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The paper presents a 'multiple agent' logic where formulas are pairs of the form (a, A), made of a proposition a and a subset of agents A. The formula (a, A) is intended to mean '(at least) all agents in A believe that a is true'. The formal similarity of such formulas with those of possibilistic logic, where propositions are associated with certainty levels, is emphasised. However, the subsets of agents are organised in a Boolean lattice, while certainty levels belong to a totally ordered scale. The semantics of a set of 'multiple agent' logic formulas is expressed by a mapping which associates a subset of agents with each interpretation (intuitively, the maximal subset of agents for whom this interpretation is possibly true). Soundness and completeness results are established. Then a joint extension of the multiple agent logic and possibilistic logic is outlined. In this extended logic, propositions are then associated with both sets of agents and certainty levels. A formula then expresses that 'all agents in set A believe that a is true at least at some level'. The semantics is then given in terms of fuzzy sets of agents that find an interpretation more or less possible. A specific feature of possibilistic logic is that the inconsistency of a knowledge base is a matter of degree. The proposed setting enables us to distinguish between the global consistency of a set of agents and their individual consistency (where both can be a matter of degree). In particular, given a set of multiple agent possibilistic formulas, one can compute the subset of agents that are individually consistent to some degree.

Keywords: possibilistic logic; possibility theory; uncertainty; multiple agent inconsistency

1. Introduction

Possibilistic logic (Dubois, Lang, & Prade, 1994; Dubois & Prade, 2004) was originally motivated by the need to manipulate statements of the form $N(a) \geq \alpha$ in a logical way (Dubois & Prade, 1987), where N is a necessity measure (see, for example, Dubois & Prade, 1998) valued in a totally ordered scale and a is a proposition in classical logic. This statement means that the proposition is an accepted belief with a minimal strength, and is syntactically encoded under the form of a pair (a, α). Such certainty-qualified statements have a clear modal flavour. Possibilistic logic can also be viewed as a special case of a labelled deductive system (Gabbay, 1996). Inference in possibilistic logic propagates certainty in a qualitative manner, using the law of the weakest link, and is inconsistency-tolerant, as it enables non-trivial reasoning to be performed from the largest consistent subset of most certain formulas. A characteristic feature of this uncertainty theory is that

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a set of propositions $\{a \in \mathcal{L} : N(a) \geq \alpha\}$ in a propositional language $\mathcal{L}$, the propositions of which are believed at least to a certain extent, is deductively closed (thanks to the min-decomposability of necessity measures with respect to conjunction). As a consequence, possibilistic logic remains very close to classical logic.

One may think of associating other 'labels' with classical logical formulas. It may be lower bounds of other measures in possibility theory, such as weak or strong possibility measures (see, for example, Dubois & Prade, 2004). It may also be a set of logical arguments in favour of a (Lafage, Lang, & Sabbadin, 1999), a set of distinct sources supporting the truth of the associated formula (Dubois, Lang, & Prade, 1992), or a set of time instants where the formula holds true for sure (Dubois, Lang, & Prade, 1991). In this paper, we investigate in detail a 'multiple agent' logic where formulas are of the form (a, A), and A is a subset of agents that are known to believe that a is true. The idea was outlined in (Dubois & Prade, 2006, 2007), and previously in (Dubois et al., 1992) in an information fusion perspective, but the underlying semantics was never studied. In contrast, with timed possibilistic logic where it is important to make sure that the knowledge base remains consistent over time, what matters in multiple agent logic is the collective consistency of subsets of agents. The formula (a, A) will be understood at the semantic level as a constraint of the form $\mathbf{N}(a) \supseteq A$, where $\mathbf{N}$ is a set-valued mapping that returns the set of agents for whom it is certain that a is true. Thus stating (a, A) amounts to saying that at least all the agents in A believe a.

The problem of processing information coming from various sources has a rather long tradition in logic, which at least dates back to Belnap (1977), whose approach was extended by Carnielli & Lima-Marques (1999). However, these authors only consider the situations where sources provide information about atomic formulas. A solution to this limitation has been proposed in Avron, Ben-Naim, & Konikowska (2009) using a new tool called non-deterministic logical matrices. The approach that we present in this paper handles information pertaining to any kind of propositional formulas, but we restrict ourselves to the case of propositions a jointly supplied by all agents in a set A, and the conjunction of formulas of the form (a, A). In other words, the language does not allow formulas like $\neg(a, A)$ ('it is false that all the agents in A believe a') or disjunctions $(a, A) \vee (b, B)$. In particular, and in contrast with Belnap (1977), we do not represent pieces of information of the form 'there is at least one agent in A who believes a', which amounts to a disjunctive statement. Handling such formulas could be achieved by getting inspiration from the so-called generalised possibilistic logic (Dubois, Prade, & Schockaert, 2012), as briefly mentioned at the end of this paper. Indeed, basic possibilistic logic, from which our multiple agent logic derives, suffers from the same limitations of handling only conjunctions of formulas of the form (a, α). Besides, the multiple agent logic can be nicely extended into a multiple agent possibilistic logic which handles formulas expressing that 'at least all agents in A believe that a is true at least at level α'. Then $\mathbf{N}$ becomes a fuzzy set-valued mapping that returns a weighted set of agents.

The paper is organised in the following way. The next section provides a brief refresher on possibilistic logic, where its main features are emphasised. Section 3 presents the multiple agent logic and its semantics, along with soundness and completeness results. Section 4 outlines the extension of multiple agent logic with certainty levels. These three sections are illustrated by a running example which shows how (in)consistency is handled in terms of subsets of agents and in terms of certainty levels. The concluding section briefly discusses prospective extensions of the proposed framework.

2. A refresher on possibilistic logic

We first describe the syntax of possibilistic logic, provide a short background on possibility theory that underlies its semantics, and then present its semantics in terms of possibility distributions and necessity measures, before ending the section with an illustrative example.

2.1. Syntax

Let $\mathcal{L}$ denote a propositional logical language. Well-formed formulas of $\mathcal{L}$, denoted $a, b, c, \ldots, a_i, \ldots$, are built from atoms, denoted $p, q, r, \ldots$ and the usual connectives $\neg$, $\wedge, \vee$ (where $a \vee b =_{def} \neg(\neg a \wedge \neg b)$) with parentheses (and), using the usual rules. $\top$ and $\bot$ denote the tautology and the contradiction respectively. We take the axioms of propositional logic (*PL*) for granted, and the modus ponens as inference rule $(a, \neg a \vee b \vdash b)$. We can equivalently use the resolution or cut rule $(\neg a \vee b, a \vee c \vdash b \vee c)$ and refutation (proving $\Gamma \vdash a$ amounts to proving $\Gamma, \neg a \vdash \bot$, where Γ stands for a collection of propositional formulas $a_1, \ldots, a_m$ and $\bot$ is the contradiction).

Language

A (basic) possibilistic logic formula is a pair (a, α), with $a \in \mathcal{L}$ and $\alpha \in \mathcal{S}$, where $\mathcal{S}$ is a bounded totally ordered scale, equipped with an ordering relation denoted by $\leq$. Then α is called the certainty level (or weight) of the propositional formula a. More precisely, (a, α) means that it is certain *at least* at level α that a is true. In the following we take $\mathcal{S} = [0, 1]$. But any bounded totally ordered set, finite or not, can be used as well. Then the complementation to 1 operation, namely $1 - (\cdot)$, used in the following should be understood as an order-reversing map. A possibilistic logic base Γ is a set of such pairs, namely $\Gamma = \{(a_i, \alpha_i) \mid i = 1, \ldots, m\}$, understood as the conjunction of pairs (a_i, α_i), with $\forall i, \alpha_i > 0$.

Axioms and inference rules

The axioms of possibilistic logic (Dubois et al., 1994), ΠL for short, are those of *PL*, where each axiom schema is now supposed to hold with the maximal certainty, i.e., is associated with level 1. It has two inference rules:

- if $\beta \leq \alpha$ then $(a, \alpha) \vdash (a, \beta)$ (level weakening);
- $(\neg a \vee b, \alpha), (a, \alpha) \vdash (b, \alpha), \forall \alpha \in (0, 1]$ (level modus ponens).

We may equivalently use the level weakening rule with the ΠL counterpart of the resolution rule:

$$(\neg a \vee b, \alpha), (a \vee c, \alpha) \vdash (b \vee c, \alpha), \quad \forall \alpha \in (0, 1] \text{ (level resolution).}$$

Using level weakening, it is then easy to see that the following inference rule is valid:

$$(\neg a \vee b, \alpha), (a \vee c, \beta) \vdash (b \vee c, \min(\alpha, \beta)) \text{ (α-β-resolution).}$$

The idea that in a reasoning chain, the certainty level of the conclusion is the smallest of the certainty levels of the formulas involved in the premises is at the basis of the syntactic approach proposed by Rescher (1976) for plausible reasoning, and would date back to Theophrastus, a follower of Aristotle.

The following inference rule, which we call *formula weakening*, also holds as a consequence of α-β-resolution:

$$\text{if } a \vdash b \text{ then } (a, \alpha) \vdash (b, \alpha), \quad \forall \alpha \in (0, 1] \text{ (formula weakening).}$$

Indeed, $a \vdash b$ expresses that $\neg a \vee b$ is valid in *PL* and thus $(\neg a \vee b, 1)$ holds, which by applying the α-β-resolution rule with (a, α) yields the result.

It turns out that any valid deduction in propositional logic is valid as well in possibilistic logic where the corresponding propositions are associated with any level $\alpha \in (0, 1]$. Thus since $a, b \vdash a \wedge b$, we have $(a, \alpha), (b, \alpha) \vdash (a \wedge b, \alpha)$. Note that we also have $(a \wedge b, \alpha) \vdash (a, \alpha)$ and $(a \wedge b, \alpha) \vdash (b, \alpha)$ by the formula weakening rule. Thus, stating $(a \wedge b, \alpha)$ is equivalent to stating (a, α) and (b, α). Thanks to this property, it is always possible to rewrite a ΠL base under the form of a collection of weighted *clauses*.

Note also that if we assume that for any propositional tautology t, i.e., such that $t \equiv \top$, (t, α) holds with any certainty level, which amounts to saying that each axiom schema holds with any certainty level, then the α-β-resolution rule entails the level weakening rule, since $(\neg a \vee a, \beta)$ together with $(a \vee c, \alpha)$ entails $(a \vee c, \beta)$ when $\beta \leq \alpha$.

