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Abstract This paper deals with the problem of outsourcing the debt for a big investment, according two situations: either the firm outsources both the investment (and the associated debt) and the exploitation to another firm (for example a private consortium), or the firm supports the debt and the investment but outsources the exploitation. We prove the existence of Stackelberg and Nash equilibria between the firms, in both situations. We compare the benefits of these contracts, theorically and numerically. We conclude with a study of what happens in case of incomplete information, in the sense that the risk aversion coefficient of each partner may be unknown by the other partner.

Keywords Outsourcing, Public Debt, Public-Private-Partnership, Nash and Stackelberg equilibria, Optimization, Partial information
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1 Introduction

With the significant increase in recent years of public debt in many developed countries, together with the associated concerns related to possible defaults of some

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of them, the question of financing public projects is more than ever at the center of economic and political considerations. To overcome this problem, leveraging on the private sector appears at first glance as a good idea. This type of Public-Private Partnership (PPP) was initiated in the United Kingdom in 1992, under the name Private Financing Initiative (PFI), and has widely been used since then, so that it represented one third of all public investments made in the UK during the period 2001-2006. It has also been used in many other countries, in particular in Europe, Canada and in the United States, to finance hospitals, prisons or stadiums among others. It is also recommended by the OECD. We refer among many other references to [3] for an overview of the extent of PPPs in Europe and in the US, to the website of the European PPP Expertise Center (EPEC) or the website of the National Council for Public-Private Partnerships, and to [7] for a global overview made by the OECD.

However, as emphasized by the recent discussions in the UK, although the benefits of this type of partnership are mainly admitted, there are still many concerns about its drawbacks (see [8] for a detailed overview). Even though some drawbacks are of political, social or behavioral natures, others are purely economic and are the ones that we are interested in. More precisely, we would like to answer the following question: from an economic point of view, and taking into account the constraints that a country faces when issuing a new amount of debt, is it optimal for this country to finance a public project via a private investment?

Although already of a big interest, this question does not need to be restricted to debt issuance by a country but can be generalized to any economic agent, be it a country or a firm. Indeed, any firm has some constraints on its debt level for several reasons. In some cases, banks will simply not allow a company to borrow enough money to sustain a very expensive project. But even if it is not the case, since the debt level appears on the company's accounts, issuing too much debt will affect the opinion and confidence of investors, and in particular deteriorate its rating. This can lead to a higher credit spread when issuing new bonds, difficulties to increase the capital of the firm, a drop of the company's stock price, dissatisfaction of shareholders or in the worst case, bankruptcy. We can cite some concrete examples where the dilemma between investing directly or resorting to outside investment can occur: owning or renting offices or factories, owning or leasing trucks, trains or planes, some industrial machines or some office materials (such as computers).

Therefore we will consider in this paper the problem of outsourcing from the debt point of view. Since the question of outsourcing some operations has already been widely studied and our aim is only to study the relevance of outsourcing an investment in order to reduce the debt of a firm (or economic agent), we will compare two situations where the operations are always outsourced. In the first one, the firm outsources both the investment/debt and the operations, while in the second one, the firm supports the debt and the investment but outsources the operations. In both situations, the investment is covered by issuing a debt at time 0 but the cost of borrowing of the outsourcer and the outsourcee may be different. More specifically, we suppose that the outsourcer faces some important constraints if he has to issue a new debt, stylized by a convex function f modeling his aversion for debt issuance. That is why he considers the possibility to outsourcing the investment to a firm which has less constraints.

In [5], Iossa, Martimort and Pouyet give some results on the comparison of the costs and benefits associated to PPP. Hillairet and Pontier [4] propose a study

on PPP and their relevance, assuming the eventuality of a default of the counterparty, but they do not take into account the government debt aversion. However, the attractivity for government of PPP contracts relies obviously on the short term opportunity gain to record infrastructure assets out of the government's book. To our knowledge, there does not exist any reference in Mathematics area. In Economics, a narrow strand of literature is dedicated to the discussion of Build-Operate-Transfer (BOT) concession contracts, which is a frequent form of PPP. Under BOT contracts the private sector builds and operates an infrastructure project for a well defined concession period and then transfers it to public authorities. The attractiveness of BOT contracts to governments stems from the possibility to limit governmental spending by shifting the investment costs to a private consortium. In [1], Auriol and Picard discuss the choice of BOT contracts when governments and consortia do not share the same information about the cost parameter during the project life. They summarize the government's financial constraint by its "shadow" cost of public funds, which reflects the macro-economic constraints that are imposed on national governments' surplues and debt levels by supranational institutions such as the I.M.F. Using linear demand functions and uniform cost distributions, they compute theoretical values of shadow costs that would entice governments to choose BOT concessions contracts. Our approach is different from the modelization and the resolution point of view.

This paper studies, in a two-period setting, two kinds of equilibria between risk averse firms. The first one is a Stackelberg equilibrium with the outsourcer as leader, which corresponds to the more classic setting for outsourcing situations. The second one is a Nash equilibrium. It may correspond to an outsourcing between two economic entities within a same group: in this case, the two entities make their decisions simultaneously and a Nash equilibrium is more favourable for the group than a Stackelberg one. For both situations where the investment is outsourced or not, Stackelberg and Nash equilibria are characterized. A theorical comparison is provided for the Nash equilibrium, from the point of view of the outsourcer: we check that the investment should be outsourced if the outsourcee has a lower cost of capital or if the outsourcer has a high debt aversion. For the Stackelberg equilibrium, the comparison is done numerically on a concrete example. The analysis is extended to an incomplete information setting in which the firms do not have perfect knowledge of the preference of their counterparty. To model the social need of the investment, the outsourcer gets a penalty if the outsourcee does not accept the contract.

The present paper is organised as follows. In Section 2 we set the problem of outsourcing between two firms and we define the optimization problems in Situation 1, in which the firm outsources both the investment/debt and the operations, and in Situation 2 , in which the firm supports the debt and the investment but outsources the operations. Section 3 provides the main results concerning Nash and Stackelberg equilibria, the comparison between the two situations, and the case of incomplete information. The proofs are postponed in Appendices. We provide in Section 4 a numerical example to better investigate the quantitative effects of the model. Appendix 5 is devoted to the proofs of existence and characterization of Nash and Stackelberg equilibria in Situation 1, then Appendix 6 does the same in Situation 2. Appendix 7 concerns the proofs of the comparison results between the two situations and Appendix 8 the results obtained in incomplete information.

2 Problem formulation

2.1 Costs and revenue

Consider two firms. Firm I is the one who wants to reduce its debt and therefore considers the possibility of outsourcing an investment to a second firm J. In any case, firm J is the one that will support the operational cost of the project, on the time horizon T. Let the operational cost on the time-interval $[0,T]$ C^{op} be given by:

$$C^{op} = \mu - \varphi(e) - \delta\psi(a), \tag{1}$$

where

- μ is the "business as usual" cost, such that $\mathbb{E}[\mu]$ represents the "average" benchmark cost (it takes into account the price of commodities, employees, rents...). We assume that μ is a non constant random variable bounded from below by a finite constant $\underline{\mu}$ on a probability space $(\Omega, F_T, \mathbb{P})$ such that

$$\forall \lambda \in \mathbb{R},\ \mathbb{E}\left[e^{\lambda\mu}\right] < +\infty. \tag{2}$$

  Notice that this implies that the function $\lambda \mapsto \mathbb{E}\left[e^{\lambda\mu}\right]$ is infinitely differentiable.
- e is a non-negative parameter and represents the effort made on the time-interval $[0,T]$ in order to reduce the operational cost such as logistics improvements, research and development, maintenance or more efficient or less workers. e will in general have a social impact for firm I,
- $\delta \in \mathbb{R}$ represents the impact of the quality of the investment on the reduction of operational costs. We do not impose any restriction on the sign of δ, since, as suggested in [5], both signs can make sense depending on the situation. Indeed, when constructing a building, using more expensive material usually brings less maintenance costs and therefore a positive delta. On the contrary, for a hospital, using more sophisticated (and expensive) machines can bring bigger maintenance costs and a negative delta.
- $a \geq 0$ is the effort done at time 0 to improve the (initial) quality of the investment, improving in the meantime the social value of the project. Depending on δ, a affects positively, negatively or does not affect the operational cost. Remark is that a has the same dimension as the effort e,
- $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, modeling the respective impacts of the efforts e and a respectively on the operational cost C^{op}, are C^1, increasing and strictly concave functions, satisfying the Inada conditions $\varphi'(0) = +\infty$ and $\varphi'(\infty) = 0$, $\psi'(0) = +\infty$ and $\psi'(\infty) = 0$. We also assume that $\varphi(\infty) + \delta^+\psi(\infty) < \underline{\mu}$ where $\underline{\mu}$ is the lower bound of the random variable μ and $\delta^+ = \max(\delta, 0)$, which ensures that $C^{op} > 0$; as a consequence, $\forall (x,y) \in \mathbb{R}_+^2$, $\varphi(x) + \delta^+\psi(y) < \underline{\mu}$. We assume furthermore that $(\varphi')^{-1}(x) \sim (\psi')^{-1}(x)$ for $x \to 0$ for technical reasons and to make the computations lighter even if we could relax this last assumption.

The parameter e is a control for firm J, while a is a control for the firm that supports the investment, I or J depending on the situation; μ represents the cost on the time-interval $[0,T]$.

The minimal investment required by the project is $D > 0$ and if initial effort are done (i.e. if $a > 0$) the total investment is the sum $D + a$. This investment is assumed to be entirely covered by issuing a debt with horizon T at time 0. To take into account the possibility that the cost of borrowing is in general not the same for different firms, we denote the respective non-negative constant interest rates of firms I and J by r_I and r_J. On the time-interval $[0, T]$, the amount to be reimbursed by the borrower $K \in \{I, J\}$ is $(1 + r_K)(D + a)$.
Finally, we need to add the remaining costs on $[0, T]$ coming from the effort e as well as the maintenance costs denoted by m :

$$C^m = e + m. \tag{3}$$

The maintenance cost m is a non-negative parameter and will have a social impact for firm I.

Since firm I gives to firm J either a rent or the right to exploit the project on $[0, T]$, in both cases we can consider a random variable R which corresponds to the endowment for firm J and the rent or shortfall for firm I, on $[0, T]$. This variable is computed using a reasonably simple rule, decided at $t = 0$ and subject to a control of firm I. In reality, in such contracts, the endowment can be indexed on the price of commodities in the case of transportation or on a real-estate index for the rent of a building. Since firm I wants to have a project of good quality as well as a well maintained project, we assume that R is a non-negative random variable and depends on both C^{op} and the maintenance cost in the following way:

$$R = \alpha + \beta C^{op} + \gamma g(m), \tag{4}$$

with $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and g is a C^1, increasing and strictly concave function on $\mathbb{R}_+$, such that $g'(0) = \infty$ and $g'(\infty) = 0$. Moreover, we assume that

$$m_0 := \inf\{m > 0 : g(m) > 0\} < +\infty. \tag{5}$$

The constants α, β and γ are controls of firm I. We do not put any randomness in the coefficients α, β and γ of R since we consider that they are defined at time $t = 0$ by a contract between firms I and J. All the randomness in R comes from the operational cost term C^{op}. Still, this model allows for an indexation on a benchmark such as the price of commodities or inflation through this dependence on operational costs.

2.2 Optimization problems

We assume that the risk aversions of firm I and J are represented respectively by the exponential utility functions $U(x) = -e^{-ux}$ and $V(x) = -e^{-vx}$, $x \in \mathbb{R}$, with $u, v > 0$.
We consider two different situations: in Situation 1, firm J supports the debt and takes care of the exploitation; its controls are a, e and m, whereas the controls of firm I are α, β and γ. In Situation 2, firm J only takes care of the exploitation, its controls are e and m, whereas the controls of firm I are a, α, β and γ. Firm I is the one that chooses between the two situations. The optimization problems for firm

J respectively in Situation 1 and 2 are $\sup_{(a,e,m)} J^1(a,e,m)$ and $\sup_{(e,m)} J^2(e,m)$ respectively, where:

$$J^1(a,e,m) = \mathbb{E}\left[V\big(\alpha + (\beta-1)C^{op} - e + \gamma g(m) - m - (1+r_J)(D+a)\big)\right] \quad (6)$$

$$J^2(e,m) = \mathbb{E}\left[V\big(\alpha + (\beta-1)C^{op} - e + \gamma g(m) - m\big)\right] \quad (7)$$

recalling $C^{op} = \mu - \varphi(e) - \delta\psi(a)$.

In these optimization problems, we have assumed that the controls of firm I are given (they have no reason to be the same in the two cases). We look for controls in the following admissible sets: e, m and a are non-negative constants. The eventuality that firm J does not accept the contract will be taken into account in the constraints of the optimization problem for firm I.

On the other hand, we consider that the project has an initial "social" value $b^a(a)$ for firm I, and a good maintenance also represents a social benefit $b^m(m)$. The benefits of the efforts on operational costs are modelled through the function b^e. We also introduce a penalization function f representing the aversion for debt issuance of firm I (firm J has no debt aversion). Those functions satisfy the following hypotheses

- $b^a : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1, increasing and concave function. $(b^a)'(0) > 0$, possibly infinite, $(b^a)'(\infty) = 0$ and $b^a(\infty) < \infty$.
- $b^m : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1, increasing and concave function, such that $(b^m)'(0) = \infty$ and $(b^m)'(\infty) = 0$.
- $b^e : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1, increasing and concave function, such that $(b^e)'(0) = \infty$ and $(b^e)'(\infty) = 0$
- f is an increasing and strictly convex function, satisfying $f'(\infty) = \infty$.

Therefore we write the optimization problem for firm I in both situations as $\sup_{(\alpha,\beta,\gamma)} I^1(\alpha,\beta,\gamma)$ and $\sup_{(a,\alpha,\beta,\gamma)} I^2(a,\alpha,\beta,\gamma)$ where:

$$I^1(\alpha,\beta,\gamma) = \mathbb{E}\left[b^a(a) + U\big(b^m(m) + b^e(e) - \alpha - \beta C^{op} - \gamma g(m)\big)\right]$$

$$I^2(a,\alpha,\beta,\gamma) = \mathbb{E}\left[b^a(a) - f\big((1+r_I)(D+a)\big) + U\big(b^m(m) + b^e(e) - \alpha - \beta C^{op} - \gamma g(m)\big)\right].$$

Hypotheses on b^a and f imply that $F(a) := b^a(a) - f((1+r_I)(D+a))$ is strictly concave, satisfies $F'(\infty) = -\infty$ and $F(\infty) = -\infty$. Finally we assume that $F'(0) > 0$, possibly infinite. The admissible sets are:
- in Situation 1, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and such that

$$\mathbb{E}\left[V\big(\alpha + (\beta-1)(\mu - \varphi(e) - \delta\psi(a)) - e + \gamma g(m) - m - (1+r_J)(D+a)\big)\right] \geq V(0); \quad (8)$$

- in Situation 2, $a \geq 0$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and such that

$$\mathbb{E}\left[V\big(\alpha + (\beta-1)(\mu - \varphi(e) - \delta\psi(a)) - e + \gamma g(m) - m\big)\right] \geq V(0). \quad (9)$$

The constraint ensures that firm J will accept the contract, since it is better or equal for it than doing nothing.

The most natural equilibrium to be considered in outsourcing situations is a Stackelberg equibrium with firm I as leader. It corresponds for example to a government which outsources the construction of a stadium, or to an industrial group

which wants to outsource its trucks/trains to a transport company. Nevertheless, within a group, a given entity may be interested in outsourcing its debt to another entity of the same group. In this situation, a Nash equilibrium is more relevant. Therefore this paper addresses those two kinds of equilibria.

Remark 1 A Stackelberg equilibrium with firm J leader is never relevant from an economical perspective, since it is never the outsourcee who makes an offer to the outsourcer. It is also not interesting from a mathematical point of view. Indeed, in Situation $i \in \{1, 2\}$, since $C^{op} = \mu - \varphi(e) - \delta\psi(a) > 0$, I^i is decreasing with respect to β, while J^i is increasing with respect to β. Therefore if firm J is the leader, for any choice of its controls, firm I's optimal controls will always bind the constraint $J^i \geq V(0)$.

3 Main results

The best responses of firm J to given controls of firm I turn out to be easily derived. That is why we first present them, before stating our main results concerning Nash and Stackelberg equilibria where these best responses appear. The proofs of the main results are postponed in Appendices 5, 6, 7, 8.

3.1 Best responses of firm J in Situations 1 and 2

Let us first consider Situation 1 and suppose that (α, β, γ) is given in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$. For firm J the optimization problem is ω by ω, and since U is increasing it writes:

$$\sup_{e\geq 0}\{(1-\beta)\varphi(e) - e\} + \sup_{m\geq 0}\{\gamma g(m) - m\} + \sup_{a\geq 0}\{\delta(1-\beta)\psi(a) - (1+r_J)a\}.$$

Since ψ, φ and g are strictly concave, the first order conditions characterize the points maximizing each function between braces and, with the convention that $(\phi')^{-1}(\infty) = 0$ for $\phi = \psi, \varphi, g$, we have :

$$m^* = (g')^{-1}(1/\gamma) \ ; \ e^* = (\varphi')^{-1}\left(\frac{1}{(1-\beta)^+}\right) \ ; \ a^* = (\psi')^{-1}\left(\frac{1+r_J}{(\delta(1-\beta))^+}\right). \tag{10}$$

Let us now consider Situation 2 and suppose that $(a, \alpha, \beta, \gamma)$ is given. Similarly we obtain that

$$m^* = (g')^{-1}(1/\gamma), \ e^* = (\varphi')^{-1}\left(\frac{1}{(1-\beta)^+}\right). \tag{11}$$

When useful to explicit the dependence of these best responses of firm J with respect to the controls of firm I, we use the notation $m^*(\gamma)$, $e^*(\beta)$ and $a^*(\beta)$.

