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Logical limits of abstract argumentation frameworks

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Dung's (1995) *argumentation framework* takes as input two abstract entities: a set of *arguments* and a *binary relation* encoding attacks between these arguments. It returns acceptable sets of arguments, called *extensions*, w.r.t. a given *semantics*. While the abstract nature of this setting is seen as a great advantage, it induces a big gap with the application that it is used to. This raises some questions about the compatibility of the setting with a logical formalism (i.e., whether it is possible to instantiate it properly from a *logical* knowledge base), and about the significance of the various semantics in the application context. In this paper we tackle the above questions. We first propose to fill in the previous gap by extending Dung's (1995) framework. The idea is to consider all the ingredients involved in an argumentation process. We start with the notion of an abstract monotonic logic which consists of a language (defining the formulas) and a consequence operator. We show how to build, in a systematic way, arguments from a knowledge base formalised in such a logic. We then recall some basic postulates that any instantiation should satisfy. We study how to choose an attack relation so that the instantiation satisfies the postulates. We show that symmetric attack relations are generally not suitable. However, we identify at least one 'appropriate' attack relation. Next, we investigate under stable, semi-stable, preferred, grounded and ideal semantics the outputs of logic-based instantiations that satisfy the postulates. For each semantics, we delimit the number of extensions an argumentation system may have, characterise the extensions in terms of subsets of the knowledge base, and finally characterise the set of conclusions that are drawn from the knowledge base. The study reveals that stable, semi-stable and preferred semantics either lead to counter-intuitive results or provide no added value w.r.t. naive semantics. Besides, naive semantics either leads to arbitrary results or generalises the coherence-based approach initially developed by Rescher and Manor (1970). Ideal and grounded semantics either coincide and generalise the free consequence relation developed by Benferhat, Dubois, and Prade (1997), or return arbitrary results. Consequently, Dung's (1995) framework seems problematic when applied over deductive logical formalisms.

Keywords: abstract argumentation frameworks; logic; postulates

1. Introduction

Argumentation has become an Artificial Intelligence keyword for the last twenty years, especially for handling inconsistency in knowledge bases (e.g., Amgoud & Cayrol, 2002; Besnard & Hunter, 2008; Simari & Loui, 1992), making decisions (e.g., Amgoud & Prade, 2009; Bonet & Geffner, 1996), modelling different types of dialogues between agents like persuasion (e.g., Amgoud, Maudet, & Parsons, 2000; Zabala, Lara, & Geffner, 1999), negotiation (e.g., Rahwan et al., 2003; Sycara, 1990) and inquiry (e.g., Black & Hunter, 2009;

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Parsons, Wooldridge, & Amgoud, 2003), and for learning concepts (e.g., Amgoud & Serrurier, 2008; Mozina, Zabkar, & Bratko, 2007).

One of the most abstract argumentation formalisms in existing literature was proposed by Dung (1995). It consists of a set of *arguments* and a *binary relation* encoding *attacks* between these arguments. *Semantics* are used for defining acceptable sets of arguments, called *extensions*. Since its original formulation, Dung's (1995) framework has become very popular because it seriously abstracts away from the application for which it can be used. Indeed, the structure and the origin of arguments and attacks are left unspecified. While this can be seen as a great advantage of the framework, two important questions are raised regarding its interplay with logic – namely, when it is applied for reasoning about *inconsistent information*:

(1) Is the framework *compatible* with a logical formalism? To put it differently, is it possible to instantiate 'properly' the framework from a *logical* knowledge base?
(2) Are the semantics *significant* when the framework is instantiated from a logical knowledge base? Are they really different? What are the counterparts of the extensions (under each semantics) in the knowledge base? Are those counterparts meaningful? What are the plausible inferences under these semantics?

In this paper we answer the above questions. For this purpose, we first propose to fill in the gap between the abstract framework and the logical knowledge base from which it is specified. The idea is to consider all the ingredients involved in an argumentation problem. We start with the notion of an *abstract monotonic logic*, as defined by Tarski (1956). According to Tarski, a monotonic logic is a set of *formulas* and a *consequence operator* that satisfies some axioms. It is worth mentioning that almost *all* well-known monotonic logics such as propositional logic, modal logic, first order logic, fuzzy logic and probabilistic logic are special cases of Tarski's notion of abstract logic. Consequently, any result that holds in the general case of a Tarskian logic obviously holds under all of these particular logics. We then show how to build, in a systematic way, arguments from a knowledge base formalised in such a logic.

A 'good' instantiation of Dung's (1995) framework is one that satisfies some basic *rationality postulates*. In Caminada and Amgoud (2007), three postulates were proposed, and it was shown that not every instantiation is acceptable since some may lead to counter-intuitive results. Examples are the instantiations proposed in Prakken and Sartor (1997) and Governatori, Maher, Antoniou, and Billington (2004). The three postulates are tailored for rule-based formalisms, i.e., logical languages that distinguish between strict rules and defeasible ones. In Amgoud (2012) and Amgoud and Besnard (2009), those postulates were generalised to any Tarskian logic. Moreover, three new and intuitive ones were proposed. We study under which conditions they are satisfied or violated. The satisfaction/violation of a postulate by an instantiation depends mainly on the properties of the attack relation. We show that this relation should be based on the inconsistency of the knowledge base. We show also that symmetric relations cannot be adopted when the knowledge base contains a ternary or n-ary ($n > 2$) minimal conflict. We do establish the existence of appropriate attack relations. To sum up, there are interesting cases that cannot be captured by Dung's (1995) framework. Nevertheless, the framework can still be properly instantiated.

In the second part of the paper, we investigate the *underpinnings* of the main semantics: *stable*, *semi-stable*, *preferred*, *grounded* and *ideal*. For this purpose, we consider only the logic-based instantiations that satisfy the postulates, since the remaining ones are not good. For each semantics, we delimit the number of extensions an instantiation may have, characterise its extensions in terms of subsets of the knowledge base over which the instantiation is built, and fully characterise its set of plausible inferences that may be drawn from the knowledge base. The results show that unlike the abstract framework, its good instantiations have a finite number of extensions under stable, semi-stable and preferred semantics. This is particularly the case when the knowledge base is finite. We show that the set of all formulas used in the supports of the arguments of a stable extension is a maximal (for set inclusion) consistent subset of the knowledge base. However, not every maximal consistent subset of the knowledge base necessarily has a corresponding stable extension. This leads to arbitrary plausible inferences. In the case of a full correspondence, stable semantics does not play any role since the same result is already ensured by the naive semantics. This means that stable semantics either returns arbitrary results or provides no added value. Besides, naive semantics either leads to arbitrary results or generalises the coherence-based approach initially developed by Rescher and Manor (1970). The situation is worse for preferred semantics. The set of formulas that are used in support of the arguments of a preferred extension is a consistent (but not necessarily maximal for set inclusion) subset of the knowledge base. Thus, arbitrary inferences may be drawn from the knowledge base. Semi-stable extensions are shown to always coincide with stable ones. Thus, semi-stable semantics provides no added value w.r.t. stable semantics. Regarding the ideal and grounded semantics, there are two cases as well. In the first case, both semantics coincide, i.e., the ideal extension of any argumentation framework satisfying the postulates coincides with the grounded extension, which itself coincides with the set of arguments that are built from the free part of a knowledge base, i.e., using the subset of formulas which are not involved in the inconsistency of the knowledge base. Consequently, under these semantics, the set of plausible conclusions is the so-called *free consequences* in the coherence-based approach for reasoning about inconsistent information (Benferhat, Dubois, & Prade, 1997). In the second case, both semantics return arbitrary conclusions.

The overall study reveals that Dung's (1995) framework can be properly instantiated; i.e., there are instantiations that satisfy some basic rationality postulates. However, stable, semi-stable, preferred, ideal and grounded semantics are not suitable. Thus, Dung's framework is problematic when applied over a logical formalism, specifically a deductive one.

The paper is structured as follows: Section 2 recalls the abstract argumentation framework of Dung (1995). Section 3 details our instantiation of Dung's framework. Section 4 defines rationality postulates that such instantiation should satisfy, and investigates when those postulates are satisfied/violated. Section 5 analyses the different acceptability semantics introduced by Dung. Section 6 compares our contribution with existing works. Finally, Section 7 concludes the paper with some remarks and perspectives.

2. Dung's (1995) abstract argumentation framework

In Dung (1995), an argumentation framework consists of a set of arguments and a binary relation expressing attacks among the arguments.

Definition 1 (Argumentation framework) *An* argumentation framework *is a pair* $(\mathcal{A}, \mathcal{R})$ *where* $\mathcal{A}$ *is a set of arguments and* $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ *is an attack relation.*

A pair $(a, b) \in \mathcal{R}$ *means that a attacks b. A set* $\mathcal{E} \subseteq \mathcal{A}$ *attacks an argument b iff* $\exists a \in \mathcal{E}$ *such that* $(a, b) \in \mathcal{R}$.

Notations: We sometimes use the infix notation $a\mathcal{R}b$ to denote $(a, b) \in \mathcal{R}$.

An argumentation framework $(\mathcal{A}, \mathcal{R})$ is a *graph*, the nodes of which are the arguments of $\mathcal{A}$ and the edges of which are the attacks in $\mathcal{R}$. The arguments are evaluated using a semantics. In Dung (1995), different acceptability semantics were proposed. Some of them were refined, for instance in Caminada (2006b) and Dung, Mancarella, and Toni (2007). The basic idea behind them is the following: for a rational agent, an argument is acceptable if the agent can defend this argument against all attacks upon it. All the arguments jointly acceptable for a rational agent will be gathered in a so-called extension. An extension must satisfy a consistency requirement and must defend all of its elements.

Definition 2 (Conflict-freeness, Defence) *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation framework and $\mathcal{E} \subseteq \mathcal{A}$.*

- $\mathcal{E}$ *is* conflict-free *iff* $\nexists a, b \in \mathcal{E}$ *such that* $(a, b) \in \mathcal{R}$.
- $\mathcal{E}$ defends *an argument* a *iff* $\forall\ b \in \mathcal{A}$, *if* $(b, a) \in \mathcal{R}$, *then* $\exists c \in \mathcal{E}$ *such that* $(c, b) \in \mathcal{R}$.

The following definition recalls the main semantics that were proposed by Dung (1995), as well as their refinements (Caminada 2006b; Dung, Mancarella, & Toni 2007). It is worth noticing that the fundamental semantics features admissible extensions. The other semantics are based on it.

Definition 3 (Acceptability semantics) *Let $\mathcal{T} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework, and $\mathcal{E} \subseteq \mathcal{A}$ be a conflict-free set.*

- $\mathcal{E}$ *is a* naive *extension iff it is a maximal (w.r.t. set $\subseteq$) conflict-free set.*
- $\mathcal{E}$ *is an* admissible *set iff it defends all of its elements.*
- $\mathcal{E}$ *is a* complete *extension iff it is an admissible set that contains any argument it defends.*
- $\mathcal{E}$ *is a* preferred *extension iff it is a maximal (w.r.t. set $\subseteq$) admissible set.*
- $\mathcal{E}$ *is a* stable *extension iff it attacks any argument in $\mathcal{A} \setminus \mathcal{E}$.*
- $\mathcal{E}$ *is a* semi-stable extension *iff it is a complete extension and the union of the set $\mathcal{E}$ and the set of all arguments attacked by $\mathcal{E}$ is maximal (w.r.t. $\subseteq$).*
- $\mathcal{E}$ *is a* grounded extension *iff it is a minimal (w.r.t. set $\subseteq$) complete extension.*
- $\mathcal{E}$ *is an* ideal extension *iff it is a maximal (w.r.t. set $\subseteq$) admissible set contained in every preferred extension.*

Notations: $\texttt{Ext}_x(\mathcal{T})$ denotes the set of all extensions of $\mathcal{T}$ under semantics x where $x \in \{n, p, s, ss\}$ and n (respectively p, s, ss) stands for naive (respectively preferred, stable and semi-stable). When we do not need to refer to a particular semantics, we write $\texttt{Ext}(\mathcal{T})$ for short. Since grounded and ideal extensions are unique for any argumentation framework $\mathcal{T}$, they will be denoted respectively by $\texttt{GE}(\mathcal{T})$ and $\texttt{IE}(\mathcal{T})$.

It is worth recalling that stable extensions are naive (respectively preferred) extensions but that the converse is not always true. Moreover, an argumentation framework has *at least* one preferred extension, but it may have no stable extensions. When stable extensions exist, they coincide with the semi-stable ones (i.e., if $|\texttt{Ext}_s(\mathcal{T})| > 0$, then $\texttt{Ext}_s(\mathcal{T}) = \texttt{Ext}_{ss}(\mathcal{T})$ for any argumentation framework $\mathcal{T}$).

Example 4 *Let us consider the argumentation framework $\mathcal{T} = (\mathcal{A}, \mathcal{R})$ such that:*

- $\mathcal{A} = \{a, b, c, d, e, f, g\}$.
- $\mathcal{R} = \{(c, b), (b, e), (e, c), (d, c), (a, d), (d, a), (a, f), (f, g)\}$.

This framework has five naive extensions:

- $\mathcal{E}_1 = \{a, c, g\}$,

- $\mathcal{E}_2 = \{d, e, f\}$,
- $\mathcal{E}_3 = \{b, d, f\}$,
- $\mathcal{E}_4 = \{a, e, g\}$, *and*
- $\mathcal{E}_5 = \{a, b, g\}$.

It has one stable/semi-stable extension $\mathcal{E}_3$ *and two preferred extensions:* $\mathcal{E}_3$ *and* $\mathcal{E}_6 = \{a, g\}$. *Both the grounded and the ideal extensions are empty, i.e.,* $\texttt{GE}(\mathcal{T}) = \texttt{IE}(\mathcal{T}) = \emptyset$.

An argumentation framework may be *infinite*, i.e., its set of arguments may be infinite. Consequently, it may have an infinite number of extensions (under a given semantics).

3. Logic-based instantiations of Dung's (1995) framework

Argumentation is an alternative approach for reasoning with inconsistent information. It follows three main steps: (i) constructing *arguments* and counterarguments from a logical knowledge base; (ii) defining the *status* of each argument; and (iii) specifying the *conclusions* to be drawn from the base. In what follows, we instantiate Dung's (1995) framework by defining all of these items. We start with an *abstract logic* as defined by Alfred Tarski (1956), from which the notions of argument and attacks between arguments are defined.

3.1. Tarski's (1956) abstract consequence operators

Tarski (1956) defines a logic as a pair $(\mathcal{L}, \mathsf{CN})$ where the members of $\mathcal{L}$ are called *well-formed formulas*, and CN is a *consequence operator*. No constraints are defined on the logical language $\mathcal{L}$. Thus, no particular connectors are required. However, the consequence operator CN is a function from $2^{\mathcal{L}}$ to $2^{\mathcal{L}}$ that should satisfy the following axioms:

(1) $X \subseteq \mathsf{CN}(X)$ (Expansion)
(2) $\mathsf{CN}(\mathsf{CN}(X)) = \mathsf{CN}(X)$ (Idempotence)
(3) $\mathsf{CN}(X) = \bigcup_{Y \subseteq_f X} \mathsf{CN}(Y)$ (Compactness)
(4) $\mathsf{CN}(\{x\}) = \mathcal{L}$ for some $x \in \mathcal{L}$ (Absurdity)
(5) $\mathsf{CN}(\emptyset) \neq \mathcal{L}$ (Coherence)

Notations: $Y \subseteq_f X$ means that Y is a finite subset of X.

Intuitively, $\mathsf{CN}(X)$ returns the set of formulas that are logical consequences of X according to the logic in question. In Tarski (1956), it was shown that CN is a closure operator, that is, CN enjoys properties such as:

Property 5 *Let* $X, X', X'' \subseteq \mathcal{L}$.

(1) $X \subseteq X' \Rightarrow \mathsf{CN}(X) \subseteq \mathsf{CN}(X')$.
(2) $\mathsf{CN}(X) \cup \mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$.
(3) $\mathsf{CN}(X) = \mathsf{CN}(X') \Rightarrow \mathsf{CN}(X \cup X'') = \mathsf{CN}(X' \cup X'')$.
(4) $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X) \cap \mathsf{CN}(X')$.

Almost all well-known monotonic logics (classical logics, intuitionistic logics, modal logics, etc.) can be viewed as special cases of Tarski's notion of an abstract logic. AI introduced non-monotonic logics, which do not satisfy monotonicity (Bobrow, 1980).

Once $(\mathcal{L}, \mathsf{CN})$ is fixed, a notion of *consistency* arises as follows:

Definition 6 (Consistency) *Let* $X \subseteq \mathcal{L}$. X *is* consistent *w.r.t. the logic* $(\mathcal{L}, \mathsf{CN})$ *iff* $\mathsf{CN}(X) \neq \mathcal{L}$. *It is* inconsistent *otherwise.*

In simple English, this says that X is consistent iff its set of consequences is not the set of all formulas. The coherence requirement (absent from Tarski's original proposal but added here to avoid considering trivial systems) forces the empty set $\emptyset$ to always be consistent – this makes sense for any reasonable logic.

One can show that if a set X is consistent, then its closure under CN is also consistent and any proper subset of X is consistent.

Property 7 *Let $X \subseteq \mathcal{L}$.*

(1) If X is consistent, then $\mathsf{CN}(X)$ is consistent as well.
(2) $\forall X' \subseteq X$, if X is consistent, then X' is consistent.
(3) $\forall X' \subseteq X$, if X' is inconsistent, then X is inconsistent.

If a set $X \subseteq \mathcal{L}$ of formulas is inconsistent, this means that it contains *minimal conflicts*.