Inference and consistency

Inference in ΠL is quite similar to the one in *PL*. We may either use the ΠL axioms and the level weakening and level modus ponens rules, or equivalently proceed by refutation (proving $\Gamma \vdash (a, \alpha)$ amounts to proving $\Gamma, (\neg a, 1) \vdash (\bot, \alpha)$ by repeated application of the α-β-resolution rule, where Γ stands for a collection of ΠL formulas $(a_1, \alpha_1), \ldots, (a_m, \alpha_m)$). Moreover, note that

$$\Gamma \vdash (a, \alpha) \quad \text{if and only if } \Gamma_\alpha \vdash (a, \alpha) \quad \text{if and only if } (\Gamma_\alpha)^* \vdash a$$

where $\Gamma_\alpha = \{(a_i, \alpha_i) \in \Gamma, \alpha_i \geq \alpha\}$ and $\Gamma^* = \{a_i \mid (a_i, \alpha_i) \in \Gamma\}$. Thus, the certainty levels stratify the knowledge base Γ into nested level cuts Γ_α, i.e., $\Gamma_\alpha \subseteq \Gamma_\beta$ if $\beta \leq \alpha$. A consequence (a, α) from Γ can only be obtained from formulas having a certainty level at least equal to α, so from formulas in Γ_α; then a is a classical consequence from the *PL* knowledge base $(\Gamma_\alpha)^*$, and $\alpha = \max\{\beta | \Gamma_\beta \vdash (a, \alpha)\}$.

The *inconsistency level* of Γ is defined by

$$inc\text{-}l(\Gamma) = \max\{\alpha \mid \Gamma \vdash (\bot, \alpha)\}.$$

The possibilistic formulas in Γ whose level is strictly above $inc\text{-}l(\Gamma)$ are safe from inconsistency, namely $inc\text{-}l(\{(a_i, \alpha_i) \mid (a_i, \alpha_i) \in \Gamma \text{ and } \alpha_i > inc\text{-}l(\Gamma)\}) = 0$. Indeed, if $\alpha > inc\text{-}l(K)$, $(\Gamma_\alpha)^*$ is consistent. In particular, we have the remarkable property that the classical consistency of Γ^* is equivalent to saying that Γ has a level of inconsistency equal to 0. Namely,

$$inc\text{-}l(\Gamma) = 0 \quad \text{if and only if } \Gamma^* \text{ is consistent.}$$

The semantics of ΠL is expressed in terms of possibility distributions, (weak) possibility measures and (strong) necessity measures. Let us first recall these notions.

2.2. Background on possibility theory

Let Ω be the set of interpretations of the propositional logic language $\mathcal{L}$. A *possibility distribution* (Zadeh, 1978) on Ω is a function π from Ω to [0,1]. It is supposed to reflect the available knowledge: $\pi(\omega)$ estimates to what extent it is possible that the interpretation ω corresponds to the real state of the world. The possibility distribution π is *normalised* if $\exists\omega \in \Omega$ such that $\pi(\omega) = 1$. Normalisation expresses that at least one interpretation in Ω is fully possible.

Based on a possibility distribution π, a function, from $\mathcal{L}$ to [0,1], called *possibility measure*, denoted by Π, is defined by Zadeh (1978)[1] as

$$\Pi(a) = \max\{\pi(\omega) \mid \omega \models a\}.$$

The possibility measure estimates the extent to which a is compatible with the available knowledge represented by π. Informally, $\Pi(a) = 0$ means that a is impossible, while $\Pi(a) = 1$ means that a is fully possible and expresses full consistency with the current knowledge. Particularly, $\Pi(a) = 0$ when a is a contradiction. $\Pi(a) = \Pi(\neg a) = 1$ expresses that both a and $\neg a$ are fully possible, which corresponds to a case of total ignorance (about a).

Possibility measures are such that $\Pi(\top) = 1$ and $\Pi(\bot) = 0$, and satisfy the characteristic axiom

$$\Pi(a \vee b) = \max(\Pi(a), \Pi(b)).$$

A *necessity measure* N (Dubois & Prade, 1980), defined from $\mathcal{L}$ to [0,1], is associated by duality with Π, namely

$$N(a) = 1 - \Pi(\neg a),$$

expressing that a is all the more necessarily true, or certain, as $\neg a$ is more impossible, and conversely the absence of certainty in favour of a ($N(a)$ small) makes $\neg a$ possible ($\Pi(\neg a)$ high). Thus N is defined from π by

$$N(a) = \min\{1 - \pi(\omega) | \omega \models \neg a\}.$$

Informally, $N(a)$ estimates to what extent a is entailed by the available knowledge. $N(a) = 1$ means that a is sure, for instance $N(a) = 1$ when a is a tautology. The case of total ignorance in terms of necessity measure is represented by $N(a) = N(\neg a) = 0$ indeed; from the available knowledge nothing enables us to say if a is rather true or rather false.

Necessity measures are such that $N(\top) = 1$ and $N(\bot) = 0$, and satisfy the characteristic axiom

$$N(a \wedge b) = \min(N(a), N(b)).$$

As a consequence, $\min(N(a), N(\neg a)) = 0$ should hold. Dually, $\max(\Pi(a), \Pi(\neg a)) = 1$. It can be checked that these two conditions hold for any a if and only if π is normalised. Thus having for some a, $\min(N(a), N(\neg a)) > 0$ (or equivalently $\max(\Pi(a), \Pi(\neg a)) < 1$) amounts to having π unnormalised, which intuitively expresses a situation of inconsistency, since one cannot be somewhat certain of both a and $\neg a$, or find both a and $\neg a$ somewhat impossible.

2.3. Semantics

We have now recalled the notions needed for presenting the semantics of ΠL (Dubois et al., 1994). Let us first consider a ΠL formula (a, α) that encodes the statement $N(a) \geq \alpha$. Its semantics is given by the following possibility distribution $\pi_{(a,\alpha)}$, defined by

$$\pi_{(a,\alpha)}(\omega) = 1 \quad \text{if } \omega \vDash a \ \text{ and } \ \pi_{(a,\alpha)}(\omega) = 1 - \alpha \quad \text{if } \omega \vDash \neg a.$$

Intuitively, the underlying idea is that any model of a should be fully possible, and that any interpretation that is a counter-model of a is all the less possible as a is more certain, i.e., as α is higher. It can easily be checked that the associated necessity measure is such that $N_{(a,\alpha)}(a) = \alpha$, and $\pi_{(a,\alpha)}$ is the least informative possibility distribution (i.e., maximising possibility degrees) such that this constraint holds. In fact, any possibility distribution π such that $\forall\omega, \pi(\omega) \leq \pi_{(a,\alpha)}(\omega)$ is such that its associated necessity measure N satisfies $N(a) \geq N_{(a,\alpha)}(a) = \alpha$ (hence is more committed).

Let us now consider a ΠL knowledge base $\Gamma = \{(a_i, \alpha_i), i = 1, \ldots, m\}$, thus corresponding to the conjunction of ΠL formulas (a_i, α_i), each representing a constraint $N(a_i) \geq \alpha_i$. The base Γ is semantically associated with the possibility distribution

$$\pi_\Gamma(\omega) = \min_{i=1,\ldots,m} \pi_{(a_i,\alpha_i)}(\omega) = \min_{i=1,\ldots,m} \max([a_i](\omega), 1 - \alpha_i),$$

where $[a_i]$ is the characteristic function of the models of a_i, namely $[a_i](\omega) = 1$ if $\omega \vDash a_i$ and $[a_i](\omega) = 0$ otherwise. Thus, the least informative induced possibility distribution π_Γ is obtained as the min-based conjunction of the fuzzy sets of interpretations (with membership functions $\pi_{(a_i,\alpha_i)}$), representing each formula. It can be checked that

$$N_\Gamma(a_i) \geq \alpha_i \quad \text{for i} = 1, \ldots, \text{m},$$

where N_Γ is the necessity measure defined from π_Γ. Note that we may only have an inequality here since Γ may, for instance, include two formulas associated with equivalent propositions, but with distinct certainty levels.

So a ΠL knowledge base is understood as a set of constraints $N(a_i) \geq \alpha_i$ for $i = 1, \ldots, m$, and the set of possibility distributions π associated with N that are compatible with this set of constraints has a largest element which is nothing but π_Γ, i.e., we have $\forall\omega, \pi(\omega) \leq \min_{i=1,\ldots,m} \pi_{(a_i,\alpha_i)} = \pi_\Gamma(\omega)$. Thus, the possibility distribution π_Γ semantically representing a ΠL base Γ is the one which assigns the largest possibility degree to each interpretation, in agreement with the semantic constraints $N(a_i) \geq \alpha_i$ for i $= 1, \ldots,$ m that are associated with the formulas (a_i, α_i) in Γ. Thus, any possibility distribution $\pi \leq \pi_\Gamma$ semantically agrees with Γ, which can be written $\pi \vDash \Gamma$.

The semantic entailment is defined by

$$\Gamma \vDash (a, \alpha) \quad \text{if and only if } \forall\omega,\ \pi_\Gamma(\omega) \leq \pi_{\{(a,\alpha)\}}(\omega).$$

It can be shown (Dubois et al., 1994) that possibilistic logic is sound and complete w.r.t. this semantics, namely

$$\Gamma \vdash (a, \alpha) \quad \text{if and only if } \Gamma \vDash (a, \alpha).$$

Moreover, we have

$$inc\text{-}l(\Gamma) = 1 - \max_{\omega\in\Omega} \pi_\Gamma(\omega),$$

which acknowledges the fact that the normalisation of π_Γ is equivalent to the classical consistency of Γ^*. Thus, an important feature of possibilistic logic is its ability to deal with inconsistency. The consistency of Γ is estimated by the extent to which there is at least one interpretation completely possible for Γ, i.e., by the quantity $cons\text{-}l(\Gamma) = 1 - inc\text{-}l(\Gamma) = \max_{\omega\in\Omega} \pi_\Gamma(\omega) = max_{\pi\vDash\Gamma} max_{\omega\in\Omega}\pi(\omega)$.

Table 1. Detailed computation of the possibility distribution in the example.

