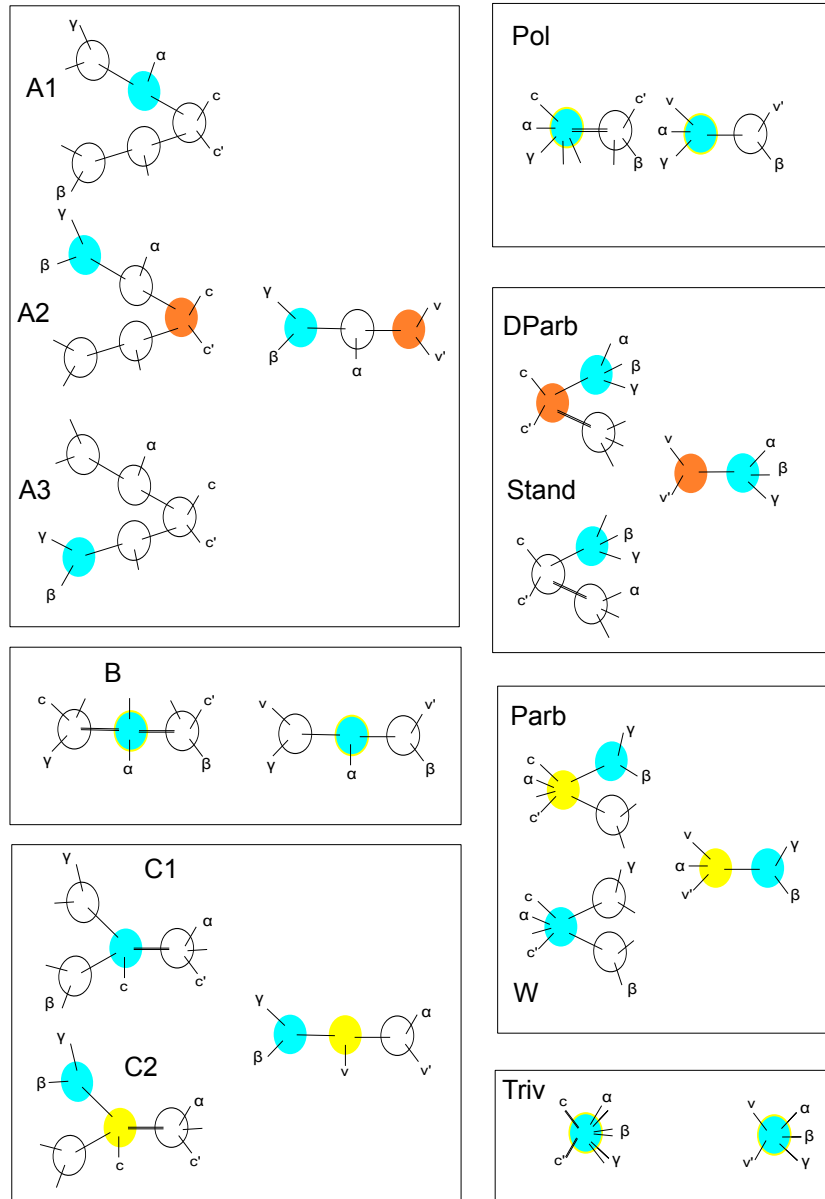
Figure 7: Enumeration of the possible configurations (after permutations of the labelings) of the convers between trees of spheres with a portrait corresponding to a rational map f of degree 2 with non super-attractive fixed points α, β and γ and two critical points c and c'. We use the notations $v := f(c)$ and $v' := f(c)$. The tree T^Y is on the left and the corresponding tree T^Z is on its right. The pre-fixed leaves of F are not labeled.

such that these rational maps are marked by (f, y, z) and such that we have $y|_X = z|_X$. All these rational maps have same portrait $\mathbf{F}$ and we will denote it by $\mathrm{Rat}_{\mathbf{F},X}$. Note that $\mathrm{Rat}_{\mathbf{F},X} \subset \mathrm{Rat}_2$ but this inclusion is strict. Indeed, this set does not contains:

- the rational maps with a critical fixed points (conjugated to a polynomial in rat$_2$);

- the rational maps with a simple parabolic point;

- the rational maps with a double parabolic point.

We are going to see that all of these missing elements appear in some way in $\partial\mathrm{Rat}_{\mathbf{F},X}$.