Definition 8 (Minimal conflict) *A set $C \subseteq \mathcal{L}$ is a* minimal conflict *iff:*

- *C is inconsistent.*
- *$\forall x \in C$, $C \backslash \{x\}$ is consistent.*

Notations: Let $X \subseteq \mathcal{L}$. $\mathcal{C}_X$ denotes the set of all minimal conflicts C such that $C \subseteq X$. $\mathtt{Max}(X)$ is the set of all maximal (for set inclusion) consistent subsets of X, $\mathtt{Free}(X) = \bigcap_{\mathcal{S}_i \in \mathtt{Max}(X)} \mathcal{S}_i$, and $\mathtt{Inc}(X) = X \setminus \mathtt{Free}(X)$.

The following properties are useful for proving our results.

Property 9 *For all $X \subseteq \Sigma \subseteq \mathcal{L}$,*

- *if X is consistent then $\mathcal{C}_X = \emptyset$;*
- *if X is consistent then $X \subseteq \mathcal{S}$ for some $\mathcal{S} \in \mathtt{Max}(\Sigma)$;*
- *if X is inconsistent then there exists at least one minimal conflict C such that $C \subseteq X$.*

The next property is true in the case that the underlying logic is *adjunctive*. Let us first define this new concept.

Definition 10 (Adjunctiveness) *A logic $(\mathcal{L}, \mathsf{CN})$ is* adjunctive *iff for all x and y in $\mathcal{L}$, there exists $z \in \mathcal{L}$ such that $\mathsf{CN}(\{z\}) = \mathsf{CN}(\{x, y\})$.*

Intuitively, an adjunctive logic infers, from the union of two formulas $\{x, y\}$, some formula(s) that can be inferred neither from x alone nor from y alone (except, of course, when y ensues from x or vice versa). In fact, most well-known logics are adjunctive.[1] A logic which is not adjunctive could for instance fail to deny $x \vee y$ from the premises $\{\neg x, \neg y\}$.

Property 11 *Let $(\mathcal{L}, \mathsf{CN})$ be* adjunctive*, $C \subseteq \mathcal{L}$ be a minimal conflict. For all $X \subset C$, if $X \neq \emptyset$, then:*

(1) $\exists x \in \mathcal{L}$ such that $\mathsf{CN}(\{x\}) = \mathsf{CN}(X)$.
(2) $\exists x_1 \in \mathsf{CN}(X)$ and $\exists x_2 \in \mathsf{CN}(C \setminus X)$ such that the set $\{x_1, x_2\}$ is inconsistent.

3.2. Tarskian logic-based instantiations

Let $(\mathcal{L}, \mathsf{CN})$ be a fixed abstract logic. From now on, we will consider a *knowledge base* Σ, which is a subset of the logical language $\mathcal{L}$ (in symbols, $\Sigma \subseteq \mathcal{L}$). This base may be infinite, however, with no loss of generality and for the sake of simplicity, it is assumed to be free of tautologies:

Assumption 12 *Let Σ be a knowledge base. For all $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$.*

The first parameter of an argumentation framework is the set of arguments. In Dung's (1995) framework, an argument is an abstract entity. In what follows, it is built from a knowledge base Σ. It gives a reason for believing a conclusion. Formally, an argument satisfies three main requirements: (i) the reason is a subset of the knowledge base, thus restricting the origin of the arguments; (ii) the reason should be consistent, thus avoiding absurd reasons; and (iii) the reason is minimal. The third requirement means that only relevant information w.r.t. the conclusion is considered.

Definition 13 (Argument) *Let Σ be a knowledge base. An* argument *is a pair (X, x) such that:*

(1) $X \subseteq \Sigma$ and $x \in \mathcal{L}$.
(2) X is consistent.
(3) $x \in \mathsf{CN}(X)$.
(4) $\nexists X' \subset X$ such that $x \in \mathsf{CN}(X')$.

An argument (X, x) is a sub-argument *of another argument (X', x') iff $X \subseteq X'$.*

Let us introduce some notations that will be used throughout the paper.

Notations: `Supp` and `Conc` are two functions that return respectively the *support* X and the *conclusion* x of an argument (X, x). `Sub` is a function that returns all the sub-arguments of a given argument. For $X \subseteq \mathcal{L}$, $\texttt{Arg}(X)$ denotes the set of all arguments that can be built from X by means of Definition 13. For a set $\mathcal{E}$ of arguments, $\texttt{Concs}(\mathcal{E}) = \{\texttt{Conc}(a) \mid a \in \mathcal{E}\}$ and $\texttt{Base}(\mathcal{E}) = \bigcup_{a \in \mathcal{E}} \texttt{Supp}(a)$.

The following property shows that the conclusion of any argument is consistent.

Property 14 *For all $(X, x) \in \texttt{Arg}(\Sigma)$, the set $\{x\}$ is consistent.*

Due to Assumption 12 ($x \notin \mathsf{CN}(\emptyset)$ for all $x \in \Sigma$), it can also be shown that each consistent formula in Σ gives birth to an argument:

Property 15 *Let Σ be a knowledge base such that for all $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$. For all $x \in \Sigma$ such that the set $\{x\}$ is consistent, $(\{x\}, x) \in \texttt{Arg}(\Sigma)$.*

Since CN is monotonic, constructing arguments is a monotonic process; additional knowledge never causes the set of arguments to shrink but only gives rise to extra arguments that may interact with the existing ones.

Property 16 $\texttt{Arg}(\Sigma) \subseteq \texttt{Arg}(\Sigma')$ *whenever $\Sigma \subseteq \Sigma' \subseteq \mathcal{L}$.*

We show now that any proper subset of a minimal conflict is the support of at least one argument. This is particularly so in the case of adjunctive logics. This result is of utmost importance as regards encoding the attack relation.

Proposition 17 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive and Σ be a knowledge base. For all non-empty proper subsets X of some minimal conflict $C \in \mathcal{C}_\Sigma$, there exists $a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Supp}(a) = X$.*

Proposition 17 is fundamental because it says that if statements from Σ contradict others then it is always possible to define an argument exhibiting the conflict.

In the sequel, we use the term *system* instead of *framework* in order to distinguish the framework of Dung (1995) from its logical instantiations, which are defined as follows.

Definition 18 (Argumentation system) *Let* $(\mathcal{L}, \mathsf{CN})$ *be a given Tarskian logic and* $\Sigma \subseteq \mathcal{L}$ *be a knowledge base. An* argumentation system *over* Σ *is a pair* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *such that* $\mathcal{R} \subseteq \texttt{Arg}(\Sigma) \times \texttt{Arg}(\Sigma)$ *(intuitively, it is an* attack *relation).*

In the previous definition, the attack relation is left unspecified. However, in Section 4 we show that it should be assigned some properties, otherwise the system may return counter-intuitive results.

The arguments of a system $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ are evaluated using one of the semantics given in Definition 3. Recall that the structure of arguments is not taken into account in those semantics. The extensions are used in order to define the conclusions that may be drawn from Σ according to the system $\mathcal{T}$. The idea is to conclude x if it is the conclusion of an argument in every extension of the system.

Definition 19 (Output) *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *and* $\texttt{Ext}(\mathcal{T})$ *its set of extensions under a given semantics. For* $x \in \mathcal{L}$, x *is a* conclusion *of* $\mathcal{T}$ *iff* $\forall \mathcal{E}_i \in \texttt{Ext}(\mathcal{T})$, $\exists a \in \mathcal{E}_i$ *such that* $\texttt{Conc}(a) = x$. *We write* $\texttt{Output}(\mathcal{T})$ *to denote the set of all conclusions of* $\mathcal{T}$.

It follows immediately from the definition that the set of conclusions exactly consists of the formulas which happen to be, in each extension, the conclusion of an argument of the extension.

Property 20 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *and* $\texttt{Ext}(\mathcal{T})$ *its set of extensions under a given semantics. It holds that*

$$\texttt{Output}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \texttt{Ext}(\mathcal{T})} \texttt{Concs}(\mathcal{E}_i).$$

Finally, it is obvious that the outputs of an argumentation system are consequences of the corresponding knowledge base under the consequence operator CN.

Property 21 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ. $\texttt{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\Sigma)$.

4. On the quality of logic-based argumentation systems

We have shown so far how to build an argumentation system from a logical knowledge base. The system is still incomplete since the attack relation is not specified. As already mentioned, Dung (1995) is silent on how to proceed in order to obtain a reasonable $\mathcal{R}$ in practice. It happens that it is in fact a delicate step. We will show next that the choice of this relation is crucial for the 'soundness' of the system. Soundness is determined by some *rationality postulates* that any system should satisfy.

The first work on rationality postulates in argumentation was undertaken by Caminada and Amgoud (2007). The authors focused *only* on *rule-based systems* (i.e., systems that distinguish between strict and defeasible rules in their underlying logical language). They proposed the following postulates that such systems should satisfy:

- *Closure:* the idea is that if a system concludes x and there is a strict rule[2] $x \to y$, then the system should also conclude y.
- *Direct consistency:* the set of conclusions of arguments of each extension should be consistent.
- *Indirect consistency:* the closure of the set of conclusions of arguments of each extension should be consistent.

As obvious as they may appear, these postulates are violated by most rule-based systems (such as Amgoud, Caminada, Cayrol, Lagasquie, & Prakken 2004; Governatori, Maher, Antoniou, and Billington 2004; Prakken 2010; Prakken & Sartor 1997. Besides, they are tailored for rule-based logics. Their counterparts for any Tarskian logic were defined in Amgoud (2012) and Amgoud and Besnard (2009). Moreover, three new postulates were proposed in Amgoud (2012). In what follows, we recall all the postulates that are necessary for our study.

4.1. Rationality postulates for logic-based argumentation systems

The first rationality postulate that an argumentation system should satisfy concerns the closure of its output. The basic idea is that the conclusions of a formalism should be 'complete'. There should be no case where a user performs some extra reasoning on her own to derive statements that the formalism apparently 'forgot' to entail. In Caminada and Amgoud (2007), closure is defined for rule-based argumentation systems. In what follows, we extend this postulate to systems that are grounded in any Tarskian logic. The idea is to define closure using the consequence operator CN.

Postulate 22 (Closure under CN) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. For all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}) = \mathsf{CN}(\mathtt{Concs}(\mathcal{E}))$. We say that $\mathcal{T}$ is closed under CN.*

In Caminada and Amgoud (2007), closure is imposed both on the extensions of a system and on its output set. The next result shows that the closure of the output set does not deserve to be a separate postulate since it immediately follows from the closure of extensions.

Proposition 23 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ is closed under CN, then $\mathtt{Output}(\mathcal{T}) = \mathsf{CN}(\mathtt{Output}(\mathcal{T}))$.*

The second rationality postulate concerns *sub-arguments*. An argument may have one or several sub-arguments, reflecting the different premises on which it is based. Thus, the acceptance of an argument should also imply the acceptance of all of its sub-parts.

Postulate 24 (Closure under sub-arguments) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. For all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, if $a \in \mathcal{E}$, then $\mathtt{Sub}(a) \subseteq \mathcal{E}$. We say that $\mathcal{T}$ is closed under sub-arguments.*

These two postulates have a great impact on the extensions of an argumentation system, as shown by the following result:

Proposition 25 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ is closed under sub-arguments and under CN, then for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}) = \mathsf{CN}(\mathtt{Base}(\mathcal{E}))$.*

The third rationality postulate concerns the *consistency* of the results. It ensures that the set of conclusions supported by each extension is consistent. The following postulate generalises the *direct* consistency postulate, which was proposed for rule-based argumentation systems in Caminada and Amgoud (2007):

Postulate 26 (Consistency) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. For all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E})$ is consistent. We say that $\mathcal{T}$ satisfies consistency.*

As for closure, in Caminada and Amgoud (2007) a postulate imposing the consistency of the output is defined. We show next that such a postulate is not necessary. Indeed, an argumentation system that satisfies Postulate 26 necessarily has a consistent output.

Proposition 27 *If an argumentation system* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *satisfies consistency, then the set* $\texttt{Output}(\mathcal{T})$ *is consistent.*

We show next that argumentation systems that satisfy both consistency and closure under sub-arguments enjoy a strong version of consistency. Indeed, the set of formulae used in the arguments of each extension is consistent. It is worth mentioning that this result is *very general*, as it holds under any semantics, any attack relation and any Tarskian logic.

Proposition 28 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that for all* $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$. *If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments, then for all* $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\texttt{Base}(\mathcal{E})$ *is consistent.*

Since the free formulas of a knowledge base (i.e., the ones that are not involved in any minimal conflict) are the 'hard' part in the base, it is natural that any argument that is built only from this part should be in every extension of an argumentation system built over the knowledge base.

Postulate 29 (Free Precedence) *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ. *For all* $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\texttt{Arg}(\texttt{Free}(\Sigma)) \subseteq \mathcal{E}$. *We say that* $\mathcal{T}$ *satisfies* free precedence.

We show next that the free formulas are drawn by any argumentation system satisfying Postulate 29.

Proposition 30 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ. *If* $\mathcal{T}$ *satisfies free precedence, then* $\texttt{Free}(\Sigma) \subseteq \texttt{Output}(\mathcal{T})$ *(under any of the reviewed semantics).*

The last postulate says that if the support and the conclusion of an argument are part of the conclusions of a given extension, then the argument should belong to the extension. Informally: if each step in the argument is good enough to be in the extension, then so is the argument itself.

Postulate 31 (Exhaustiveness) *An argumentation system* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *over a knowledge base* Σ *satisfies* exhaustiveness *iff for all* $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, *for all* $(X, x) \in \texttt{Arg}(\Sigma)$, *if* $X \cup \{x\} \subseteq \texttt{Concs}(\mathcal{E})$, *then* $(X, x) \in \mathcal{E}$.

The following result shows that when this postulate is satisfied, then extensions are closed in terms of arguments.

Proposition 32 *If an argumentation system* $\mathcal{T}$ *is closed under both* CN *and sub-arguments and satisfies the exhaustiveness postulate, then* $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$ *(under any of the reviewed semantics).*

The five postulates are generally independent. However, in the case of naive and stable semantics, closure under the consequence operator CN is induced from closure under sub-arguments and consistency. This is particularly the case when the attack relation is based on inconsistency.

Definition 33 (Conflict-dependent) *An attack relation* $\mathcal{R}$ *is* conflict-dependent *iff for all* $a, b \in \texttt{Arg}(\Sigma)$, *if* $a\mathcal{R}b$ *then* $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ *is inconsistent.*

The above definition says that $\mathcal{R}$ should show no attack from a to b unless Σ provides evidence (according to CN) that the supports of a and b conflict with each other. That is, being conflict-dependent ensures that, when passing from Σ to $(\mathtt{Arg}(\Sigma), \mathcal{R})$, no conflict is 'invented' in $\mathcal{R}$. Note that all the attack relations that are used in existing structured argumentation systems are conflict-dependent (see Gorogiannis & Hunter, 2011, for a summary of existing relations).

Proposition 34 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive and stable semantics), then it is closed under* CN *(under naive and stable semantics).*

It was shown in Amgoud (2012) that the five postulates are *compatible*, i.e., they can be satisfied altogether by an argumentation system. This is particularly witnessed in the argumentation system studied in Cayrol (1995). This system is grounded in propositional logic (a Tarskian logic) and uses the *assumption attack* relation defined in Elvang-Gøransson, Fox, and Krause (1993). According to this relation, an argument attacks another if its conclusion is the negation of an element of the support of the second argument. This relation was generalised to any Tarskian logic in Amgoud and Besnard (2010) as follows:

Definition 35 (Assumption attack relation) *Let $(\mathcal{L}, \mathsf{CN})$ be a Tarskian logic. An argument (X, x) attacks another argument (X', x') iff $\exists y \in X'$ such that the set $\{x, y\}$ is inconsistent. This relation will be denoted by $\mathcal{R}_{as}$.*

It was shown in Amgoud (2012) that any argumentation system $(\mathtt{Arg}(\Sigma), \mathcal{R}_{as})$ satisfies the five postulates. This is certainly a positive result as it shows that Dung's (1995) abstract framework can be correctly instantiated with logical formalisms. However, in the next section we show that there is a broad class of natural instantiations that are not possible since they violate the postulates.

4.2. On the violation of consistency postulate

In subsection 3.2 we provided a clear definition of an argument and how it is built from a knowledge base Σ. However, there is still no indication on how the attack relation $\mathcal{R}$ is chosen and how it is related to Σ. Moreover, in Caminada and Amgoud (2007) it was shown that there are some instantiations of Dung's (1995) framework that violate the consistency postulate. This means that the choice of the attack relation has a direct impact on the postulates. This also means that conflict-freeness is not sufficient to ensure consistency. Thus, an attack relation should enjoy some basic properties. The first one concerns its *origin*. We show that an attack relation should be based on inconsistency, and thus conflict-dependent.

When the attack relation is conflict-dependent, then it is empty when the knowledge base is consistent.

Proposition 36 *Let $(\mathtt{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent. If Σ is consistent, then $\mathcal{R} = \emptyset$.*

It follows that when the attack relation is conflict-dependent, if a set of arguments is such that its corresponding base (set-theoretic union of supports) is consistent then it is a conflict-free set:

Proposition 37 *Let* $(\texttt{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent.* $\forall \mathcal{E} \subseteq \texttt{Arg}(\Sigma)$, *if* $\texttt{Base}(\mathcal{E})$ *is consistent, then* $\mathcal{E}$ *is conflict-free.*

It is also worth pointing out that an attack relation which is conflict-dependent exhibits no self-attacks.

Proposition 38 *Let* $(\texttt{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent. For all* $a \in \texttt{Arg}(\Sigma)$, $(a, a) \notin \mathcal{R}$.