ω	$\pi_{\{(\neg p\vee q,.8)\}}$	$\pi_{\{(\neg p\vee r,.9)\}}$	$\pi_{\{(\neg p\vee\neg r,.1)\}}$	$\pi_{\{(\neg q\vee r,.6)\}}$	$\pi_{\{(p,.3)\}}$	$\pi_{\{(q,.7)\}}$	$\pi_{\{(\neg q,.2)\}}$	$\pi_{\{(r,.8)\}}$	π_Γ
pqr	1	1	0.9	1	1	1	0.8	1	0.8
$pq\neg r$	1	0.1	1	0.4	1	1	0.8	0.2	0.1
$p\neg qr$	0.2	1	0.9	1	1	0.3	1	1	0.2
$p\neg q\neg r$	0.2	0.1	1	1	1	0.3	1	0.2	0.1
$\neg pqr$	1	1	1	1	0.7	1	0.8	1	0.7
$\neg pq\neg r$	1	1	1	0.4	0.7	1	0.8	0.2	0.2
$\neg p\neg qr$	1	1	1	1	0.7	0.3	1	1	0.3
$\neg p\neg q\neg r$	1	1	1	1	0.7	0.3	1	0.2	0.2

2.4. An example in possibilistic logic

Let us illustrate the previously introduced notions on the following ΠL base Γ, which is in clausal form (p, q, r are atoms):
$\{(\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (\neg q, 0.2), (r, 0.8)\}$.

First, it can be checked that $inc\text{-}l(\Gamma) = 0.2$.

Thus, the sub-base $\Gamma_{0.3} = \{(\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (r, 0.8)\}$ is safe from inconsistency, and its deductive closure is consistent, i.e., $\nexists a$, $\nexists \alpha > 0$, $\nexists \beta > 0$ such that $\Gamma_{0.3} \vdash (a, \alpha)$ and $\Gamma_{0.3} \vdash (\neg a, \beta)$. By contrast, $\Gamma_{0.1} \vdash (\neg r, 0.1)$ and $\Gamma_{0.1} \vdash (r, 0.8)$. Note also that while $(\neg p \vee r, 0.9), (p, 0.3) \vdash (r, 0.3)$, we clearly have $\Gamma \vdash (r, 0.8)$ as well. This illustrates the fact that in possibilistic logic we are interested in practice in the proofs leading to the highest certainty levels. Besides, in case Γ contains $(r, 0.2)$ rather than $(r, 0.8)$, then $(r, 0.2)$ would be of no use, since subsumed by $(r, 0.3)$. Indeed, it can be checked that $\Gamma \setminus \{(r, 0.8)\}$ and $(\Gamma \setminus \{(r, 0.8)\}) \cup \{(r, 0.2)\}$ are associated with the same possibility distribution.

The possibility distribution associated with Γ, whose computation is detailed in Table 1, is given by $\pi_\Gamma(pqr) = 0.8$; $\pi_\Gamma(\neg pqr) = 0.7$; $\pi_\Gamma(\neg p\neg qr) = 0.3$; $\pi_\Gamma(p\neg qr) = \pi_\Gamma(\neg pq\neg r) = \pi_\Gamma(\neg p\neg q\neg r) = 0.2$; $\pi_\Gamma(pq\neg r) = \pi_\Gamma(p\neg q\neg r) = 0.1$.

As can be seen $cons\text{-}l(\Gamma) = \max_{\omega\in\Omega} \pi_\Gamma(\omega) = 0.8$ and $inc\text{-}l(\Gamma) = 1 - 0.8 = 0.2$. Similarly, $inc\text{-}l(\Gamma \setminus \{(\neg q, 0.2)\}) = 0.1$ and $inc\text{-}l(\Gamma \setminus \{(\neg q, 0.2), (\neg p \vee \neg r, 0.1)\}) = 0$.

3. Multiple agent logic

We now introduce a multiple agent logic (*ma-L*) which parallels possibilistic logic in many respects, starting with the syntactic aspects, then introducing a set-valued counterpart of the notions of possibility distribution, possibility measure, and necessity measure, before presenting the semantics that relies on these notions, and ending with an illustrative example.

3.1. Syntax

Language

$\mathcal{L}$ still denotes a propositional logic language. Let All denote the finite set of all agents considered. A subset of agents is denoted by capital letters A, B, or by indexed letters A_i for $i = 1, \ldots, m$. Clearly, the set of subsets of agents equipped with the usual set operations, i.e., $(2^{ALL}, \cap, \cup, \overline{}, \subseteq)$, is a Boolean algebra, thus only partially ordered, which contrasts with the scale $\mathcal{S}$ used in possibilistic logic.

A multiple agent propositional formula (*ma*-formula) is a pair (a, A), where a is a classical propositional formula of $\mathcal{L}$ and A is a non-empty subset of All, i.e., $A \subseteq All$. The intuitive meaning of formula (a, A) is that *at least all* the agents in A believe that a is true. In spite of the obvious parallel with possibilistic logic (where propositions are associated with levels expressing the strength with which the propositions are believed to be true), (a, A) should not be just understood as another way of expressing the strength of the support in favour of a (the larger A, the stronger the support), but rather as a piece of information linking a proposition with a group of agents. A multiple agent knowledge base (*ma*-base) is simply a finite set $\Gamma = \{(a_i, A_i), i = 1, \ldots, m\}$, viewed as the conjunction of *ma*-formulas. Γ° denotes the set of classical formulas obtained from Γ by ignoring the sets of agents: $\Gamma^\circ = \{a_i \mid (a_i, A_i) \in \Gamma, i = 1, \ldots, m\}$.

Inference rules and axioms

Ma-L has two inference rules:

- if $B \subseteq A$ then $(a, A) \vdash (a, B)$ (subset weakening);
- $(\neg a \vee b, A), (a, A) \vdash (b, A), \forall A \in 2^{ALL} \setminus \emptyset$ (subset modus ponens).

The axioms of *ma-L* are those of *PL* where each axiom schema is now supposed to hold for the maximal set of agents, i.e., is associated with subset All. We may equivalently use the subset weakening rule with the *ma-L* counterpart of the resolution rule:

$$(\neg a \vee b, A), (a \vee c, A) \vdash (b \vee c, A),\ \forall A \in 2^{ALL} \setminus \emptyset \text{ (subset resolution).}$$

Using subset weakening (since $A \cap B \subseteq A$ and $A \cap B \subseteq B$), it is then easy to see that the following inference rule is valid:

$$\text{if } A \cap B \neq \emptyset,\ \text{then } (\neg a \vee b, A), (a \vee c, B) \vdash (b \vee c, A \cap B) \text{ (}A\text{-}B\text{-resolution).}$$

Clearly, if $A \cap B = \emptyset$, the information resulting from applying the rule does not belong to the language, and would make little sense: it is of no use to put vacuous formulas of the form $(a, \emptyset)$ in a *ma*-base as it corresponds to information possessed by (at least) no agent. As a consequence, we also have a formula weakening rule:

$$\text{if } a \vdash b \text{ then } (a, A) \vdash (b, A), \quad \forall A \in 2^{ALL} \setminus \emptyset \text{ (ma-formula weakening).}$$

As for ΠL, any valid deduction in propositional logic is valid as well in *ma-L* where the corresponding propositions are associated with a subset $A \in 2^{ALL} \setminus \emptyset$. Thus, similarly to ΠL, stating $(a \wedge b, A)$ is equivalent to stating (a, A) and (b, A). Thanks to this property, it is always possible to rewrite a *ma*-base under the form of a collection of *clauses* labelled by subsets of agents.

Inference and consistency

Inference in *ma-L* is similar to that of *PL*. We may either use the *ma-L* axioms and the subset weakening and subset modus ponens rules, or equivalently proceed by refutation (proving $\Gamma \vdash (a, A)$ amounts to proving $\Gamma, (\neg a, All) \vdash (\bot, A)$ by repeated application of the A-B-resolution rule, where Γ stands for a collection of *ma*-formulas $(a_1, A_1), \ldots, (a_m, A_m)$).

Since 2^{ALL} is not totally ordered like $\mathcal{S}$ is, we cannot 'slice' Γ into layers. Still, one can define the restriction of Γ to a subset $A \subseteq All$ as

$$\Gamma_A = \{(a_i, A_i \cap A) \mid A_i \cap A \neq \emptyset \text{ and } (a_i, A_i) \in \Gamma\}.$$

Moreover, an *inconsistency subset* of agents for Γ can be defined as

$$inc\text{-}s(\Gamma) = \bigcup\{A \subseteq All \mid \Gamma \vdash (\bot, A)\} \text{ and } inc\text{-}s(\Gamma) = \emptyset \quad \text{if } \nexists A \text{ s.t. } \Gamma \vdash (\bot, A).$$

Note that in this definition, $A = \emptyset$ is not forbidden. For instance, let $\Gamma = \{(p, A), (q, B), (\neg p \vee q, C), (\neg q, D)\}$, then $inc\text{-}s(\Gamma) = (A \cap C \cap D) \cup (B \cap D)$, and obviously $inc\text{-}s(\Gamma_{\overline{A \cap B \cap C \cap D}}) = \emptyset$.

Clearly, *it is not the case* that the consistency of Γ ($inc\text{-}s(\Gamma) = \emptyset$) implies that Γ° is consistent. This feature contrasts with possibilistic logic. Just consider the example $\Gamma = \{(a, A), (\neg a, \overline{A})\}$, then $inc\text{-}s(\Gamma) = A \cap \overline{A} = \emptyset$ while Γ° is inconsistent. This is because there is nothing anomalous with agents that contradict each other. Yet the consistency of Γ° does entail $inc\text{-}s(\Gamma) = \emptyset$.

The semantics of *ma-L* is expressed in terms of set-valued possibility distributions, set-valued possibility measures and set-valued necessity measures, which are now introduced.

3.2. A set-valued possibility theory

A multiple agent possibility distribution (*ma*-distribution) is a function $\boldsymbol{\pi}$ from a set of interpretations Ω to 2^{All}, the set of subsets of agents. Then $\boldsymbol{\pi}(\omega)$ represents the subset of agents in All who find ω possible. If $\boldsymbol{\pi}(\omega) = \emptyset$, it means that all agents agree that ω is impossible. A *ma*-distribution is *ma*-normalised if $\exists \omega \in \Omega, \boldsymbol{\pi}(\omega) = All$. This expresses a collective (or social) consistency since there exists at least one interpretation that all agents find possible.