To describe the Nash and Stackelberg equilibria, we introduce the continuous mapping $C : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R}_+ \to \mathbb{R}_+$ and $\tilde{B} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$C(\beta) := \frac{1}{v}\ln \mathbb{E}\left[e^{v(1-\beta)(\mu - \varphi(e^*(\beta)))}\right], \tag{12}$$

$$B(m) := e^{u(Id-b^m)(m)} e^{u(Id-b^e)(e^*(\frac{v}{u+v}))} e^{(u+v)C(\frac{v}{u+v})}, \tag{13}$$

$$\tilde{B}(\beta, m) := e^{u(Id-b^m)(m)} e^{u(Id-b^e)\circ e^*(\beta)} e^{uC(\beta)} \mathbb{E}\left[e^{u\beta(\mu - \varphi\circ e^*(\beta))}\right]. \tag{14}$$

3.2 Nash equilibria

To describe the Nash equilibria, we need the following technical result about the function g :

Lemma 1 *The function* $G : m \mapsto \frac{g(m)}{g'(m)} - m$ *is continuous, satisfies* $\lim_{m\to\infty} \frac{g(m)}{g'(m)} - m = +\infty$ *and is decreasing from* 0 *to* $-m_0$ *on* $[0, m_0]$ *(where* m_0 *is defined in (5)) and increasing from* $-m_0$ *to* $+\infty$ *on* $[m_0, +\infty)$ *thus admitting an continuous inverse* $G^{-1} : [-m_0, +\infty) \to [m_0, +\infty)$.

Theorem 1 *In Situation 1, there exists an infinite number of Nash equilibria, namely the vectors* $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m}, \hat{a})$ *satisfying*

$$\hat{\beta} = \tfrac{v}{u+v},\ \hat{e} = (\varphi')^{-1}(\tfrac{u+v}{u}),\ \hat{a} = (\psi')^{-1}\left(\tfrac{(u+v)(1+r_J)}{\delta+u}\right),\ \hat{\gamma} = \tfrac{1}{g'(\hat{m})}, \quad (15)$$

$$\hat{\alpha} = C(\hat{\beta}) + (1+r_J)(D+\hat{a}) + \hat{e} - \tfrac{u}{u+v}\delta\psi(\hat{a}) - G(\hat{m}), \quad (16)$$

(where C *is defined in (12)) for* $\hat{m}$ *varying in*
$\hat{\mathcal{M}}_1(r_J) := [0, G^{-1}(C(\hat{\beta}) + (1+r_J)(D+\hat{a}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}))]$.
The corresponding optimal values for firms J *and* I *are respectively* $V(0)$ *and*

$$\hat{I}^1(\hat{m}) = b^a(\hat{a}) - e^{-u\delta\psi(\hat{a})}e^{u(1+r_J)(D+\hat{a})}B(\hat{m}),$$

where B *is defined in* (13).

Remark 2 – Although there exists an infinite number of Nash equilibria, the controls β, e and a are the same in all these equilibria.
– Since $\hat{\mu} - \varphi(\hat{e}) - \delta\psi(\hat{a}) \geq 0$, one has $C(\hat{\beta}) - \frac{u}{u+v}\delta\psi(\hat{a}) \geq 0$ so that $[0, G^{-1}((\varphi')^{-1}(\frac{u+v}{u}) + D)] \subset \bigcap_{r_J\geq 0} \hat{\mathcal{M}}_1(r_J)$.
– It is natural to wonder whether there exists in Situation 1 a Nash equilibrium among the infinite family of such equilibria exhibited in Theorem 1 which maximizes $\hat{I}^1$. This function depends on the Nash equilibrium only through the term $b^m(\hat{m}) - \hat{m}$ which has to be maximized. The function $\hat{m} \mapsto b^m(\hat{m}) - \hat{m}$ being concave, it admits a unique maximum on the interval $\hat{\mathcal{M}}_1(r_J)$ where $\hat{m}$ associated with a Nash equilibrium varies. When $[(b^m)']^{-1}(1) \in \hat{\mathcal{M}}_1(r_J)$ (which is the case for the numerical example investigated in Section 4), then $\sup_{\hat{m}\in\hat{\mathcal{M}}_1(r_J)} \hat{I}^1(\hat{m}) = \hat{I}^1([(b^m)']^{-1}(1))$ and the optimal value $[(b^m)']^{-1}(1)$ of $\hat{m}$ will turn out to be the optimal control m in the Stackelberg equilibria (see Theorems 5 and 6 below).
Otherwise, $\sup_{\hat{m}\in\hat{\mathcal{M}}_1(r_J)} \hat{I}^1(\hat{m}) = \hat{I}^1(G^{-1}(C(\hat{\beta})+(1+r_J)(D+\hat{a})+\hat{e}-\frac{u}{u+v}\delta\psi(\hat{a})))$.

Theorem 2 *Let* $F(a) = b^a(a) - f((1+r_I)(D+a))$. *In Situation 2, there exists an infinite number of Nash equilibria namely the vectors* $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m}, \hat{a})$ *defined by*

$$\hat{m} \geq 0,\ \hat{\beta} = \tfrac{v}{u+v},\ \hat{e} = (\varphi')^{-1}\left(\tfrac{u+v}{u}\right),\ \hat{\gamma} = \tfrac{1}{g'(\hat{m})}, \quad (17)$$

$$\hat{a} \in \arg\max_{a\geq 0}\left[F(a) - e^{-u\delta\psi(a)}B(\hat{m})\right], \quad (18)$$

$$\hat{\alpha} = C(\hat{\beta}) + \hat{e} - \tfrac{u}{u+v}\delta\psi(\hat{a}) - G(\hat{m}), \quad (19)$$

and such that $C(\hat{\beta}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}) - G(\hat{m}) \geq 0$, *a condition that is satisfied when* $\hat{m} \leq G^{-1}((\varphi')^{-1}(\frac{u+v}{u}))$. *Moreover,* $\hat{\alpha} + \hat{\gamma} > 0$ *and if* $\delta \geq 0$, *then* $\hat{a}$ *is positive and*

unique for each $\hat{m}$.
The corresponding optimal values for firms J and I are respectively $V(0)$ and

$$\hat{I}^2(\hat{m}) = F(\hat{a}) - e^{-u\delta\psi(\hat{a})}B(\hat{m}).$$

Let $\hat{\mathcal{M}}_2(r_I)$ denote the set of $\hat{m} \geq 0$ for which there exists $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{a})$ such that $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m}, \hat{a})$ is a Nash equilibrium in Situation 2.

Remark 3 – Notice that the order the different controls are determined is important, since some of them depend on the other ones. Indeed $\hat{\beta}$ depends on no other control and therefore should be determined first, leading to the value of $\hat{e}$. Then one should fix $\hat{m}$, in order to have $\hat{\gamma}$, which allows then to determine $\hat{a}$, and once this is done, we can find $\hat{\alpha}$. Although $\hat{\alpha}$ and $\hat{\gamma}$ essentially play the same role, the fact that $\hat{\gamma}$ only depends on $\hat{m}$ makes this order important. If one chooses $\hat{\alpha}$ first, then the determination of $\hat{a}$ is not clear, since then $\hat{a}$ depends on $\hat{m}$, while $\hat{m}$ depends on $\hat{a}$ and $\hat{\alpha}$.

– We expect that, as in Situation 1, when $[(b^m)']^{-1}(1) \in \hat{\mathcal{M}}_2(r_I)$ (which is the case for the numerical example investigated in Section 4), then $\sup_{\hat{m}\in\hat{\mathcal{M}}_2(r_I)} \hat{I}^2(\hat{m}) = \hat{I}^1([(b^m)']^{-1}(1))$. Indeed, a formal differentiation of $\hat{I}^2(\hat{m})$ with respect to $\hat{m}$ leads to $(\hat{I}^2)'(\hat{m}) = -e^{-u\delta\psi(\hat{a})}B'(\hat{m})$ since, because of the first order optimality condition related to (18), one should not need to take care of the dependence of $\hat{a}$ on $\hat{m}$. Moreover, one easily checks that the unique solution to $B'(m) = 0$ is $m = [(b^m)']^{-1}(1)$. This is illustrated in Figure 1.

We now compare the respective optimal values $\hat{I}^1(\hat{m})$ and $\hat{I}^2(\hat{m})$ for firm I in Situations 1 and 2 for the same maintenance effort $\hat{m} \in \hat{\mathcal{M}}_1(r_J) \cap \left\{\bigcap_{r_I\geq 0} \hat{\mathcal{M}}_2(r_I)\right\}$. According to Remarks 2 and 3, the same value $\hat{m} = [(b^m)']^{-1}(1)$ is likely to maximize $\hat{I}^1(\hat{m})$ and $\hat{I}^2(\hat{m})$. Therefore choosing the same maintenance effort $\hat{m}$ for the comparison is natural. Notice also that, by Theorem 2 and Remark 2, $[0, G^{-1}((\varphi')^{-1}(\frac{u+v}{u}))] \subset \hat{\mathcal{M}}_1(r_J) \cap \left\{\bigcap_{r_I\geq 0} \hat{\mathcal{M}}_2(r_I)\right\}$. Let $\hat{a}_1(r_J) = (\psi')^{-1}\left(\frac{(u+v)(1+r_J)}{\delta+u}\right)$ (resp. $\hat{a}_2(r_I)$) denote the value of the control a in the Nash equilibrium with $m = \hat{m}$ in Situation 1 (resp. in Situation 2 when $\delta \geq 0$).

We are going to exhibit cases in which Situation 1 (meaning outsourcing (respectively Situation 2, meaning debt issuance) is the more profitable for firm I.

Theorem 3 *Let rate $r_J \geq 0$ be fixed and $\hat{m} \in \hat{\mathcal{M}}_1(r_J) \cap \left\{\bigcap_{r_I\geq 0} \hat{\mathcal{M}}_2(r_I)\right\}$. In case of rate r_I satisfying*

$$r_I \leq \frac{f^{-1}\left[B(\hat{m})e^{-u\delta\psi(\hat{a}_1(r_J))}\left(e^{u(1+r_J)(D+\hat{a}_1(r_J))} - 1\right)\right]}{D + \hat{a}_1(r_J)} - 1, \qquad (20)$$

we have $\hat{I}^2(\hat{m}) \geq \hat{I}^1(\hat{m})$ and the better contract for firm I is the second one, meaning debt issuance.

Condition (20) has a clear economical interpretation. The right-hand side does not depend on r_I. Therefore for a fixed r_J, debt issuance is the best choice for firm I as soon as its interest rate r_I is small enough. Note the impact of the function f modeling its debt aversion : the larger f, the smaller the threshold on r_I in condition (20), look at Figure 2.

Theorem 4 *We assume $\delta > 0$. Let rate $r_J \geq 0$ be fixed (thus $\hat{a}_1(r_J)$ is fixed) and $\hat{m} \in \hat{\mathcal{M}}_1(r_J) \cap \left\{\bigcap_{r_I \geq 0} \hat{\mathcal{M}}_2(r_I)\right\}$. In case of rate r_I satisfying*

$$(1+r_I)(D+\hat{a}_2(r_I)) \geq f^{-1}\left[B(\hat{m})e^{-u\delta\psi(\hat{a}_2(r_I))}\left(e^{u(1+r_J)(D+\hat{a}_2(r_I))}-1\right)\right] \quad (21)$$

and one of the following:

$$\begin{aligned}(1+r_I)f'[(1+r_I)(D+\hat{a}_1(r_J))] &> (b^a)'(\hat{a}_1(r_J)) + u\delta\psi'(\hat{a}_1(r_J))B(\hat{m})e^{-u\delta\psi(\hat{a}_1(r_J))},\\ &\textit{or}\\ \psi'(\hat{a}_2(r_I)) &> \frac{(u+v)(1+r_J)}{u\delta}\end{aligned} \quad (22)$$

one has $\hat{I}^1(\hat{m}) \geq \hat{I}^2(\hat{m})$ and the better contract for firm I is the first one, meaning outsourcing.

The economical interpretation of condition (22) is natural. Indeed, the right-hand sides of the inequalities do not depend on r_I whereas the left-hand sides are increasing functions of r_I. Hence (22), leading to optimality of outsourcing for firm I, is satisfied as soon as its interest rate r_I is large enough. Besides, we see that the more convex f is, the smaller is the threshold on r_I in the first inequality of condition (22).

Unfortunately, we have not been able to check that the condition (21) for optimality is satisfied for large r_I, but Figure 3 gives a rule of decision between debt issuance and outsourcing.

3.3 Stackelberg equilibria

Depending on Situation 1 or 2 and on the sign of δ, the optimal β will be characterized as solution of different equations. To specify those equations, we need to introduce the functions

$$h(\lambda) = \frac{\mathbb{E}\left[\mu e^{\lambda\mu}\right]}{\mathbb{E}\left[e^{\lambda\mu}\right]}, \quad (23)$$

$$S(\beta) := \frac{\frac{\beta}{1-\beta} + (b^e)' \circ (\varphi')^{-1}\left(\frac{1}{(1-\beta)^+}\right)}{(1-\beta)^2\varphi'' \circ (\varphi')^{-1}\left(\frac{1}{(1-\beta)^+}\right)}, \quad (24)$$

$$\widetilde{S}(\beta) := \frac{1+r_J}{\delta(1-\beta)^2(\psi'')((\psi')^{-1}\left(\frac{1+r_J}{(\delta(1-\beta))^+}\right))}\left((1+r_J)\frac{\beta}{1-\beta} + \frac{(b^a)' \circ (\psi')^{-1}\left(\frac{1+r_J}{(\delta(1-\beta))^+}\right)}{ue^{(Id-b^m))((b^m)')^{-1}(1)}k(\beta)}\right), \quad (25)$$

where $k(\beta)$ is a positive function of β defined as follows,

$$k(\beta) := e^{u(Id-b^e)\circ e^*(\beta)}e^{-u\delta\psi\circ a^*(\beta)}e^{u(1+r_J)(D+a^*(\beta))}e^{uC(\beta)}\mathbb{E}\left[e^{u\beta\left(\mu-\varphi\circ e^*(\beta)\right)}\right] \quad (26)$$

with $e^*(\beta)$ and $a^*(\beta)$ defined in (10) and $C(\beta)$ in (12).

We consider the following equations

$$h(u\beta) - h(v(1-\beta)) = S(\beta), \quad (27)$$

$$h(u\beta) - h(v(1-\beta)) = S(\beta) + \widetilde{S}(\beta), \tag{28}$$

$$h(u\beta) - h(v(1-\beta)) = \widetilde{S}(\beta). \tag{29}$$

Theorem 5 *In Situation 1, there exists at least one Stackelberg equilibrium with firm I as the leader. Moreover, if there exists a Stackelberg equilibrium* $(\hat{e}, \hat{m}, \hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ *with* $\hat{\alpha} > 0$*, then it is characterized by :*

$$\hat{e} = (\varphi')^{-1}\Big(\frac{1}{(1-\hat{\beta})^+}\Big),\ \hat{a} = (\psi')^{-1}\left(\frac{1+r_J}{(\delta(1-\hat{\beta}))^+}\right), \hat{m} = \left[(b^m)'\right]^{-1}(1),\ \hat{\gamma} = 1/g'(\hat{m}),$$

$$\hat{\alpha} = C(\hat{\beta}) + (1+r_J)(D+\hat{a}) + \hat{e} - (1-\hat{\beta})\delta\psi(\hat{a}) - G(\hat{m}).$$

If $\delta > 0$ *then* $\hat{\beta}$ *is a solution of (28) and is less than* $\frac{v}{u+v}$*.*
If $\delta < 0$ *then either* $\hat{\beta}$ *is less than* $\frac{v}{u+v}$ *and solves (27) or* $\hat{\beta}$ *is larger than one and solves (29).*
The corresponding optimal values for firms J and I are respectively $V(0)$ *and* $I^1(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$*.*

Theorem 6 *In Situation 2, there exists at least one Stackelberg equilibrium with firm I as the leader. Moreover, if there exists a Stackelberg equilibrium* $(\hat{e}, \hat{m}, \hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ *with* $\hat{\alpha} > 0$*, then it satisfies:*
$\hat{\beta}$ *is a solution of (27),* $\hat{e} = (\varphi')^{-1}\Big(\frac{1}{(1-\hat{\beta})^+}\Big)$*,* $\hat{m} = \left[(b^m)'\right]^{-1}(1)$*,* $\hat{\gamma} = 1/g'(\hat{m})$*,* $\hat{a} \in \arg\max_{a\geq 0} F(a) - e^{-u\delta\psi(a)}\tilde{B}(\hat{\beta}, \hat{m})$ *and* $\hat{\alpha} = C(\hat{\beta}) + \hat{e} - (1-\hat{\beta})\delta\psi(\hat{a}) - G(\hat{m})$*, where the mappings C and* $\tilde{B}$ *are defined by (12)-(14). In particular,* $\hat{\beta} < \frac{v}{u+v}$*.*
Moreover, if $\delta \geq 0$*, then* $\arg\max_{a\geq 0} F(a) - e^{-u\delta\psi(a)}\tilde{B}(\hat{\beta}, \hat{m})$ *is a singleton and* $\hat{a} > 0$*.*
The corresponding optimal values for firms J and I are respectively $V(0)$ *and* $I^2(\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$*.*

Proposition 2 below states that Equation (27) which appears in the characterization of $\hat{\beta}$ when $\hat{\alpha} > 0$ in both Situations 1 and 2 always admits a solution.