Let us now consider the following example of an argumentation system that is built from a propositional knowledge base and uses the symmetric attack relation known as *rebut* (Elvang-Gøransson, Fox, & Krause, 1993). According to this relation, *an argument a attacks another argument b iff* $\texttt{Conc}(a) \equiv \neg\texttt{Conc}(b)$ *(in the case that* $(\mathcal{L}, \mathsf{CN})$ *is propositional logic). This relation will be denoted by* $\mathcal{R}_{re}$.

Example 39 *Let* $(\mathcal{L}, \mathsf{CN})$ *be propositional logic and* $\Sigma = \{x, y, x \to \neg y\}$. *Let us consider the following set of arguments:*

- $a_1 = (\{x\}, x)$.
- $a_2 = (\{y\}, y)$.
- $a_3 = (\{x \to \neg y\}, x \to \neg y)$.
- $a_4 = (\{x, x \to \neg y\}, \neg y)$.
- $a_5 = (\{y, x \to \neg y\}, \neg x)$.
- $a_6 = (\{x, y\}, x \wedge y)$.

The rebut relation is as follows: $\{(a_1, a_5), (a_5, a_1), (a_2, a_4), (a_4, a_2), (a_3, a_6), (a_6, a_3)\}$. *The set* $\{a_1, a_2, a_3\}$ *– as a finite representation (Amgoud, Besnard, & Vesic, 2011) for all of its 'mates', i.e., the arguments* $(\{x\}, \ldots)$ *and* $(\{y\}, \ldots)$ *and* $(\{x \to \neg y\}, \ldots)$ *– is an admissible extension of the system* $(\texttt{Arg}(\Sigma), \mathcal{R}_{re})$. *However, the set* $\{\texttt{Conc}(a_1), \texttt{Conc}(a_2), \texttt{Conc}(a_3)\}$ *is inconsistent. Similarly, the set* $\{a_4, a_5, a_6\}$ *is (a finite representation of) another admissible extension whose set of conclusions is inconsistent.*

This example shows that an admissible set of arguments may fail to have a consistent set of conclusions. The problem encountered with the rebut relation is due to the fact that it is binary, in compliance with Dung's (1995) definitions of the attack relation as binary. Thus, the ternary conflict between a_1, a_2 and a_3 is not captured. Particularly, symmetric attack relations are crippled by non-binary minimal conflicts. Indeed, we show that when the attack relation is symmetric, Postulate 26 is violated.

Proposition 40 *Let* $(\mathcal{L}, \mathsf{CN})$ *be adjunctive and* Σ *be a knowledge base such that* $\exists C \in \mathcal{C}_\Sigma$ *and* $|C| > 2$. *If* $\mathcal{R}$ *is conflict-dependent and symmetric, then the argumentation system* $(\texttt{Arg}(\Sigma), \mathcal{R})$ *violates consistency.*

This result shows a broad class of attack relations that cannot be used in argumentation: the symmetric ones. Relations like rebut or a union of rebut and any other conflict-dependent attack relation would lead to the violation of consistency, namely when there exist n-ary ($n > 2$) minimal conflicts in the knowledge base. Consequently, the symmetric systems studied in Coste, Devred, and Marquis (2005) cannot be adopted in a concrete application.

5. The outcomes of logic-based argumentation systems

The aim of this section is to investigate the underpinnings of the different acceptability semantics introduced in Dung (1995), Dung, Mancarella, and Toni (2007), and

Caminada (2006b) and to check whether they make sense in a concrete application. Recall that those semantics are defined without considering either the internal structure or the origin of arguments and attacks. In this section, we fully characterise for the first time both the extensions and the output set of any Tarskian logic-based argumentation system under naive, stable, semi-stable, preferred, grounded and ideal semantics. For this purpose, we consider only systems that enjoy the rationality postulates introduced in the previous section (as the other systems are regarded as ill-fated instantiations of Dung's 1995 framework).

5.1. Naive semantics

In this section, we characterise the outputs of an argumentation system under naive semantics. We show that the naive extensions of *any* argumentation system that satisfies consistency and closure under sub-arguments *always* return maximal (for set inclusion) consistent subsets of Σ.

Theorem 41 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathtt{Base}(\mathcal{E}) \in \mathtt{Max}(\Sigma)$.*
- *For all $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_n(\mathcal{T})$, if $\mathtt{Base}(\mathcal{E}_i) = \mathtt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

The next theorem confirms that *any* maximal consistent subset of Σ defines a naive extension of an argumentation system which satisfies consistency and closure under sub-arguments. This is the case when the logic $(\mathcal{L}, \mathsf{CN})$ is adjunctive.

Theorem 42 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_n(\mathcal{T})$.*
- *For all $\mathcal{S}_i, \mathcal{S}_j \in \mathtt{Max}(\Sigma)$, if $\mathtt{Arg}(\mathcal{S}_i) = \mathtt{Arg}(\mathcal{S}_j)$ then $\mathcal{S}_i = \mathcal{S}_j$.*
- *For all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathcal{S} = \mathtt{Base}(\mathtt{Arg}(\mathcal{S}))$.*

It follows that any argumentation system that satisfies the two postulates 24 and 26 enjoys a full correspondence between the maximal consistent subsets of Σ and the naive extensions of the system.

Corollary 43 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics) iff there is a bijection between the naive extensions of $\mathcal{T}$ and the elements of $\mathtt{Max}(\Sigma)$.*

A direct consequence of the previous result is that the number of naive extensions of an argumentation system is less than or equal to the number of maximal consistent sub-bases of the knowledge base over which the system is built. Thus, if the knowledge base is finite, then the system has a finite number of naive extensions.

Corollary 44 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).*

- $|\mathtt{Ext}_n(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|$.
- *If Σ is finite, then $\mathcal{T}$ has a finite number of naive extensions.*

The following result characterises the case where an argumentation system has an empty naive extension. It shows that the knowledge base contains only inconsistent formulae.

Corollary 45 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics). If $\mathtt{Ext}_n(\mathcal{T}) = \{\emptyset\}$, then for all $x \in \Sigma$, $\mathsf{CN}(\{x\})$ is inconsistent.*

Let us now characterise the set of inferences that may be drawn from a knowledge base Σ by any argumentation system under naive semantics. It coincides with the set of inferences that are drawn from *some* maximal consistent subsets of Σ.

Theorem 46 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics). $\mathtt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ where $\mathcal{S}_i$ ranges over $\{\mathcal{S}_i \in \mathtt{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \mathtt{Ext}_n(\mathcal{T})$ and $\mathcal{S}_i = \mathtt{Base}(\mathcal{E}_i)\}$.*

When the number of naive extensions of an argumentation system is less than the number of maximal consistent subsets of the knowledge base over which the system is built, the system returns arbitrary conclusions.

Example 47 *Assume that $(\mathcal{L}, \mathsf{CN})$ is non-adjunctive, $\Sigma = \{x, \neg x \wedge y\}$ and that this base has two maximal consistent subsets:*

- $\mathcal{S}_1 = \{x\}$.
- $\mathcal{S}_2 = \{\neg x \wedge y\}$.

According to Theorem 41, any argumentation system $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ satisfying the postulates and whose attack relation $\mathcal{R}$ is conflict-dependent will have one or two naive extensions: $\mathcal{E}_1 = \mathtt{Arg}(\mathcal{S}_1)$ and $\mathcal{E}_2 = \mathtt{Arg}(\mathcal{S}_2)$. Assume that $\mathtt{Ext}_n(\mathcal{T}) = \{\mathcal{E}_1\}$. It follows that $x \in \mathtt{Output}(\mathcal{T})$ and $\neg x \notin \mathtt{Output}(\mathcal{T})$. If $\mathtt{Ext}_n(\mathcal{T}) = \{\mathcal{E}_2\}$, $x \notin \mathtt{Output}(\mathcal{T})$ and $\neg x \in \mathtt{Output}(\mathcal{T})$. Both results are arbitrary.

In the case of adjunctive logics, the output of an argumentation system is the set of conclusions that follow from *all* the maximal consistent subsets of Σ.

Corollary 48 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).*

$$\mathtt{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \mathtt{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i).$$

Example 47 (Cont): Assume now that $(\mathcal{L}, \mathsf{CN})$ is propositional logic (which is adjunctive). The base $\Sigma = \{x, \neg x \wedge y\}$ has two maximal consistent subsets:

- $\mathcal{S}_1 = \{x\}$.
- $\mathcal{S}_2 = \{\neg x \wedge y\}$.

According to Corollary 43, any argumentation system $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ satisfying the postulates and whose attack relation $\mathcal{R}$ is conflict-dependent will have exactly two naive extensions: $\mathtt{Arg}(\mathcal{S}_1)$ and $\mathtt{Arg}(\mathcal{S}_2)$. Moreover, $\mathtt{Output}(\mathcal{T}) = \mathsf{CN}(\mathcal{S}_1) \cap \mathsf{CN}(\mathcal{S}_2)$.

In short, under naive semantics, *any* 'good' instantiation of Dung's (1995) abstract framework returns exactly the formulas that are drawn (with CN) by all the maximal consistent subsets of the knowledge base Σ. So whichever attack relation is chosen, the

$(\mathcal{L}, \mathsf{CN})$	Output
Non-adjunctive	Arbitrary conclusions may be drawn from Σ.
Adjunctive	Generalised universal conclusions of Rescher and Manor (1970).

Figure 1. Reasoning under naive semantics.

result will be the same. It is worth recalling that the output set contains exactly the so-called *universal conclusions* in the *coherence-based approach* developed in Rescher and Manor (1970) for reasoning from inconsistent propositional bases. Indeed, Rescher and Manor take as input a (possibly inconsistent) propositional knowledge base, then compute all of its maximal (for set inclusion) consistent subsets. The universal conclusions to be drawn from the base are the formulae that follow logically from all of these subsets. Thus, argumentation systems generalise (under naive semantics) this approach to any *adjunctive Tarskian logic*. As a consequence, the argumentation approach is syntax-dependent, and may thus lead to undesirable results, as discussed in the following example.

Example 47 (Cont): Assume again that $(\mathcal{L}, \mathsf{CN})$ is propositional logic. Thus, $\mathtt{Output}(\mathcal{T}) = \mathsf{CN}(\mathcal{S}_1) \cap \mathsf{CN}(\mathcal{S}_2)$. Note that $y \notin \mathtt{Output}(\mathcal{T})$.

Assume that x stands for 'Sunny day' and y for 'It is cloudy'. The fact that $y \notin \mathtt{Output}(\mathcal{T})$ seems reasonable. Assume now that y stands for 'The temperature is 18 degrees'. In this case, y should be inferred from Σ according to the idea that it is not part of the conflict.

Should $\neg x \wedge y$ instead be written as two formulas, namely $\neg x$ and y, then y is out of the conflict and is inferred.

Figure 1 summarises the main results under naive semantics.

5.2. Stable and semi-stable semantics

We show that the stable extensions of *any* argumentation system satisfying consistency and closure under sub-arguments return maximal consistent subsets of Σ. This means that if one instantiates Dung's (1995) framework and does not get maximal consistent subsets with stable extensions, then the instantiation certainly violates one or both of the two key postulates: consistency and closure under sub-arguments.

Theorem 49 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics) and $\mathtt{Ext}_s(\mathcal{T}) \neq \emptyset$, then:*

- *For all $\mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$, $\mathtt{Base}(\mathcal{E}) \in \mathtt{Max}(\Sigma)$.*
- *For all $\mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_s(\mathcal{T})$, if $\mathtt{Base}(\mathcal{E}_i) = \mathtt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

This result characterises the stable extensions of a large class of argumentation systems, namely the ones that are built using (adjunctive and non-adjunctive) Tarskian logics. However, it does not guarantee that each maximal consistent subset of Σ has a corresponding stable extension in an argumentation system $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$. To put it differently, it does not guarantee a bijection from the set $\mathtt{Ext}_s(\mathcal{T})$ to the set $\mathtt{Max}(\Sigma)$. The bijection (thus the equality $|\mathtt{Ext}_s(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$) broadly depends on the attack relation that is chosen.

Let $\Re_s$ be the set of *all* attack relations that ensure the postulates under stable semantics:

$$\Re_s = \bigcup_{\Sigma \subseteq \mathcal{L}} \{\mathcal{R} \subseteq \texttt{Arg}(\Sigma) \times \texttt{Arg}(\Sigma) \mid \mathcal{R} \text{ is conflict-dependent and } (\texttt{Arg}(\Sigma), \mathcal{R}) \text{ satisfies Postulates 22, 24, 26, 29 and 31 under stable semantics}\}.$$

This set contains three *disjoint* subsets of attack relations: $\Re_s = \Re_{s_1} \cup \Re_{s_2} \cup \Re_{s_3}$:

- $\Re_{s_1}$: the relations which lead to $|\texttt{Ext}_s(\mathcal{T})| = 0$.
- $\Re_{s_2}$: the relations which ensure $0 < |\texttt{Ext}_s(\mathcal{T})| < |\texttt{Max}(\Sigma)|$.
- $\Re_{s_3}$: the relations which ensure $|\texttt{Ext}_s(\mathcal{T})| = |\texttt{Max}(\Sigma)|$.

Let us analyse each category of attack relations in turn. The following result shows that the set $\Re_{s_1}$ is empty, meaning that there is no attack relation which prevents the existence of stable extensions. In other words, any argumentation system satisfying the rationality postulates has at least one stable extension. It is worth recalling that in the general case, Dung (1995) has shown that stable semantics does not guarantee the existence of extensions. This was considered a weakness of this semantics.

Theorem 50 *It holds that* $\Re_{s1} = \emptyset$.

What about the attack relations of the category $\Re_{s_2}$? Systems that use these relations choose a proper subset of the maximal consistent subsets of Σ and make inferences from them. Their output sets are as follows:

Theorem 51 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s_2}$. $\texttt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i$ *ranges over* $\{\mathcal{S}_i \in \texttt{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \texttt{Ext}_s(\mathcal{T})$ *and* $\mathcal{S}_i = \texttt{Base}(\mathcal{E}_i)\}$.

These attack relations lead to an unjustified discrimination between maximal consistent subsets of a knowledge base. Unfortunately, this is fatal for the argumentation systems which use them as they may return arbitrary results. Note that the situation is similar to the one encountered under naive semantics when the logic is non-adjunctive (see Example 47).

Attack relations of the category $\Re_{s3}$ induce a one-to-one correspondence between the stable extensions of an argumentation system and the maximal consistent subsets of the knowledge base over which it is built.

Theorem 52 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s3}$. *For all* $\mathcal{S} \in \texttt{Max}(\Sigma)$, $\texttt{Arg}(\mathcal{S}) \in \texttt{Ext}_s(\mathcal{T})$.

The stable extensions of any argumentation system using an attack relation of the category $\Re_{s3}$ coincide with the naive extensions. They even coincide with the preferred extensions of the system, meaning that this system is *coherent* (Dung, 1995).

Theorem 53 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s3}$. *The equality* $\texttt{Ext}_n(\mathcal{T}) = \texttt{Ext}_s(\mathcal{T})$ *holds. If* $\mathcal{T}$ *satisfies the postulates under preferred semantics, then* $\texttt{Ext}_s(\mathcal{T}) = \texttt{Ext}_p(\mathcal{T})$.

An argumentation system using an attack relation of this category leads to exactly the same result under naive semantics. It returns the universal conclusions (of the coherence-based approach) under any monotonic logic as opposed to only propositional logic, as in Rescher and Manor (1970). Finally, it is worth mentioning that the set $\Re_{s3}$ is not empty. Indeed, the *assumption attack relation* ($\mathcal{R}_{as}$) recalled in Definition 35 is one of its elements. In Cayrol (1995), it was shown that there is a full correspondence between the stable

$\mathcal{R} \in \Re_{s_1}$	Impossible.
$\mathcal{R} \in \Re_{s_2}$	Arbitrary conclusions are drawn from Σ.
$\mathcal{R} \in \Re_{s_3}$	$\text{Ext}_n(\mathcal{T}) = \text{Ext}_s(\mathcal{T}) = \text{Ext}_{ss}(\mathcal{T}) = \text{Ext}_p(\mathcal{T})$.

Figure 2. Reasoning under stable semantics.

extensions of an argumentation system (defined over propositional logic) and the maximal consistent subsets of the propositional knowledge base over which it is built. This result was generalised to any Tarskian logic in Amgoud and Besnard (2010). Consequently, any argumentation system using $\mathcal{R}_{as}$ is coherent.

Corollary 54 *For all $\Sigma \subseteq \mathcal{L}$, the argumentation system $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R}_{as})$ is coherent.*

From the previous results, it follows that any argumentation system satisfying the four postulates has stable extensions. Moreover, it is possible to delimit their maximum number.

Corollary 55 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). It holds that*

$$0 < |\mathtt{Ext}_s(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|.$$

It follows that when the knowledge base is finite, the number of stable extensions is finite as well.

Corollary 56 *If Σ is finite, then the set $\mathtt{Ext}_s(\mathcal{T})$ is finite, whenever $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ satisfies consistency and closure under sub-arguments (under stable semantics).*

To sum up, there are two possible categories of attack relations that lead to the satisfaction of the rationality postulates: $\Re_{s_2}$ and $\Re_{s_3}$. Relations of $\Re_{s_2}$ should be avoided as they lead to arbitrary results. Relations of $\Re_{s_3}$ lead to 'correct' results, but argumentation systems based on them return exactly the same results under naive semantics. This means that stable semantics does not play any particular role in the logic-based argumentation systems studied in the paper. Thus, stable semantics either leads to undesirable results or provides no added value w.r.t. naive semantics. Figure 2 summarises the different situations that may be encountered under this semantics.