We take $\mathcal{F}$ a dynamical system between trees of spheres with portrait $\mathbf{F}$. The cover $\mathcal{F}$ has degree 2. According to lemma 4.2, $\mathcal{F}$ has two critical leaves c and c' and all the other critical vertices lie on the path connecting them. Figure 7 represent all the different possible applications of combinatorial trees for such a $\mathcal{F}$ (after a change of the fixed or critical leaves labels).

The vertex w_0 separating the three fixed leaves is represented in cyan. It is surrounded by some yellow when it is critical and fixed, ie in the configurations **B, Pol** and **Triv**. According to corollary 2.10, the covers are converging in rat$_2$ in those cases. In the cases **Pol** and **Triv**, we recognize the limits which are respectively the class of the polynomial maps and of the rational maps which have no super-attractve fixed points and no parabolic fixed points. In the case **B**, we recognize the class of the polynomial maps with a super -attractive fixed point, ie the class of $z \to z^2$.

The vertex w_0 is not fixed in the configurations **A1,C1** and **W** so there is no rescaling limits in these cases.

Denote by v_0 the critical vertex which is the closest to w_0.

In the configurations **C2** and **Parb**, the vertex v_0 and its image are on the path $[w_0, \alpha]$, so we can apply the annulus lemma in the non critical case and conclude that v_0 is fixed, thus according to corollary 2.10 we are in the adherence of rat$_2$. In the case **C2** we remark that the critical point which is the attaching point of the branch of c' on v_0 is fixed and it is the limit of a fixed point so we are in the case of the polynomial maps class and there is in addition a double fixed point so this polynomial is conjugated to $z \to z^2 + 1/4$. In the case **Parb** there is again a double fixed point and a third non critical fixed point so we are in the parabolic rational map class which have no super-atractive fixed point.

The configurations **DParb, A2** and **A3** are a bit more complicated to identify. For this suppose that these covers a dynamical limits of dynamical systems between spheres covers with portrait $\mathbf{F}$. Then according to the last lemma of [A1], after passing to a subsequence and the changing the portrait, we can suppose that v_0 and $F(v_0)$ are in T^X. Suppose that w_0, v_0 and $F(v_0)$ are on a same path then, as the vertex w_0 is fixed (and the associated cover is the identity because it fixes the attaching points of the branches containing the fixed points and they have degree one), according to the annuli lemma we have $v_0 = F(v_0)$

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and thus from corollary 2.10, the sequence converges uniformly to $[f_{v_0}]$ in rat_2. So they are in the adherence of rat_2 and the limit is a class of non polynomial covers with a triple fixed point in the cases **DParb** and **A2** or double fixed point in the case **A3**. Reciprocally, we know that such elements in Rat_2 are limits of dynamical systems between spheres covers with portrait **F** and it is clear that we obtain such dynamical covers as their limits.

Suppose by contradiction that w_0, v_0 and $F(v_0)$ are not on a same path then, using the annuli lemma in the non critical case, we prove that the vertex separating them is fixed and the corresponding cover is the identity and it follows that $v_0 = F(v_0)$ which is absurd.

In configuration **Stand**, the cover associated to w_0 is not the identity because the attaching point of the branch containing α doesn't maps to itself. We deduce that the vertex v_0 is not fixed. If this cover is a dynamical limits of dynamical systems between spheres covers with portrait **F** and if there is a rescaling limit, then it has period more than 1 and, according to Theorem 1, the map f_{w_0} has finite order so the multipliers at the fixed points β and γ are conjugated roots of unity.

Comments.

First we can see that a main missing tool is a sufficient condition for a dynamical system in order to be a dynamical limit of dynamical systems between trees of spheres.

The second comment is that we can do is that we the assumptions we did, all the elements of $\mathrm{Rat}_2 \setminus \mathrm{Rat}_{\mathbf{F},X}$ can be found in $\partial\mathrm{Rat}_{\mathbf{F},X}$.

The third important remark is that our space contains the elements than J.Milnor's compactification and contains in addition a blow up at the points corresponding to the rational map with a triple fixed point. Moreover, our space recover the fact that dynamically interesting points can be find we at the limit the multipliers of two fixed points are conjugated roots of unity.

Note that J.Milnor did the same kind of remarks about the space of bicritical rational maps of degree $d > 2$ and from Theorem 1 we can do the same remark about the multipliers in this case.

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