An analytical comparison is not so easy, but Figures 4 and 5 allow a numerical comparison between debt issuance and outsourcing.

3.4 Incomplete information

In this section we consider the previous equilibrium problems when the firms do not have a perfect knowledge of the preferences of the other firm. More precisely, we still assume that the firms' utility functions are $U(x) = -e^{-ux}$ and $V(x) = -e^{-vx}$ respectively, but firm I perceives v as a $(0, +\infty)$-valued random variable with known distribution and independent from μ that we denote $\mathcal{V}$ and firm J perceives u as a random variable with known distribution and independent from μ that we denote $\mathcal{U}$. According to Section (3.1), firm J optimal controls are functions of the controls β, γ fixed by firm I that do not depend on the risk aversion parameters u. Therefore, equations (10) and (11) still hold in incomplete information and incomplete information on the risk aversion parameter u has no impact on the

equilibria. In contrast, the uncertainty on the parameter v has an impact as the acceptation of the contract by firm J depends on it. To model the social need of the investment, we introduce a (social) penalty $p \in \mathbb{R} \cup \{+\infty\}$ that firm I gets if firm J does not accept the contract.

3.4.1 Stackelberg equilibrium, firm I is leader

We first introduce the events $\mathcal{A}^i,\ i = 1, 2$: "firm J accepts the contract" in Situation i.
The optimization problem for firm I is

$$u_I^1 := -p \vee \sup_{(\alpha,\beta,\gamma)} \{\tilde{I}^1(\alpha,\beta,\gamma)\mathbb{P}\left(\mathcal{A}^1(\alpha,\beta,\gamma)\right) - p(1 - \mathbb{P}\left(\mathcal{A}^1(\alpha,\beta,\gamma)\right))\}, \quad (30)$$

in Situation 1 and in Situation 2, it becomes :

$$u_I^2 := -p \vee \sup_{(a,\alpha,\beta,\gamma)} \{\tilde{I}^2(a,\alpha,\beta,\gamma)\mathbb{P}\left(\mathcal{A}^2(a,\alpha,\beta,\gamma)\right) - p(1 - \mathbb{P}\left(\mathcal{A}^2(a,\alpha,\beta,\gamma)\right))\}. \quad (31)$$

The functions

$$\tilde{I}^1(\alpha,\beta,\gamma) := b^a(a^*(\beta)) - \mathbb{E}\left[e^{-u\left([b^m-\gamma g](m^*(\gamma))+[b^e+\beta\varphi](e^*(\beta))-\alpha-\beta(\mu-\delta\psi(a^*(\beta)))\right)}\right],$$

$$\tilde{I}^2(a,\alpha,\beta,\gamma) := F(a) - \mathbb{E}\left[e^{-u\left([b^m-\gamma g](m^*(\gamma))+[b^e+\beta\varphi](e^*(\beta))-\alpha-\beta(\mu-\delta\psi(a))\right)}\right],$$

where $e^*,\ m^*$ and a^* have been defined in (10), are the social gain that firm I respectively gets in Situations 1 and 2 if firm J accepts the contract. Notice that the supremum is taken with $-p$ to model the possibility for firm J not to enter the game and that $p = +\infty$ corresponds to the case where firm I absolutely wants that firm J accepts the contract.
In order to characterize the acceptance set $\mathcal{A}^i$, we introduce

$$\tilde{J}^1(v,\alpha,\beta,\gamma) := -e^{v(1+r_J)(D+a^*(\beta))} \times \mathbb{E}e^{-v\left(\alpha+(\beta-1)(\mu-\delta\psi(a^*(\beta))-\varphi(e^*(\beta)))-e^*(\beta)+[\gamma g-Id](m^*(\gamma))\right)} \quad (32)$$

and

$$\tilde{J}^2(v,a,\alpha,\beta,\gamma) := -\mathbb{E}e^{-v\left(\alpha+(\beta-1)(\mu-\delta\psi(a)-\varphi(e^*(\beta)))-e^*(\beta)+[\gamma g-Id](m^*(\gamma))\right)}. \quad (33)$$

Firm J accepts the contract if and only if $-\tilde{J}^i(\mathcal{V},.) \leq 1$, thus $\mathcal{A}^i(.) = \{-\tilde{J}^i(\mathcal{V},.) \leq 1\}$.
We define the value function of the problem with complete information that firm J's risk aversion is equal to v

$$u^1(v) := \sup_{\{(\alpha,\beta,\gamma)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+ : -\tilde{J}^1(v,\alpha,\beta,\gamma)\leq 1\}} \tilde{I}^1(\alpha,\beta,\gamma)$$

$$u^2(v) := \sup_{\{(a,\alpha,\beta,\gamma)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+ : -\tilde{J}^2(v,a,\alpha,\beta,\gamma)\leq 1\}} \tilde{I}^2(a,\alpha,\beta,\gamma)$$

These value functions are respectively obtained for the Stackelbreg equilibria given in Theorems 5 and 6. We have the following result:

Theorem 7 *Let*

$$w_I^i := -p \vee \sup_{v>0}\{u^i(v)\mathbb{P}(\mathcal{V} \leq v) - p\big(1 - \mathbb{P}(\mathcal{V} \leq v)\big)\}. \tag{34}$$

We have $w_I^i \leq u_I^i$ *and when either* $p < +\infty$ *or* $\exists v \in (0, +\infty)$, $\mathbb{P}(\mathcal{V} > v) = 0$ *then* $w_I^i = u_I^i$.

Theorem 7 has an important interpretation. Indeed, it means that in order to solve (30) or (31), firm I first solves its problem for any given v as if the information was complete or in other words as in Section 3.3, and then "chooses" the level v that would bring the greatest social expectation in (34).

Theorem 8 *Let* $v_0 := \inf\{v > 0 : \mathbb{P}(\mathcal{V} \leq v) > 0\}$. *If* $\lim_{v \to v_0^+} u^i(v) \leq -p$ *then the fact that the two firms do not enter into any contract is a Stackelberg equilibrium in Situation* i.
Otherwise, if $v_1 := \sup\{v > 0 : \mathbb{P}(\mathcal{V} > v) > 0\} < +\infty$ *then the optimization problem (34) has a solution* $v^\star \in (0, v_1] \cap [v_0, v_1]$ *(equal to* v_1 *when* $p = +\infty$*) and any Stackelberg equilibrium for the problem with complete information and risk aversion* $v^\star$ *for firm* J *is a Stackelberg equilibrium for the problem with incomplete information.*

3.4.2 Nash equilibrium

We did not succeed in finding sufficient conditions for the existence of a Nash equilibrium with incomplete information. Nevertheless, we obtain necessary conditions that are similar for both situations:

Theorem 9 *Assume existence of a Nash equilibrium* $\hat{c} = (\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m})$ *such that the value for firm* I *is greater than* $-p$ *and let* $\hat{v} := \sup\{v > 0 : -J(v, \hat{c}) \leq 1\}$ *with* $J(v, \hat{c})$ *defined (using* $C^{op} = \mu - \varphi(e) - \delta\psi(a)$*) in Situations 1 and 2 respectively as*

$$J(v, a, \alpha, \beta, \gamma, e, m) := -\mathbb{E}\left[e^{-v\left(\alpha + (\beta-1)C^{op} - e + \gamma g(m) - m - (1+r_J)(D+a)\right)}\right],$$

$$J(v, a, \alpha, \beta, \gamma, e, m) := -\mathbb{E}\left[e^{-v\left(\alpha + (\beta-1)C^{op} - e + \gamma g(m) - m\right)}\right].$$

Then $\hat{v} > 0$, $\hat{e} = (\varphi')^{-1}\left(\frac{1}{(1-\hat{\beta})^+}\right)$, $\hat{\gamma} = \frac{1}{g'(\hat{m})}$ *and in Situation 1,* $\hat{a} = (\psi')^{-1}\left(\frac{1+r_J}{(\delta(1-\hat{\beta}))^+}\right)$.
If $\hat{v} < +\infty$*, then* $\hat{c}$ *is a Nash equilibrium for the problem with complete information and risk aversion* $\hat{v}$ *for firm* J *and for each* $v < \hat{v}$, $\mathbb{P}(\mathcal{V} \leq v) < \mathbb{P}(\mathcal{V} \leq \hat{v})$.
If $\hat{v} = +\infty$*, then for each* $v \in (0, +\infty)$, $\mathbb{P}(\mathcal{V} \leq v) < 1$.

Remark 4 If there is a vector $(v_1, \cdots, v_n)$ of elements of $(0, +\infty)$ such that $\sum_{k=1}^n \mathbb{P}(\mathcal{V} = v_k) = 1$, one deduces that if there exists a Nash equilibrium for the problem with incomplete information, then $\exists i$ such that $\hat{v} = v_i$.

4 Numerical Example

We investigate a numerical example to better quantify and compare the two different situations (Situation 1: outsourcing, Situation 2 : debt issuance) and the two equilibria (Nash and Stackelberg equilibria). We have chosen the following numerical values:

- the risk aversion parameters are $u = v = 1$,
- the impact of the quality of the investment on the operational costs is $\delta = 1$,
- the minimal investment is $D = 1$,
- the random cost μ follows a uniform distribution on the interval $[\underline{\mu}, \overline{\mu}]$, with $\underline{\mu} = 1$ and $\overline{\mu} = 2$,
- the aversion to debt issuance is $f(x) = e^{1.6x} - 1$ (or $e^{1.7x} - 1$),
- the benefits functions are $b^a(x) = \frac{x}{(1+x)}$ and $b^m(x) = b^e(x) = \sqrt{x}$,
- the impact of the maintenance cost m on the rent R is modeled by the function $g(x) = \sqrt{x}$, and the impact of the efforts a and e on the operational costs by the functions $\phi(x) = \psi(x) = \frac{\underline{\mu}}{3(1+\pi/4)}(\sqrt{x}\mathbf{1}_{x\leq 1} + (\arctan(x) + 1 - \pi/4)\mathbf{1}_{x>1})$ (thus satisfying the condition $\varphi(\infty) + \delta^+\psi(\infty) < \underline{\mu}$),
- the interest rates r_I and r_J over the period $[0, T]$ (when borrowing 1 initially, firm $K \in \{I, J\}$ has to reimburse $(1 + r_K)$ on the time-interval $[0, T]$) vary in the interval $[0, 1[$.

4.1 Nash equilibrium

Dependency on the maintenance costs $\hat{m}$ in Nash equilibrium We now investigate the Nash equilibria in both situations. As stated in Theorem 1 and Theorem 2, the optimal value functions $\hat{I}^1$ and $\hat{I}^2$ depend on the optimal maintenance costs $\hat{m}$. Figure 1 gives an insight of this dependency, for two different values of r_J in Situation 1 and for a larger value of r_I in Situation 2. We notice that outsourcing is worthless for average maintenance costs (between 5% and 50% of D for $r_J = 45\%$) We also observe that the smaller r_J, the larger is the interval of values $\hat{m}$ for which outsourcing is better than debt issuance.

The maintenance costs $\hat{m}$ numerically optimizing the value functions $\hat{I}^1$ and $\hat{I}^2$ is $[(b^m)']^{-1}(1) = \frac{1}{4}$ (which is also the maintenance costs in Stackelberg equilibrium, whatever the situation) as expected from Remarks 2 and 3. Therefore, in the forthcoming figures, the Nash equilibrium is computed for this optimal maintenance costs $\hat{m} = \frac{1}{4}$.

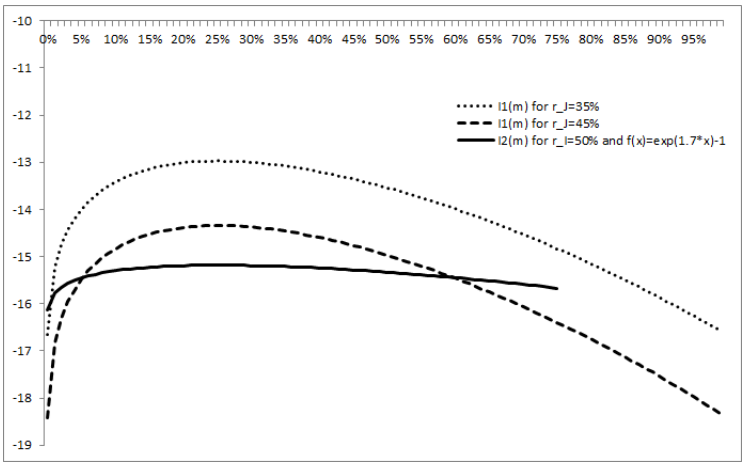

Fig. 1 Value functions $\hat{I}^1$ and $\hat{I}^2$ as functions of $\hat{m}$

Dependency on the interest rates r_I and r_J in Nash equilibrium Figure 2 gives, for Nash equilibrium, the optimal value function $\hat{I}^1$ in Situation 1 (outsourcing) as a function of $r_J \in [0, 1[$, and $\hat{I}^2$ in Situation 2 (debt issuance) as a function of $r_I \in [0, 1[$ and for two differents functions of debt aversion ($f(x) = e^{1.6x} - 1$ or $e^{1.7x} - 1$). We notice that the greater the debt aversion function f, the smaller

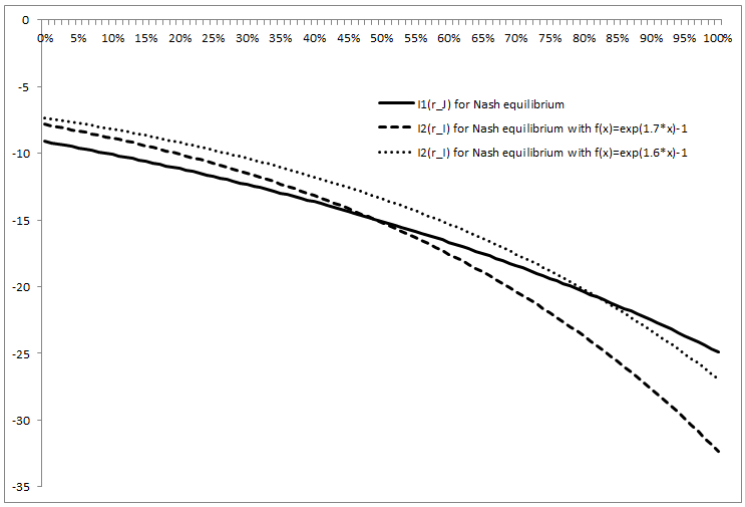

Fig. 2 $\hat{I}^1(r_J)$ and $\hat{I}^2(r_I)$ in Nash equilibrium

the value of $r_I = r_J$ at which it becomes more favourable to outsource (50% for $f(x) = e^{1.7x} - 1$, 80% for $e^{1.6x} - 1$).

Outsourcing or not in Nash equilibrium? Figure 3 gives a decision criterion of outsourcing or not, function of $r_I \in [0, 1[$ in the x-axis and $r_J \in [0, 1[$ in the y-

axis. The grey area corresponds to the region where it is optimal to issue debt, while the black area corresponds to the region where it is optimal to outsource. As expected, if it is optimal to outsource for a given couple (r_I, r_J), then it remains optimal to outsource for all couples (r'_I, r_J) with $r'_I > r_I$. If it is optimal to issue debt for a given couple (r_I, r_J), then it remains optimal to issue debt for all couples (r_I, r'_J) with $r'_J > r_J$.

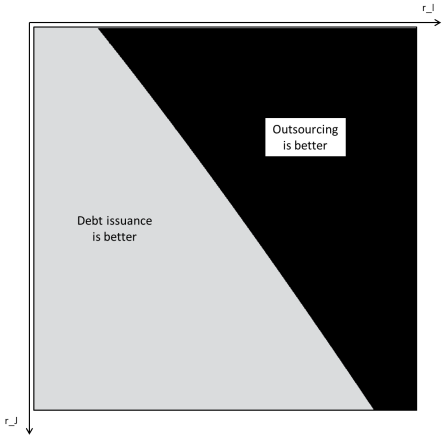

Fig. 3 Decision areas in Nash equilibrium

4.2 Stackelberg equilibrium

Dependency on the interest rates r_I and r_J in Stackelberg equilibrium Figure 4 below gives, for Stackelberg equilibrium, the optimal value function $\hat{I}^1$ in Situation 1 (outsourcing) as a function of $r_J \in [0, 1[$, and $\hat{I}^2$ in Situation 2 (debt issuance) as a function of $r_I \in [0, 1[$ and for two different functions of debt aversion ($f(x) = e^{1.6x} - 1$ or $e^{1.7x} - 1$). The conclusions are the same as the ones in Figure 2 for

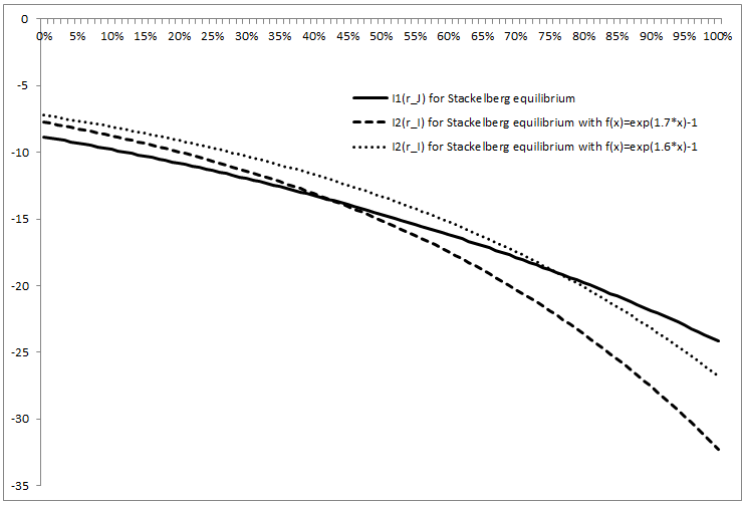

Fig. 4 $\hat{I}^1(r_J)$ and $\hat{I}^2(r_I)$ in Stackelberg equilibrium

Nash equilibrium, and we notice that the value functions $\hat{I}^1$ and $\hat{I}^2$ are slightly greater in Stackelberg equilibrium than in Nash equilibrium. Moreover, the range of interest rate r for which outsourcing is more favourable is slightly wider in Stackelberg equilibrium than in Nash equilibrium (43% for $f(x) = e^{1.7x} - 1$, 73% for $e^{1.6x} - 1$). Indeed, being a leader in Stackelberg equilibrium, firm I has a more favourable position to get better conditions to outsource, in comparison with Nash equilibrium.