From the definitions of the two categories $\Re_{s2}$ and $\Re_{s3}$, stable extensions exist. Besides, it was shown in Caminada (2006b) that when this is the case, semi-stable extensions coincide with the stable ones.

Corollary 57 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). The equality $\mathtt{Ext}_s(\mathcal{T}) = \mathtt{Ext}_{ss}(\mathcal{T})$ holds.*

Thus, in practice semi-stable semantics does not offer added value w.r.t. stable semantics, which is itself problematic.

5.3. Preferred semantics

Preferred semantics was mainly proposed in Dung (1995) as an alternative to stable semantics, since the latter does not guarantee (for abstract frameworks) the existence of extensions. In this section, we study the outcomes of logic-based argumentation systems under preferred semantics and check whether it offers added value w.r.t. stable semantics in the context of handling inconsistency in knowledge bases.

We have previously shown in Proposition 28 that the extensions (under any admissibility-based semantics) of an argumentation system satisfying the postulates are made up of consistent subsets of the knowledge base over which the system is defined. Thus, the subset $\texttt{Base}(\mathcal{E})$ computed from any preferred extension $\mathcal{E}$ is a subset of maximal consistent sub-base of the knowledge base at hand.

Theorem 58 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). For all $\mathcal{E} \in \texttt{Ext}_p(\mathcal{T})$, there exists $\mathcal{S} \in \texttt{Max}(\Sigma)$ such that $\texttt{Base}(\mathcal{E}) \subseteq \mathcal{S}$.*

Unlike stable extensions, the subsets of a knowledge base that are computed from preferred extensions are not necessarily maximal (for set inclusion). This is due to the existence of *undecided* arguments under preferred semantics. In Caminada (2006a) another way of defining Dung's (1995) semantics was provided. It consists of labelling the nodes of the graph corresponding to the argumentation system with three possible values: {in, out, undec}. An argument is labelled *in* iff all of its attackers are labelled *out*, and labelled *out* iff one of its attackers is labelled *in*. Finally, it is labelled *undec* iff it is not possible to assign either in or out. When the subset of Σ which is computed from a preferred extension is not maximal then some formulae of Σ appear only in support of undecided arguments.

This does not mean that a preferred extension can never return a maximal consistent subset. Remember that stable extensions exist, thus, there is at least one preferred extension whose base is maximal for set inclusion.

Corollary 59 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under preferred semantics). There exists $\mathcal{E} \in \texttt{Ext}_p(\mathcal{T})$ such that $\texttt{Base}(\mathcal{E}) \in \texttt{Max}(\Sigma)$.*

The following result shows that the subsets computed from the preferred extensions of an argumentation system are pairwise different.

Theorem 60 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). For all $\mathcal{E}_i, \mathcal{E}_j \in \texttt{Ext}_p(\mathcal{T})$, if $\texttt{Base}(\mathcal{E}_i) \subseteq \texttt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

We show that every maximal consistent subset of a knowledge base is captured by at most one preferred extension.

Theorem 61 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). Let $\mathcal{S} \in \texttt{Max}(\Sigma)$. For all $\mathcal{E}_i, \mathcal{E}_j \in \texttt{Ext}_p(\mathcal{T})$, if $\texttt{Base}(\mathcal{E}_i) \subseteq \mathcal{S}$ and $\texttt{Base}(\mathcal{E}_j) \subseteq \mathcal{S}$, then $\mathcal{E}_i = \mathcal{E}_j$.*

The previous result allows us to delimit the maximum number of preferred extensions a system may have. Like stable semantics, it is the number of maximal (for set inclusion) consistent subsets of the knowledge base at hand.

Theorem 62 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). It holds that*

$$1 \leq |\texttt{Ext}_p(\mathcal{T})| \leq |\texttt{Max}(\Sigma)|.$$

When a knowledge base is finite, each argumentation system enjoying the rationality postulates has a finite number of preferred extensions.

Corollary 63 *If a knowledge base Σ is finite, then for all $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics), $\mathtt{Ext}_p(\mathcal{T})$ is finite.*

Let us characterise the inferences that are drawn from a knowledge base Σ by an argumentation system $\mathcal{T}$ satisfying the rationality postulates under preferred semantics. Let $\Re_p$ be the set of *all* attack relations that ensure the postulates under preferred semantics:

$$\Re_p = \bigcup_{\Sigma \subseteq \mathcal{L}} \{\mathcal{R} \subseteq \mathtt{Arg}(\Sigma) \times \mathtt{Arg}(\Sigma) \mid \mathcal{R} \text{ is conflict-dependent and } (\mathtt{Arg}(\Sigma), \mathcal{R}) \text{ satisfies Postulates 22, 24, 26, 29 and 31 under preferred semantics}\}.$$

In his seminal paper, Dung (1995) has shown that the stable extensions of an argumentation system are also preferred extensions of the system. Consequently, the set $\Re_p$ is a subset of $\Re_s$.

Property 64 *It holds that $\Re_p \subseteq \Re_s$.*

The set $\Re_p$ thus contains three *disjoint* subsets of attack relations: $\Re_p = \Re_{p_1} \cup \Re_{p_2} \cup \Re_{p_3}$:

- $\Re_{p_1}$: the relations which are in $\Re_p \cap \Re_{s_1}$.
- $\Re_{p_2}$: the relations which are in $\Re_p \cap \Re_{s_2}$.
- $\Re_{p_3}$: the relations which are in $\Re_p \cap \Re_{s_3}$.

Let us analyse each category of attack relations in turn. The first set is empty (i.e., $\Re_{p_1} = \emptyset$) since we have shown previously that there is no attack relation which prevents an argumentation system from having stable extensions ($\Re_{s_1} = \emptyset$).

Attack relations of the category $\Re_{p_3}$ lead to coherent argumentation systems (their stable extensions coincide with the preferred extensions) as shown in Theorem 53. Moreover, the preferred extensions coincide with the naive ones, meaning that preferred semantics provides no added value w.r.t. naive semantics.

Theorem 65 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system. If $\mathcal{R} \in \Re_{p_3}$ then:*

- *For all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_p(\mathcal{T})$.*
- $|\mathtt{Ext}_p(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$.

The output of an argumentation system is in this case the same as under naive semantics, i.e., the universal conclusions given in Corollary 48.

Corollary 66 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$.*

$$\mathtt{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \mathtt{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i).$$

Let us now analyse the attack relations of the second category, $\Re_{p_2}$. Remember that in this case stable semantics chooses only some maximal consistent subsets of the knowledge base at hand. Four situations may be encountered:

(1) The stable extensions and the preferred extensions of an argumentation system coincide. Thus, preferred semantics provides no added value w.r.t. stable semantics.

$\mathcal{R} \in \Re_{p_1}$	Impossible.
$\mathcal{R} \in \Re_{p_2}$	Arbitrary conclusions are drawn from Σ.
$\mathcal{R} \in \Re_{p_3}$	$\mathrm{Ext}_n(\mathcal{T}) = \mathrm{Ext}_s(\mathcal{T}) = \mathrm{Ext}_{ss}(\mathcal{T}) = \mathrm{Ext}_p(\mathcal{T})$.

Figure 3. Reasoning under preferred semantics.

Moreover, it leads to arbitrary results, as discussed in the previous subsection (i.e., when $\mathcal{R} \in \Re_{s_2}$).

(2) The preferred extensions consider additional but *not all* maximal consistent subsets (other than the ones chosen by stable semantics). This case is similar to the previous one and the argumentation system returns arbitrary results.

(3) The preferred extensions return *all* the maximal consistent subsets of the knowledge base. This means that stable semantics chooses *some* maximal consistent subsets and preferred semantics considers the remaining ones. This case collapses in the case of attack relations of the category $\Re_{p_3}$. We have seen that the result of preferred semantics is already ensured by naive and stable semantics in this case.

(4) Some of the preferred extensions provide *non-maximal* consistent subsets of the knowledge base. In this case, the result of the argumentation system is arbitrary.

To sum up, attack relations of the category $\Re_{p_2}$ may lead either to *arbitrary* results or to results which can be provided by naive semantics. The results are characterised in the following theorem.

Theorem 67 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{p_2}$. $\mathtt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i$ *ranges over* $\{\mathcal{S}_i \in \mathtt{Cons}(\Sigma) \mid \exists \mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})$ *and* $\mathcal{S}_i = \mathtt{Base}(\mathcal{E}_i)\}$ *and* $\mathtt{Cons}(\Sigma) = \{\mathcal{S} \mid \mathcal{S} \subseteq \Sigma,\ \mathcal{S}$ *is consistent and* $\mathtt{Free}(\Sigma) \subseteq \mathcal{S}\}$.

The results of this section show that reasoning under preferred semantics is not recommended, since it leads either to arbitrary results or to the results got under naive semantics. Figure 3 summarises the different situations that may be encountered under this semantics.

5.4. Grounded and ideal semantics

This section analyses the outputs of argumentation systems under existing sceptical semantics, namely grounded and ideal. Grounded semantics was proposed by Dung (1995). It ensures a unique extension for every argumentation system, and is based on a sceptical principle. It starts with non-attacked arguments, to which are added the arguments they defend. This reinstatement process is repeated until a fixpoint is reached. Argumentation systems that do not have non-attacked arguments have empty grounded extensions. In Dung, Mancarella, and Toni (2007), this semantics was extended to the so-called ideal semantics. The new semantics returns a unique extension which is an admissible set of arguments contained by every preferred extension of an argumentation system. The following properties were shown in Dung, Mancarella, and Toni (2007).

Property 68 (Dung, Mancarella, and Toni, 2007) *Let* $\mathcal{T}$ *be an argumentation system.*

- $\mathcal{T}$ *admits a unique ideal extension.*
- $\mathtt{GE}(\mathcal{T}) \subseteq \mathtt{IE}(\mathcal{T}) \subseteq \bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathcal{E}_i$.
- *If* $\bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathcal{E}_i$ *is admissible, then* $\mathtt{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathcal{E}_i$.

The following example, borrowed from Dung, Mancarella, and Toni (2007), shows some differences between ideal and grounded semantics.

Example 69 *Let us consider the argumentation framework* $\mathcal{T} = (\mathcal{A}, \mathcal{R})$*, where*

- $\mathcal{A} = \{a, b, c, d\}$.
- $\mathcal{R} = \{(a, a), (a, b), (b, a), (c, d), (d, c)\}$.

It can be checked that:

- $\texttt{GE}(\mathcal{T}) = \emptyset$,
- $\texttt{Ext}_p(\mathcal{T}) = \{\{b, c\}, \{b, d\}\}$, *and*
- $\texttt{IE}(\mathcal{T}) = \{b\}$.

Before analysing the argumentation systems' outputs under ideal and grounded semantics, we provide a result of great importance. It shows that the set of arguments built from $\texttt{Free}(\Sigma)$ is an admissible extension of any argumentation system whose attack relation is conflict-dependent. Thus, this is true even for systems that do not satisfy the free precedence postulate.

Theorem 70 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent.*

- *For all* $a \in \texttt{Arg}(\texttt{Free}(\Sigma))$, *a neither attacks nor is attacked by another argument in* $\texttt{Arg}(\Sigma)$.
- $\texttt{Arg}(\texttt{Free}(\Sigma))$ *is an admissible extension of* $\mathcal{T}$.

Since ideal semantics is based on preferred semantics, we analyse the two cases that may be encountered with the latter. We start with the case of an argumentation system that uses an attack relation of the category $\Re_{p_3}$. We show that the ideal extension of such a system coincides with the intersection of all of its preferred extensions. Moreover, it is exactly the set of arguments built from the free part of the knowledge base.

Theorem 71 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$.

$$\texttt{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i = \texttt{Arg}(\texttt{Free}(\Sigma)).$$

Since arguments of $\texttt{Arg}(\texttt{Free}(\Sigma))$ are not attacked by any argument, then they belong to the grounded extension of the argumentation system. Consequently, grounded and ideal extensions coincide.

Corollary 72 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$. $\texttt{IE}(\mathcal{T}) = \texttt{GE}(\mathcal{T}) = \texttt{Arg}(\texttt{Free}(\Sigma))$.

From the previous results, it is possible to characterise the set of conclusions drawn from a knowledge base using grounded and ideal semantics. It is the set of all formulae that follow using the consequence operator CN from $\texttt{Free}(\Sigma)$.

Theorem 73 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$. *The output of* $\mathcal{T}$ *under grounded/ideal semantics is:*

$$\texttt{Output}(\mathcal{T}) = \mathsf{CN}(\texttt{Free}(\Sigma)).$$

$\mathcal{R} \in \Re_{p_2}$	Arbitrary conclusions are drawn from Σ.
$\mathcal{R} \in \Re_{p_3}$	$\text{Output}(\mathcal{T}) = \mathsf{CN}(\text{Free}(\Sigma))$.

Figure 4. Reasoning under grounded and ideal semantics.

It is worth pointing out that, in this case, argumentation systems generalise the *free consequences* proposed by Benferhat, Dubois, and Prade (1997) for reasoning about inconsistent propositional knowledge bases. Indeed, argumentation systems consider not only propositional logic but also any other Tarskian logic.

Recall that the assumption attack relation leads to coherent argumentation systems, thus their ideal and grounded semantics coincide.

Corollary 74 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R}_{as})$. *For any* $\Sigma \subseteq \mathcal{L}$, $\mathtt{Output}(\mathcal{T}) = \mathsf{CN}(\mathtt{Free}(\Sigma))$.

Let us now consider the case where the attack relation of an argumentation system is of the category $\Re_{p_2}$. Here again, since the attack relation is conflict-dependent, the set of arguments $\mathtt{Arg}(\mathtt{Free}(\Sigma))$ is contained by both ideal and grounded extensions.

Corollary 75 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_2}$. *The inclusions* $\mathtt{Arg}(\mathtt{Free}(\Sigma)) \subseteq \mathtt{GE}(\mathcal{T}) \subseteq \mathtt{IE}(\mathcal{T}) \subseteq \mathcal{S}$ *hold for some* $\mathcal{S} \in \mathtt{Max}(\Sigma)$.

In the case of the above inclusions being strict, i.e., $\mathtt{Arg}(\mathtt{Free}(\Sigma)) \subset \mathtt{GE}(\mathcal{T})$ (respectively $\mathtt{Arg}(\mathtt{Free}(\Sigma)) \subset \mathtt{IE}(\mathcal{T})$), we show that the argumentation system $\mathcal{T}$ returns arbitrary conclusions.

Theorem 76 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies the five postulates under grounded (respectively ideal) semantics. If* $\mathtt{Arg}(\mathtt{Free}(\Sigma)) \subset \mathtt{GE}(\mathcal{T})$ *(respectively* $\mathtt{Arg}(\mathtt{Free}(\Sigma)) \subset \mathtt{IE}(\mathcal{T})$*) then there exists* $C \in \mathcal{C}_\Sigma$ *such that there exist* $x, x' \in C$ *and* $x \in \mathtt{Output}(\mathcal{T})$ *and* $x' \notin \mathtt{Output}(\mathcal{T})$.

To sum up, ideal and grounded semantics either coincide and return as output the set of all formulas that follow from the safe part of a knowledge base, i.e., $\mathsf{CN}(\mathtt{Free}(\Sigma))$, or may both return arbitrary results. Figure 4 summarises the different situations encountered under these two semantics.

6. Related work

This paper investigated the compatibility of Dung's (1995) argumentation framework with logical formalisms. For this purpose, it showed how to instantiate the framework from a logical knowledge base, namely how to define the arguments in a systematic way. Then, it proposed some basic postulates that the logical instantiations should satisfy, and studied under which conditions the postulates may be violated. Next, it investigated the outputs of such instantiations under various semantics.

There are some works in the literature which are in some ways related to ours. In Caminada and Amgoud (2007), rationality postulates were proposed for instantiations that use a particular language (it distinguishes between strict rules and defeasible rules). In our work, we extended some of those postulates to Tarskian logics and proposed three new ones. In Caminada and Amgoud (2007), the authors investigated when the ASPIC system satisfies the postulates. In Gorogiannis and Hunter (2011), the authors focused on some argumentation systems that are defined using propositional logic. They studied when those systems satisfy some of the rationality postulates presented in this paper. Our work is more general, since we considered a larger class of logics and did not focus on particular attack relations. Our results hold for any attack relation that is conflict-dependent. Note that all the attack relations that were studied in Gorogiannis and Hunter (2011) are conflict-dependent.

The second part of our paper on the outputs of argumentation systems under various semantics is novel. There is almost no work on the topic, with the exception of Cayrol (1995). Cayrol studied the underpinnings of stable semantics for one particular argumentation system: the one that is grounded in propositional logic (a particular case of Tarski's logics) and uses the 'assumption attack' relation. She showed that there is a one-to-one correspondence between the stable extensions of the system and the maximal consistent subsets of the knowledge base over which the system is built. In our paper, this result is generalised to any Tarskian logic and any attack relation. Moreover, we have shown that this result is already ensured by naive semantics. Thus, in the case of the system studied by Cayrol, stable semantics is useless. Finally, we have shown that this particular system is coherent, i.e., its stable extensions coincide with its preferred ones. Our work is more general, since it presented a *complete* view of the outputs of argumentation systems not only under stable semantics but also under various other semantics.

7. Conclusion

The paper investigated Dung's (1995) argumentation framework. It started by pointing out its main limits, namely the gap between the abstract framework and the application for which it may be used, the lack of rationality postulates that would describe the kind of results expected from the framework, the lack of methodology for defining arguments and the attacks between them, and finally an overview of the underpinnings of the different acceptability semantics, as well as the basic concepts of the framework such as defence and conflict-freeness. The paper gave an answer to each of these issues.