Associated with $\boldsymbol{\pi}$, we can define a function from $\mathcal{L}$ to 2^{All} that will be called a multiple agent possibility measure (*ma*-possibility measure) by analogy, for obvious reasons, in the following way:

$$\boldsymbol{\Pi}(a) = \bigcup_{\omega \models a} \boldsymbol{\pi}(\omega).$$

It is the set of agents for whom a is possibly true. Note that there is an equivalent definition of $\boldsymbol{\Pi}(a)$. Namely, instead of focusing on the sets $\boldsymbol{\pi}(\omega)$ of agents that consider each interpretation possible, one can represent the same information by focusing on the set of interpretations E_k considered possible by each agent $k \in All$, obviously defined as $E_k = \{\omega, k \in \boldsymbol{\pi}(\omega)\}$. Dually we have that $\boldsymbol{\pi}(\omega) = \{k \in All, \omega \in E_k\}$. Then it is easy to see that

$$\boldsymbol{\Pi}(a) = \bigcup_{\omega \models a} \boldsymbol{\pi}(\omega) = \{k, E_k \cap [a] \neq \emptyset\}.$$

By convention, $\boldsymbol{\Pi}(\bot) = \emptyset$. Moreover, if $\boldsymbol{\pi}$ is *ma*-normalised, then $\boldsymbol{\Pi}(\top) = All$. However, $\boldsymbol{\Pi}(\top) = All$ does not entail that $\boldsymbol{\pi}$ is *ma*-normalised. Just consider the example where $\Omega = \{p, \neg p\}$, $\boldsymbol{\pi}(p) = A$, $\boldsymbol{\pi}(\neg p) = \overline{A}$. Then $\boldsymbol{\Pi}(\top) = A \cup \overline{A} = All$, while $\boldsymbol{\pi}$ is *not ma*-normalised. This situation departs from usual valued possibility theory in finite settings, where $\boldsymbol{\Pi}(\top) = 1$ is equivalent to $\exists \omega, \pi(\omega) = 1$.

Besides, the condition $\boldsymbol{\Pi}(\top) = All$ is equivalent to $\bigcup_{\omega \in \Omega} \boldsymbol{\pi}(\omega) = All$. This condition means that the set of agents who find at least one interpretation possible is precisely the whole set of agents All. This means that each agent is *individually* consistent, even if altogether the agents are not necessarily mutually so. For this reason, the condition

$$\boldsymbol{\Pi}(\top) = \bigcup_{\omega \in \Omega} \boldsymbol{\pi}(\omega) = All$$

is called the i-normalisation of $\boldsymbol{\pi}$. Clearly, $\boldsymbol{\Pi}(\top) \subset All$ if and only if $\exists k \in All : E_k = \emptyset$, and the set of individually inconsistent agents is $I = All \backslash \boldsymbol{\Pi}(\top)$. Clearly $\boldsymbol{\Pi}(a) \cup \boldsymbol{\Pi}(\neg a) = \boldsymbol{\Pi}(\top)$ (the set of self-consistent agents), while $\boldsymbol{\Pi}(a) \cap \boldsymbol{\Pi}(\neg a)$ is the set of agents that ignore everything about a.

We define a multiple agent necessity measure (*ma*-necessity measure) as usual by duality, $\mathbf{N} : \mathcal{L} \to 2^{All}$, namely $N(\top) = All$ (tautologies are certain for any agent), and

$$\mathbf{N}(a) = \overline{\boldsymbol{\Pi}(\neg a)} = \bigcap_{\omega \models \neg a} \overline{\boldsymbol{\pi}(\omega)}.$$

Note that if $\boldsymbol{\Pi}(\top) \neq All$, we may fail to have the inclusion $\forall a \in \mathcal{L}, \mathbf{N}(a) \subseteq \boldsymbol{\Pi}(a)$. Indeed, while $\boldsymbol{\Pi}(a) \subseteq \boldsymbol{\Pi}(\top)$, $\mathbf{N}(a)$ includes the set I of inconsistent agents. So, $\mathbf{N}(a) \subseteq \boldsymbol{\Pi}(a)$ for all a if and only if $I = \emptyset$. However we always have that $\mathbf{N}(a) \cap \boldsymbol{\Pi}(\neg a) = \emptyset$.

In contrast, and as suggested by Smets (1988) for belief functions, we can define the set of agents that believe a as

$$\mathbf{Bel}(a) = \{k : \emptyset \neq E_k \subseteq [a]\},$$

which gathers the subset of self-consistent agents who are sure that a is true. We obviously have the desirable inclusion $\forall a \in \mathcal{L}, \mathbf{Bel}(a) \subseteq \boldsymbol{\Pi}(a)$, namely the consistent agents who are sure that a is true are among the ones who think that a is possibly true. It is easy to see that

$$\mathbf{Bel}(a) = \boldsymbol{\Pi}(\top) \setminus \boldsymbol{\Pi}(\neg a) = \boldsymbol{\Pi}(\top) \cap \mathbf{N}(a),$$

and $\mathbf{Bel}(a) \cap \mathbf{Bel}(\neg a) = \emptyset$. Conversely, $\mathbf{N}(a) = \mathbf{Bel}(a) \cup I = \{k : E_k \subseteq [a]\}$. In other words, $\mathbf{N}(a)$ corresponds to the idea of logical implicability, and accounts for the fact that in classical logic everything follows from contradictions, while $\mathbf{Bel}(a)$ corresponds to a more intuitive idea of non-trivial belief.

The following properties are easy to check:

- If $a \models b$ then $\boldsymbol{\Pi}(a) \subseteq \boldsymbol{\Pi}(b)$, $\mathbf{N}(a) \subseteq \mathbf{N}(b)$, and $\mathbf{Bel}(a) \subseteq \mathbf{Bel}(b)$;
- $\boldsymbol{\Pi}(a \vee b) = \boldsymbol{\Pi}(a) \cup \boldsymbol{\Pi}(b)$;
- $\mathbf{N}(a \wedge b) = \mathbf{N}(a) \cap \mathbf{N}(b)$;
- $\mathbf{Bel}(a \wedge b) = \mathbf{Bel}(a) \cap \mathbf{Bel}(b)$.

Thus, the set of agents who think that $a \wedge b$ is certainly true is the intersection of the set of agents who think that a is certainly true, and of the set of agents who think that b is certainly true. Moreover, we have $\mathbf{N}(a \vee b) \supseteq \mathbf{N}(a) \cup \mathbf{N}(b)$ (and likewise for **Bel**): the set of agents who are certain about $a \vee b$ is larger (in the broad sense) than the union of the agents who are certain about a and of the agents who are certain about b.

If $\boldsymbol{\Pi}(\top) = All$ (i-normalisation), then $\mathbf{Bel}(a) = \overline{\boldsymbol{\Pi}(\neg a)} = \mathbf{N}(a)$. Hence, under i-normalisation, $\mathbf{Bel}(a) \cup \boldsymbol{\Pi}(\neg a) = All$, i.e., any agent finds either a certainly true or $\neg a$ possibly true, but not both (since in any case, $\mathbf{N}(a) \cap \boldsymbol{\Pi}(\neg a) = \emptyset$). Otherwise there would exist one agent who is inconsistent. But it does not hold that $\mathbf{N}(a) \neq \emptyset$ implies that $\boldsymbol{\Pi}(a) = All$, in contrast with the usual possibility theory.

As a summary, Figure 1 pictures the relations between the set $\boldsymbol{\Pi}(a)$ of agents who think that a is possible, the set $\boldsymbol{\Pi}(\neg a)$ of agents who think that $\neg a$ is possible, the set $\mathbf{I}$ of agents who are individually inconsistent, the set $\mathbf{Bel}(a)$ of consistent agents who are certain of a, and the set $\mathbf{Bel}(\neg a)$ of consistent agents who are certain of $\neg a$.

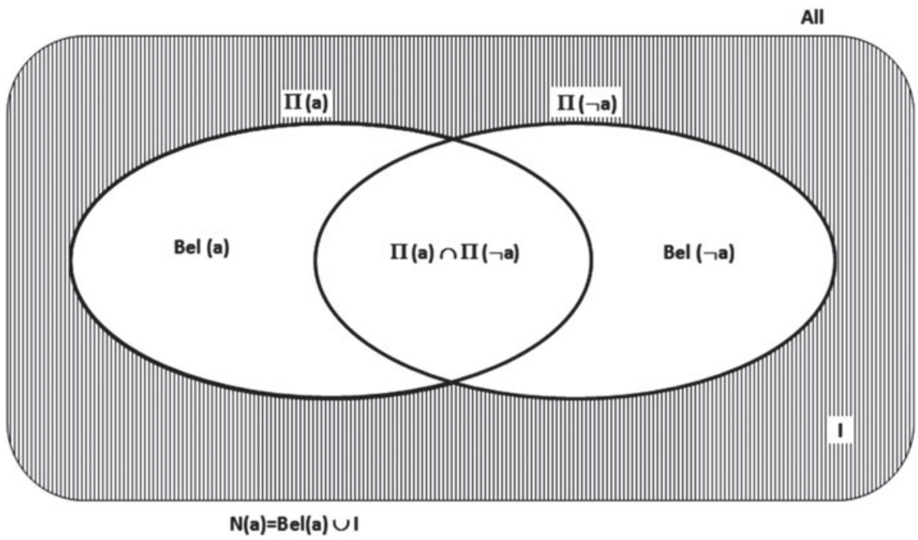

Figure 1. The different sets of agents according to their attitude w.r.t. a.

3.3. *Possibilistic semantics of the multiple agent logic*

We have now introduced the notions needed for presenting the semantics of *ma*-L. Let us first consider a *ma*-L formula (a, A), which represents the piece of information 'at least all agents in A believe a'. In other words, the agents in A find any interpretation of $\neg a$ impossible. This means that the maximal set of agents who think that $\neg a$ is possible is $\overline{A}$ (these agents, like those in A, also find the interpretations of a possible, according to this knowledge). Thus, the fact that all agents in A believe a does not prevent *all* agents from finding the models of a possible. This leads to the following semantic representation of formula (a, A) by the *ma*-distribution $\boldsymbol{\pi}_{\{(a,A)\}}$:

$$\forall \omega \in \Omega, \boldsymbol{\pi}_{\{(a,A)\}}(\omega) = \begin{cases} All & \text{if } \omega \models a \\ \overline{A} & \text{if } \omega \models \neg a, \end{cases}$$

where Ω is the set of interpretations associated with $\mathcal{L}$. It can be checked that the associated *ma*-necessity measure is such that $\mathbf{N}_{\{(a,A)\}}(a) = A$. In fact, any possibility distribution $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} \subseteq \boldsymbol{\pi}_{\{(a,A)\}}$, i.e., $\forall \omega \in \Omega, \boldsymbol{\pi}(\omega) \subseteq \boldsymbol{\pi}_{\{(a,A)\}}(\omega)$, has its associated *ma*-necessity measure $\mathbf{N}$ satisfying $\mathbf{N}(a) \supseteq \mathbf{N}_{\{(a,A)\}}(a) = A$.