Outsourcing or not in Stackelberg equilibrium? Figure 5 gives a decision criteria of outsourcing or not, function of r_I in the x-axis and r_J in the y-axis. The grey area corresponds to the region where it is optimal to issue debt, while the black area corresponds to the region where it is optimal to outsource. In comparison with Figure 3 dedicated to Nash equilibria, the boundary between the outsourcing and debt issuance regions has the same shape but the outsourcing region is slighly bigger.

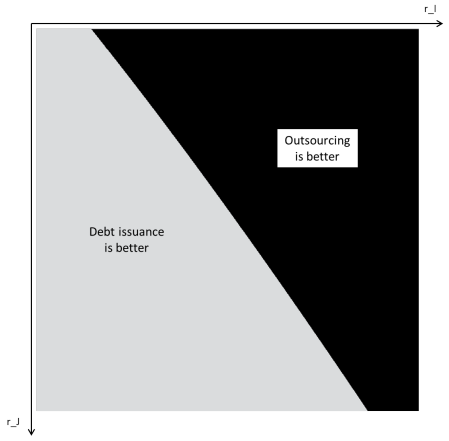

Fig. 5 Decision areas in Stackelberg equilibrium

4.3 Comparison Nash/Stackelberg equilibria

We first compare the value functions of firm I between Nash and Stackelberg equilibria, in the situation of outsourcing the debt (Figure 6) and in the situation of debt issuance (Figure 7). In both situations, the value function is higher in Stackelberg equilibrium than in Nash equilibrium, which can be interpreted by the fact that firm I has a leader position in Stackelberg equilibrium. The difference is a little less significant in Situation 2 than in Situation 1.

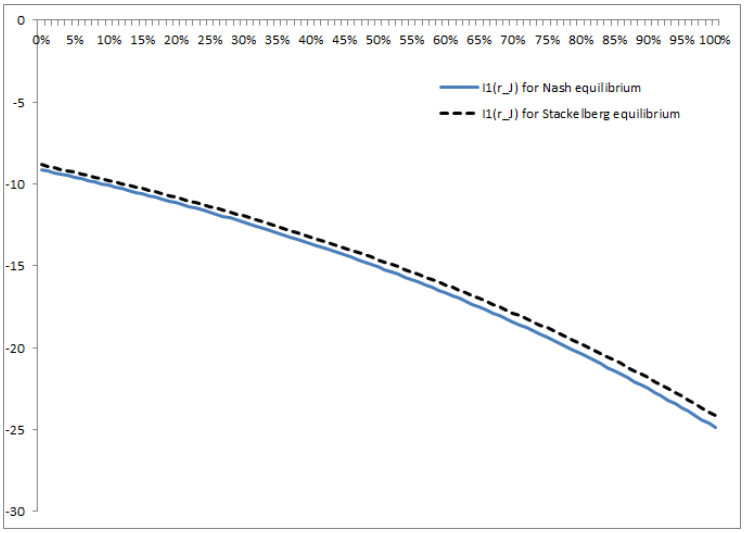

Fig. 6 Value function $\hat{I}^1$ (outsourcing) in Nash and Stackelberg equilibria

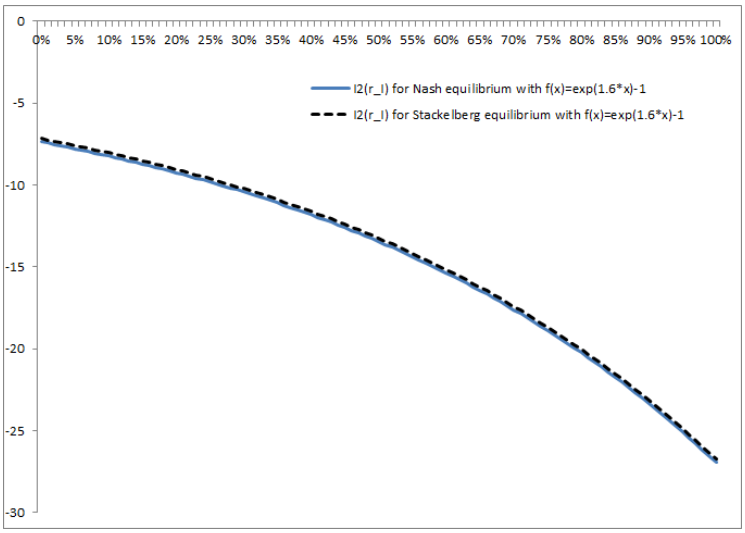

Fig. 7 Value function $\hat{I}^2$ (debt issuance) in Nash and Stackelberg equilibria

5 Proofs in Situation 1

5.1 Best responses in Situation 1

Let a, e and m be given and constant. Then we get the following optimization problem for firm I:

$$\sup_{\alpha,\beta,\gamma} \mathbb{E}\left[b^a(a) - e^{-u\left(b^m(m)+b^e(e)-\alpha-\beta(\mu-\varphi(e)-\delta\psi(a))-\gamma g(m)\right)}\right]$$

such that $\alpha \geq 0,\ \gamma \geq 0$, and using $V(0) = -1$,

$$\mathbb{E}\left[e^{-v\left(\alpha+(\beta-1)(\mu-\varphi(e)-\delta\psi(a))-e+\gamma g(m)-m-(1+r_J)(D+a)\right)}\right] \leq 1. \tag{35}$$

Proposition 1 *Let $a \geq 0$, $e \geq 0$ and $m \geq 0$ be given and constant. Then there exist optimal triplets (α, β, γ) for the above problem. Moreover (α, β, γ) is optimal if and only if it satisfies: $\beta = \beta^* := \frac{v}{u+v} \in (0,1)$ and $\alpha + \gamma g(m) = C_e(\beta^*) - \frac{u}{u+v}\delta\psi(a) + e + m + (1+r_J)(D+a)$ with $\alpha \geq 0$ and $\gamma \geq 0$, where*

$$C_e(\beta) := \frac{1}{v}\ln \mathbb{E}\left[e^{v(1-\beta)(\mu-\varphi(e))}\right]. \tag{36}$$

Proof We first need the following lemmas :

Lemma 2 *The function $h(\lambda) = \frac{\mathbb{E}\left[\mu e^{\lambda\mu}\right]}{\mathbb{E}\left[e^{\lambda\mu}\right]}$ is increasing, thus the equation $h(u\beta) = h(v(1-\beta))$ admits the unique solution $\beta^* := \frac{v}{u+v} \in (0,1)$.*

Proof We compute using Cauchy-Schwarz inequality

$$\left(\mathbb{E}e^{\lambda\mu}\right)^2 h'(\lambda) = \mathbb{E}\mu^2 e^{\lambda\mu}\mathbb{E}e^{\lambda\mu} - \left(\mathbb{E}\mu e^{\lambda\mu}\right)^2 > 0,$$

The inequality is strict since μ is not constant $d\mathbb{P}$ a.s.. Therefore β satisfies $h(u\beta) = h(v(1-\beta))$ if and only if $u\beta = -v(\beta-1)$, so that the only solution is $\beta^* := \frac{v}{u+v} \in (0,1)$.

□

Lemma 3 *The functions $C_e(\beta)$ and $C(\beta) = C_{e^*(\beta)}(\beta)$ defined in (36) and (12) are such that*

$$\forall \beta < 1,\ \forall a, e \geq 0,\ \forall \delta \in \mathbb{R},\ C_e(\beta) - \delta^+(1-\beta)\psi(a) > 0 \tag{37}$$

$$\forall \beta < 1,\ \forall a \geq 0,\ \forall \delta \in \mathbb{R},\ C(\beta) - \delta^+(1-\beta)\psi(a) > 0 \tag{38}$$

$$\forall \beta \in \mathbb{R},\ C(\beta) \geq (1-\beta)\Big(\mathbb{E}[\mu]\ - \varphi(e^*(\beta))\Big) \tag{39}$$

Proof Since $v > 0$, $C_e(\beta) - \delta^+(1-\beta)\psi(a) = \frac{1}{v}\ln \mathbb{E}\left[e^{v(1-\beta)(\mu-\varphi(e)-\delta^+\psi(a))}\right]$ and $\mu - \varphi(e) - \delta^+\psi(a) > 0$ a.s., the two first statements are consequences of the monotonicity of $x \mapsto \exp v(1-\beta)x$. The last one is a consequence of Jensen's inequality. □

Let a, e and m be given and $C^{op} = \mu - \varphi(e) - \delta\psi(a)$, we introduce:

$$K(\alpha,\beta,\gamma) := -\mathbb{E}\left[e^{-u\left(b^m(m)+b^e(e)-\alpha-\beta C^{op}-\gamma g(m)\right)}\right]$$

$$E := \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3;\ \mathbb{E}\left[e^{-v\left(\alpha+(\beta-1)C^{op}-e+\gamma g(m)-m-(1+r_J)(D+a)\right)}\right] \leq 1\Big\}.$$

We will first solve the problem of maximization of K on E, forgetting about the constraints $\alpha \geq 0$ and $\gamma \geq 0$, and we will then see that it allows to solve the original constrained problem. Let us therefore consider the following problem: $\sup_{(\alpha,\beta,\gamma)\in E} K(\alpha,\beta,\gamma)$. Since K is concave and E is a closed convex set, the first order conditions for the Lagrangian associated to this problem are also sufficient conditions. The Lagrangian is given by:

$$L(\alpha,\beta,\gamma,\lambda) := -\mathbb{E}\left[e^{-u\left(b^m(m)+b^e(e)-\alpha-\beta C^{op}-\gamma g(m)\right)}\right]$$
$$-\lambda\Big(\mathbb{E}\left[e^{-v\left(\alpha+(\beta-1)C^{op}-e+\gamma g(m)-m-(1+r_J)(D+a)\right)}\right]-1\Big).$$

Hypothesis (2) implies that L is differentiable and the following are null:

$$\frac{\partial L}{\partial \alpha} = -\mathbb{E}\left[ue^{-u()}\right] + \lambda\mathbb{E}\left[ve^{-v()}\right], \tag{40}$$

$$\frac{\partial L}{\partial \beta} = -\mathbb{E}\left[ue^{-u()}C^{op}\right] + \lambda\mathbb{E}\left[ve^{-v()}C^{op}\right], \tag{41}$$

$$\frac{\partial L}{\partial \gamma} = -\mathbb{E}\left[ue^{-u()}g(m)\right] + \lambda\mathbb{E}\left[ve^{-v()}g(m)\right]. \tag{42}$$

Since $g(m)$ is a constant, equation (40) implies equation (42). Furthermore, since $C^{op} = \mu - \varphi(e) - \delta\psi(a) > 0$, then $\lambda > 0$ and the constraint is always binding. This is natural since K is decreasing with respect to α, β and γ, while

$$(\alpha,\beta,\gamma) \mapsto -\mathbb{E}\left[e^{-v\left(\alpha+(\beta-1)C^{op}-e+\gamma g(m)-m-(1+r_J)(D+a)\right)}\right]$$

is increasing with respect to α, β and γ. Therefore, at an interior point of E denoted (α, β, γ), for sufficiently small $\varepsilon > 0$, for example $(\alpha, \beta - \varepsilon, \gamma)$ is still in E, while $K(\alpha, \beta - \varepsilon, \gamma) > K(\alpha, \beta, \gamma)$, so that (α, β, γ) cannot be a maximum of K. Therefore we also have:

$$\mathbb{E}\left[e^{-v\left(\alpha+(\beta-1)C^{op}-e+\gamma g(m)-m-(1+r_J)(D+a)\right)}\right] = 1. \tag{43}$$

Combining (40) and (41), we get:

$$\mathbb{E}\left[\mu e^{u\beta\mu}\right]\mathbb{E}\left[e^{-v(\beta-1)\mu}\right] = \mathbb{E}\left[e^{u\beta\mu}\right]\mathbb{E}\left[\mu e^{-v(\beta-1)\mu}\right].$$

This equation is equivalent to $h(u\beta) - h(v(1-\beta) = 0$ which admits the unique solution $\beta^* = \frac{v}{u+v} \in (0,1)$ (cf. Lemma 2).
We have

$$\mathbb{E}\left[e^{v(1-\beta^*)(\mu\ -\varphi(e)-\delta\psi(a))}\right] = e^{v(C_e(\beta^*)-\frac{u}{u+v}\delta\psi(a))}, \tag{44}$$

which together (43) yields the following necessary and sufficient condition for optimality:

$$\alpha + \gamma g(m) = C_e(\beta^*) - \frac{u}{u+v}\delta\psi(a) + e + m + (1+r_J)(D+a). \tag{45}$$

Since $1 - \beta^* > 0$, by (37), $C_e(\beta^*) - \frac{u}{u+v}\delta\psi(a) > 0$, $C_e(\beta^*) - \frac{u}{u+v}\delta\psi(a) + e + m + (1+r_J)(D+a) > 0$ and the set
$\{(\alpha,\gamma) \in [0,+\infty)^2,\ \alpha + \gamma g(m) = C_e(\beta^*) - \frac{u}{u+v}\delta\psi(a) + e + m + (1+r_J)(D+a)\}$
is not empty. Therefore the optimal (α, β, γ) for the problem:

$$\sup_{(\alpha,\beta,\gamma)\in E\cap(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)} K(\alpha,\beta,\gamma),$$

are exactly the elements of

$$\Big\{(\alpha,\beta,\gamma) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+;\ \beta = \frac{v}{u+v},$$
$$\alpha + \gamma g(m) = C_e(\beta) - \frac{u}{u+v}\delta\psi(a) + e + m + (1+r_J)(D+a)\Big\}.$$

□

5.2 Nash equilibrium in Situation 1

Proof of Lemma 1 The function g is assumed to be increasing, strictly concave and such that $g'(0) = \infty$, $g'(\infty) = 0$ and (5) holds. Since g is increasing and concave, we compute for any $x \in [0, m]$:

$$g(m) - g(x) = \int_x^m g'(u)du \geq (m-x)g'(m).$$

Since g is strictly concave and $g'(\infty) = 0$, $g'(m) > 0$ and we have for $m \geq x$:

$$\frac{g(m)}{g'(m)} - m \geq \frac{g(x)}{g'(m)} - x.$$

By (5) and monotonicity of g, for $x > m_0$, $g(x) > 0$ and $\lim_{m\to+\infty} \frac{g(x)}{g'(m)} - x = +\infty$ so that $\lim_{m\to+\infty} \frac{g(m)}{g'(m)} - m = +\infty$. Since $G'(m) = -\frac{gg''(m)}{(g'(m))^2}$ has the same sign as $g(m)$ by strict concavity of g, one easily concludes. □

Proof of Theorem 1 The characterization conditions (15) and (16) for a Nash equilibrium follow from the optimal expressions (10) and Proposition 1. Thus, the only thing to check is the existence of an infinite number of solutions in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ to these equations.
By (38) and since $(1 + r_J)(D + \hat{a}) + \hat{e} > 0$,

$$C(\hat{\beta}) + (1 + r_J)(D + \hat{a}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}) > 0.$$

Therefore there exists infinitely many couples $(\alpha, m) \in \mathbb{R}_+^2$ such that

$$\alpha + G(m) = C(\hat{\beta}) + (1 + r_J)(D + \hat{a}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a})$$

namely the couples $\left(C(\hat{\beta}) + (1 + r_J)(D + \hat{a}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}) - G(x), x\right)$ where $x \in [0, G^{-1}(C(\hat{\beta}) + (1 + r_J)(D + \hat{a}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}))]$. □

5.3 Stackelberg equilibrium in Situation 1, firm I is leader

As a preliminary, we prove Proposition 2, useful for Stackelberg equilibria in both situations.