The paper extended Dung's (1995) argumentation framework by taking into account the logic from which arguments are built. The new framework is general since it is grounded in any abstract logic in Tarski's sense. Thus, a wide variety of logics can be used – even those that have not yet been considered in argumentation, such as temporal logic, modal logic, etc. The extension has two main advantages: first, it enforces the framework to avoid unsound conclusions; second, it relates the different notions of Dung's approach, like the attack relation and conflict-freeness, to the knowledge base at hand.

In Caminada and Amgoud (2007), three rationality postulates were defined for rule-based argumentation systems. The paper generalised the two postulates on direct consistency and closure to any argumentation framework built over a monotonic logic, and proposed three new postulates on sub-arguments, free precedence and exhaustiveness. It then showed that indirect consistency is always satisfied if direct consistency is ensured. Moreover, it showed that if consistency (respectively closure) is satisfied by the different extensions, then it is also satisfied by the output of the framework.

The paper then presented a formal methodology for defining arguments from a knowledge base, and for eliciting an appropriate attack relation. By appropriate, we mean an attack relation that satisfies the postulates. It showed that an attack relation should be grounded in the minimal conflicts that occur in the knowledge base at hand. An important result shows that when ternary or more minimal conflicts occur in the knowledge base, symmetric attack relations should be avoided since they lead to the violation of direct consistency.

Using well-behaved attack relations, the paper analysed the different acceptability semantics introduced in Dung (1995) in terms of the subsets that are returned by each extension. The results of the analysis are very surprising and, unfortunately, disappointing. In fact, they show to what extent the rationality of Dung's approach is at stake. Moreover, it behaves in a completely arbitrary way. The first important result shows that maximal conflict-free sets of arguments are sufficient in order to derive reasonable conclusions from a knowledge base. Indeed, there is a one-to-one correspondence between maximal consistent subsets of a knowledge base and maximal conflict-free sets of arguments. This means that the different acceptability semantics defined in the literature are not necessary, and the notion of defence is useless. It is also shown that under naive semantics, argumentation systems generalise the coherence-based approach of Rescher and Manor (1970) to any Tarskian logic. This is particularly the case for adjunctive logics.

Remember that stable extensions are maximal conflict-free sets of arguments. Does this mean that stable semantics is appropriate? The answer is unfortunately 'no'. Indeed, stable extensions amount to either an arbitrary pick of *some* maximal consistent subsets of the knowledge base, or to the consideration of all of them. In the first case, they lead to arbitrary inferences whereas in the second case they lead to the result already ensured by naive semantics. The case of preferred semantics is even worse, and so this semantics should be avoided. The corresponding extensions represent *some* consistent subsets but not necessarily maximal ones. Thus, they lead to arbitrary inferences. There are, however, two pieces of good news. The first is that the number of (stable/preferred/naive) extensions is finite as soon as the knowledge base is finite. The second is that stable extensions always exist, meaning that semi-stable semantics proposed in Caminada (2006b) is useless since it does not provide added value w.r.t. stable semantics. Reasoning under ideal and grounded semantics may also be problematic. There are two possible situations: (i) the situation where the two extensions (grounded and ideal) coincide with the set of arguments built from the free part of a knowledge base; and (ii) the situation where both semantics lead to arbitrary results.

One of the main reasons for this defective behaviour of Dung's (1995) approach is the attack relation. Indeed, we have shown that in order to ensure reasonable results, this relation should capture the minimal conflicts that occur in the knowledge base at hand. A minimal conflict is an inconsistent set of formulas. It is well known that the notion of inconsistency is not oriented. Thus, it *should* be captured by a symmetric attack relation. However, we have also shown that, due to the binary character of the attack relation, if it is symmetric, then consistency often fails. Which is when n-ary (other than binary) minimal conflicts occur in the knowledge base. The second reason is the definition of semantics without taking into account the application to which the system is applied, and thus without considering the structure of arguments.

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Notes

1. Some fragments of well-known logics fail to be adjunctive, e.g., the pure implicational fragment of classical logic as it is negationless, disjunctionless, and, of course, conjunctionless.
2. A strict rule $x \to y$ means that if x holds, then y holds with no exception whatsoever.

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Appendix

Property 5 *Let* $X, X', X'' \subseteq \mathcal{L}$.

(1) $X \subseteq X' \Rightarrow \mathsf{CN}(X) \subseteq \mathsf{CN}(X')$.
(2) $\mathsf{CN}(X) \cup \mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$.
(3) $\mathsf{CN}(X) = \mathsf{CN}(X') \Rightarrow \mathsf{CN}(X \cup X'') = \mathsf{CN}(X' \cup X'')$.
(4) $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X) \cap \mathsf{CN}(X')$.

Proof Let $X, X', X'' \subseteq \mathcal{L}$.

(1) Assume that $X \subseteq X'$. According to the compactness axiom, it holds that $\mathsf{CN}(X) = \bigcup_{Y \subseteq_f X} \mathsf{CN}(Y)$. $\bigcup_{Y \subseteq_f X} \mathsf{CN}(Y) \subseteq \bigcup_{Y \subseteq_f X'} \mathsf{CN}(Y) = \mathsf{CN}(X')$ since $Y \subseteq_f X \subset X'$.

(2) $X \subseteq X \cup X'$, thus by monotonicity, $\mathsf{CN}(X) \subseteq \mathsf{CN}(X \cup X')$ (a). Similarly, $X' \subseteq X \cup X'$ thus $\mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$ (b). From (a) and (b), $\mathsf{CN}(X) \cup \mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$.

(3) Assume that $\mathsf{CN}(X) = \mathsf{CN}(X')$. According to the expansion axiom, the following inclusions hold: $X' \subseteq \mathsf{CN}(X')$ and $X'' \subseteq \mathsf{CN}(X'')$. Thus, $X' \cup X'' \subseteq \mathsf{CN}(X') \cup \mathsf{CN}(X'') = \mathsf{CN}(X) \cup \mathsf{CN}(X'')$ (a). Moreover, $X \subseteq X \cup X''$ thus $\mathsf{CN}(X) \subseteq \mathsf{CN}(X \cup X'')$ (a'). Similarly, $X'' \subseteq X \cup X''$ thus $\mathsf{CN}(X'') \subseteq \mathsf{CN}(X \cup X'')$ (b'). From (a')

and (b'), $\mathsf{CN}(X) \cup \mathsf{CN}(X'') \subseteq \mathsf{CN}(X \cup X'')$. From (a), it follows that $X' \cup X'' \subseteq \mathsf{CN}(X \cup X'')$. By monotonicity, $\mathsf{CN}(X' \cup X'') \subseteq \mathsf{CN}(\mathsf{CN}(X \cup X''))$. Finally, by the idempotence axiom, the inclusion $\mathsf{CN}(X' \cup X'') \subseteq \mathsf{CN}(X \cup X'')$ holds. To show that $\mathsf{CN}(X \cup X'') \subseteq \mathsf{CN}(X' \cup X'')$, the same reasoning is applied by starting with X instead of X'.

(4) $X \cap X' \subseteq X$ thus $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X)$. Similarly, $X \cap X' \subseteq X'$ thus $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X')$. Consequently, $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X) \cap \mathsf{CN}(X')$. ■

Property 7 *Let $X \subseteq \mathcal{L}$.*

(1) If X is consistent, then $\mathsf{CN}(X)$ is consistent as well.
(2) $\forall X' \subseteq X$, if X is consistent, then X' is consistent.
(3) $\forall X' \subseteq X$, if X' is inconsistent, then X is inconsistent.

Proof Let $X \subseteq \mathcal{L}$. Assume that X is consistent, then $\mathsf{CN}(X) \neq \mathcal{L}$ (1).

(1) Let us now assume that $\mathsf{CN}(X)$ is inconsistent. This means that $\mathsf{CN}(\mathsf{CN}(X)) = \mathcal{L}$. However, according to the idempotence axiom, $\mathsf{CN}(\mathsf{CN}(X)) = \mathsf{CN}(X)$. Thus, $\mathsf{CN}(X) = \mathcal{L}$, which contradicts (1).
(2) Assume that $\exists X' \subseteq X$ such that X' is inconsistent. This means that $\mathsf{CN}(X') = \mathcal{L}$. However, since $X' \subseteq X$ then $\mathsf{CN}(X') \subseteq \mathsf{CN}(X)$ (according to the monotonicity axiom). Thus, $\mathcal{L} \subseteq \mathsf{CN}(X)$. Since $\mathsf{CN}(X) \subseteq \mathcal{L}$, then $\mathsf{CN}(X) = \mathcal{L}$. Thus, X is inconsistent. Contradiction.
(3) Let $X' \subseteq X$. Assume that X' is inconsistent. Since $X' \subseteq X$ thus $\mathsf{CN}(X') \subseteq \mathsf{CN}(X)$ (by the monotonicity axiom). Since X' is inconsistent, $\mathsf{CN}(X') = \mathcal{L}$. Consequently, $\mathsf{CN}(X) = \mathcal{L}$ which means that X is inconsistent. ■

Property 9 *For all $X \subseteq \Sigma \subseteq \mathcal{L}$,*

- *if X is consistent then $\mathcal{C}_X = \emptyset$;*
- *if X is consistent then $X \subseteq \mathcal{S}$ for some $\mathcal{S} \in \mathtt{Max}(\Sigma)$;*
- *if X is inconsistent then there exists at least one minimal conflict C such that $C \subseteq X$.*

Proof The first and third items are obvious. Let us now prove the second item. Let us construct a maximal consistent subset of Σ that contains X. Consider an enumeration $s_1, s_2, \ldots$ of Σ. Define a series $S_0, S_1, S_2, \ldots$ of subsets of Σ as follows: S_0 is X, and $S_{n+1} = S_n \cup \{s_{n+1}\}$ if $S_n \cup \{s_{n+1}\}$ is consistent, otherwise $S_{n+1} = S_n$. By construction, S_n is consistent for all $n \geq 0$. Let $\mathcal{S} = \bigcup_{n \geq 0} S_n$. Trivially, $X \subseteq \mathcal{S}$. We now show $\mathcal{S} \in \mathtt{Max}(\Sigma)$. Assume first that $\mathcal{S}$ is inconsistent, i.e., $\mathsf{CN}(\mathcal{S}) = \mathcal{L}$ according to Definition 6. Tarski's absurdity axiom yields that $x \in \mathsf{CN}(\mathcal{S})$ for some x satisfying $\mathsf{CN}(\{x\}) = \mathcal{L}$. Since $S_n \subseteq S_{n+1}$, Tarski's compactness axiom means that $x \in \mathsf{CN}(\mathcal{S})$ iff $x \in \mathsf{CN}(S_k)$ for some k. However, $x \in \mathsf{CN}(S_k)$ means that $x \in S_k$ is inconsistent – a contradiction. Hence, we have shown that $\mathcal{S}$ is consistent. By construction, it is a subset of Σ. It remains to show that it is a maximal consistent subset of Σ. Let y be in $\Sigma \setminus \mathcal{S}$. Assume that $\mathcal{S} \cup \{y\}$ is consistent. Since $y \in \Sigma$, y is some s_h in the above enumeration. Should $\mathcal{S} \cup \{y\}$ be consistent, so would $S_h \cup \{y\}$ be, in view of Tarski's compactness axiom (monotony direction). However, $S_h \cup \{y\}$ consistent would mean that $y \in S_{h+1}$ hence $y \in \mathcal{S}$, a contradiction. ■

Property 11 *Let $(\mathcal{L}, \mathsf{CN})$ be* adjunctive*, $C \subseteq \mathcal{L}$ be a minimal conflict. For all $X \subset C$, if $X \neq \emptyset$, then:*

(1) $\exists x \in \mathcal{L}$ such that $\mathsf{CN}(\{x\}) = \mathsf{CN}(X)$.
(2) $\exists x_1 \in \mathsf{CN}(X)$ and $\exists x_2 \in \mathsf{CN}(C \setminus X)$ such that the set $\{x_1, x_2\}$ is inconsistent.

Proof Let C be a minimal conflict. Consider $X \subset C$ such that $X \neq \emptyset$.

We prove the first item of the property by induction, after we first take care to show that X is finite. By Tarski's requirements, there exists $x_0 \in \mathcal{L}$ such that $\mathsf{CN}(\{x_0\}) = \mathcal{L}$. Since C is a conflict, $\mathsf{CN}(C) = \mathsf{CN}(\{x_0\})$. As a consequence, $x_0 \in \mathsf{CN}(C)$. However, $\mathsf{CN}(C) = \bigcup_{C' \subseteq_f C} \mathsf{CN}(C')$ by Tarski's requirements. Thus, $x_0 \in \mathsf{CN}(C)$ means that there exists $C' \subseteq_f C$ such that $x_0 \in \mathsf{CN}(C')$. This says that C' is a conflict. Since C is a minimal conflict, $C = C'$ and it follows that C is finite. Of course, so is X: let us write $X = \{x_1, \ldots, x_n\}$. *Base step:* $n = 1$. Taking x to be x_1 is enough. *Induction step:* Assume the lemma is true up to rank $n-1$. As CN is a closure operator, $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\mathsf{CN}(\{x_1, \ldots, x_{n-1}\}) \cup \{x_n\})$. The induction hypothesis entails $\exists x \in \mathcal{L}$ such that $\mathsf{CN}(\mathsf{CN}(\{x_1, \ldots, x_{n-1}\}) \cup \{x_n\}) = \mathsf{CN}(\mathsf{CN}(\{x\}) \cup \{x_n\})$. Then, $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\{x, x_n\})$. Hence, there exists $y \in \mathcal{L}$ such that $\mathsf{CN}(\{x, x_n\}) = \mathsf{CN}(\{y\})$ because $(\mathcal{L}, \mathsf{CN})$ is adjunctive. Since $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\{x, x_n\})$ was just proved, it follows that $\mathsf{CN}(\{y\}) = \mathsf{CN}(\{x_1, \ldots, x_n\})$.

Take $X_1 = X$ and $X_2 = C \setminus X_1$. Since X is a non-empty proper subset of C, so are both X_1 and X_2. Then, the first bullet of this property can be applied to X_1 and X_2. As a result, $\exists x_1 \in \mathcal{L}$ such that $\mathsf{CN}(\{x_1\}) = \mathsf{CN}(X_1)$ and $\exists x_2 \in \mathcal{L}$ such that $\mathsf{CN}(\{x_2\}) = \mathsf{CN}(X_2)$. The expansion axiom gives $\{x_1\} \subseteq \mathsf{CN}(\{x_1\})$ and $\{x_2\} \subseteq \mathsf{CN}(\{x_2\})$. Thus, $x_1 \in \mathsf{CN}(X_1)$ and $x_2 \in \mathsf{CN}(X_2)$. Using the expansion axiom again, $X_1 \subseteq \mathsf{CN}(X_1)$ and $X_2 \subseteq \mathsf{CN}(X_2)$. Thus, $X_1 \cup X_2 \subseteq \mathsf{CN}(X_1) \cup \mathsf{CN}(X_2) = \mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\})$. It follows that $C \subseteq \mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\})$. Using Property 5, $\mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\}) \subseteq \mathsf{CN}(\{x_1, x_2\})$, thus $C \subseteq \mathsf{CN}(\{x_1, x_2\})$. Since C is inconsistent, Property 7 gives that $\mathsf{CN}(\{x_1, x_2\})$ is inconsistent as well. By the definition of inconsistency, it follows that $\mathsf{CN}(\mathsf{CN}(\{x_1, x_2\})) = \mathcal{L}$. Applying the idempotence axiom, $\mathsf{CN}(\{x_1, x_2\}) = \mathcal{L}$, thus the set $\{x_1, x_2\}$ is inconsistent. ■

Property 14 *For all* $(X, x) \in \texttt{Arg}(\Sigma)$*, the set* $\{x\}$ *is consistent.*

Proof Let $(X, x) \in \texttt{Arg}(\Sigma)$. From Definition 13, the support X is consistent. Thus, according to Property 7, $\mathsf{CN}(X)$ is consistent as well. Since $x \in \mathsf{CN}(X)$, then $\{x\} \subseteq \mathsf{CN}(X)$ thus $\{x\}$ is consistent. ■

Property 15 *Let* Σ *be a knowledge base such that for all* $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$*. For all* $x \in \Sigma$ *such that the set* $\{x\}$ *is consistent,* $(\{x\}, x) \in \texttt{Arg}(\Sigma)$.

Proof Let $x \in \Sigma$ be such that the set $\{x\}$ is consistent. Since $x \notin \mathsf{CN}(\emptyset)$, then $\{x\}$ is a minimal set such that $x \in \mathsf{CN}(\{x\})$. It follows that $(\{x\}, x) \in \texttt{Arg}(\Sigma)$. ■

Property 16 $\texttt{Arg}(\Sigma) \subseteq \texttt{Arg}(\Sigma')$ *whenever* $\Sigma \subseteq \Sigma' \subseteq \mathcal{L}$.

Proof Let $\Sigma \subseteq \Sigma' \subseteq \mathcal{L}$. Assume that $(X, x) \in \texttt{Arg}(\Sigma)$ and $(X, x) \notin \texttt{Arg}(\Sigma')$. Since $(X, x) \notin \texttt{Arg}(\Sigma')$, then there are four possible cases:

(1) X is not a subset of Σ'. This is impossible since $(X, x) \in \texttt{Arg}(\Sigma)$, thus $X \subseteq \Sigma \subseteq \Sigma'$.
(2) X is inconsistent. This is impossible since $(X, x) \in \texttt{Arg}(\Sigma)$.
(3) $x \notin \mathsf{CN}(X)$. This is impossible since $(X, x) \in \texttt{Arg}(\Sigma)$ hence $x \in \mathsf{CN}(X)$.
(4) X is not minimal. This means that $\exists X' \subset X$ that satisfies conditions 1–3 of Definition 13. This is impossible since $(X, x) \in \texttt{Arg}(\Sigma)$. ■

Property 20 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *and* $\texttt{Ext}(\mathcal{T})$ *its set of extensions under a given semantics. It holds that* $\texttt{Output}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \texttt{Ext}(\mathcal{T})} \texttt{Concs}(\mathcal{E}_i)$.