More generally, the *ma*-distribution $\boldsymbol{\pi}_\Gamma$ semantically associated with a set of *ma*-formulas $\Gamma = \{(a_i, A_i), i = 1, \ldots, m\}$ is given by

$$\boldsymbol{\pi}_\Gamma(\omega) = \begin{cases} All & \text{if } \forall (a_i, A_i) \in \Gamma, \omega \models a_i \\ \bigcap\{\overline{A_i} : (a_i, A_i) \in \Gamma, \omega \models \neg a_i\} & \text{otherwise.} \end{cases}$$

Thus, the value $\boldsymbol{\pi}_\Gamma(\omega)$ of the *ma*-distribution for ω is obtained as the intersection of the different subsets $\overline{A_i}$ of agents that still find ω possible according to the different formulas (a_i, A_i) violated by this interpretation: the larger the set of agents who find the interpretation impossible, the smaller the maximal set of those that may find it possible. It can be checked that

$$\mathbf{N}_\Gamma(a_i) \supseteq A_i \quad \text{for i} = 1, \ldots, \text{m},$$

where $\mathbf{N}_\Gamma$ is the *ma*-necessity measure defined from $\boldsymbol{\pi}_\Gamma$. Note that we may only have an inequality here since Γ may, for instance, include two formulas associated with equivalent propositions, but with distinct subsets of agents.

Thus, $\boldsymbol{\pi}_\Gamma$ is the largest *ma*-distribution satisfying the set of formulas Γ in the sense that it allocates to each interpretation the maximal subset of agents that may find it possible according to the constraints expressed by the formulas (a_i, A_i), namely the constraints $\mathbf{N}_\Gamma(a_i) \supseteq A_i$. Any *ma*-distribution $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} \subseteq \boldsymbol{\pi}_\Gamma$ (i.e., $\forall \omega \in \Omega, \boldsymbol{\pi}(\omega) \subseteq \boldsymbol{\pi}_\Gamma(\omega)$) semantically agrees with Γ, which can be written $\boldsymbol{\pi} \vDash \Gamma$. The semantic entailment is defined by

$$\Gamma \vDash (a, A) \quad \text{if and only if } \forall \omega, \boldsymbol{\pi}_\Gamma(\omega) \subseteq \boldsymbol{\pi}_{\{(a,A)\}}(\omega).$$

Thus, if $\boldsymbol{\pi} \vDash \Gamma$ then $\boldsymbol{\pi} \vDash \{(a, A)\}$.

Proposition 1. *ma-L is sound and complete w.r.t. this semantics, namely*

$$\Gamma \vdash (a, A) \textit{ if and only if } \Gamma \vDash (a, A).$$

Proof. (sketch) Assume Γ is put under clausal form. This can be done without loss of information since the equivalence of $\{(a \wedge b, A)\}$ with $\{(a, A), (b, A)\}$ is semantically expressed by $\mathbf{N}(a \wedge b) = \mathbf{N}(a) \cap \mathbf{N}(b) \supseteq A \Leftrightarrow \mathbf{N}(a) \supseteq A$ and $\mathbf{N}(b) \supseteq A$. Then the *ma-L* syntactic inference $\Gamma \vdash (a, A)$ amounts to a finite number of applications of the A-B-resolution rule to $\Gamma \cup \{(\neg a, All)\}$, put in clausal form, leading to $(\bot, A)$.

For proving the soundness of the rule we have to check that, if $A_1 \cap A_2 \neq \emptyset$,

$$\forall \omega, \quad \pi(\omega)_{\{(c_1, A_1), (c_2, A_2)\}} \subseteq \pi(\omega)_{\{(c_3, A_1 \cap A_2)\}}$$

where (c_1, A_1) and (c_2, A_2) are *ma-L* clauses, and c_3 is any classical resolvent of c_1 and c_2, then $(c_3, A_1 \cap A_2)$ is a resolvent of (c_1, A_1) and (c_2, A_2) by the A-B-resolution rule $(c_1, A_1), (c_2, A_2) \vdash (c_3, A_1 \cap A_2)$. We have

$$\pi_{\{(c_1,A_1),(c_2,A_2)\}}(\omega) = \begin{cases} All & if\, \omega \models c_1 \wedge c_2 \\ \overline{A_1} \cap \overline{A_2} & if\, \omega \models \neg(c_1 \wedge c_2) \end{cases} \quad \text{and}$$

$$\pi_{\{(c_3,A_1 \cap A_2)\}}(\omega) = \begin{cases} All & if\, \omega \models c_3 \\ \overline{A_1 \cap A_2} & if\, \omega \models \neg c_3 \end{cases}.$$

It holds since the situations $\pi_{\{(c_1,A_1),(c_2,A_2)\}}(\omega) = All$ and $\pi_{\{(c_3,A_1 \cap A_2)\}}(\omega) = \overline{A_1} \cup \overline{A_2}$ cannot occur, as by hypothesis $\omega \models c_1 \wedge c_2 \Rightarrow \omega \models c_3$. It is easy to see that in the other situations the inclusion holds. So the A-B-resolution rule is sound.

For completeness, suppose $\Gamma \nvdash (a, A)$. Then $\nexists \Delta \subseteq \Gamma, \Delta \vdash (a, A)$ using A-B refutation, with all formulas in Δ. Hence for all such $\Delta \subseteq \Gamma$, either $\Delta^\circ \nvdash a$, or $\Delta^\circ \vdash a$ but $A \not\subseteq \cap_{(a_i, A_i) \in \Delta} A_i$.

In the first case $\Delta^\circ \not\models a$ (completeness of PL), hence $\Delta \not\models (a, A)$ as well.

In the second case $\exists (i, k) : k \in A, k \notin A_i$. Let $\Delta^\circ_{\{k\}} = \{a_j : k \in A_j, (a_j, A_j) \in \Delta\}$, clearly $\Delta^\circ_{\{k\}} \nvdash a$ since it does not contain a_i and a_i is needed to derive a. Hence $\Delta^\circ_{\{k\}} \not\models a$ (completeness of PL). But then $\Delta_{\{k\}} \not\models (a, \{k\})$ either, and $\Delta \not\models (a, A)$. Since this is the case for all subsets of Γ, $\Gamma \not\models (a, A)$. ■

Remark. *Another proof could proceed by equivalences:*
$\Gamma \vdash (a, A) \iff \Gamma \vdash (a, \{k\}), \forall k \in A \iff \Gamma_{\{k\}} \vdash (a, \{k\}), \forall k \in A \iff \Gamma^\circ_{\{k\}} \vdash a, \forall k \in A \iff \Gamma^\circ_{\{k\}} \models a, \forall k \in A$ *(completeness of PL). The latter stands for* $E_k = [\Gamma^\circ_{\{k\}}] \subseteq [a]$. *Then again* $\Gamma^\circ_{\{k\}} \models a \iff \Gamma_{\{k\}} \models (a, \{k\})$ *(this is* $\{(\omega, k) : \omega \in$

$E_k\} \subseteq \{(\omega, k) : \omega \in [a]\})$; then $\Gamma_{\{k\}} \models (a, \{k\}) \iff \Gamma \models (a, \{k\})$ (adding irrelevant agents to $\Gamma_{\{k\}}$). And since all of these derivations hold for all $k \in A$, this is equivalent to $\Gamma \models (a, A)$.

As defined in the previous subsection, there exist two forms of normalisation for a *ma*-distribution $\boldsymbol{\pi}$, the *ma*-normalisation and the *i*-normalisation, the first one entailing the second one:

- $\exists \omega \in \Omega$ such that $\boldsymbol{\pi}(\omega) = All$ (*ma*-normalisation);
- $\bigcup\{\boldsymbol{\pi}(\omega), \omega \in \Omega\} = All$ (*i*-normalisation).

Proposition 2. *Given a ma-base $\Gamma = \{(a_i, A_i), i = 1, \ldots, m\}$, the ma-normalisation of $\boldsymbol{\pi}_\Gamma$ is equivalent to the consistency of Γ°.*

Proof. $\boldsymbol{\pi}_\Gamma$ is *ma*-normalised if and only if $\exists \omega \in \Omega$ such that $\boldsymbol{\pi}_\Gamma(\omega) = All$, if and only if $\omega \models a_i, \forall i = 1, \ldots, m$, that is, Γ° is consistent. ■

Proposition 3. *Given a ma-base $\Gamma = \{(a_i, A_i), i = 1, \ldots, m\}$, the i-normalisation of $\boldsymbol{\pi}_\Gamma$ is equivalent to the consistency of $(\Gamma_k)^\circ = \{a_i \mid (a_i, A_i) \in \Gamma,\ k \in A_i\}$, for all agents $k \in \bigcup_{i=1,\ldots,m} A_i$.*

Proof. Indeed, assume $\exists k$ such that $(\Gamma_k)^\circ$ is inconsistent; then this agent k has an inconsistent set of beliefs, which contradicts *i*-normalisation. ■

More generally, $cons\text{-}s(\Gamma) = \bigcup_{\omega \in \Omega} \boldsymbol{\pi}_\Gamma(\omega) = \boldsymbol{\Pi}_\Gamma(\top)$ is the subset of agents who are individually consistent in Γ, while its complement $inc\text{-}s(\Gamma) = \bigcap_{\omega \in \Omega} \overline{\boldsymbol{\pi}_\Gamma(\omega)} = I$ is the subset of agents who are individually inconsistent. Lastly, $inc\text{-}s(\Gamma)$ as just defined is precisely equal to $\bigcup\{A \subseteq All \mid \Gamma \vdash (\bot, A)\}$, since they respectively correspond to the semantic and syntactic ways of computing the subset of individually inconsistent agents.