Proposition 2 *Equation (27) admits at least one solution $\hat{\beta}$. Moreover, all solutions are smaller than $\frac{v}{u+v}$.*

Proof Let us recall (27): $h(u\beta) - h(v(1-\beta)) = S(\beta)$. By Lemma 2, as β goes from $-\infty$ to $+\infty$, the left-hand side of this equation is increasing from $h(-\infty) - h(+\infty) < 0$ to $h(+\infty) - h(-\infty) > 0$ and is null for $\beta = \frac{v}{u+v}$.
For $\beta < 1$, since $\varphi'(e^*(\beta)) = \frac{1}{1-\beta}$, we have $S(\beta) = (\varphi'(e^*(\beta)) + (b^e)'(e^*(\beta)) - 1)\frac{(\varphi'(e^*(\beta)))^2}{\varphi''(e^*(\beta))}$ so that, by concavity of φ, the sign of $S(\beta)$ is equal to the one of $1 - \varphi'(e^*(\beta)) - (b^e)'(e^*(\beta))$. Remember that when β goes from $-\infty$ to 1, $e^*(\beta)$ is decreasing from $+\infty$ to 0, φ and b^e are concave, so $\beta \mapsto 1-\varphi'(e^*(\beta))-(b^e)'(e^*(\beta))$ is decreasing, from 1 ($\varphi'(+\infty) = (b^e)'(+\infty) = 0$) to $-\infty$ ($\varphi'(0) = (b^e)'(0) = +\infty$). Since $\varphi'+(b^e)'$ is monotonic, there exists a unique β_0 such that $\varphi'(\beta_0)+(b^e)'(\beta_0) = 1$, so $\beta < \beta_0 \Rightarrow S(\beta) > 0$, $\beta > \beta_0 \Rightarrow S(\beta) < 0$. As a consequence, there exists a solution $\hat{\beta}$ to (27). For $\beta \geq 0$, $S(\beta)$ is negative since in (24), the numerator is positive whereas the denominator is negative by concavity of φ. Hence $\beta_0 < 0$. Moreover $S(\frac{v}{u+v}) < 0$, so any solution $\hat{\beta}$ belongs to $(\beta_0, \frac{v}{u+v})$. □

Proof of Theorem 5
Firm I has to find (α, β, γ) maximising

$$b^a(a^*(\beta)) - \mathbb{E}\left[e^{-u\left(b^m(m^*(\gamma))+b^e(e^*(\beta))-\alpha-\beta(\mu\ -\varphi(e^*(\beta))-\delta\psi(a^*(\beta)))-\gamma g(m^*(\gamma))\right)}\right].$$

Since the inverse function of $m^*(\gamma) = (g')^{-1}(1/\gamma)$ is the increasing bijection $\gamma^*(m) = \frac{1}{g'(m)}$ from $\mathbb{R}^+$ onto itself, the maximisers are the triplets $(\alpha, \beta, \gamma^*(m))$ with (α, β, m) maximising

$$\tilde{I}^1(\alpha,\beta,m) := b^a(a^*(\beta)) - \mathbb{E}\left[e^{-u\left(b^m(m)+b^e(e^*(\beta))-\alpha-\beta(\mu\ -\varphi(e^*(\beta))-\delta\psi(a^*(\beta)))-\frac{g}{g'}(m)\right)}\right]$$

under the constraint

$$-\tilde{J}^1(\alpha,\beta,m) := \mathbb{E}\left[e^{-v(\alpha+(\beta-1)(\mu\ -\varphi(e^*(\beta))-\delta\psi(a^*(\beta)))-e^*(\beta)+\frac{g}{g'}(m)-m-(1+r_J)(D+a^*(\beta)))}\right] \leq 1,$$

where, by a slight abuse of notations, we still denote by $\tilde{I}^1$ and $\tilde{J}^1$ the functions obtained by applying the change of variable $(\alpha, \beta, \gamma) \to (\alpha, \beta, m)$ to the ones introduced in Section 3.4.1. We also recall the application $C : \mathbb{R} \to \mathbb{R}$ defined in (12)

$$C(\beta) = \frac{1}{v} \ln \mathbb{E}\left[e^{v(1-\beta)(\mu\ -\varphi\circ e^*(\beta))}\right].$$

Setting

$$\mathcal{A} := \Big\{(\alpha,\beta,m) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+;\ \ -\tilde{J}^1(\alpha,\beta,m) \leq 1\}, \tag{46}$$

the optimization problem for firm I then writes:

$$\sup_{(\alpha,\beta,m)\in\mathcal{A}} \tilde{I}^1(\alpha,\beta,m).$$

We will prove the existence of a maximizer for this problem, and therefore of a Stackelberg equilibrium, by checking that we can restrict the set $\mathcal{A}$ to a compact subset. Notice first that $\mathcal{A} \neq \emptyset$. In fact, one can easily check that for any $\beta \in \mathbb{R}$ and $m \geq 0$, one can choose α large enough so that $(\alpha, \beta, m) \in \mathcal{A}$.

Proposition 3 *We have* $\sup_{(\alpha,m)\in\mathbb{R}^2_+} \tilde{I}^1(\alpha,\beta,m) \to -\infty$ *when* $\beta \to \infty$. *Moreover, there exists* $\bar{\beta} \in \mathbb{R}$, *not depending on* $v > 0$, *such that the supremum over* $\mathcal{A}$ *is attained if and only if the supremum over* $\mathcal{A} \cap \{\beta \in (-\infty, \bar{\beta}]\}$ *is attained, and both supremum are equal.*

Proof To prove this proposition, we need the following result which applies to functions φ and ψ :

Lemma 4 *For any increasing, strictly concave* C^1 *and bounded function* ϕ, $\phi'(x) = \circ(1/x)$ *when* $x \to \infty$ *and* $y(\phi')^{-1}(y) \to 0$ *when* $y \to 0$.

Proof Integrating by parts, we get for $x \geq 1$,

$$\int_1^x u\phi''(du) = x\phi'(x) - \phi'(1) - \phi(x) + \phi(1) \tag{47}$$

where $\phi''(du)$ denotes the negative measure equal to the second order distribution derivative of ϕ. Since ϕ is increasing and concave, the terms $-x\phi'(x)$ and $\int_1^x u\phi''(du)$ are non-positive on $[1, +\infty)$. The boundedness of ϕ then implies their

boundedness on $[1,+\infty)$. Since $\int_1^x u\phi''(du)$ and $\phi(x)$ are monotonic and bounded, they admit finite limits when $x \to \infty$. By (47), $x\phi'(x)$ admits a finite limit as well, denoted ℓ. Since ϕ is bounded, ϕ' is integrable on $[1,+\infty)$, which implies $\ell = 0$ and gives the result. Let $x = (\phi')^{-1}(y)$. When $y \to 0$, x goes to ∞ and $y(\phi')^{-1}(y) = \phi'(x)x$ goes to 0.

□

Since b^m is such that $b^{m\prime}(\infty) = 0$, $\lim_{m\to+\infty} m - b^m(m) = +\infty$ thus, using the first assertion in Lemma 1,

$$\lim_{m\to+\infty} \frac{g}{g'}(m) - b^m(m) = +\infty. \tag{48}$$

Hence $e^{u\left(\frac{g}{g'}(m)-b^m(m)\right)}$ goes to infinity when $m \to \infty$ and there exists a constant $c > 0$ not depending on $v > 0$ such that for any $m \geq 0$, $e^{u\left(\frac{g}{g'}(m)-b^m(m)\right)} \geq c$.
For any $\beta \geq 1$, $e^*(\beta) = 0$ and since $\tilde{I}^1$ is decreasing with respect to α, we have for $(\alpha, m) \in \mathbb{R}_+^2$,

$$\begin{aligned}\tilde{I}^1(\alpha,\beta,m) &\leq \tilde{I}^1(0,\beta,m)\\ &\leq b^a(a^*(\beta)) - ce^{-ub^e(0)}\,\mathbb{E}\left[e^{u\beta(\mu\ -\varphi(0)-\delta\psi(a^*(\beta)))}\right]\\ &\to -\infty \text{ when } \beta\to\infty.\end{aligned} \tag{49}$$

Indeed, if $\delta > 0$, then $a^*(\beta) = (\psi')^{-1}\left(\frac{1+r_J}{(\delta(1-\beta))^+}\right) = 0$ for $\beta \geq 1$ and the result is obvious.
Otherwise for $\delta \leq 0$, $b^a(a^*(\beta)) = o(\beta)$ when $\beta \to \infty$ (indeed $b^a(x) = o(x)$ when $x \to \infty$ and $a^*(\beta) = o(\beta)$ when $\beta \to \infty$, see Lemma 4), $\mu\ -\varphi(0) - \delta\psi(a^*(\beta)) > \mu\ -\varphi(+\infty) - \delta^+\psi(+\infty) > 0$. Therefore $e^{u\beta(\mu\ -\varphi(0)-\delta\psi(a^*(\beta)))}$ goes to $+\infty$ faster than $b^a(a^*(\beta))$ and, by Fatou Lemma,

$$\liminf_{\beta\to\infty} \mathbb{E}\left[e^{u\beta(\mu\ -\varphi(0)-\delta\psi(a^*(\beta)))-\ln(b^a(a^*(\beta)))}\right] = \infty$$

so that (49) holds.
Since $((1+r_J)D, 1, 0) \in \mathcal{A}$ for any $v > 0$, one deduces the existence of $\bar{\beta} \in \mathbb{R}$ not depending on v such that

$$\sup_{(\alpha,\beta,m)\in\mathbb{R}_+\times(\bar{\beta},+\infty)\times\mathbb{R}_+} \tilde{I}^1(\alpha,\beta,m) < \sup_{(\alpha,\beta,m)\in\mathcal{A}} \tilde{I}^1(\alpha,\beta,m)$$

and the supremum over $\mathcal{A}$ is attained if and only if the supremum over $\mathcal{A} \cap \{\beta \in (-\infty, \bar{\beta}]\}$ is attained. Moreover, if the suprema are attained, they are equal. □

We now decompose the optimisation on $\mathcal{A} \cap \{\beta \in (-\infty, \bar{\beta}]\}$ according to the positivity of α.

Lemma 5 *Let*

$$\mathcal{B} := \left\{(\alpha,\beta,m) \in \mathbb{R}_+ \times (-\infty,\bar{\beta}] \times \mathbb{R}_+;\ \ -\tilde{J}^1(\alpha,\beta,m) = 1\right\}, \tag{50}$$

$$\mathcal{C} := \left\{(0,\beta,m);\ \beta \in (-\infty,\bar{\beta}],\ m \geq 0,\ \ -\tilde{J}^1(0,\beta,m) \leq 1\right\}. \tag{51}$$

The supremum over $\mathcal{A}$ is attained if and only if the supremum over $\mathcal{B} \cup \mathcal{C}$ is attained.

Proof Since $\tilde{I}^1$ is decreasing with respect to α and $\tilde{J}^1$ is continuous, if $\alpha > 0$ and $-\tilde{J}^1(\alpha,\beta,m) < 1$, then there exists $\varepsilon > 0$ such that $\tilde{I}^1(\alpha-\varepsilon,\beta,m) > \tilde{I}^1(\alpha,\beta,m)$, while $-\tilde{J}^1(\alpha-\varepsilon,\beta,m) \leq 1$. Therefore the supremum over $\mathcal{A}\cap\{\alpha>0\}$ is the same as the supremum over

$$\tilde{\mathcal{B}} := \Big\{(\alpha,\beta,m) \in (0,+\infty)\times(-\infty,\bar{\beta}]\times\mathbb{R}_+;\ \tilde{J}^1(\alpha,\beta,m) = -1\Big\}.$$

Noticing then that $\mathcal{A}$ is closed and that $\mathcal{B} := Cl(\tilde{\mathcal{B}})$, we easily conclude. □

Lemmas 6 and 7 are devoted to the supremum over $\mathcal{B}$. In Lemma 6, we check that the supremum is attained on a compact subset $\mathcal{K}$ not depending on $v > 0$. In Lemma 7, we derive optimality conditions satisfied by a maximizer with positive α.

Lemma 6 *The function k defined in (26):*

$$k : \beta \to e^{u(Id-b^e)\circ e^*(\beta)} e^{-u\delta\psi\circ a^*(\beta)} e^{u(1+r_J)(D+a^*(\beta))} e^{uC(\beta)} \mathbb{E}\left[e^{u\beta\left(\mu-\varphi\circ e^*(\beta)\right)}\right]$$

depends on $v > 0$ through the function $C(\beta)$ defined in (12), goes to ∞ uniformly in $v > 0$ when $\beta \to -\infty$.
Moreover, in the set $\mathcal{B}$, α is a continuous function of (β, m) and there exists a compact set $\mathcal{K} \subset \mathcal{B}$ not depending on $v > 0$ such that $\sup_{\mathcal{B}} \tilde{I}^1 = \sup_{\mathcal{K}} \tilde{I}^1$. In particular the supremum on $\mathcal{B}$ is attained.

Proof Let us consider $\sup_{\mathcal{B}} \tilde{I}^1$. Notice that $C(\beta)$ given by (12) depends on $v > 0$. Since $\tilde{J}^1(\alpha,\beta,m) = -1$ on $\mathcal{B}$, we know that

$$\alpha = \alpha(\beta,m) := C(\beta) - (1-\beta)\delta\psi(a^*(\beta)) + e^*(\beta) + m - \frac{g}{g'}(m) + (1+r_J)(D+a^*(\beta)),$$

so that

$$\tilde{I}^1(\alpha(\beta,m),\beta,m) = b^a(a^*(\beta)) - e^{u(Id-b^m)(m)} k(\beta). \tag{52}$$

In $\tilde{I}^1(\alpha(\beta,m),\beta,m)$, we have $b^a(a^*(\beta))$ minus the product of two positive functions, the first one depending only on m, the second one, k, only on β. Since $(b^m)'(\infty) = 0$, $e^{u(Id-b^m)(m)} \to \infty$ when $m \to \infty$, therefore the first function is bounded from below by a positive constant c and goes to infinity when m goes to infinity.
Let us now examine the function k. By Jensen's inequality, $\mathbb{E}\left[e^{u\beta(\mu-\varphi(e^*(\beta)))}\right] \geq e^{u\beta(\mathbb{E}[\mu]-\varphi(e^*(\beta)))}$; then, using (39) and $e^{u(1+r_J)(D+a^*(\beta))} \geq 1$ it yields

$$k(\beta) \geq e^{u(Id-b^e)(e^*(\beta))} e^{u\left(\mathbb{E}[\mu]\ -\varphi(e^*(\beta))-\delta\psi(a^*(\beta))\right)}.$$

Using $\mathbb{E}[\mu\,] \geq \varphi(x)+\delta^+\psi(y)$ for any x, y, we deduce that $k(\beta) \geq e^{u(Id-b^e)\circ(\varphi')^{-1}(\frac{1}{(1-\beta)^+})}$ where the right-hand side does not depend on $v > 0$ and goes to infinity when $\beta \mapsto -\infty$ since $b^e(x) = o(x)$ when $x \to \infty$ (recall that $(b^e)'(\infty) = 0$). Finally using $b^a(x) = o(x)$ and $(\varphi')^{-1}(x) \sim (\psi')^{-1}(x)$ for $x \to 0$, we conclude when $\beta \to -\infty$:

$$\tilde{I}^1(\alpha,\beta,m) \leq b^a \circ (\psi')^{-1}\Big(\frac{1+r_J}{\delta^+(1-\beta)}\Big) - ce^{u(Id-b^e)\circ(\varphi')^{-1}(\frac{1}{1-\beta})} \to -\infty.$$

Since $\alpha(1,0) = (1+r_J)D > 0$ and $\tilde{I}^1(\alpha(1,0),1,0)$ do not depend on $v > 0$ and $(\alpha(1,0),1,0) \in \mathcal{B}$ for any $v > 0$, we deduce the existence of $\underline{\beta} > -\infty$ and $\bar{m}$ in $[0,+\infty)$ not depending on v such that $\sup_{\mathcal{B}} \tilde{I}^1 = \sup_{\mathcal{B}\cap\{\beta\in[\underline{\beta},\bar{\beta}],m\in[0,\bar{m}]\}} \tilde{I}^1$.

To conclude that $\sup_{\mathcal{B}} \tilde{I}^1$ is attained on a compact $\mathcal{K} \subset \mathcal{B}$ not depending on v, one still has to make sure that α remains bounded when v varies. Since for any α,

$$\sup_{\beta\in[\underline{\beta},\bar{\beta}],m\in[0,\bar{m}]} \tilde{I}^1(\alpha,\beta,m) \leq \sup_{\beta\in[\underline{\beta},\bar{\beta}]} b^a(a^*(\beta))$$
$$- e^{u\alpha} \inf_{\beta\in[\underline{\beta},\bar{\beta}],m\in[0,\bar{m}]} \mathbb{E}\left[e^{-u\left(b^m(m)+b^e(e^*(\beta))-\beta(\mu\ -\varphi(e^*(\beta))-\delta\psi(a^*(\beta)))-\frac{g}{g'}(m)\right)}\right]$$

where the right-hand side tends to $-\infty$ as $\alpha \to \infty$, one concludes that there exists a compact set $\mathcal{K} \subset \mathcal{B}$ not depending on $v > 0$ such that $\sup_{\mathcal{B}} \tilde{I}^1 = \sup_{\mathcal{K}} \tilde{I}^1$. □

Lemma 7 *Let $(\hat{\alpha},\hat{\beta},\hat{m})$ with $\hat{\alpha} > 0$ be such that the maximum on $\mathcal{B}$ is attained at $(\hat{\alpha},\hat{\beta},\hat{m})$. Then necessarily*

$$\hat{m} = [(b^m)']^{-1}(1) > 0, \tag{53}$$

and there are two different cases, depending on the sign of δ :
• *If $\delta > 0$,*

$$h(u\hat{\beta}) - h(v(1-\hat{\beta})) = S(\hat{\beta}) + \widetilde{S}(\hat{\beta}), \text{ and } \hat{\beta} < 1. \tag{28}$$

• *If $\delta < 0$,*

$$h(u\hat{\beta}) - h(v(1-\hat{\beta})) = S(\hat{\beta}), \text{ and } \hat{\beta} < 1, \tag{27}$$

or

$$h(u\hat{\beta}) - h(v(1-\hat{\beta})) = \widetilde{S}(\hat{\beta}), \text{ and } \hat{\beta} \geq 1. \tag{29}$$

with h, S and $\widetilde{S}$ respectively defined in (23), (24) and (25).