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ.

(1) Let $x \in \texttt{Output}(\mathcal{T})$. Thus, for all $\mathcal{E}_i \in \texttt{Ext}(\mathcal{T})$, $\exists a_i \in \mathcal{E}_i$ such that $\texttt{Conc}(a_i) = x$. It follows that $x \in \texttt{Concs}(\mathcal{E}_i)$, $\forall \mathcal{E}_i$ and hence $x \in \cap\texttt{Concs}(\mathcal{E}_i)$.
(2) Assume that $x \in \cap\texttt{Concs}(\mathcal{E}_i)$. Thus, $\forall \mathcal{E}_i \in \texttt{Ext}(\mathcal{T})$, $\exists a_i \in \mathcal{E}_i$ such that $\texttt{Conc}(a_i) = x$. Consequently, $x \in \texttt{Output}(\mathcal{T})$.

■

Property 21 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. $\texttt{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\Sigma)$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. Let $x \in \texttt{Output}(\mathcal{T})$. Thus, $\exists a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Conc}(a) = x$. Besides, $x \in \mathsf{CN}(\texttt{Supp}(a))$ and $\texttt{Supp}(a) \subseteq \Sigma$. By monotonicity of CN, $\mathsf{CN}(\texttt{Supp}(a)) \subseteq \mathsf{CN}(\Sigma)$. Thus, $x \in \mathsf{CN}(\Sigma)$. ■

Property 64 *It holds that $\Re_p \subseteq \Re_s$.*

Proof Let $\mathcal{R} \in \Re_p$. Thus, for all $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$, $\mathcal{T}$ satisfies all the postulates, i.e., for all $\mathcal{E} \in \texttt{Ext}_p(\mathcal{T})$,

- $\texttt{Concs}(E)$ is consistent;
- $\texttt{Concs}(E) = \mathsf{CN}(\texttt{Concs}(E))$;
- for all $a \in \mathcal{E}$, $\texttt{Sub}(a) \subseteq \mathcal{E}$;
- $\texttt{Arg}(\texttt{Free}(\Sigma)) \subseteq \mathcal{E}$;
- for all $(X, x) \in \texttt{Arg}(\Sigma)$, if $X \cup \{x\} \subseteq \texttt{Concs}(\mathcal{E})$, then $(X, x) \in \mathcal{E}$.

Since $\texttt{Ext}_s(\mathcal{T}) \subseteq \texttt{Ext}_p(\mathcal{T})$, the previous properties are satisfied by all the stable extensions of $\mathcal{T}$. Consequently, $\mathcal{R} \in \Re_s$. ■

Proposition 17 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive and Σ be a knowledge base. For all non-empty proper subsets X of some minimal conflict $C \in \mathcal{C}_\Sigma$, there exists $a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Supp}(a) = X$.*

Proof Let $C \in \mathcal{C}_\Sigma$ and $X \subseteq C$ such that X is non-empty. Assume $\nexists a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Supp}(a) = X$ – i.e., there exists no x such that X is a minimal consistent set satisfying $x \in \mathsf{CN}(X)$. So, for all $x \in \mathcal{L}$, if $x \in \mathsf{CN}(X)$ then $\exists Y \subset X$ such that $x \in \mathsf{CN}(Y)$. In short, for all $x \in \mathsf{CN}(X)$, there exists $Y \subset X$ such that $x \in \mathsf{CN}(Y)$. Property 11 says that there exists $z \in \mathcal{L}$ such that $\mathsf{CN}(\{z\}) = \mathsf{CN}(X)$. However, $z \notin \mathsf{CN}(Y)$ for all $Y \subset X$ otherwise C would fail to be minimal (because $Y \cup (C \setminus X) \subset C$ while $z \in \mathsf{CN}(Y)$ implies $\mathsf{CN}(C) = \mathsf{CN}(X \cup (C \setminus X)) = \mathsf{CN}(\{z\} \cup (C \setminus X)) \subseteq \mathsf{CN}(Y \cup (C \setminus X))$). A contradiction arises. ■

Proposition 23 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ is closed under CN, then $\texttt{Output}(\mathcal{T}) = \mathsf{CN}(\texttt{Output}(\mathcal{T}))$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ, and $\texttt{Ext}(\mathcal{T})$ be its set of extensions under a given semantics. Assume that $\mathcal{T}$ satisfies closure under CN. From the Expansion axiom, it follows that $\texttt{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\texttt{Output}(\mathcal{T}))$.

Assume now that $x \in \mathsf{CN}(\texttt{Output}(\mathcal{T}))$. Thus, $\exists x_1, \ldots, x_n \in \texttt{Output}(\mathcal{T})$ such that $x \in \mathsf{CN}(\{x_1, \ldots, x_n\})$. From Property 20, $x_1, \ldots, x_n \in \cap\texttt{Concs}(\mathcal{E}_i)$ where $\mathcal{E}_i \in \texttt{Ext}(\mathcal{T})$. From Property 5, it holds that $\mathsf{CN}(\{x_1, \ldots, x_n\}) \subseteq \mathsf{CN}(\cap\texttt{Concs}(\mathcal{E}_i))$. Again, from Property 5, $x \in \mathsf{CN}(\texttt{Concs}(\mathcal{E}_1)) \cap \ldots \cap \mathsf{CN}(\texttt{Concs}(\mathcal{E}_n))$. Since $\mathcal{T}$ satisfies closure under CN, then for each $\mathcal{E}_i$ it holds that $\mathsf{CN}(\texttt{Concs}(\mathcal{E}_i)) = \texttt{Concs}(\mathcal{E}_i)$. Thus, $x \in \texttt{Concs}(\mathcal{E}_1) \cap \ldots \cap \texttt{Concs}(\mathcal{E}_n)$. From Property 20, it holds that $x \in \texttt{Output}(\mathcal{T})$. ■

Proposition 25 *Let $\mathcal{T} = (\texttt{Args}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ is closed under sub-arguments and under* CN*, then for all $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$,* $\texttt{Concs}(\mathcal{E}) = \mathsf{CN}(\texttt{Base}(\mathcal{E}))$.

Proof Assume that $\mathcal{T} = (\texttt{Args}(\Sigma), \mathcal{R})$ is closed under sub-arguments and under CN. From Property 4 in Amgoud (2012), since $\mathcal{T}$ is closed under sub-arguments, then it follows that $\texttt{Base}(\mathcal{E}) \subseteq \texttt{Concs}(\mathcal{E})$. By monotonicity of CN, we get $\mathsf{CN}(\texttt{Base}(\mathcal{E})) \subseteq \mathsf{CN}(\texttt{Concs}(\mathcal{E}))$. Since $\mathcal{T}$ is closed under CN, then $\mathsf{CN}(\texttt{Base}(\mathcal{E})) \subseteq \texttt{Concs}(\mathcal{E})$. Besides, by definition of $\texttt{Concs}(\mathcal{E})$, $\texttt{Concs}(\mathcal{E}) \subseteq \bigcup \mathsf{CN}(\texttt{Supp}(a_i))$ with $a_i \in \mathcal{E}$. From Property 5, it follows that $\texttt{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\bigcup \texttt{Supp}(a_i))$, thus $\texttt{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\texttt{Base}(\mathcal{E}))$. ∎

Proposition 27 *If an argumentation system $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ satisfies consistency, then the set $\texttt{Output}(\mathcal{T})$ is consistent.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system built over a knowledge base Σ. Assume that $\mathcal{T}$ satisfies consistency. Thus, $\forall \mathcal{E}_i \in \texttt{Ext}(\mathcal{T})$, $\texttt{Concs}(\mathcal{E}_i)$ is consistent. Let $\mathcal{E}$ be a given extension in the set $\texttt{Ext}(\mathcal{T})$. Since $\cap\texttt{Concs}(\mathcal{E}_i) \subseteq \texttt{Concs}(\mathcal{E})$, then $\cap\texttt{Concs}(\mathcal{E}_i)$ is consistent as well. Besides, from Property 20, $\texttt{Output}(\mathcal{T}) = \cap\texttt{Concs}(\mathcal{E}_i)$. It follows that $\texttt{Output}(\mathcal{T})$ is consistent. ∎

Proposition 28 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that for all $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments, then for all $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\texttt{Base}(\mathcal{E})$ is consistent.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. Assume that $\mathcal{T}$ satisfies consistency and closure under sub-arguments. From closure under sub-arguments, it follows that for all $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\texttt{Base}(\mathcal{E}) \subseteq \texttt{Concs}(\mathcal{E})$ (from Property 4 in Amgoud, 2012). Since $\mathcal{T}$ satisfies consistency, the set $\texttt{Concs}(\mathcal{E})$ is consistent. From Property 7, it follows that $\texttt{Base}(\mathcal{E})$ is consistent. ∎

Proposition 30 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ satisfies free precedence, then $\texttt{Free}(\Sigma) \subseteq \texttt{Output}(\mathcal{T})$ (under any of the reviewed semantics).*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ, and let $\texttt{Ext}(\mathcal{T})$ be its set of extensions under any of the reviewed semantics. Assume that $\mathcal{T}$ satisfies Postulate 29. Thus, for all $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\texttt{Arg}(\texttt{Free}(\Sigma)) \subseteq \mathcal{E}$. Assume that $\texttt{Free}(\Sigma) \neq \emptyset$. Thus, for all $x \in \texttt{Free}(\Sigma)$, $(\{x\}, x) \in \texttt{Arg}(\texttt{Free}(\Sigma))$ (indeed, $x \notin \mathsf{CN}(\emptyset)$) and thus, $(\{x\}, x) \in \mathcal{E}$ (for all $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$). Consequently, $\texttt{Free}(\Sigma) \subseteq \texttt{Concs}(\mathcal{E})$ (for all $\mathcal{E} \in \texttt{Ext}$). So, $\texttt{Free}(\Sigma) \subseteq \texttt{Output}(\mathcal{T})$. ∎

Proposition 32 *If an argumentation system $\mathcal{T}$ is closed under both* CN *and sub-arguments and satisfies the exhaustiveness postulate, then $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$ (under any of the reviewed semantics).*

Proof Let $\mathcal{T}$ be an argumentation system that satisfies exhaustiveness and that is closed under both CN and sub-arguments. Let $\texttt{Ext}(\mathcal{T})$ be its extensions under any of the reviewed semantics. Let $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$. From the definition of $\texttt{Arg}$ and $\texttt{Base}$, it follows that $\mathcal{E} \subseteq \texttt{Arg}(\texttt{Base}(\mathcal{E}))$.

Assume now $a \in \texttt{Arg}(\texttt{Base}(\mathcal{E}))$ and let $a = (X, x)$. Thus, $X \subseteq \texttt{Base}(\mathcal{E})$. By monotonicity of CN, $\mathsf{CN}(X) \subseteq \mathsf{CN}(\texttt{Base}(\mathcal{E}))$. From Proposition 25, since $\mathcal{T}$ is closed under both CN and sub-arguments, then $\texttt{Concs}(\mathcal{E}) = \mathsf{CN}(\texttt{Base}(\mathcal{E}))$. Thus, $\mathsf{CN}(X) \subseteq \texttt{Concs}(\mathcal{E})$. Besides, $X \subseteq \mathsf{CN}(X)$ (from the Expansion Axiom of CN) and $x \in \mathsf{CN}(X)$ (from the definition of an argument), thus, $X \cup \{x\} \subseteq \texttt{Concs}(\mathcal{E})$. By exhaustiveness of $\mathcal{T}$, $a \in \mathcal{E}$. ∎

Proposition 34 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive and stable semantics), then it is closed under* CN *(under naive and stable semantics).*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ is closed under sub-arguments and satisfies consistency. Assume also that $\mathcal{T}$ violates closure under CN. Thus, $\exists \mathcal{E} \in \texttt{Ext}_n(\mathcal{T})$ such that $\texttt{Concs}(\mathcal{E}) \neq \mathsf{CN}(\texttt{Concs}(\mathcal{E}))$. This means that $\exists x \in \mathsf{CN}(\texttt{Concs}(\mathcal{E}))$ and $x \notin \texttt{Concs}(\mathcal{E})$. Besides, $\mathsf{CN}(\texttt{Concs}(\mathcal{E})) \subseteq \mathsf{CN}(\texttt{Base}(\mathcal{E}))$. Thus, $x \in \mathsf{CN}(\texttt{Base}(\mathcal{E}))$. Since CN verifies compactness, then $\exists X \subseteq \texttt{Base}(\mathcal{E})$ such that X is finite and $x \in \mathsf{CN}(X)$. Moreover, from Proposition 28, $\texttt{Base}(\mathcal{E})$ is consistent. Then, X is consistent as well (from Property 7). Consequently, the pair (X, x) is an argument. Besides, since $x \notin \texttt{Concs}(\mathcal{E})$ then $(X, x) \notin \mathcal{E}$. This means that $\exists a \in \mathcal{E}$ such that $a\mathcal{R}(X, x)$ or $(X, x)\mathcal{R}a$. Finally, since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup X$ is inconsistent and consequently $\texttt{Base}(\mathcal{E})$ is inconsistent. This contradicts the assumption.

Since $\texttt{Ext}_s(\mathcal{T}) \subseteq \texttt{Ext}_n(\mathcal{T})$, then since $\mathcal{T}$ satisfies closure under CN under naive semantics, then it also satisfies the postulate under stable semantics. ∎

Proposition 36 *Let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent. If Σ is consistent, then $\mathcal{R} = \emptyset$.*

Proof Let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent. Assume that Σ is consistent. Then, for all $a, b \in \texttt{Arg}(\Sigma)$, $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is consistent. Since $\mathcal{R}$ is conflict-dependent, then $(a, b) \notin \mathcal{R}$. Thus, $\mathcal{R} = \emptyset$. ∎

Proposition 37 *Let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent. $\forall \mathcal{E} \subseteq \texttt{Arg}(\Sigma)$, if $\texttt{Base}(\mathcal{E})$ is consistent, then $\mathcal{E}$ is conflict-free.*

Proof Let $\mathcal{E} \subseteq \texttt{Arg}(\Sigma)$. Since $\texttt{Base}(\mathcal{E})$ is consistent, then so is $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ for all a and b in $\mathcal{E}$ (according to Property 7). Hence, there exists no minimal conflict $C \subseteq \texttt{Supp}(a) \cup \texttt{Supp}(b)$. By the definition of $\mathcal{R}$ being conflict-dependent, $(a, b) \notin \mathcal{R}$ ensues. ∎

Proposition 38 *Let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent. For all $a \in \texttt{Arg}(\Sigma)$, $(a, a) \notin \mathcal{R}$.*

Proof Assume that $\mathcal{R}$ is conflict-dependent and $a \in \texttt{Arg}(\Sigma)$ such that$(a, a) \in \mathcal{R}$. Since $\mathcal{R}$ is conflict-dependent, then $\exists C \in \mathcal{C}_\Sigma$ such that $C \subseteq \texttt{Supp}(a)$. This means that $\texttt{Supp}(a)$ is inconsistent. This contradicts the fact that a is an argument. ∎

Proposition 40 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive and Σ be a knowledge base such that $\exists C \in \mathcal{C}_\Sigma$ and $|C| > 2$. If $\mathcal{R}$ is conflict-dependent and symmetric, then the argumentation system $(\texttt{Arg}(\Sigma), \mathcal{R})$ violates consistency.*

Proof Let C be a minimal conflict in a knowledge base Σ. Consider a partition $\{X_1, X_2, X_3\}$ of C. Due to Proposition 17, there exist a_1, a_2, a_3 in $\mathtt{Arg}(\Sigma)$ such that $\mathsf{CN}(\mathtt{Supp}(a_i)) = \mathsf{CN}(X_i)$ for $i = 1...3$. For $\{a_1, a_2, a_3\}$ not to be an admissible extension, either it is not conflict-free or it fails to defend its elements. Assume that $\{a_1, a_2, a_3\}$ is not conflict-free, i.e., $a_i \mathcal{R} a_j$ for some i and j in $\{1, 2, 3\}$. Since $\mathcal{R}$ is conflict-dependent, there exists $C' \in \mathcal{C}_\Sigma$ such that $C' \subseteq \mathtt{Supp}(a_i) \cup \mathtt{Supp}(a_j)$. Hence, $\mathsf{CN}(\mathtt{Supp}(a_i) \cup \mathtt{Supp}(a_j)) = \mathcal{L}$. However, $\mathsf{CN}(\mathtt{Supp}(a_i) \cup \mathtt{Supp}(a_j)) = \mathsf{CN}(\mathsf{CN}(\mathtt{Supp}(a_i)) \cup \mathsf{CN}(\mathtt{Supp}(a_j))) = \mathsf{CN}(\mathsf{CN}(X_i) \cup \mathsf{CN}(X_j)) = \mathsf{CN}(X_i \cup X_j)$, meaning that $X_i \cup X_j$ is an inconsistent proper subset of C, contradicting $C \in \mathcal{C}_\Sigma$. Otherwise, assume that $\{a_1, a_2, a_3\}$ fails to defend its elements. It obviously cannot be the case because $\mathcal{R}$ is symmetric: if $a \mathcal{R} a_i$ then $a_i \mathcal{R} a$. ■

Theorem 41 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathtt{Base}(\mathcal{E}) \in \mathtt{Max}(\Sigma)$.*
- *For all $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_n(\mathcal{T})$, if $\mathtt{Base}(\mathcal{E}_i) = \mathtt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. Let $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$. From Proposition 28, $\mathtt{Base}(\mathcal{E})$ is consistent.