3.4. A multiple agent logic example

Let us consider the following *ma*-base:

$\Gamma = \{(\neg p \vee q, A), (\neg p \vee q, C), (\neg p \vee r, A), (\neg p \vee \neg r, B), (\neg q \vee r, All), (p, All), (q, A), (\neg q, D), (r, C)\}$.

Note that the classical *PL* base Γ° coincides with Γ^* in the example in Section 2.4.

The *ma*-distribution associated with Γ is given by

$\boldsymbol{\pi}_\Gamma(pqr) = \overline{B} \cap \overline{D}$;

$\boldsymbol{\pi}_\Gamma(pq\neg r) = \boldsymbol{\pi}_\Gamma(\neg pqr) = \boldsymbol{\pi}_\Gamma(\neg pq\neg r) = \boldsymbol{\pi}_\Gamma(\neg p\neg qr) = \boldsymbol{\pi}_\Gamma(\neg p\neg q\neg r) = \emptyset$;

$\boldsymbol{\pi}_\Gamma(p\neg qr) = \overline{A} \cap \overline{B} \cap \overline{C}$;

$\boldsymbol{\pi}_\Gamma(p\neg q\neg r) = \overline{A} \cap \overline{C}$.

Its detailed computation can be found in Table 2.

The set of agents who are individually consistent is given by

$$cons\text{-}s(\Gamma) = \boldsymbol{\Pi}_\Gamma(\top) = (\overline{A} \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap \overline{C}) \cup (\overline{B} \cap \overline{D}) = (\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{D})$$

and then the set of agents who are individually inconsistent is

$$I = inc\text{-}s(\Gamma) = (A \cup C) \cap (B \cup D).$$

If we now ask who believes r, it can easily be seen that the answer is $\mathbf{N}(r) = A \cup C \cup D$, since $\Gamma \vdash (r, A)$, $\Gamma \vdash (r, C)$, $\Gamma \vdash (r, D)$, and thus $\Gamma \vdash (r, A \cup C \cup D)$. If we want to know who believes r and is consistent, then we have to compute $\mathbf{Bel}(r)$ in the form

$$(A \cup C \cup D) \setminus inc\text{-}s(\Gamma) = (A \cup C \cup D) \setminus ((A \cup C) \cap (B \cup D)) = (\overline{B} \cap C \cap \overline{D}) \cup (A \cap \overline{B} \cap \overline{C} \cap \overline{D}).$$

Table 2. Detailed computation of the *ma*-distribution in the example.

ω	$\pi_{(\neg p \vee q, A)}$	$\pi_{(\neg p \vee q, C)}$	$\pi_{(\neg p \vee r, A)}$	$\pi_{(\neg p \vee \neg r, B)}$	$\pi_{(\neg q \vee r, All)}$	$\pi_{(p, All)}$	$\pi_{(q, A)}$	$\pi_{(\neg q, D)}$	$\pi_{(r, C)}$	π_{Γ}
pqr	All	All	All	$\overline{B}$	All	All	All	$\overline{D}$	All	$\overline{B} \cap \overline{D}$
$pq\neg r$	All	All	$\overline{A}$	All	$\emptyset$	All	All	$\overline{D}$	$\overline{C}$	$\emptyset$
$p\neg qr$	$\overline{A}$	$\overline{C}$	All	$\overline{B}$	All	All	$\overline{A}$	All	All	$\overline{A} \cap \overline{B} \cap \overline{C}$
$p\neg q\neg r$	$\overline{A}$	$\overline{C}$	$\overline{A}$	All	All	All	$\overline{A}$	All	$\overline{C}$	$\overline{A} \cap \overline{C}$
$\neg pqr$	All	All	All	All	All	$\emptyset$	All	$\overline{D}$	All	$\emptyset$
$\neg pq\neg r$	All	All	All	All	$\emptyset$	$\emptyset$	All	$\overline{D}$	$\overline{C}$	$\emptyset$
$\neg p\neg qr$	All	All	All	All	All	$\emptyset$	$\overline{A}$	All	All	$\emptyset$
$\neg p\neg q\neg r$	All	All	All	All	All	$\emptyset$	$\overline{A}$	All	$\overline{C}$	$\emptyset$

Besides, note that in Γ, we have four distinct symbols pertaining to subsets of agents, namely A, B, C, D. This induces a partition of the set of agents into $2^4 = 16$ subsets of indistinguishable agents sharing the same opinion (which correspond to the interpretations of the language induced by these symbols). For each subset S in this partition, one can compute the propositional part $(\Gamma_S)^\circ$ of the restriction Γ_S of Γ to S. The result is given below. Then, the propositional bases that are consistent are marked with $*$. The facts that $\Gamma \vdash (r, A \cup C \cup D)$ and $inc\text{-}s(\Gamma) = (A \cup C) \cap (B \cup D)$ can be checked using these *PL* bases.

$$
\begin{aligned}
\Gamma^\circ_{A\cap B\cap C\cap D} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q, \neg p \vee \neg r, r, \neg q\}\\
\Gamma^\circ_{A\cap B\cap C\cap \overline{D}} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q, \neg p \vee \neg r, r\}\\
\Gamma^\circ_{A\cap B\cap \overline{C}\cap D} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q, \neg p \vee \neg r, \neg q\}\\
\Gamma^\circ_{A\cap B\cap \overline{C}\cap \overline{D}} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q, \neg p \vee \neg r\}\\
\Gamma^\circ_{A\cap \overline{B}\cap C\cap D} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q, r, \neg q\}\\
\Gamma^\circ_{A\cap \overline{B}\cap C\cap \overline{D}} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q, r\}*\\
\Gamma^\circ_{A\cap \overline{B}\cap \overline{C}\cap D} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q, \neg q\}\\
\Gamma^\circ_{A\cap \overline{B}\cap \overline{C}\cap \overline{D}} &= \{\neg p \vee q, \neg p \vee r, \neg q \vee r, p, q\}*\\
\Gamma^\circ_{\overline{A}\cap B\cap C\cap D} &= \{\neg p \vee q, \neg q \vee r, p, \neg p \vee \neg r, r, \neg q\}\\
\Gamma^\circ_{\overline{A}\cap B\cap C\cap \overline{D}} &= \{\neg p \vee q, \neg q \vee r, p, \neg p \vee \neg r, r\}\\
\Gamma^\circ_{\overline{A}\cap B\cap \overline{C}\cap D} &= \{\neg q \vee r, p, \neg p \vee \neg r, \neg q\}*\\
\Gamma^\circ_{\overline{A}\cap B\cap \overline{C}\cap \overline{D}} &= \{\neg q \vee r, p, \neg p \vee \neg r\}*\\
\Gamma^\circ_{\overline{A}\cap \overline{B}\cap C\cap D} &= \{\neg p \vee q, \neg q \vee r, p, r, \neg q\}\\
\Gamma^\circ_{\overline{A}\cap \overline{B}\cap C\cap \overline{D}} &= \{\neg p \vee q, \neg q \vee r, p, r\}*\\
\Gamma^\circ_{\overline{A}\cap \overline{B}\cap \overline{C}\cap D} &= \{\neg q \vee r, p, \neg q\}*\\
\Gamma^\circ_{\overline{A}\cap \overline{B}\cap \overline{C}\cap \overline{D}} &= \{\neg q \vee r, p\}*
\end{aligned}
$$

Lastly, using here the obvious identity $\Gamma^\circ_{S\cup T} = \Gamma^\circ_S \cap \Gamma^\circ_T$, we could compute the sets of beliefs of any given subset of agents. However, this would be an extremely costly process. This points out that beyond the proof that $\Gamma \vdash (a, A)$ by establishing that $\Gamma \cup \{(\neg a, All)\} \vdash (\bot, A)$ using the A-B-resolution rule, we need a dual inference mechanism where A is 'fixed', rather than a, taking advantage of the 'symmetrical' roles played by a and A.

4. Outline of a multiple agent possibilistic logic

We now introduce a multiple agent *possibilistic* logic (*ma*-Π-*L*) which extends both possibilistic logic and the multiple agent of the previous section.

One may think of two different ways for building a joint generalisation of the two settings. A first idea would be to embed Π-*L* inside *ma-L*. This would mean having in the syntax formulas of the form $((a, \alpha), A)$, understood as (at least) all the agents in A have the uncertain piece of belief (a, α). *Ma-L* would then correspond to formulas of the form $((a, 1), A)$. The other way to have a graded generalisation of *ma-L*, outlined in this section, is to consider that the language is now made of formulas of the form (a, F) where F is a *fuzzy* set of agents. The degree of membership $\mu_F(k)$ is the minimal degree of certainty of a for this agent. Note that in general, F is not normalised, i.e., $\max_k \mu_F(k) \neq 1$. Clearly, All is partitioned by F into subgroups of agents F_i having the same certainty level α_i associated with proposition a (including the subgroup of agents having zero certainty); then, F can be viewed as a weighted union $\bigcup_i \alpha_i/F_i$, where all the α_i are distinct and strictly positive, and the F_is are classical, mutually disjoint subsets.

We start with the syntactic aspects of *ma*-Π-*L*, then introduce a fuzzy set-valued counterpart of the notions of possibility distribution, possibility measure, and necessity measure, before presenting the semantics that rely on these notions, and ending with an illustrative example.

4.1. Syntax

A *ma*-Π-*L* formula is a pair (a, F) where a is a proposition in $\mathcal{L}$ and F is a fuzzy subset of All. Namely, F belongs to the complete distributive lattice $L = [0, 1]^{All}$, where L is equipped with the *fuzzy set* max-based union ∪, min-based intersection ∩, and inclusion $F \subseteq G \iff \mu_F \leq \mu_G$. The following equivalence is expected:

$$(a, F), (a, F') \vdash\dashv (a, F \cup F').$$

This generalises the syntactic equivalence of *ma*-*L* $(a, A), (a, B) \vdash\dashv (a, A \cup B)$. We also expect the resolution rule, if F and G do not have disjoint supports,

$$(\neg p \vee q, F); (p \vee r, G) \vdash (q \vee r, F \cap G),$$

and the weakening rule, if $F \subseteq G$ then $(a, F) \vdash (a, G)$.

In the following we use particular fuzzy sets $F = (\alpha/A)$ such that $(\alpha/A)(k) = \alpha$ if $k \in A$, and $(\alpha/A)(k) = 0$ if $k \in \overline{A}$. Thus, we restrict ourselves to formulas of the form $(a, \alpha/A)$ that encode the piece of information 'at least all agents in A believe a at least at level α', and formulas with more complex weights $(a, \bigcup_i \alpha_i/F_i)$.