Proof Let $(\hat{\alpha},\hat{\beta},\hat{m})$ be such that the maximum on $\mathcal{B}$ is attained at $(\hat{\alpha},\hat{\beta},\hat{m})$. Since $(b^m)'(0) = +\infty$, $m \mapsto m - b^m(m)$ is decreasing in a neighborhood of 0, so that from (52), $\hat{m} > 0$. Assume moreover that $\hat{\alpha} > 0$. Then the mapping $(\beta,m) \mapsto \tilde{I}^1(\alpha(\beta,m),\beta,m)$ where $\alpha(\beta,m)$ is defined at the beginning of the proof of Lemma 6 admits a local maximum at $(\hat{\beta},\hat{m})$ and therefore the first order conditions are satisfied (notice that $\bar{\beta}$ may be increased), i.e. $\frac{\partial}{\partial\beta}\tilde{I}^1(\alpha(\hat{\beta},\hat{m}),\hat{\beta},\hat{m}) = \frac{\partial}{\partial m}\tilde{I}^1(\alpha(\hat{\beta},\hat{m}),\hat{\beta},\hat{m}) = 0$. The second one gives

$$\hat{m} = [(b^m)']^{-1}(1) > 0. \tag{54}$$

The computation of $\hat{\beta}$ is more tricky and depends on the coefficient δ.

1) $\underline{\delta > 0}$: $\frac{\partial}{\partial\beta}\tilde{I}^1(\alpha(\hat{\beta},\hat{m}),\hat{\beta},\hat{m}) =$

$$\begin{cases} \bullet\ -e^{u(Id-b^m)(\hat{m})}\mathbb{E}\left[u\nu e^{u[\beta\mu+C(\beta)]}\right] \text{ if } \beta \geq 1 \\ \bullet (b^a)' \circ a^*(\beta)\frac{1+r_J}{\delta(1-\beta)^2\psi''\circ a^*(\beta)} \\ -u\mathbb{E}\Big(\frac{\beta(1+r_J)}{\beta-1}(a^*)'(\beta) - \frac{\beta}{1-\beta}(e^*)'(\beta) - (b^e)'\circ e^*(\beta)(e^*)'(\beta) + \nu\Big)e^{u(Id-b^m)(\hat{m})} \\ e^{u[(Id-b^e)\circ e^*(\beta)-\delta\psi\circ a^*(\beta)+\beta(\mu-\varphi\circ e^*(\beta))+(1+r_J)(D+a^*(\beta))+C(\beta)]} \text{ if } \beta < 1 \end{cases}$$

where $\nu := \mu - h(v(1-\beta))$. Then we get

$$\frac{\partial}{\partial\beta}\tilde{I}^1(\alpha(\hat{\beta},\hat{m}),\hat{\beta},\hat{m}) = 0 \text{ iff } \begin{cases} h(u\beta) - h(v(1-\beta)) = 0 \text{ if } \beta \geq 1, \\ h(u\beta) - h(v(1-\beta)) = S(\beta) + \widetilde{S}(\beta) \text{ if } \beta < 1, \end{cases}$$

recalling

$$S(\beta) := \frac{\frac{\beta}{1-\beta} + (b^e)' \circ (\varphi')^{-1}\Big(\frac{1}{1-\beta}\Big)}{(1-\beta)^2\varphi'' \circ (\varphi')^{-1}\Big(\frac{1}{1-\beta}\Big)}.$$

$$\widetilde{S}(\beta) := \frac{1+r_J}{\delta(1-\beta)^2(\psi'')(a(\beta))}\left((1+r_J)\frac{\beta}{1-\beta} + \frac{(b^a)'(a(\beta))}{ue^{u(Id-b^m)(\hat{m})}k(\beta)}\right)$$

As seen in Lemma 2, for $\beta \geq 1$, $h(u\beta) - h(v(1-\beta)) > 0$, thus $\hat{\beta} < 1$ and we study the equation (28)

$$h(u\beta) - h(v(1-\beta)) = S(\beta) + \widetilde{S}(\beta), \text{ for } \beta < 1.$$

The left hand side is positive for $\beta > \frac{v}{u+v}$. The functions S and $\widetilde{S}$ are negative on $[0,1]$ thus $\hat{\beta} < \frac{v}{u+v}$.

2) $\underline{\delta < 0}$: $\frac{\partial}{\partial\beta}\tilde{I}^1(\alpha(\hat{\beta},\hat{m}),\hat{\beta},\hat{m}) =$

$$\begin{cases} \bullet (b^a)' \circ a^*(\beta)\frac{1+r_J}{\delta(1-\beta)^2\psi'' \circ a^*(\beta)} - u\mathbb{E}\Big[(\frac{\beta(1+r_J)}{\beta-1}(a^*)'(\beta) + \nu) \\ \times e^{u[(Id-b^m)(\hat{m}) - b^e(0) - \delta\psi\circ a^*(\beta) + \beta(\mu - \varphi(0)) + (1+r_J)(D+a^*(\beta)) + C(\beta)]}\Big] \text{ if } \beta \geq 1 \\ \\ \bullet - u\mathbb{E}\Big[\big(-\frac{\beta}{1-\beta}(e^*)'(\beta) - (b^e)' \circ e^*(\beta)(e^*)'(\beta) + \nu\big) \\ \times e^{u[(Id-b^m)(\hat{m}) + (Id-b^e)\circ e^*(\beta) - \delta\psi(0) + \beta(\mu - \varphi\circ e^*(\beta)) + (1+r_J)D + C(\beta)]}\Big] \text{ if } \beta < 1. \end{cases}$$

Thus,

$$\frac{\partial}{\partial\beta}\tilde{I}^1(\alpha(\hat{\beta},\hat{m}),\hat{\beta},\hat{m}) = 0 \text{ iff } \begin{cases} h(u\beta) - h(v(1-\beta)) = \widetilde{S}(\beta) \text{ if } \beta \geq 1 \\ h(u\beta) - h(v(1-\beta)) = S(\beta) \text{ if } \beta < 1. \end{cases}$$

Thus the optimal β on $\mathcal{B}$ is either the solution of equation (27) (which is less than $\frac{v}{u+v}$) or the solution of

$$h(u\beta) - h(v(1-\beta)) = \widetilde{S}(\beta), \text{ for } \beta \geq 1.$$

□

The next lemma is devoted to the optimisation over $\mathcal{C}$.

Lemma 8 *The supremum of $(\alpha,\beta,m) \mapsto \tilde{I}^1(\alpha,\beta,m)$ on $\mathcal{C}$ defined in (51) is attained.*

Proof Let us then consider $\sup_{\mathcal{C}} \tilde{I}^1$. We have $\alpha = 0$ on $\mathcal{C}$. Since $(b^m)'(\infty) = 0$ and, by Lemma 1, $G(m) = \frac{g}{g'}(m) - m$ is increasing on $[m_0, +\infty)$, the mapping $m \mapsto b^m - \frac{g(m)}{g'(m)}$ is decreasing for sufficiently large m. Thus there is a constant $\bar{m} \in [m_0, +\infty)$ not depending on $v > 0$ such that for any $\beta \in \mathbb{R}$, $m \mapsto \tilde{I}^1(0, \beta, m)$ is decreasing for $m \geq \bar{m}$. Therefore, writing

$$\mathcal{C}_1 := \mathcal{C} \cap \{m \in [0, \bar{m}]\} \quad \text{and} \quad \mathcal{C}_2 := \mathcal{C} \cap \{m \geq \bar{m} \text{ and } \tilde{J}^1(0, \beta, m) = -1\},$$

the supremum of $\tilde{I}^1$ over $\mathcal{C}$ is attained iff the supremum of $\tilde{I}^1$ over $\mathcal{C}_1 \cup \mathcal{C}_2$ is attained.

(i) We prove that $\mathcal{C}_1$ is compact. The condition $-\tilde{J}^1(0, \beta, m) \leq 1$ is equivalent to

$$C(\beta) - (1 - \beta)\delta\psi(a^*(\beta)) + e^*(\beta) + m - \frac{g}{g'}(m) + (1 + r_J)(D + a^*(\beta)) \leq 0.$$

If $\beta < 1$, by (38), it implies $e^*(\beta) + m - \frac{g}{g'}(m) + (1 + r_J)(D + a^*(\beta)) \leq 0$. Since $\lim_{\beta \to -\infty} e^*(\beta) = +\infty$ and $a^*(\beta) \geq 0$, while $m \in [0, \bar{m}]$, this implies that there exists $\underline{\beta} \in \mathbb{R}$ not depending on $v > 0$, such that $\mathcal{C}_1 \subset \{0\} \times [\underline{\beta}, \bar{\beta}] \times [0, \bar{m}]$, $\bar{\beta}$ being defined in Proposition 3. Since $\mathcal{C}_1$ is closed by continuity of $\tilde{J}^1$, it is compact and therefore the supremum over $\mathcal{C}_1$ is attained.

(ii) On $\mathcal{C}_2$, we have

$$C(\beta) - (1 - \beta)\delta\psi(a^*(\beta)) + e^*(\beta) + (1 + r_J)(D + a^*(\beta)) = \frac{g(m)}{g'(m)} - m = G(m).$$

Since $m \geq \bar{m} \geq m_0$, using Lemma 1 and the inverse function G^{-1}, $m = H(\beta) := G^{-1}\big[C(\beta) - (1 - \beta)\delta\psi(a^*(\beta)) + e^*(\beta) + (1 + r_J)(D + a^*(\beta))\big]$. Thus we have $\sup_{\mathcal{C}_2} \tilde{I}^1 = \sup_{\mathcal{C}_2 \cap \{\beta \in (-\infty, \bar{\beta}]\}} \tilde{I}^1\big(0, \beta, H(\beta)\big)$.
Let us now prove that $\tilde{I}^1(0, \beta, H(\beta)) \to -\infty$ uniformly in $v > 0$ when $\beta \to -\infty$. When $\beta \to -\infty$, $e^*(\beta) \to \infty$, while $a^*(\beta) \geq 0$, $C(\beta) - (1 - \beta)\delta\psi(a^*(\beta)) \geq 0$ by (38), so that $C(\beta) - (1 - \beta)\delta\psi(a^*(\beta)) + e^*(\beta) + (1 + r_J)(D + a^*(\beta)) \to \infty$ uniformly in $v > 0$, and therefore $m = H(\beta) \to \infty$. We recall that $b^m(m) = \circ(m)$, $a^*(\beta) = \circ(-\beta)$ and $e^*(\beta) = \circ(-\beta)$ (see Lemma 4) and $b^e \circ e^*(\beta) = \circ(-\beta)$, when $\beta \to -\infty$. Using

$$\frac{g \circ H(\beta)}{g' \circ H(\beta)} - H(\beta) = C(\beta) - (1 - \beta)\delta\psi(a^*(\beta)) + e^*(\beta) + (1 + r_J)(D + a^*(\beta)),$$

we deduce that for all $v > 0$,

$$\begin{aligned} &-u\left(b^m \circ H(\beta) + b^e(e^*(\beta)) - \beta(\mu\ - \varphi(e^*(\beta)) - \delta\psi(a^*(\beta))) - \frac{g \circ H(\beta)}{g' \circ H(\beta)}\right) \\ &= -u\big(b^m \circ H(\beta) - H(\beta) + (b^e - Id)(e^*(\beta)) - \beta(\mu\ - \varphi(e^*(\beta)) - \delta\psi(a^*(\beta))) \\ &\qquad - C(\beta) + (1 - \beta)\delta\psi(a^*(\beta)) - (1 + r_J)(D + a^*(\beta))\big) \qquad (55) \\ &\geq -u\beta\left(\mathbb{E}[\mu]\ - \mu\ \right) + o(-\beta), \text{ as } -\beta \to \infty, \qquad (56) \end{aligned}$$

where the last inequality follows from (39) , $o(-\beta)$ being uniform in $v > 0$. Since μ is not $\mathbb{P}$ constant,

$$\exists \varepsilon > 0, \exists \mathcal{D} \subset \Omega \text{ s.t. } \mathbb{P}(\mathcal{D}) \geq \varepsilon \text{ and } \mu + \varepsilon \leq \mathbb{E}[\mu] \text{ on } \mathcal{D}. \tag{57}$$

Using (56), we compute for $-\beta$ sufficiently large not depending on $v > 0$:

$$\tilde{I}^1(0, \beta, H(\beta)) \leq b^a(a^*(\beta)) - \mathbb{E}[1_{\mathcal{D}} e^{-\frac{u\beta}{2}(\mathbb{E}[\mu] - \mu)}] \leq b^a(a^*(\beta)) - \varepsilon e^{-\frac{u\beta\varepsilon}{2}}.$$

Since $b^a(a^*(\beta)) = o(-\beta)$ when $\beta \to -\infty$, the right-hand side goes to $-\infty$ uniformly in $v > 0$ when $\beta \to -\infty$. Since $(0, 1, G^{-1}((1 + r_J)D)) \in \mathcal{C}$ for any $v > 0$ and $\tilde{I}^1(0, 1, G^{-1}((1 + r_J)D))$ does not depend on $v > 0$, one deduces the existence of $\underline{\beta} > -\infty$ such that if $\sup_{\mathcal{C}} \tilde{I}^1 = \sup_{\mathcal{C}_2} \tilde{I}^1$ then $\sup_{\mathcal{C}} \tilde{I}^1 = \sup_{\mathcal{C}_2 \cap \{\beta \in [\underline{\beta}, \bar{\beta}]\}} \tilde{I}^1$. Now

$$\begin{aligned}
\sup_{\beta \in [\underline{\beta}, \bar{\beta}]} \tilde{I}^1(0, \beta, m) \leq & \sup_{\beta \in [\underline{\beta}, \bar{\beta}]} b^a(a^*(\beta)) \\
& - e^{u\left(\frac{g}{g'}(m) - b^m(m)\right)} \inf_{\beta \in [\underline{\beta}, \bar{\beta}]} \mathbb{E}\left[e^{-u\left(b^e(e^*(\beta)) - \beta(\mu - \varphi(e^*(\beta)) - \delta\psi(a^*(\beta)))\right)}\right]
\end{aligned}$$

where, by (48), the right-hand-side tends to $-\infty$ as $m \to \infty$. Hence there exists $\bar{m} \in [0, +\infty)$ not depending on $v > 0$ such that if $\sup_{\mathcal{C}} \tilde{I}^1 = \sup_{\mathcal{C}_2} \tilde{I}^1$ then $\sup_{\mathcal{C}} \tilde{I}^1 = \sup_{\mathcal{C}_2 \cap \{\beta \in [\underline{\beta}, \bar{\beta}], m \in [0, \bar{m}]\}} \tilde{I}^1$. □

To conclude the proof of Theorem 5, Lemmas 5, 6 and 8 prove that the maximum of $\tilde{I}^1$ over $\mathcal{A}$ is attained at $(\hat{\alpha}, \hat{\beta}, \hat{m})$ which belongs either to $\mathcal{B}$ or to $\mathcal{C}$, and a Stackelberg equilibrium exists. Moreover if $\hat{\alpha} > 0$, then $(\hat{\alpha}, \hat{\beta}, \hat{m}) \in \mathcal{B}$ and the maximum of $\tilde{I}^1$ over $\mathcal{B}$ is attained at $(\hat{\alpha}, \hat{\beta}, \hat{m})$. The equilibrium characterization given in the statement of Theorem 5 then follows from Lemma 7. □

6 Proofs in Situation 2

6.1 Best responses in Situation 2

Let $e \geq 0$ and $m \geq 0$ be given and constant. We recall that $F(a) = b^a(a) - f((1 + r_I)(D + a))$ is assumed to be strictly concave, $F'(0) > 0$, possibly infinite, and $F'(\infty) = -\infty$. We introduce

$$K_a(\alpha, \beta, \gamma) := -\mathbb{E}\left[e^{-u\left(b^m(m) + b^e(e) - \alpha - \beta C^{op} - \gamma g(m)\right)}\right]$$

Then we get the following optimization problem:

$$\begin{aligned}
& \sup_{a, \alpha, \beta, \gamma} F(a) + K_a(\alpha, \beta, \gamma) \text{ with } a \geq 0,\ \alpha \geq 0,\ \gamma \geq 0, \\
& \text{and} \qquad \mathbb{E}\left[e^{-v\left(\alpha + (\beta - 1)(\mu - \varphi(e) - \delta\psi(a)) - e + \gamma g(m) - m\right)}\right] \leq 1.
\end{aligned} \tag{58}$$

We have the following result:

Proposition 4 *Let $e \geq 0$ and $m \geq 0$ be given and constant and let $\beta^* := \frac{v}{u+v}$. There exists optimal controls and $(a, \alpha, \beta, \gamma)$ is optimal if and only if it satisfies: $\beta = \beta^*$, $a \in \arg\max_{a \geq 0} \left[F(a) - e^{-u\delta\psi(a)} e^{-u(b^m(m)+b^e(e)-e-m)} e^{(u+v)C_e(\beta^*)} \right]$ and*

$$\alpha + \gamma g(m) = C_e(\beta^*) - \frac{u}{u+v}\delta\psi(a) + e + m \tag{59}$$

with $\alpha \geq 0$ and $\gamma \geq 0$ and $C_e(\beta) := \frac{1}{v} \ln \mathbb{E}\left[e^{v(1-\beta)(\mu-\varphi(e))} \right]$.
Last, if $\delta \geq 0$, then $a^* > 0$ and is unique.