Assume now that $\mathtt{Base}(\mathcal{E})$ is not maximal (for set inclusion) consistent. Thus, $\exists x \in \Sigma \setminus \mathtt{Base}(\mathcal{E})$ such that $\mathtt{Base}(\mathcal{E}) \cup \{x\}$ is consistent. This means that $\{x\}$ is consistent. Thus, $\exists a \in \mathcal{A}$ such that $\mathtt{Supp}(a) = \{x\}$. Since $x \notin \mathtt{Base}(\mathcal{E})$, then $a \notin \mathcal{E}$. Since $\mathcal{E}$ is a naive extension, then $\exists b \in \mathcal{E}$ such that $a \mathcal{R} b$ or $b \mathcal{R} a$. Since $\mathcal{R}$ is conflict-dependent, then $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is inconsistent. But, $\mathtt{Supp}(b) \subseteq \mathtt{Base}(\mathcal{E})$, this would mean that $\mathtt{Base}(\mathcal{E}) \cup \{x\}$ is inconsistent. Contradiction.

Let $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$. It is obvious that $\mathcal{E} \subseteq \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$ since the construction of arguments is monotonic. Let $a \in \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$. Thus, $\mathtt{Supp}(a) \subseteq \mathtt{Base}(\mathcal{E})$. Assume that $a \notin \mathcal{E}$, then $\exists b \in \mathcal{E}$ such that $a \mathcal{R} b$ or $b \mathcal{R} a$. Since $\mathcal{R}$ is conflict-dependent, then $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is inconsistent. Besides, $\mathtt{Supp}(a) \cup \mathtt{Supp}(b) \subseteq \mathtt{Base}(\mathcal{E})$. This means that $\mathtt{Base}(\mathcal{E})$ is inconsistent. Contradiction.

Now let $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_n(\mathcal{T})$. Assume that $\mathtt{Base}(\mathcal{E}_i) = \mathtt{Base}(\mathcal{E}_j)$. Then, $\mathtt{Arg}(\mathtt{Base}(\mathcal{E}_i)) = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_j))$. Besides, from the second bullet of the theorem, $\mathcal{E}_i = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_i))$ and $\mathcal{E}_j = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_j))$. Consequently, $\mathcal{E}_i = \mathcal{E}_j$. ■

Theorem 42 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_n(\mathcal{T})$.*
- *For all $\mathcal{S}_i, \mathcal{S}_j \in \mathtt{Max}(\Sigma)$, if $\mathtt{Arg}(\mathcal{S}_i) = \mathtt{Arg}(\mathcal{S}_j)$ then $\mathcal{S}_i = \mathcal{S}_j$.*
- *For all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathcal{S} = \mathtt{Base}(\mathtt{Arg}(\mathcal{S}))$.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies Postulates 24 and 26.

Let $\mathcal{S} \in \mathtt{Max}(\Sigma)$, and assume that $\mathtt{Arg}(\mathcal{S}) \notin \mathtt{Ext}_n(\mathcal{T})$. Since $\mathcal{R}$ is conflict-dependent and $\mathcal{S}$ is consistent, it follows from Proposition 37 that $\mathtt{Arg}(\mathcal{S})$ is conflict-free. Thus, $\mathtt{Arg}(\mathcal{S})$ is not maximal for set inclusion. So, $\exists a \in \mathcal{A}$ such that $\mathtt{Arg}(\mathcal{S}) \cup \{a\}$ is conflict-free. There are two possibilities:

(1) $\mathcal{S} \cup \texttt{Supp}(a)$ is consistent. But since $\mathcal{S} \in \texttt{Max}(\Sigma)$, then $\texttt{Supp}(a) \subseteq \mathcal{S}$, and this would mean that $a \in \texttt{Arg}(\mathcal{S})$.
(2) $\mathcal{S} \cup \texttt{Supp}(a)$ is inconsistent. Thus, $\exists C \in \mathcal{C}_\Sigma$ such that $C \subseteq \mathcal{S} \cup \texttt{Supp}(a)$. Let $X_1 = C \cap \mathcal{S}$ and $X_2 = C \cap \texttt{Supp}(a)$ with $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ (since $\mathcal{S}$ and $\texttt{Supp}(a)$ are consistent). From Property 11, $\exists x_1 \in \mathsf{CN}(X_1)$ and $\exists x_2 \in \mathsf{CN}(X_2)$ such that the set $\{x_1, x_2\}$ is inconsistent. Note that (X_1, x_1) and (X_2, x_2) are arguments. Moreover, $(X_1, x_1) \in \texttt{Arg}(\mathcal{S})$ and $(X_2, x_2) \in \texttt{Sub}(a)$. Besides, since $\texttt{Arg}(\mathcal{S}) \cup \{a\}$ is conflict-free, then $\exists \mathcal{E} \in \texttt{Ext}(\mathcal{T})$ such that $\texttt{Arg}(\mathcal{S}) \cup \{a\} \subseteq \mathcal{E}$. Thus, $(X_1, x_1) \in \mathcal{E}$. Since $\mathcal{T}$ is closed under sub-arguments then $(X_2, x_2) \in \mathcal{E}$. Thus, $\{x_1, x_2\} \subseteq \texttt{Concs}(\mathcal{E})$. From Property 7, it follows that $\texttt{Concs}(\mathcal{E})$ is inconsistent. This contradicts the fact that $\mathcal{T}$ satisfies consistency.

Now let $\mathcal{S}_i, \mathcal{S}_j \in \texttt{Max}(\Sigma)$ be such that $\texttt{Arg}(\mathcal{S}_i) = \texttt{Arg}(\mathcal{S}_j)$. Assume that $\mathcal{S}_i \neq \mathcal{S}_j$, thus $\exists x \in \mathcal{S}_i$ and $x \notin \mathcal{S}_j$. Besides, if $\mathcal{S}_i$ is consistent, so is the set $\{x\}$. Consequently, $\exists a \in \mathcal{A}$ such that $\texttt{Supp}(a) = \{x\}$. It follows also that $a \in \texttt{Arg}(\mathcal{S}_i)$ and thus $a \in \texttt{Arg}(\mathcal{S}_j)$. By definition of an argument, $\texttt{Supp}(a) \subseteq \mathcal{S}_j$. Contradiction.

Let $\mathcal{S} \in \texttt{Max}(\Sigma)$. Since $\mathcal{S}$ is consistent, then $\forall x \in \mathcal{S}$, it holds that the set $\{x\}$ is consistent as well (from Property 7). Then, $(\{x\}, x)$ is an argument in $\texttt{Arg}(\mathcal{S})$ (from Property 15). Thus, $\mathcal{S} \subseteq \texttt{Base}(\texttt{Arg}(\mathcal{S}))$. Conversely, let $x \in \texttt{Base}(\texttt{Arg}(\mathcal{S}))$. By definition of the function $\texttt{Base}$, $\exists a \in \texttt{Arg}(\mathcal{S})$ such that $x \in \texttt{Supp}(a)$. Besides, by definition of an argument, $\texttt{Supp}(a) \subseteq \mathcal{S}$. Consequently, $x \in \mathcal{S}$. ■

Theorem 46 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent, $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).* $\texttt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where $\mathcal{S}_i$ ranges over $\{\mathcal{S}_i \in \texttt{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \texttt{Ext}_n(\mathcal{T})$ and $\mathcal{S}_i = \texttt{Base}(\mathcal{E}_i)\}$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. From Proposition 34, $\mathcal{T}$ enjoys closure under CN (under naive semantics). From Proposition 25, for all $\mathcal{E} \in \texttt{Ext}_n(\mathcal{T})$, $\texttt{Concs}(\mathcal{E}) = \mathsf{CN}(\texttt{Base}(\mathcal{E}))$. Besides, from Theorem 41, for all $\mathcal{E}_i \in \texttt{Ext}_n(\mathcal{T})$, $\exists! \mathcal{S}_i \in \texttt{Max}(\Sigma)$ such that $\texttt{Base}(\mathcal{E}_i) = \mathcal{S}_i$. Thus, $\texttt{Concs}(\mathcal{E}_i) = \mathsf{CN}(\mathcal{S}_i)$. By definition, $\texttt{Output}(\mathcal{T}) = \bigcap \texttt{Concs}(\mathcal{E}_i)$, thus $\texttt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$. ■

Theorem 49 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics) and $\texttt{Ext}_s(\mathcal{T}) \neq \emptyset$, then:*

- *For all $\mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$, $\texttt{Base}(\mathcal{E}) \in \texttt{Max}(\Sigma)$.*
- *For all $\mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \texttt{Ext}_s(\mathcal{T})$, if $\texttt{Base}(\mathcal{E}_i) = \texttt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Let $\mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$. Since $\mathcal{T}$ satisfies Postulates 22, 24 and 26, then $\texttt{Base}(\mathcal{E})$ is consistent (from Proposition 28). Assume now that $\texttt{Base}(\mathcal{E})$ is not maximal for set inclusion. Thus, $\exists x \in \Sigma \setminus \texttt{Base}(\mathcal{E})$ such that $\texttt{Base}(\mathcal{E}) \cup \{x\}$ is consistent. This means that $\{x\}$ is consistent. Thus, from Property 15, $\exists a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Supp}(a) = \{x\}$. Since $x \notin \texttt{Base}(\mathcal{E})$, then $a \notin \mathcal{E}$. Since $\mathcal{E}$ is a stable extension, then $\exists b \in \mathcal{E}$ such that $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is inconsistent. But, $\texttt{Supp}(b) \subseteq \texttt{Base}(\mathcal{E})$, which would mean that $\texttt{Base}(\mathcal{E}) \cup \{x\}$ is inconsistent. Contradiction.

Let $\mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$. It is obvious that $\mathcal{E} \subseteq \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$ since the construction of arguments is monotonic. Let $a \in \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$. Thus, $\mathtt{Supp}(a) \subseteq \mathtt{Base}(\mathcal{E})$. Assume that $a \notin \mathcal{E}$, then $\exists b \in \mathcal{E}$ such that $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is inconsistent. Besides, $\mathtt{Supp}(a) \cup \mathtt{Supp}(b) \subseteq \mathtt{Base}(\mathcal{E})$. This means that $\mathtt{Base}(\mathcal{E})$ is inconsistent. Contradiction.

Now let $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_s(\mathcal{T})$. Assume that $\mathtt{Base}(\mathcal{E}_i) = \mathtt{Base}(\mathcal{E}_j)$. Then, $\mathtt{Arg}(\mathtt{Base}(\mathcal{E}_i)) = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_j))$. Besides, from bullet 2 of the theorem, $\mathcal{E}_i = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_i))$ and $\mathcal{E}_j = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_j))$. Consequently, $\mathcal{E}_i = \mathcal{E}_j$. ■

Theorem 50 *It holds that* $\Re_{s1} = \emptyset$.

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s1}$. Thus, $\mathtt{Output}(\mathcal{T}) = \emptyset$. Assume that $\mathtt{Free}(\Sigma) \neq \emptyset$. Thus, $\mathcal{T}$ violates free precedence postulate. This contradicts the fact that $\mathcal{R} \in \Re_{s1}$. ■

Theorem 51 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s_2}$. $\mathtt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i$ *ranges over* $\{\mathcal{S}_i \in \mathtt{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \mathtt{Ext}_s(\mathcal{T})$ *and* $\mathcal{S}_i = \mathtt{Base}(\mathcal{E}_i)\}$.

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s2}$. From Proposition 25, for all $\mathcal{E}_i \in \mathtt{Ext}_s(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}_i) = \mathsf{CN}(\mathtt{Base}(\mathcal{E}_i))$. Thus, $\mathtt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathtt{Base}(\mathcal{E}_i))$ with $\mathcal{E}_i \in \mathtt{Ext}_s(\mathcal{T})$. From Theorem 49, for all $\mathcal{E}_i \in \mathtt{Ext}_s(\mathcal{T})$, $\exists! \mathcal{S}_i \in \mathtt{Max}(\Sigma)$ such that $\mathtt{Base}(\mathcal{E}_i) = \mathcal{S}_i$. Thus, $\mathtt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$. ■

Theorem 52 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s3}$. *For all* $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_s(\mathcal{T})$.

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. Let $\mathcal{S} \in \mathtt{Max}(\Sigma)$. Since $|\mathtt{Ext}_s(\mathcal{T})(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$, then from Theorem 49, $\exists \mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$ such that $\mathtt{Base}(\mathcal{E}) = \mathcal{S}$. Besides, from the same theorem, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$, thus $\mathcal{E} = \mathtt{Arg}(\mathcal{S})$. Consequently, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_s(\mathcal{T})$. ■

Theorem 53 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s3}$. *The equality* $\mathtt{Ext}_n(\mathcal{T}) = \mathtt{Ext}_s(\mathcal{T})$ *holds. If* $\mathcal{T}$ *satisfies the postulates under preferred semantics, then* $\mathtt{Ext}_s(\mathcal{T}) = \mathtt{Ext}_p(\mathcal{T})$.

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. From Theorem 52, for all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_s(\mathcal{T})$. From Theorem 41, $|\mathtt{Ext}_n(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|$. Thus, $\mathtt{Ext}_n(\mathcal{T}) = \mathtt{Ext}_s(\mathcal{T})$. Assume now that $\mathcal{T}$ satisfies the postulates under preferred semantics, then from Theorem 60, $|\mathtt{Ext}_p(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$ and $\mathtt{Ext}_s(\mathcal{T}) = \mathtt{Ext}_p(\mathcal{T})$. ■

Theorem 58 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies consistency and closure under sub-arguments (under preferred semantics). For all* $\mathcal{E} \in \mathtt{Ext}_p(\mathcal{T})$, *there exists* $\mathcal{S} \in \mathtt{Max}(\Sigma)$ *such that* $\mathtt{Base}(\mathcal{E}) \subseteq \mathcal{S}$.

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments. Let $\mathcal{E} \in \mathtt{Ext}_p(\mathcal{T})$. Due to consistency and closure under sub-arguments, $\mathtt{Base}(\mathcal{E})$ is consistent (cf. Proposition 28). From Property 9, there exists $\mathcal{S} \in \mathtt{Max}(\Sigma)$ such that $\mathtt{Base}(\mathcal{E}) \subseteq \mathcal{S}$.

■

Theorem 60 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). For all $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_p(\mathcal{T})$, if $\mathtt{Base}(\mathcal{E}_i) \subseteq \mathtt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments. Assume that $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_p(\mathcal{T})$ and $\mathtt{Base}(\mathcal{E}_i) \subseteq \mathtt{Base}(\mathcal{E}_j)$.

We first show that $\forall x \in \{i, j\}$, $\mathtt{Arg}(\mathtt{Base}(\mathcal{E}_x))$ is conflict-free. Assume that $\mathtt{Arg}(\mathtt{Base}(\mathcal{E}_x))$ is not conflict-free. Thus, $\exists a, b \in \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_x))$ such that $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is inconsistent. Besides, $\mathtt{Supp}(a) \cup \mathtt{Supp}(b) \subseteq \mathtt{Base}(\mathcal{E}_x)$. Thus, $\mathtt{Base}(\mathcal{E}_x)$ is inconsistent. This contradicts Proposition 28.