In the syntactic equivalence between

$$(a, \alpha/A), (a, \beta/B) \vdash\dashv (a, (\alpha/A) \cup (\beta/B)),$$

the weight $(\alpha/A) \cup (\beta/B)$ is provably the same as

$$\alpha/(A \cap \overline{B}) \cup \max(\alpha, \beta)/(A \cap B) \cup \beta/(\overline{A} \cap B).$$

The result is equivalent to the set of elementary formulas $\{(a, \alpha/A \cap \overline{B}), (a, \max(\alpha, \beta)/A \cap B), (a, \beta/\overline{A} \cap B)\}$, with the proviso that we omit formulas weighted by empty sets.

A *multiple agent possibilistic logic base* (*ma*-Π-*L* base) is then defined as a finite set (i.e., conjunctions) of *ma*-Π formulas. Let $\Gamma^{*\circ}$ denote the set of classical formulas obtained from Γ by ignoring the fuzzy sets of agents: if $\Gamma = \{(a_i, \alpha_i/A_i), i = 1, \ldots, m\}$ then $\Gamma^{*\circ} = \{a_i, i = 1, \ldots, m\}$.

The resolution rule becomes in *ma*-Π-*L*

$$(\neg p \vee q, \alpha/A); (p \vee r, \beta/B) \vdash (q \vee r, \min(\alpha, \beta)/(A \cap B)).$$

When $\alpha = 1 = \beta$, we retrieve the *ma*-*L* resolution rule, identifying (a, A) with $(a, 1/A)$. When $A = All = B$, we retrieve the possibilistic resolution rule.

Inference in *ma*-Π-*L* is very similar to the one in *ma*-*L*. We proceed by refutation, i.e., proving $\Gamma \vdash (a, F)$ amounts to proving $\Gamma, (\neg a, All) \vdash (\bot, F)$ by repeated application of the above resolution rule, also using the equivalence $(a, G), (a, G') \vdash\dashv (a, G \cup G')$ where G and G' are fuzzy sets. Then the fuzzy inconsistency subset for Γ is now obtained as $inc\text{-}sl(\Gamma) = \bigcup\{\alpha/A \mid \Gamma \vdash (\bot, \alpha/A)\}$. It yields the fuzzy set of individually inconsistent agents.

4.2. Multiple agent possibility theory

A *graded* multiple agent possibility distribution (*ma*-$\boldsymbol{\Pi}$-distribution) is a function $\boldsymbol{\pi}$ from a set of interpretations Ω to $[0, 1]^{All}$, the set of fuzzy subsets of agents. Then $\boldsymbol{\pi}(\omega)$ represents the fuzzy subset of agents in All who find ω possible to some extent. Associated with $\boldsymbol{\pi}$, we can define a function from $\mathcal{L}$ to $[0, 1]^{All}$, which will be called the graded multiple agent possibility measure (*ma*-$\boldsymbol{\Pi}$-possibility measure) for obvious reasons, in the following way (now using the max-based fuzzy set union):

$$\boldsymbol{\Pi}(a) = \bigcup_{\omega \in \Omega} \{\boldsymbol{\pi}(\omega), \omega \models a\}.$$

$\boldsymbol{\Pi}(a)$ is the fuzzy set of agents who think that it is possible to some extent that a is true. By duality, $\mathbf{N}(a) = \overline{\boldsymbol{\Pi}(\neg a)} = \bigcap_{\omega \in \Omega} \overline{\{\boldsymbol{\pi}(\omega), \omega \models \neg p\}}$ where fuzzy set complementation is defined as usual by $\overline{F}(k) = 1 - F(k)$, and we use the min-based fuzzy set intersection. $\mathbf{N}(a)$ is the fuzzy set of agents who are certain to some extent that a is true, or are inconsistent to some degree.

The *ma*-normalisation continues to be defined as $\exists \omega \in \Omega, \boldsymbol{\pi}(\omega) = All$ (All is clearly the same as $1/All$), and is still equivalent to the consistency of $\Gamma^{*\circ}$. The *i*-normalisation is still defined by $\boldsymbol{\Pi}(\top) = \bigcup_{\omega \in \Omega} \boldsymbol{\pi}(\omega) = All$; it still means that all the agents are individually consistent. More generally, the degree of membership of an agent k to the fuzzy set $\boldsymbol{\Pi}(\top)$ is nothing but the level to which the possibilistic logic base made of the uncertain pieces of belief held by the agent is consistent (in the Π-L sense).

4.3. Semantics

Let us consider a *ma*-Π-L formula $(a, \alpha/A)$, which expresses that a is certain at least at level α for at least all agents in A. So the set of agents in A finds any interpretation of a completely possible. Furthermore, the other agents in $\overline{A}$ are free to find the interpretation of a completely possible. Then, the maximal set of agents who find any interpretation of a completely possible is $A \cup \overline{A} = All$. Besides, the maximal set of agents who find any interpretation of $\neg a$ possible are the agents in A at least at level $1 - \alpha$, and the agents in $\overline{A}$ at least at level 1. This leads to the following semantic representation of $(a, \alpha/A)$:

$$\boldsymbol{\pi}_{\{(a,\alpha/A)\}}(\omega) = \begin{cases} 1/All & \text{if } \omega \models a \\ \{(1-\alpha)/A \cup 1/\overline{A}\} & \text{if } \omega \models \neg a. \end{cases}$$

In agreement with the syntactic equivalence $(a, \alpha/A), (a, \beta/B) \vdash\dashv (a, (\alpha/A) \cup (\beta/B))$, if $\omega \models \neg a$, $\boldsymbol{\pi}_{\{(a,\alpha/A),(a,\beta/B)\}}(\omega) = \boldsymbol{\pi}_{\{(a,\alpha/A)\}}(\omega) \cap \boldsymbol{\pi}_{\{(a,\beta/B)\}}(\omega) = ((1-\alpha)/A \cup 1/\overline{A}) \cap ((1-\beta)/B \cup 1/\overline{B}) = (1-\alpha)/(A \cap \overline{B}) \cup (1 - \max(\alpha, \beta))/(A \cap B) \cup (1-\beta)/(\overline{A} \cap B) \cup 1/(\overline{A} \cap \overline{B})$, which defines $\boldsymbol{\pi}_{\{(a,(\alpha/A) \cup (\beta/B))\}}(\omega)$.

More generally, the *ma*-Π-distribution associated with $\Gamma = \{(a_i, \alpha_i/A_i), i = 1, \ldots, m\}$ is the mapping π from Ω to L:

$$\boldsymbol{\pi}_\Gamma(\omega) = \begin{cases} 1/All & \text{if } \forall (a_i, \alpha_i/A_i) \in \Gamma, \omega \models a_i \\ \bigcap \{(1-\alpha_i)/A_i \cup 1/\overline{A_i} \mid (a_i, \alpha_i/A_i) \in \Gamma, \omega \models \neg a_i\} & \text{otherwise.} \end{cases}$$

Thus, the semantics of a base $\Gamma = \{(a_i, \alpha_i/A_i) \mid i = 1, \ldots, m\}$ is now in terms of a fuzzy set-valued distribution, for which it can be checked that $\mathbf{N}(a_i) \supseteq \alpha_i/A_i$, where $\mathbf{N}(a) = \overline{\boldsymbol{\Pi}(\neg a)}$ and $\boldsymbol{\Pi}(a) = \bigcup_{\omega:\ \omega \models a} \pi_\Gamma(\omega)$.

Soundness and completeness of *ma*-Π-L can be conjectured on the basis of the soundness and completeness of Π-L and of *ma*-L, and is a matter left for further research.

4.4. *A multiple agent possibilistic logic example*

Let us consider the following *ma*-Π-*L* base, which combines the certainty levels and the agent subsets of the two previous examples on the same propositional formulas:

$\Gamma = \{(\neg p \vee q, 0.6/A), (\neg p \vee q, 0.8/C), (\neg p \vee r, 0.9/A), (\neg p \vee \neg r, 0.1/B), (\neg q \vee r, 0.6/All), (p, 0.3/All), (q, 0.7/A), (r, 0.8/C), (\neg q, 0.2/D)\}$.

The *ma*-Π-distribution associated with Γ is given by

$\pi_\Gamma(pqr) = (0.9/B \cup 1/\overline{B}) \cap (0.8/D \cup 1/\overline{D})$

$\pi_\Gamma(pq\neg r) = (0.1/A \cup 0.4/\overline{A}) \cap (0.2/C \cup 0.4/\overline{C}) \cap (0.8/D \cup 0.4/\overline{D})$

$\pi_\Gamma(p\neg qr) = (0.3/A \cup 1/\overline{A}) \cap (0.9/B \cup 1/\overline{B}) \cap (0.2/C \cup 1/\overline{C})$

$\pi_\Gamma(p\neg q\neg r) = (0.1/A \cup 1/\overline{A}) \cap (0.2/C \cup 1/\overline{C})$

$\pi_\Gamma(\neg pqr) = (0.7/ALL)$

$\pi_\Gamma(\neg pq\neg r) = (0.2/C \cup 0.4/\overline{C}) \cap (0.4/D \cup 0.4/\overline{D})$

$\pi_\Gamma(\neg p\neg qr) = (0.3/A \cup 0.7/\overline{A})$

$\pi_\Gamma(\neg p\neg q\neg r) = (0.3/A \cup 0.7/\overline{A}) \cap (0.2/C \cup 0.7/\overline{C})$

Its detailed computation can be found in Tables 3 and 4.

As in the previous example, one can compute the possibilistic logic part $(\Gamma_S)^\circ$ for each subset of the partition of the set of agents, and compute its inconsistency level, which is also indicated below. When this inconsistency level is equal to 0, it is marked with ∗, and we retrieve exactly the same seven cases as in the second example, as expected (since the example here coincides with the second example when we ignore the certainty levels). These inconsistency levels thus correspond to the levels of inconsistency of the different subgroups of agents.