Remark 5 Notice that $\beta^* \in (0,1)$ and any optimal control $(a, \alpha, \beta, \gamma)$ is such that $\alpha + \gamma > 0$.

Proof Let $a \geq 0$ be given for the moment and

$$E_a := \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3;\ \mathbb{E}\left[e^{-v\left(\alpha + (\beta-1)C^{op} - e + \gamma g(m) - m\right)} \right] \leq 1 \right\}.$$

In comparison with the definition of the set E introduced in the proof of Proposition 1, the term $(1 + r_J)(D + a)$ does not appear in the constraint defining the set E_a.
We will first solve, for fixed a, the problem of maximization of K on E_a, following the same steps as in Proposition 1 : the optimal $(\alpha^*, \beta^*, \gamma^*)$ for the problem

$$\sup_{(\alpha,\beta,\gamma) \in E_a \cap (\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)} K(\alpha, \beta, \gamma),$$

are exactly the elements of

$$\left\{ (\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+;\ \beta = \frac{v}{u+v},\ \alpha + \gamma g(m) = C_e(\beta) - \frac{u}{u+v}\delta\psi(a) + e + m \right\}$$

and the constraint is always binding. Using $\beta^* = \frac{v}{u+v}$, we get $\mathbb{E}[e^{u\beta^*(\mu-\varphi(e))}] = e^{vC_e(\beta^*)}$ so that, with the equality $\alpha^* + \gamma^* g(m) = C_e(\beta^*) - \frac{u}{u+v}\delta\psi(a) + e + m$,

$$K_a(\alpha^*, \beta^*, \gamma^*) = -e^{-u\delta\psi(a)} e^{-u(b^m(m)+b^e(e)-e-m)} e^{(u+v)C_e(\beta^*)}.$$

Let us then consider $\sup_{a \geq 0} F(a) + K_a(\alpha^*, \beta^*, \gamma^*)$. Since $F(a) + K_a(\alpha^*, \beta^*, \gamma^*) \leq F(a)$ and $\lim_{a \to \infty} F(a) = -\infty$, $a \mapsto F(a) + K_a(\alpha^*, \beta^*, \gamma^*)$ attains its maximum on $\mathbb{R}_+$, either at $a^* = 0$ or at a point where the first order condition is satisfied. If $\delta \geq 0$, this function is strictly concave, increasing for small a, so that there exists a unique maximum $a^* > 0$ that is the unique solution of

$$F'(a) - u\delta\psi'(a) K_a(\alpha^*, \beta^*, \gamma^*) = 0.$$

□

Remark 6 If $\delta < 0$, then the function $a \mapsto F(a) + K_a(\alpha^*, \beta^*, \gamma^*)$ is decreasing for small a, since $\psi'(0) = +\infty$, so that $a = 0$ is a local maximum. So the maximum is attained either at $a^* = 0$ or at a solution of $F'(a) + u\delta B_e(\beta^*, m)\psi'(a) e^{-u\delta\psi(a)} = 0$.

6.2 Nash equilibrium

From the previous results, the proof of Theorem 2 follows easily.

Proof of Theorem 2

The characterization conditions (17), (18) and (19) follow from Proposition 4 and the optimal expressions (11). Thus, the only thing to check is that there exists an infinite number of solutions to these equations. Since $\hat{\beta} < 1$, by (38), $C(\hat{\beta}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}) \geq \hat{e} > 0$. Therefore there exists infinitely many couples $(\alpha, m) \in \mathbb{R}_+^2$ such that

$$\alpha + G(m) = C(\hat{\beta}) + +\hat{e} - \frac{u}{u+v}\delta\psi(\hat{a})$$

namely the couples $\left(C(\hat{\beta}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}) - G(x), x\right)$ where $x \in [0, G^{-1}(C(\hat{\beta}) + \hat{e} - \frac{u}{u+v}\delta\psi(\hat{a}))]$. □

6.3 Stackelberg equilibrium in Situation 2, firm I is leader

The best response for firm J is given by (11), but now the optimization problem for firm I has changed. We recall the continuous mappings $m^* : \mathbb{R}_+ \to \mathbb{R}_+$, $e^* : \mathbb{R} \to \mathbb{R}_+$, $C : \mathbb{R} \to \mathbb{R}$ and $\tilde{B} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined in (11) (12) and (14),

$$e^*(\beta) = (\varphi')^{-1}\left(\tfrac{1}{(1-\beta)^+}\right),\ m^*(\gamma) = (g')^{-1}(1/\gamma).$$
$$C(\beta) = \tfrac{1}{v}\ln\mathbb{E}\left[e^{v(1-\beta)(\mu\ -\varphi\circ e^*(\beta))}\right].$$

We are now ready to prove the existence of a Stackelberg equilibrium, as stated in Theorem 6.

Proof of Theorem 6.

Given β and γ, the optimal controls for firm J are given by e^* and m^*. Once again, since $\gamma = \frac{1}{g'(m)}$ yields a bijection between m and γ on $\mathbb{R}^+$, we only deal with m. Writing:

$$\tilde{I}^2(a,\alpha,\beta,m) :=$$
$$F(a) - \mathbb{E}\left[e^{-u\left(b^m(m)+b^e(e^*(\beta))-\alpha-\beta(\mu\ -\varphi(e^*(\beta))-\delta\psi(a))-\frac{g}{g'}(m)\right)}\right],$$
$$\tilde{J}^2(a,\alpha,\beta,m) := -\mathbb{E}\left[e^{-v\left(\alpha+(\beta-1)(\mu\ -\varphi(e^*(\beta))-\delta\psi(a))-e^*(\beta)+\frac{g}{g'}(m)-m\right)}\right],$$
$$\mathcal{A} := \left\{(a,\alpha,\beta,m) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+;\ \ -\tilde{J}^2(a,\alpha,\beta,m) \leq 1\right\},$$

the optimization problem for firm I then writes:

$$\sup_{(a,\alpha,\beta,m)\in\mathcal{A}} \tilde{I}^2(a,\alpha,\beta,m).$$

We will prove the existence of a maximizer for this problem, and therefore of a Stackelberg equilibrium, by proving that we can restrict the set $\mathcal{A}$ to a compact subset. Notice first that $\mathcal{A} \neq \emptyset$. Indeed, since $\tilde{J}^2(a,\alpha,1,0) = -e^{-v\alpha}$, for any $a \geq 0$ and $\alpha > 0$, $(a,\alpha,1,0) \in \mathcal{A}$. In fact, one can easily check that for any $a \geq 0$, $\beta \in \mathbb{R}$

and $m \geq 0$, one can choose α large enough so that $(a, \alpha, \beta, m) \in \mathcal{A}$.

The proof will use the following lemmas, very similar to Proposition 3 and Lemmas 5, 6, 7 and 8. Nevertheless, we cannot deduce them from the previous ones because the involved functions are not defined on the same spaces.

Lemma 9 *There exists $\bar{a} \in (0, +\infty)$ and $\bar{\beta} \in \mathbb{R}$ not depending on v such that the supremum over $\mathcal{A}$ is attained if and only if the supremum over $\{(a, \alpha, \beta, m) \in \mathcal{A}$ such that $a \in [0, \bar{a}], \beta \in (-\infty, \bar{\beta}]\}$ is attained.*

Proof For any $v > 0$ and any $(a, \alpha) \in \mathbb{R}_+ \times \mathbb{R}_+$, $(a, \alpha, 1, 0) \in \mathcal{A}$. We have $\tilde{I}^2(a, \alpha, \beta, m) \leq F(a)$. Since $\lim_{x \to \infty} F(x) = -\infty$ (cf. the beginning of Subsection 6.1) there exists $\bar{a} > 0$ not depending on v, such that the supremum over $\mathcal{A}$ is the same as the supremum over $\mathcal{A} \cap \{a \in [0, \bar{a}]\}$. Notice that $\tilde{I}^2$ is decreasing with respect to α, so that $\tilde{I}^2(a, \alpha, \beta, m) \leq \tilde{I}^2(a, 0, \beta, m)$. Using (48), we get $e^{u\left(\frac{g}{g'}(m) - b^m(m)\right)} \to +\infty$ when $m \to +\infty$. Therefore, there exists a constant $c > 0$ such that for any $m \geq 0$, $e^{u\left(\frac{g}{g'}(m) - b^m(m)\right)} \geq c$.
On the other hand, for any $\beta \geq 1$, since $e^*(\beta) = 0$, we compute:

$$\tilde{I}^2(a, \alpha, \beta, m) \leq \tilde{I}^2(a, 0, \beta, m) \leq F(a) - c\, e^{-ub^e(0)} \mathbb{E}\left[e^{u\beta(\mu\ -\varphi(0) - \delta\psi(a))}\right]$$

which goes to $-\infty$ when $\beta \to \infty$, uniformly in $a \in [0, \bar{a}]$. □

Lemma 10 *Let*

$$\mathcal{B} := \left\{(a, \alpha, \beta, m) \in [0, \bar{a}] \times \mathbb{R}_+ \times (-\infty, \bar{\beta}] \times \mathbb{R}_+;\ \tilde{J}^2(a, \alpha, \beta, m) = -1\right\}, \quad (60)$$

$$\mathcal{C} := \left\{(a, 0, \beta, m);\ a \in [0, \bar{a}],\ \beta \in (-\infty, \bar{\beta}],\ m \geq 0,\ \tilde{J}^2(a, 0, \beta, m) \geq -1\right\}. \quad (61)$$

The supremum over $\mathcal{A}$ is attained if and only if the supremum over $\mathcal{B} \cup \mathcal{C}$ is attained.

Proof Since $\tilde{I}^2$ is decreasing with respect to α and $\tilde{J}^2$ is continuous, if $\alpha > 0$ and $-\tilde{J}^2(a, \alpha, \beta, m) < 1$, then there exists $\varepsilon > 0$ such that $\tilde{I}^2(a, \alpha - \varepsilon, \beta, m) > \tilde{I}^2(a, \alpha, \beta, m)$, while $-\tilde{J}^2(a, \alpha - \varepsilon, \beta, m) \leq 1$. Thus in case of optimum satisfying $\alpha > 0$, the constraint is binding. □

Lemma 11 *In set $\mathcal{B}$, α is a continuous function of (a, β, γ), so there exists a compact $\mathcal{K} \subset \mathcal{B}$ not depending on $v > 0$ such that $\sup_{\mathcal{B}} \tilde{I}^2 = \sup_{\mathcal{K}} \tilde{I}^2$, and in particular the maximum on $\mathcal{B}$ is attained.*

Proof Let us consider $\sup_{\mathcal{B}} \tilde{I}^2$. Notice that C defined by (12) depends on v. Since $\tilde{J}^2(a, \alpha, \beta, m) = -1$ on $\mathcal{B}$, we have

$$\alpha = \alpha(a, \beta, m) := C(\beta) - (1 - \beta)\delta\psi(a) + e^*(\beta) + m - \frac{g}{g'}(m),$$

therefore we have:

$$\sup_{(a,\alpha,\beta,m)\in\mathcal{B}} \tilde{I}^2(a,\alpha,\beta,m)$$
$$= \sup_{\{(a,\beta,m)\in[0,\bar{a}]\times(-\infty,\bar{\beta}]\times\mathbb{R}_+;\ \alpha(a,\beta,m)\geq 0\}} F(a) - e^{-u\delta\psi(a)}\tilde{B}(\beta,m)$$
$$= \sup_{\{(a,\beta,m)\in[0,\bar{a}]\times(-\infty,\bar{\beta}]\times\mathbb{R}_+;\ \alpha(a,\beta,m)\geq 0\}} F(a) - e^{u(Id-b^m)(m)}e^{-u\delta\psi(a)}\tilde{k}(\beta) \quad (62)$$

with

$$\tilde{k}(\beta) = e^{u(Id-b^e)\circ e^*(\beta)}e^{uC(\beta)}\mathbb{E}\left[e^{u\beta\left(\mu\ -\varphi\circ e^*(\beta)\right)}\right].$$

In (62), we have $F(a)$ minus the product of three positive functions, the first one depending only on m, the second one only on a and the third one, $\tilde{k}$, only on β. By (48), the first function $e^{u(m-b^m(m))}$ goes to ∞ when $m\to\infty$ so it is bounded from below by a positive constant. Since $\delta^+\psi$ is bounded form above, the second one is also bounded from below by a positive constant.
As in Lemma 6, we prove that $\tilde{k}(\beta)\mapsto\infty$ uniformly in $v>0$ when $\beta\mapsto-\infty$. Since $(0,0,1,0)\in\mathcal{B}$ for any $v>0$, we deduce the existence of $\bar{m}\in[0,+\infty)$ and $\underline{\beta}\in(-\infty,\bar{\beta}]$ not depending on $v>0$ such that $\sup_{\mathcal{B}}\tilde{I}^2=\sup_{\mathcal{B}\cap\{\beta\in[\underline{\beta},\bar{\beta}],m\in[0,\bar{m}]}\tilde{I}^2$.To conclude the proof, we remark that

$$\sup_{a\in[0,\bar{a}],\beta\in[\underline{\beta},\bar{\beta}],m\in[0,\bar{m}]}\tilde{I}^2(a,\alpha,\beta,m)\leq\sup_{a\in[0,\bar{a}]}F(a)$$
$$-e^{u\alpha}\inf_{a\in[0,\bar{a}],\beta\in[\underline{\beta},\bar{\beta}],m\in[0,\bar{m}]\}}\mathbb{E}\left[e^{-u\left(b^m(m)+b^e(e^*(\beta))-\beta(\mu\ -\varphi(e^*(\beta))-\delta\psi(a))-\frac{g}{g'}(m)\right)}\right]$$

tends to $-\infty$ as $\alpha\to\infty$. □

Lemma 12 *Assume that there exists a Stackelberg equilibrium $(\hat{e},\hat{m},\hat{a},\hat{\alpha},\hat{\beta},\hat{\gamma})$ with $\hat{\alpha}>0$. Then necessarily*

$$\hat{\gamma}=\tfrac{1}{g'(\hat{m})},\ \textit{where}\ \hat{m}=[(b^m)']^{-1}(1)>0, \quad (63)$$
$$\hat{\beta}<\tfrac{v}{u+v}\ \textit{and}\ \hat{\beta}\ \textit{is a solution of (27).} \quad (64)$$

Proof If there exists a Stackelberg equilibrium $(\hat{e},\hat{m},\hat{a},\hat{\alpha},\hat{\beta},\hat{\gamma})$, then the supremum of $\tilde{I}^2$ on $\mathcal{A}$ is attained at $(\hat{a},\hat{\alpha},\hat{\beta},\hat{\gamma})$. Lemma 10 and the hypothesis $\hat{\alpha}>0$ yield that $\tilde{J}^2(\hat{a},\hat{\alpha},\hat{\beta},\hat{\gamma})=-1$. Since $(b^m)'(0)=+\infty$, $m\mapsto m-b^m(m)$ is decreasing in a neighborhood of 0, so that from (62), $\hat{m}>0$. The mapping $(\beta,m)\mapsto\tilde{I}^2(\hat{a},\alpha(\hat{a},\beta,m),\beta,m)$ where $\alpha(a,\beta,m)$ is the function introduced at the beginning of the proof of Lemma 11 admits a local maximum at $(\hat{\beta},\hat{m})$. Therefore the first order conditions are satisfied (notice that $\bar{\beta}$ may be increased), i.e. $\frac{\partial}{\partial\beta}\tilde{I}^2(\hat{a},\alpha(\hat{a},\hat{\beta},\hat{m}),\hat{\beta},\hat{m})=\frac{\partial}{\partial m}\tilde{I}^2(\hat{a},\alpha(\hat{a},\hat{\beta},\hat{m}),\hat{\beta},\hat{m})=0$. The partial derivative with respect to m yields

$$\hat{m}=[(b^m)']^{-1}(1)>0.$$

On the other hand, using $e^*(\beta))=(\varphi')^{-1}(\frac{1}{(1-\beta)^+})$, we compute:

$$C'(\beta)=\begin{cases}-\frac{\mathbb{E}\left[\mu\ e^{v(1-\beta)\mu}\ \right]}{\mathbb{E}\left[e^{v(1-\beta)\mu}\ \right]}=\varphi(0)-h(v(1-\beta))\text{ if }\beta\geq 1\\ \varphi\circ e^*(\beta)-(e^*)'(\beta)-h(v(1-\beta))\text{ if }\beta<1.\end{cases}$$

Then:

$$\tilde{k}'(\beta) = \begin{cases} \mathbb{E}\left[u\Big(\mu + \varphi(0) \ - h(v(1-\beta))\Big)e^{u\left(\beta\mu \ + C(\beta)\right)}\right] \ \text{if } \beta \geq 1 \\ u\mathbb{E}\Big[\Big(- \frac{\beta}{1-\beta}(e^*)'(\beta) - (b^e)' \circ e^*(\beta)(e^*)'(\beta) + \mu \ - h(v(1-\beta))\Big) \\ \qquad e^{u\left((Id - b^e)\circ e^*(\beta) + \beta(\mu \ - \varphi \circ e^*(\beta)) + C(\beta)\right)}\Big] \ \text{if } \beta < 1. \end{cases}$$

As seen in Lemma 2, for $\beta \geq 1$, $h(u\beta) - h(v(1-\beta)) > 0$, therefore $\tilde{k}'(\beta) > 0$ for $\beta \geq 1$. As a consequence, the equation $\frac{\partial}{\partial \beta}\tilde{I}^2 = \tilde{k}'(\beta) = 0$ implies that $\hat{\beta}$ solves (27). By Proposition 2, $\hat{\beta} < \frac{u}{u+v}$. □

Lemma 13 *The supremum of $(a, \alpha, \beta, m) \mapsto \tilde{I}^2(a, \alpha, \beta, m)$ on $\mathcal{C}$ defined by (61) is attained.*

Proof Like in the proof of Lemma 8, there exists $\bar{m} \geq m_0$ not depending on $v > 0$ such that $\tilde{I}^2$ is decreasing with respect to m on $[\bar{m}, +\infty)$. Thus in case $m \geq \bar{m}$, the optimum has to bind the constraint. Therefore, writing

$$\mathcal{C}_1 := \{(a, 0, \beta, m) \in \mathcal{C} : m \in [0, \bar{m}]\},$$

$$\mathcal{C}_2 := \{(a, 0, \beta, m) \in \mathcal{C} : m \geq \bar{m} \text{ and } \tilde{J}^2(a, 0, \beta, m) = -1\},$$

the supremum of $\tilde{I}^2$ over $\mathcal{C}$ is attained iff the supremum of $\tilde{I}^2$ over $\mathcal{C}_1 \cup \mathcal{C}_2$ is attained.

