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11th World Congress on Computational Mechanics (WCCM XI)
5th European Conference on Computational Mechanics (ECCM V)
6th European Conference on Computational Fluid Dynamics (ECFD VI)
July 20–25, 2014, Barcelona, Spain

A NITSCHE FINITE ELEMENT METHOD FOR DYNAMIC CONTACT

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Key words: *unilateral contact, elastodynamics, finite elements, Nitsche's method, time-marching schemes, stability*

We present a new approximation of elastodynamic frictionless contact problems based both on the finite element method and on an adaptation of Nitsche's method which was initially designed for Dirichlet's condition [4]. The corresponding space semi-discretized weak form reads:

$$\left\{\begin{array}{l} \text{Find a displacement } \mathbf{u}^h : [0,T] \to \mathbf{V}^h \text{ such that for } t \in [0,T] : \\ \langle \rho \ddot{\mathbf{u}}^h(t), \mathbf{v}^h \rangle + A_{\Theta\gamma_h}(\mathbf{u}^h(t), \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_h} \left[P_{\gamma_h}(\mathbf{u}^h(t))\right]_+ P_{\Theta\gamma_h}(\mathbf{v}^h)\, d\Gamma \;=\; L(t)(\mathbf{v}^h), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \mathbf{u}^h(0,\cdot) = \mathbf{u}_0^h, \qquad \dot{\mathbf{u}}^h(0,\cdot) = \dot{\mathbf{u}}_0^h. \end{array}\right. \tag{1}$$

In the above formulation $\mathbf{V}^h$ is a finite element space built from standard Lagrange finite elements, piecewise linear or quadratic, $A_{\Theta\gamma_h}(\mathbf{u}^h, \mathbf{v}^h) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h)\; d\Omega - \displaystyle\int_{\Gamma_C} \Theta\gamma_h\, \sigma_n(\mathbf{u}^h)\sigma_n(\mathbf{v}^h)\, d\Gamma$ and $P_{\Theta\gamma_h}(\mathbf{v}^h) := v_n^h - \Theta\gamma_h\, \sigma_n(\mathbf{v}^h)$. The linear form $L(\cdot)$ stands for prescribed body and boundary forces. The domain of the elastic body is denoted by Ω and the contact boundary by Γ_C ; T is the final time of simulation ; ρ is the density of the elastic material ; the notation v_n^h stands for the normal component on Γ_C of $\mathbf{v}^h$, and $\sigma_n(\mathbf{v}^h)$ is the normal stress on Γ_C ; $\mathbf{u}_0^h$ (resp. $\dot{\mathbf{u}}_0^h$) is an approximation of the initial displacement $\mathbf{u}_0$ (resp. the initial velocity $\dot{\mathbf{u}}_0$). The notation $[\cdot]_+$ stands for the positive part of a scalar quantity, and $\langle \cdot, \cdot \rangle$ stands for the $L^2(\Omega)$ inner product. The parameter γ_h

is a positive piecewise constant function on the contact interface Γ_C ($\gamma_h(x) = \gamma_0 h_T(x)$, where γ_0 is a positive constant and $h_T(x)$ is the size of the mesh element T). Note the additional numerical parameter Θ which can be freely chosen in $\mathbb{R}$. As in Nitsche's method for (static) unilateral contact, values of interest for Θ are $-1, 0, 1$ [1].

A main interesting characteristic is that this approximation produces well-posed space semi-discretizations, for any value of $\gamma_0 > 0$, contrary to standard finite element discretizations. We study associated energy conservation properties and manage to prove that :

$$\frac{d}{dt}E^h_\Theta(t) = (\Theta - 1)\int_{\Gamma_C} \frac{1}{\gamma_h}\,[P_{\gamma_h}(\mathbf{u}^h(t))]_+ \dot{u}^h_n(t)\,d\Gamma$$

where $E^h_\Theta(t) := E^h(t) - \Theta R^h(t)$ is an augmented energy associated to the semi-discrete solution $\mathbf{u}^h(t)$. The mechanical energy is given by $E^h(t) := \frac{1}{2}\left[\rho\|\dot{\mathbf{u}}^h(t)\|^2_{0,\Omega} + \frac{1}{2}a(\mathbf{u}^h(t), \mathbf{u}^h(t))\right]$ and $R^h(t) := \frac{1}{2}\left[\|{\gamma_h}^{\frac{1}{2}}\sigma_n(\mathbf{u}^h(t))\|^2_{0,\Gamma_C} - \|{\gamma_h}^{-\frac{1}{2}}[P_{\gamma_h}(\mathbf{u}^h(t))]_+\|^2_{0,\Gamma_C}\right]$ is an extra term, which represents, roughly speaking, the non-fulfillment of the contact conditions at the semi-discrete level. Note in particular that the symmetric variant ($\Theta = 1$) conserves the discrete augmented energy E^h_Θ.

Various time-discretizations for (1) are then considered, with the families of θ-schemes and Newmark schemes. We also introduce a new hybrid scheme which is second-order accurate and numerically stable without any restriction on the time-step in the case $\Theta = 1$. This new scheme is inspired from [2, 3], and introduces much less dissipation than unconditionally stable variants of θ-schemes and Newmark schemes. We study theoretically the well-posedness of each discrete scheme as well as its energy conservation properties. We finally achieve the corresponding numerical experiments.

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