The global inconsistency level of Γ when we ignore the subset of agents, i.e., $inc\text{-}l((\Gamma)^\circ)$ is equal to 0.2 as in the first example. Thus, the 0.3-level cut of Γ, namely the set of *ma*-formulas $\{(a_i, A_i) \mid (a_i, \alpha_i/A_i) \in \Gamma \text{ and } \alpha_i \geq 0.3\}$ is collectively consistent. Besides, keeping in mind that we have $cons\text{-}s(\Gamma) = (\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{D})$ in the second example, it is clear that here, for instance, $inc\text{-}l(\Gamma_{\overline{B}\cap\overline{D}}) = 0$, i.e., the agents in $\overline{B} \cap \overline{D}$ hold consistent possibilistic belief bases individually.

$$\Gamma_{A\cap B\cap C\cap D} = \{(\neg p \vee q, 0.6), (\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (r, 0.8), (\neg q, 0.2)\} \quad inc(\Gamma_{A\cap B\cap C\cap D}) = 0.2$$

$$\Gamma_{A\cap B\cap C\cap\overline{D}} = \{(\neg p \vee q, 0.6), (\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (r, 0.8)\} \quad inc(\Gamma_{A\cap B\cap C\cap\overline{D}}) = 0.1$$

$$\Gamma_{A\cap B\cap\overline{C}\cap D} = \{(\neg p \vee q, 0.6), (\neg p \vee r, 0.9), (\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (\neg q, 0.2)\} \quad inc(\Gamma_{A\cap B\cap\overline{C}\cap D}) = 0.2$$

$$\Gamma_{A\cap B\cap\overline{C}\cap\overline{D}} = \{(\neg p \vee q, 0.6), (\neg p \vee r, 0.9), (\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7)\} \quad inc(\Gamma_{A\cap B\cap\overline{C}\cap\overline{D}}) = 0.1$$

$$\Gamma_{A\cap\overline{B}\cap C\cap D} = \{(\neg p \vee q, 0.6), (\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (r, 0.8), (\neg q, 0.2)\} \quad inc(\Gamma_{A\cap\overline{B}\cap C\cap D}) = 0.2$$

$$\Gamma_{A\cap\overline{B}\cap C\cap\overline{D}} = \{(\neg p \vee q, 0.6), (\neg p \vee q, 0.8), (\neg p \vee r, 0.9), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (r, 0.8)\} *$$

$$\Gamma_{A\cap\overline{B}\cap\overline{C}\cap D} = \{(\neg p \vee q, 0.6), (\neg p \vee r, 0.9), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7), (\neg q, 0.2)\} \quad inc(\Gamma_{A\cap\overline{B}\cap\overline{C}\cap D}) = 0.2$$

$$\Gamma_{A\cap\overline{B}\cap\overline{C}\cap\overline{D}} = \{(\neg p \vee q, 0.6), (\neg p \vee r, 0.9), (\neg q \vee r, 0.6), (p, 0.3), (q, 0.7)\} *$$

Table 3. Detailed computation of the *ma*-Π-distribution in the example, part 1.

ω	$\boldsymbol{\pi}_{(\neg p \vee q,.6/A)}$	$\boldsymbol{\pi}_{(\neg p \vee q,.8/C)}$	$\boldsymbol{\pi}_{(\neg p \vee r,.9/A)}$	$\boldsymbol{\pi}_{(\neg p \vee \neg r,.1/B)}$	$\boldsymbol{\pi}_{(\neg q \vee r,.6/All)}$	$\boldsymbol{\pi}_{(q,.7/A)}$
pqr	$1/All$	$1/All$	$1/All$	$(0.9/B \cup 1/\overline{B})$	$1/All$	$1/All$
$pq\neg r$	$1/All$	$1/All$	$(0.1/A \cup 1/\overline{A})$	$1/All$	$0.4/All$	$1/All$
$p\neg qr$	$(0.4/A \cup 1/\overline{A})$	$(0.2/C \cup 1/\overline{C})$	$1/All$	$(0.9/B \cup 1/\overline{B})$	$1/All$	$(0.3/A \cup 1/\overline{A})$
$p\neg q\neg r$	$(0.4/A \cup 1/\overline{A})$	$(0.2/C \cup 1/\overline{C})$	$(0.1/A \cup 1/\overline{A})$	$1/All$	$1/All$	$(0.3/A \cup 1/\overline{A})$
$\neg pqr$	$1/All$	$1/All$	$1/All$	$1/All$	$1/All$	$1/All$
$\neg pq\neg r$	$1/All$	$1/All$	$1/All$	$1/All$	$0.4/All$	$1/All$
$\neg p\neg qr$	$1/All$	$1/All$	$1/All$	$1/All$	$1/All$	$(0.3/A \cup 1/\overline{A})$
$\neg p\neg q\neg r$	$1/All$	$1/All$	$1/All$	$1/All$	$1/All$	$(0.3/A \cup 1/\overline{A})$

Table 4. Detailed computation of the *ma*-Π-distribution in the example, part 2.

ω	$\boldsymbol{\pi}_{(p,.3/All)}$	$\boldsymbol{\pi}_{(\neg q,.2/D)}$	$\boldsymbol{\pi}_{(r,.8/C)}$	$\boldsymbol{\pi}_{\Gamma}$
pqr	$1/All$	$(0.8/D \cup 1/\overline{D})$	$1/All$	$(0.9/B \cup 1/\overline{B}) \cap (0.8/D \cup 1/\overline{D})$
$pq\neg r$	$1/All$	$(0.8/D \cup 1/\overline{D})$	$(0.2/C \cup 1/\overline{C})$	$(.1/A \cup 1/\overline{A}) \cap (.4/All) \cap (.2/C \cup 1/\overline{C}) \cap (.8/D \cup 1/\overline{D})$
$p\neg qr$	$1/All$	$1/All$	$1/All$	$(0.3/A \cup 1/\overline{A}) \cap (0.9/B \cup 1/\overline{B}) \cap (0.2/C \cup 1/\overline{C})$
$p\neg q\neg r$	$1/All$	$1/All$	$(0.2/C \cup 1/\overline{C})$	$(0.1/A \cup 1/\overline{A}) \cap (0.2/C \cup 1/\overline{C})$
$\neg pqr$	$0.7/All$	$(0.8/D \cup 1/\overline{D})$	$1/All$	$(0.7/All) \cap (0.8/D \cup 1/\overline{D})$
$\neg pq\neg r$	$0.7/All$	$(0.8/D \cup 1/\overline{D})$	$(0.2/C \cup 1/\overline{C})$	$(0.4/All) \cap (0.2/C \cup 1/\overline{C}) \cap (0.8/D \cup 1/\overline{D})$
$\neg p\neg qr$	$0.7/All$	$1/All$	$1/All$	$(0.7/All) \cap (0.3/A \cup 1/\overline{A})$
$\neg p\neg q\neg r$	$0.7/All$	$1/All$	$(0.2/C \cup 1/\overline{C})$	$(0.7/All) \cap (0.3/A \cup 1/\overline{A}) \cap (0.2/C \cup 1/\overline{C})$

$$\Gamma_{\overline{A}\cap B\cap C\cap D} = \{(\neg p \vee q, 0.8), (\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (r, 0.8), (\neg q, 0.2)\} \quad inc(\Gamma_{\overline{A}\cap B\cap C\cap D}) = 0.2$$

$$\Gamma_{\overline{A}\cap B\cap C\cap \overline{D}} = \{(\neg p \vee q, 0.8), (\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (r, 0.8)\} \quad inc(\Gamma_{\overline{A}\cap B\cap C\cap \overline{D}}) = 0.1$$

$$\Gamma_{\overline{A}\cap B\cap \overline{C}\cap D} = \{(\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (\neg q, 0.2)\} *$$

$$\Gamma_{\overline{A}\cap B\cap \overline{C}\cap \overline{D}} = \{(\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3)\} *$$

$$\Gamma_{\overline{A}\cap \overline{B}\cap C\cap D} = \{(\neg p \vee q, 0.8), (\neg q \vee r, 0.6), (p, 0.3), (r, 0.8), (\neg q, 0.2)\} \quad inc(\Gamma_{\overline{A}\cap \overline{B}\cap C\cap D}) = 0.2$$

$$\Gamma_{\overline{A}\cap \overline{B}\cap C\cap \overline{D}} = \{(\neg p \vee \neg r, 0.1), (\neg q \vee r, 0.6), (p, 0.3), (r, 0.8)\} *$$

$$\Gamma_{\overline{A}\cap \overline{B}\cap \overline{C}\cap D} = \{(\neg q \vee r, 0.6), (p, 0.3), (\neg q, 0.2)\} *$$

$$\Gamma_{\overline{A}\cap \overline{B}\cap \overline{C}\cap \overline{D}} = \{(\neg q \vee r, 0.6), (p, 0.3)\} *$$

Since this base without its certainty levels coincides with the *ma*-base of the second example, we keep the same $inc\text{-}s(\Gamma) = (A \cup C) \cap (B \cup D)$, but it is now partitioned into the subset of agents that are inconsistent at level 0.2, namely $(A \cap D) \cup (\overline{A} \cap C \cap D)$ and the subset of agents that are inconsistent at level 0.1 only, namely $(A \cap B \cap \overline{D}) \cup (\overline{A} \cap B \cap C \cap \overline{D})$.

5. Concluding remarks

The paper has presented a multiple agent logic and outlined its possibilistic extension that enables a rich handling of inconsistency both in terms of subsets of agents and in terms of levels of certainty. In particular, two formulas such as $(\neg a, A)$ and (a, B) are contradictory only if $A \cap B \neq \emptyset$, i.e., if there exists an agent that believes both a and $\neg a$. There are many issues that have still to be studied. Beyond obvious computational issues, the kind of symmetrical roles played by a and A in (a, A) has to be investigated, in particular for jointly exploiting some possible pieces of knowledge about groups of agents such as $A \subseteq B$, or even $A \cap B \neq \emptyset$.

One may think of several future lines of research. In particular, the multiple agent extension of the generalised possibilistic logic (Dubois et al., 2012) would allow us to consider the disjunction and the negation of formulas like $(a, \alpha/A)$, and then to syntactically encode statements like '*at most* all agents in some subset believe a to at least degree α', or '*there exists* at least one agent in subset A who believes a' to at least degree α.

Note

1. In this paper, we do not differentiate between functions f from the language $\mathcal{L}$ to 2^{All} and functions $2^{\Omega} \to 2^{All}$, interpreting $f(a)$ as $f([a])$, that is, $f(a) = f(b)$ whenever a and b are logically equivalent.

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