Assume that $\mathcal{E}_i \setminus \mathcal{E}_j \neq \emptyset$. Let $\mathcal{E} = \mathcal{E}_j \cup (\mathcal{E}_i \setminus \mathcal{E}_j)$. $\mathcal{E} \subseteq \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_j))$. Thus, $\mathcal{E}$ is conflict-free (since $\mathtt{Arg}(\mathtt{Base}(\mathcal{E}_j))$ is conflict-free). Moreover, $\mathcal{E}$ defends any element in $\mathcal{E}_j$ (since $\mathcal{E}_j \in \mathtt{Ext}_p(\mathcal{T})$) and any element in $\mathcal{E}_i \setminus \mathcal{E}_j$ (since $\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})$). Thus, $\mathcal{E}$ is an admissible set. This contradicts the fact that $\mathcal{E}_j \in \mathtt{Ext}_p(\mathcal{T})$. The same reasoning holds for the case $\mathcal{E}_j \setminus \mathcal{E}_i \neq \emptyset$ and having $\mathcal{E} = \mathcal{E}_i \cup \mathcal{E}_j \setminus \mathcal{E}_i$. ■

Theorem 61 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). Let $\mathcal{S} \in \mathtt{Max}(\Sigma)$. For all $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_p(\mathcal{T})$, if $\mathtt{Base}(\mathcal{E}_i) \subseteq \mathcal{S}$ and $\mathtt{Base}(\mathcal{E}_j) \subseteq \mathcal{S}$, then $\mathcal{E}_i = \mathcal{E}_j$.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). Assume that there exist two distinct preferred extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ such that for some $\mathcal{S} \in \mathtt{Max}(\Sigma)$, both $\mathtt{Base}(\mathcal{E}_1) \subseteq \mathcal{S}$ and $\mathtt{Base}(\mathcal{E}_2) \subseteq \mathcal{S}$ hold. Then, $\mathtt{Base}(\mathcal{E}_1) \cup \mathtt{Base}(\mathcal{E}_2) \subseteq \mathcal{S}$ and $\mathtt{Base}(\mathcal{E}_1) \cup \mathtt{Base}(\mathcal{E}_2)$ is consistent. Since $\mathcal{R}$ is conflict-dependent, $a\mathcal{R}b$ would demand that $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ be inconsistent. Therefore, $a\mathcal{R}b$ is impossible for a and b both in $\mathcal{E}_1 \cup \mathcal{E}_2$. That is, $\mathcal{E}_1 \cup \mathcal{E}_2$ is conflict-free. Since $\mathcal{E}_1$ is an extension, it defends all arguments in $\mathcal{E}_1$. Since $\mathcal{E}_2$ is an extension, it defends all arguments in $\mathcal{E}_2$. Hence, $\mathcal{E}_1 \cup \mathcal{E}_2$ defends all arguments in $\mathcal{E}_1 \cup \mathcal{E}_2$. That is, $\mathcal{E}_1 \cup \mathcal{E}_2$ is an admissible set, and there exists a preferred extension $\mathcal{E}_3$ that contains it (possibly improperly). According to Theorem 60, $\mathtt{Base}(\mathcal{E}_1) \subseteq \mathtt{Base}(\mathcal{E}_1 \cup \mathcal{E}_2) = \mathtt{Base}(\mathcal{E}_3)$ yield $\mathcal{E}_1 = \mathcal{E}_3$, and, similarly, $\mathtt{Base}(\mathcal{E}_2) \subseteq \mathtt{Base}(\mathcal{E}_1 \cup \mathcal{E}_2) = \mathtt{Base}(\mathcal{E}_3)$ yield $\mathcal{E}_2 = \mathcal{E}_3$. Therefore, $\mathcal{E}_1 = \mathcal{E}_2$, contradicting the assumption that $\mathcal{E}_1$ and $\mathcal{E}_2$ are distinct. ■

Theorem 62 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). It holds that $1 \leq |\mathtt{Ext}_p(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|$.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). From Theorem 58, for all $\mathcal{E} \in \mathtt{Ext}_p(\mathcal{T})$, $\exists \mathcal{S} \in \mathtt{Max}(\Sigma)$ such that $\mathtt{Base}(\mathcal{E}) \subseteq \mathcal{S}$. From Theorem 61, there cannot exist two distinct preferred extensions $\mathcal{E}_i$ and $\mathcal{E}_j$ such that for some $\mathcal{S} \in \mathtt{Max}(\Sigma)$, both $\mathtt{Base}(\mathcal{E}_i) \subseteq \mathcal{S}$ and $\mathtt{Base}(\mathcal{E}_j) \subseteq \mathcal{S}$. Thus, every $\mathcal{S} \in \mathtt{Max}(\Sigma)$ is captured by at most one preferred extension of $\mathcal{T}$. It follows that $1 \leq |\mathtt{Ext}_p(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|$. ■

Theorem 65 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system. If $\mathcal{R} \in \Re_{p_3}$ then:*

- *for all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_p(\mathcal{T})$;*
- $|\mathtt{Ext}_p(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$.

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p3}$. From Theorem 52, for all $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_s(\mathcal{T})$. Thus, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_p(\mathcal{T})$.

Since $\mathcal{R} \in \Re_{p3}$, thus $|\mathtt{Ext}_s(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$. Assume now that there exists $\mathcal{E} \in \mathtt{Ext}_p(\mathcal{T})$ such that $\mathtt{Base}(\mathcal{E}) \notin \mathtt{Max}(\Sigma)$. Thus, $\exists \mathcal{S} \in \mathtt{Max}(\Sigma)$ such that $\mathtt{Base}(\mathcal{E}) \subseteq \mathcal{S}$. From the previous result, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_p(\mathcal{T})$. From Theorem 60, $\mathcal{E} = \mathtt{Arg}(\mathcal{S})$. Consequently, $|\mathtt{Ext}_p(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$. ■

Theorem 67 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{p2}$. $\mathtt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ where $\mathcal{S}_i$ ranges over $\{\mathcal{S}_i \in \mathtt{Cons}(\Sigma) \mid \exists \mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})$ and $\mathcal{S}_i = \mathtt{Base}(\mathcal{E}_i)\}$.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p2}$. Let $\mathcal{E} \in \mathtt{Ext}_p(\mathcal{T})$. From Proposition 25, since $\mathcal{T}$ is closed both under CN and under sub-arguments, then $\mathtt{Concs}(\mathcal{E}) = \mathsf{CN}(\mathtt{Base}(\mathcal{E}))$. From Definition 19, $\mathtt{Output}(\mathcal{T}) = \bigcap \mathtt{Concs}(\mathcal{E}_i)$, $\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})$. Thus, $\mathtt{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathtt{Base}(\mathcal{E}_i))$, $\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})$. Besides, $\mathtt{Free}(\Sigma) \subseteq \mathtt{Base}(\mathcal{E})$ for all $\mathcal{E} \in \mathtt{Ext}_p(\mathcal{T})$ and from Proposition 28, $\mathtt{Base}(\mathcal{E})$ is consistent. Thus, $\mathtt{Base}(\mathcal{E}) \in \mathtt{Cons}(\Sigma)$. ■

Theorem 70 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent.*

- *For all $a \in \mathtt{Arg}(\mathtt{Free}(\Sigma))$, a neither attacks nor is attacked by another argument in $\mathtt{Arg}(\Sigma)$.*
- *$\mathtt{Arg}(\mathtt{Free}(\Sigma))$ is an admissible extension of $\mathcal{T}$.*

Proof Let $(\mathtt{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent. Let $a \in \mathtt{Arg}(\mathtt{Free}(\Sigma))$. Assume that $\exists b \in \mathcal{A}$ such that $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\exists C \in \mathcal{C}_\Sigma$ such that $C \subseteq \mathtt{Supp}(a) \cup \mathtt{Supp}(b)$. By definition of an argument, both $\mathtt{Supp}(a)$ and $\mathtt{Supp}(b)$ are consistent. Then, $C \cap \mathtt{Supp}(a) \neq \emptyset$. This contradicts the fact that $\mathtt{Supp}(a) \subseteq \mathtt{Free}(\Sigma)$. Thus, $\mathtt{Arg}(\mathtt{Free}(\Sigma))$ is *conflict-free and can never be attacked*. Consequently, $\mathtt{Arg}(\mathtt{Free}(\Sigma))$ is an admissible extension of $\mathcal{T}$. ■

Theorem 71 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p3}$.*

$$\mathtt{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathcal{E}_i = \mathtt{Arg}(\mathtt{Free}(\Sigma)).$$

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p3}$. Since $\mathcal{T}$ satisfies exhaustiveness, then from Proposition 32, for all $\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})$, $\mathcal{E}_i = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}_i))$. So,

$$\bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathcal{E}_i = \mathtt{Arg}\left(\bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathtt{Base}(\mathcal{E}_i)\right).$$

From Theorem 65, for all $\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})$, $\mathtt{Base}(\mathcal{E}_i) \in \mathtt{Max}(\Sigma)$. Moreover, $|\mathtt{Ext}_p(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$. Thus, $\bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathtt{Base}(\mathcal{E}_i) = \mathtt{Free}(\Sigma)$. Consequently,

$$\bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathcal{E}_i = \mathtt{Arg}(\mathtt{Free}(\Sigma)).$$

From Theorem 70, $\mathtt{Arg}(\mathtt{Free}(\Sigma))$ is an admissible set of $\mathcal{T}$. Thus, from Property 68,

$$\mathtt{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \mathtt{Ext}_p(\mathcal{T})} \mathcal{E}_i = \mathtt{Arg}(\mathtt{Free}(\Sigma)).$$ ■

Theorem 73 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies the five postulates (under preferred semantics). If $|\texttt{Ext}_p(\mathcal{T})| = |\texttt{Max}(\Sigma)|$, then the output of $\mathcal{T}$ under grounded/ideal semantics is: $\texttt{Output}(\mathcal{T}) = \textsf{CN}(\texttt{Free}(\Sigma))$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies the five postulates (under preferred semantics). Assume that $|\texttt{Ext}_p(\mathcal{T})| = |\texttt{Max}(\Sigma)|$. Then, from Corollary 72, $\texttt{IE}(\mathcal{T}) = \texttt{GE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i = \texttt{Arg}(\texttt{Free}(\Sigma))$.

Let us first show that $\mathcal{T}$ is closed under $\textsf{CN}$ under grounded/ideal semantics, that is

$$\texttt{Concs}\left(\bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i\right) = \textsf{CN}\left(\texttt{Concs}\left(\bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i\right)\right).$$

Let $x \in \textsf{CN}\left(\texttt{Concs}\left(\bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i\right)\right)$. Thus, for all $\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})$, $x \in \textsf{CN}(\texttt{Concs}(\mathcal{E}_i))$. Since $\mathcal{T}$ satisfies closure under $\textsf{CN}$, then $x \in \texttt{Concs}(\mathcal{E}_i)$. Besides, since $x \in \textsf{CN}\left(\texttt{Concs}\left(\bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i\right)\right)$, then there exists a finite set $x_1, \ldots, x_n \in \texttt{Concs}\left(\bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i\right)$ such that $x \in \textsf{CN}(\{x_1, \ldots, x_n\})$. Moreover, $\textsf{CN}(\{x_1, \ldots, x_n\}) \subseteq \textsf{CN}\left(\texttt{Concs}\left(\bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i\right)\right)$ (by monotonicity of $\textsf{CN}$). Thus, $x \in \textsf{CN}\left(\texttt{Concs}\left(\bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i\right)\right)$.

Let us now show that $\mathcal{T}$ is closed under sub-arguments under grounded/ideal semantics. Assume that $a \in \texttt{IE}(\mathcal{T})$. Thus, $a \in \bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i$. Since $\mathcal{T}$ is closed under sub-arguments (under preferred semantics), then for all $\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})$, $\texttt{Sub}(a) \subseteq \mathcal{E}_i$. Thus, $\texttt{Sub}(a) \subseteq \bigcap_{\mathcal{E}_i \in \texttt{Ext}_p(\mathcal{T})} \mathcal{E}_i$. So, $\texttt{Sub}(a) \subseteq \texttt{IE}(\mathcal{T})$.

Since $\mathcal{T}$ satisfies closure under sub-arguments and $\textsf{CN}$ (under ideal/grounded semantics), then from Proposition 25, $\texttt{Concs}(\texttt{IE}(\mathcal{T})) = \texttt{Concs}(\texttt{GE}(\mathcal{T})) = \texttt{Output}(\mathcal{T}) = \textsf{CN}(\texttt{Free}(\Sigma))$. ■

Theorem 76 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies the five postulates under grounded (respectively ideal) semantics. If $\texttt{Arg}(\texttt{Free}(\Sigma)) \subset \texttt{GE}(\mathcal{T})$ (respectively $\texttt{Arg}(\texttt{Free}(\Sigma)) \subset \texttt{IE}(\mathcal{T})$) then there exists $C \in \mathcal{C}_\Sigma$ such that there exist $x, x' \in C$ and $x \in \texttt{Output}(\mathcal{T})$ and $x' \notin \texttt{Output}(\mathcal{T})$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies the five postulates under grounded semantics. Assume that $\texttt{Arg}(\texttt{Free}(\Sigma)) \subset \texttt{GE}(\mathcal{T})$. Thus, there exists an argument $a \in \texttt{GE}(\mathcal{T})$ such that $a \notin \texttt{Arg}(\texttt{Free}(\Sigma))$. Thus, $\texttt{Supp}(a) \nsubseteq \texttt{Free}(\Sigma)$. So, there exists $x \in \texttt{Inc}(\Sigma)$ such that $x \in \texttt{Supp}(a)$. Then, there exists $C \in \mathcal{C}_\Sigma$ such that $x \in C$. Since $\mathcal{T}$ satisfies closure under $\textsf{CN}$ and under sub-arguments, then $\texttt{Output}(\mathcal{T}) = \texttt{Concs}(\texttt{GE}(\mathcal{T})) = \textsf{CN}(\texttt{Base}(\texttt{GE}(\mathcal{T})))$. Since $x \in \texttt{Supp}(a)$, then $x \in \texttt{Base}(\texttt{GE}(\mathcal{T}))$. From the expansion axiom, $x \in \textsf{CN}(\texttt{Base}(\texttt{GE}(\mathcal{T})))$. Since $\mathcal{T}$ satisfies consistency, then $C \nsubseteq \texttt{Concs}(\texttt{GE}(\mathcal{T}))$. Thus, there exists $x' \in C$ such that $x' \notin \texttt{Concs}(\texttt{GE}(\mathcal{T}))$.

The same proof holds for ideal semantics. ■

Corollary 43 *Let $(\mathcal{L}, \textsf{CN})$ be adjunctive. Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics) iff there is a bijection between the naive extensions of $\texttt{Ext}_n(\mathcal{T})$ and the elements of $\texttt{Max}(\Sigma)$.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies Postulates 24 and 26. Then, from Theorems 41 and 42, it follows that there is a one-to-one correspondence between $\mathtt{Max}(\Sigma)$ and $\mathtt{Ext}_n(\mathcal{T})$.

Assume now that there is a one-to-one correspondence between $\mathtt{Max}(\Sigma)$ and $\mathtt{Ext}_n(\mathcal{T})$. Then, $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ is consistent. Consequently, $\mathcal{T}$ satisfies consistency. From Amgoud (2012), $\mathcal{T}$ is also closed under sub-arguments. ■

Corollary 44 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).*

- $|\mathtt{Ext}_n(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|$
- *If Σ is finite, then $\mathcal{T}$ has a finite number of naive extensions.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. From Theorem 41, it follows that $|\mathtt{Ext}_n(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|$. If Σ is finite, then it has a finite number of maximal consistent subsets. Thus, the number of naive extensions is finite as well. ■

Corollary 45 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics). If $\mathtt{Ext}_n(\mathcal{T}) = \{\emptyset\}$, then for all $x \in \Sigma$, $\mathsf{CN}(\{x\})$ is inconsistent.*

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. Assume that $\mathtt{Ext}_n(\mathcal{T}) = \{\emptyset\}$. Thus, $\mathtt{Base}(\emptyset) = \emptyset$. From Theorem 41, $\emptyset \in \mathtt{Max}(\Sigma)$. Thus, for all $x \in \Sigma$, $\mathsf{CN}(\{x\})$ is inconsistent. ■

Corollary 48 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics). $\mathtt{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \mathtt{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i)$.*

Proof It follows immediately from Theorem 42 and Theorem 46. ■

Corollary 55 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). It holds that*

$$0 < |\mathtt{Ext}_s(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|.$$

Proof Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence. From Theorem 49, $|\mathtt{Ext}_s(\mathcal{T})| \leq |\mathtt{Max}(\Sigma)|$. From Theorem 50 $\Re_{s1} = \emptyset$, then $|\mathtt{Ext}_s(\mathcal{T})| > 0$. ■

Corollary 56 *If Σ is finite, then the set $\mathtt{Ext}_s(\mathcal{T})$ is finite, whenever $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ satisfies consistency and closure under sub-arguments (under stable semantics).*

Proof It follows from Corollary 55. ■

Corollary 57 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). The equality $\texttt{Ext}_s(\mathcal{T}) = \texttt{Ext}_{ss}(\mathcal{T})$ holds.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). From Corollary 55, $|\texttt{Ext}_s(\mathcal{T})| > 0$. From Caminada (2006b), $\texttt{Ext}_s(\mathcal{T}) = \texttt{Ext}_{ss}(\mathcal{T})$. ■

Corollary 59 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under preferred semantics). There exists $\mathcal{E} \in \texttt{Ext}_p(\mathcal{T})$ such that $\texttt{Base}(\mathcal{E}) \in \texttt{Max}(\Sigma)$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under preferred semantics). Thus, $\mathcal{T}$ satisfies the same postulates under stable semantics since $\texttt{Ext}_s(\mathcal{T}) \subseteq \texttt{Ext}_p(\mathcal{T})$. Then, $|\texttt{Ext}_s(\mathcal{T})| > 0$. Thus, $\exists \mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$ and thus $\mathcal{E} \in \texttt{Ext}_p(\mathcal{T})$. From Theorem 49, $\texttt{Base}(\mathcal{E}) \in \texttt{Max}(\Sigma)$. ■

Corollary 63 *If a knowledge base Σ is finite, then for all $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments, $\texttt{Ext}_p(\mathcal{T})$ is finite.*

Proof This follows from the compactness of the knowledge base and Theorem 62. ■

Corollary 66 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$.*

$$\texttt{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \texttt{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i).$$

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$. From Theorem 65, for all $\mathcal{E} \in \texttt{Ext}_p(\mathcal{T})$, $\texttt{Base}(\mathcal{E}) \in \texttt{Max}(\Sigma)$. Moreover, since $\mathcal{T}$ is closed under both CN and sub-arguments, then $\texttt{Concs}(\mathcal{E}) = \mathsf{CN}(\texttt{Base}(\mathcal{E}))$. Thus, $\texttt{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \texttt{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i)$. ■

Corollary 72 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$. $\texttt{IE}(\mathcal{T}) = \texttt{GE}(\mathcal{T}) = \texttt{Arg}(\texttt{Free}(\Sigma))$.*

Proof Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$. From Theorem 71, $\texttt{IE}(\mathcal{T}) = \texttt{Arg}(\texttt{Free}(\Sigma))$. Moreover, from Theorem 70, $\texttt{Arg}(\texttt{Free}(\Sigma)) \subseteq \texttt{GE}(\mathcal{T})$ (since arguments of $\texttt{Arg}(\texttt{Free}(\Sigma))$ are not attacked). Thus, $\texttt{GE}(\mathcal{T}) = \texttt{IE}(\mathcal{T})$. ■

Corollary 75 *Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_2}$. The inclusions $\texttt{Arg}(\texttt{Free}(\Sigma)) \subseteq \texttt{GE}(\mathcal{T}) \subseteq \texttt{IE}(\mathcal{T}) \subseteq \mathcal{S}$ hold for some $\mathcal{S} \in \texttt{Max}(\Sigma)$.*

Proof Follows from Theorem 70, Property 9 and Property 68. ■