(i) We prove that $\mathcal{C}_1$ is compact. The condition $-\tilde{J}^2(a, 0, \beta, m) \leq 1$ is equivalent to $C(\beta) - (1-\beta)\delta\psi(a) + e^*(\beta) + m - \frac{g}{g'}(m) \leq 0$. For $\beta < 1$, by (38), it implies $e^*(\beta) + m - \frac{g}{g'}(m) \leq 0$. Since $\lim_{\beta \to -\infty} e^*(\beta) = +\infty$, while $m \in [0, \bar{m}]$, this implies the existence of $\underline{\beta} \in (-\infty, \bar{\beta}]$ such that $\mathcal{C}_1 \subset [0, \bar{a}] \times \{0\} \times [\underline{\beta}, \bar{\beta}] \times [0, \bar{m}]$ where $\bar{m}$ and $\underline{\beta}$ do not depend on v. Since $\mathcal{C}_1$ is closed by continuity of $\tilde{J}^2$, it is compact and therefore the supremum over $\mathcal{C}_1$ is attained.

(ii) On $\mathcal{C}_2$, we have $C(\beta) - (1-\beta)\delta\psi(a) + e^*(\beta) = \frac{g(m)}{g'(m)} - m = G(m)$ with $m \geq \bar{m} \geq m_0$ and therefore, by Lemma 1, $m = H(a, \beta) := G^{-1}\big(C(\beta) - (1-\beta)\delta\psi(a) + e^*(\beta)\big)$. We thus have $\sup_{\mathcal{C}_2} \tilde{I}^2 = \sup_{\mathcal{C}_2 \cap \{a \in [0, \bar{a}], \beta \in (-\infty, \bar{\beta}]\}} \tilde{I}^2\big(a, 0, \beta, H(a, \beta)\big)$. Adapting the proof of Lemma 8, we check that when $\beta \to -\infty$, $\tilde{I}^2(a, 0, \beta, H(a, \beta)) \to -\infty$ uniformly in $(a, v) \in [0, \bar{a}] \times (0, +\infty)$.
Since $(0, 0, 1, G^{-1}(0)) \in \mathcal{C}$ for any $v > 0$ and $\tilde{I}^2(0, 0, 1, G^{-1}(0))$ does not depend on $v > 0$, one deduces the existence of $\underline{\beta} \in (-\infty, \bar{\beta}]$ not depending on $v > 0$ such that if $\sup_{\mathcal{C}} \tilde{I}^2 = \sup_{\mathcal{C}_2} \tilde{I}^2$ then $\sup_{\mathcal{C}} \tilde{I}^2 = \sup_{\mathcal{C}_2 \cap \{\beta \in [\underline{\beta}, \bar{\beta}]\}} \tilde{I}^2$. Now

$$\sup_{a \in [0, \bar{a}], \beta \in [\underline{\beta}, \bar{\beta}]} \tilde{I}^2(a, 0, \beta, m) \leq \sup_{a \in [0, \bar{a}]} F(a)$$
$$- e^{u\left(\frac{g}{g'}(m) - b^m(m)\right)} \inf_{a \in [0, \bar{a}], \beta \in [\underline{\beta}, \bar{\beta}]} \mathbb{E}\left[e^{-u\left(b^e(e^*(\beta)) - \beta(\mu \ - \varphi(e^*(\beta)) - \delta\psi(a))\right)}\right]$$

where, by (48), the right-hand-side tends to $-\infty$ as $m \to \infty$. Hence there exists $\bar{m} \in [0, +\infty)$ not depending on $v > 0$ such that if $\sup_{\mathcal{C}} \tilde{I}^2 = \sup_{\mathcal{C}_2} \tilde{I}^2$ then $\sup_{\mathcal{C}} \tilde{I}^2 = \sup_{\mathcal{C}_2 \cap \{\beta \in [\underline{\beta}, \bar{\beta}], m \in [0, \bar{m}]\}} \tilde{I}^2$. □

End of the proof of Theorem 6.
In conclusion, the maximum over $\mathcal{A}$ is attained, either on $\mathcal{B}$ or $\mathcal{C}$, and a Stackelberg equilibrium exists. Moreover if $\alpha > 0$, the characterization given in the statement of the theorem follows from Lemma 12. The last assertion follows from the end of the proof of Proposition 4. □

Remark 7 (i) Notice that $\tilde{I}^2$ is not in general concave w.r.t β or γ. Consider for example $b^m(x) = x^{0.8}/0.8$ and $g(x) = x^{0.5}/0.5$.
(ii) Assume that equation (27) admits a unique solution, namely $\hat{\beta}$. Then, from Lemma 12, any Stackelberg equilibrium $(\hat{e}, \hat{m}, \hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ with $\hat{\alpha} > 0$ satisfies : $\hat{e} = (\varphi')^{-1}\Big(\frac{1}{1-\hat{\beta}}\Big)$, $\hat{m} = \big[(b^m)'\big]^{-1}(1)$, $\hat{\gamma} = 1/g'(\hat{m})$,
$\hat{a} \in \arg\max_{a\geq 0} F(a) - e^{-u\delta\psi(a)}\tilde{B}(\hat{\beta}, \hat{m})$, $\hat{\alpha} = C(\hat{\beta}) - (1-\hat{\beta})\delta\psi(\hat{a}) + \hat{e} + \hat{m} - \hat{\gamma}g(\hat{m})$, where C and $\tilde{B}$ are respectively defined by (12) and (14).
In particular, if $\delta \geq 0$, an adaptation of the end of the proof of Proposition 4 proves that $\hat{a}$ is unique.

7 Proofs of Comparison results

7.1 Comparison between both Nash equilibria

We summarize Nash equilibria in both situations : in Situation 1, the Nash equilibria is characterized by

$$\hat{\beta} = \frac{v}{u+v},\ \hat{e} = (\varphi')^{-1}(\frac{u+v}{u}),\ \hat{a}_1 = (\psi')^{-1}\left(\frac{(u+v)(1+r_J)}{\delta^+ u}\right),\ \hat{m} \geq 0,$$

$$\hat{\gamma} = \frac{1}{g'(\hat{m})},\ \hat{\alpha}_1 + \frac{g(\hat{m})}{g'(\hat{m})} + \frac{u}{u+v}\delta\psi(\hat{a}_1) = (1+r_J)(D+\hat{a}_1) + \hat{m} + \hat{e} + C(\hat{\beta})$$

which leads to the optimal value for firm I:

$$\hat{I}^1(\hat{m}) = b^a(\hat{a}_1) - B(\hat{m})e^{u[(1+r_J)(D+\hat{a}_1)-\delta\psi(\hat{a}_1)]},$$

where we recall

$$B(\hat{m}) = e^{-u(b^m(\hat{m})-\hat{m}+b^e(\hat{e})-\hat{e})}e^{(u+v)C(\hat{\beta})},$$

$$e^{vC(\hat{\beta})} = \mathbb{E}\left[\exp(\frac{uv}{u+v}(\mu\ - \varphi(\hat{e})))\right].$$

In Situation 2, the Nash equilibria are characterized by

$$\hat{\beta} = \frac{v}{u+v},\ \hat{a}_2 \in \arg\max_{a\geq 0}\Big(b^a(a) - f[(1+r_I)(D+a)] - e^{-u\delta\psi(a)}B(\hat{m})\Big),$$

$$\hat{e} = (\varphi')^{-1}(\frac{u+v}{u}),\ \hat{\alpha}_2 + \frac{g(\hat{m})}{g'(\hat{m})} + \frac{u}{u+v}\delta\psi(\hat{a}_2) = \hat{m} + \hat{e} + C(\hat{\beta}),\ \hat{m} \geq 0,\ \hat{\gamma} = \frac{1}{g'(\hat{m})},$$

which leads to the optimal value for firm I:

$$\hat{I}^2(\hat{m}) = b^a(\hat{a}_2) - f[(1+r_I)(D+\hat{a}_2)] - B(\hat{m})e^{-u\delta\psi(\hat{a}_2)}.$$

The following proposition gives the monotonicity of the optimal initial effort $\hat{a}_1$ in Situation 1 (respectively $\hat{a}_2$ in Situation 2) as a function of the interest rate r_J (respectively r_I).

Proposition 5 *The application $r_J \mapsto \hat{a}_1(r_J)$ is non increasing.*
At least in case $\delta > 0$, the application $r_I \mapsto \hat{a}_2(r_I)$ is non increasing.

Proof The monotonicity property of the function $r_J \mapsto \hat{a}_1(r_J)$ is a trivial consequence of the definition of $\hat{a}_1$, since by hypothesis the function ψ' is non increasing. In case $\delta < 0$, $\hat{a}_1(r_J) = 0$.
The second assertion is a consequence of the characterization of $\hat{a}_2(r_I)$ as the unique solution of the equation (see the end of the proof of Proposition 4) $H(\hat{a}_2(r_I), r_I) = 0$ where

$$H(a, r_I) = (b^a)'(a) + u\delta\psi'(a)B(\hat{m})e^{-u\delta\psi(a)} - (1 + r_I)f'[(1 + r_I)(D + a)]. \quad (65)$$

Thus $\frac{d\hat{a}_2}{dr_I}(r_I) = -\frac{\partial_{r_I} H}{\partial_a H}(\hat{a}_2(r_I), r_I)$. Notice that

$$\partial_{r_I} H = -f'[(1 + r_I)(D + a)] - (1 + r_I)(D + a)f''[(1 + r_I)(D + a)] < 0$$

since f is convex non decreasing. Therefore the sign of $\frac{d\hat{a}_2}{dr_I}$ is the one of $\partial_a H =$

$$(b^a)''(a) + u\delta B(\hat{m})e^{-u\delta\psi(a)}[\psi''(a) - u\delta(\psi'(a))^2] - (1+r_I)^2 f''[(1+r_I)(D+a)] < 0 \quad (66)$$

since b^a and ψ are concave and f is convex. □

We now prove a sufficient condition under which the best situation is the second one (debt issuance), as stated in Theorem 3.

Proof of Theorem 3
The key of the proof is the remark that, since in Situation 2 a is a control of firm I, the optimal value $\hat{I}^2(\hat{m})$ obtained by this firm in any Nash equilibrium $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m}, \hat{a}_2(r_I))$ is larger than $I^2(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m}, \hat{a}_1(r_J))$. Hence

$$\begin{aligned}\hat{I}^1(\hat{m}) - \hat{I}^2(\hat{m}) &\leq \hat{I}^1(\hat{m}) - I^2(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m}, \hat{a}_1(r_J)) \\ &= f((1 + r_I)(D + \hat{a}_1(r_J))) - B(\hat{m})e^{-u\delta\psi(\hat{a}_1(r_J))}(e^{u(1+r_J)(D+\hat{a}_1(r_J))} - 1).\end{aligned}$$

Condition (20) is equivalent to non-positivity of the right-hand side. □

In the case $\delta > 0$, we now prove a sufficient condition under which the best situation is the first one (outsourcing), as stated in Theorem 4.

Proof of Theorem 4
Let $\hat{m}$ be the same parameter in both situations. By a slight abuse of notations, we introduce the function

$$I^1 : \ a \mapsto b^a(a) - B(\hat{m})e^{u((1+r_J)(D+a)-\delta\psi(a))}$$

which is such that $\hat{I}^1(\hat{m}) = I^1(\hat{a}_1(r_J))$. Let us check that (22) implies that $I^1(\hat{a}_2(r_J)) < I^1(\hat{a}_1(r_J))$. Since $\delta > 0$, the second condition in (22) implies that $\hat{a}_1(r_J) > \hat{a}_2(r_I)$ by the definition of $\hat{a}_1(r_J)$ and the monotonicity of ψ'. In terms of the function H defined by (65), the first condition in (22) writes $H(\hat{a}_1(r_J), r_I) < 0$. Since $H(\hat{a}_2(r_I), r_I) = 0$ and the function $a \mapsto H(a, r_I)$ is decreasing by (66), this also implies that $\hat{a}_1(r_J) > \hat{a}_2(r_I)$.

Since $a \mapsto e^{u((1+r_J)(D+a)-\delta\psi(a))}$ is convex by composition of the convex function $a \mapsto u\left((1+r_J)(D+a)-\delta\psi(a)\right)$ with the increasing and convex exponential function, I^1 is concave from the concavity of b^a. Now $(I^1)'(\hat{a}_1(r_J)) =$

$$b_a{}'\left((\psi')^{-1}(\frac{(u+v)(1+r_J)}{\delta u})\right) + B(\hat{m})e^{u((1+r_J)(D+a)-\delta\psi(a))}u(1+r_J) > 0.$$

From the concavity of I^1 and the inequality $\hat{a}_1(r_J) > \hat{a}_2(r_I)$, we deduce that $I^1(\hat{a}_2(r_J)) < I^1(\hat{a}_1(r_J)) = \hat{I}^1(\hat{m})$. As a consequence,

$$\begin{aligned}&\hat{I}^1(\hat{m}) - \hat{I}^2(\hat{m}) > I^1(\hat{a}_2(r_J)) - \hat{I}^2(\hat{m})\\ &= f((1+r_I)(D+\hat{a}_2(r_I))) - B(\hat{m})e^{-u\delta\psi(\hat{a}_2(r_I))}(e^{u(1+r_J)(D+\hat{a}_2(r_I))} - 1).\end{aligned}$$

Condition (21) is equivalent to the non-negativity of the right-hand side, which concludes the proof. □

8 Proofs of incomplete information results

In the incomplete information framework, the firms do not have a perfect knowledge of the preferences of the other firm. More precisely, we still assume that the firms' utility functions are $U(x) = -e^{-ux}$ and $V(x) = -e^{-vx}$ respectively, but firm I perceives v as a $(0,+\infty)$-valued random variable $\mathcal{V}$ independent of μ with a known distribution. The penalty that firm I gets if firm J does not accept the contract is denoted by $p \in \mathbb{R} \cup \{+\infty\}$.

8.1 Stackelberg equilibrium in incomplete information, firm I is leader

(cf. (30) and (31))

$$u_I^i := -p \vee \sup_{c^i}\{I^i(c^i)\mathbb{P}\left(\mathcal{A}^i(c^i)\right) - p(1-\mathbb{P}\left(\mathcal{A}^i(c^i)\right))\}, \tag{67}$$

where c^i is the control of firm I in Situation i : $c^1 = (\alpha,\beta,\gamma)$ and $c^2 = (a,\alpha,\beta,\gamma)$ and $\mathcal{A}^i(c^i)$ is the event "firm J accepts the contract". Firm J accepts the contract if and only if $-\tilde{J}^i(\mathcal{V},c^i) \leq 1$, therefore $\mathcal{A}^i(c^i) = \{\tilde{J}^i(\mathcal{V},c^i) \leq 1\}$ where $\tilde{J}^1$ and $\tilde{J}^2$ are respectively defined in (32) and (33). The next lemma aims at expliciting this acceptance set. The functions $v \mapsto -\tilde{J}^i(v,c^i)$ are infinitely differentiable according to Hypothesis (2) and convex since their second order derivative is non-negative. Since $-\tilde{J}^i(0,c^i) = 1$, one deduces that

Lemma 14 *For $c^1 = (\alpha,\beta,\gamma) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ (respectively for $c^2 = (a,\alpha,\beta,\gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$), $\bar{v}(c^i) := \sup\{v \geq 0 : -\tilde{J}^i(v,c^i) \leq 1\}$ belongs to $[0,+\infty]$. If $\bar{v}(c^i) \in [0,+\infty)$, then $\{v \geq 0 : -\tilde{J}^i(v,c^i) \leq 1\} = [0,\bar{v}(c^i)]$ and $-\tilde{J}^i(\bar{v}(c^i),c^i) = 1$. If $\bar{v}(c^i) = +\infty$, then $\{v \geq 0 : -\tilde{J}^i(v,c^i) \leq 1\} = [0,+\infty)$.*

Recall the value function of the problem with complete information that firm J's risk aversion is equal to v:

$$u^i(v) := \sup_{\{c^i : -\tilde{J}^i(v,c^i)\leq 1\}} \tilde{I}^i(c^i)$$

and $w_I^i = -p \vee \sup_{v>0}\{u^i(v)\mathbb{P}(\mathcal{V}\leq v) - p\big(1-\mathbb{P}(\mathcal{V}\leq v)\big)\}$ as defined in Theorem 7. We now are able to prove Theorem 7.

Proof of Theorem 7
We first prove that $u_I^i \geq w_I^i$. We only need to do so when $w_I^i > -p$ which implies that

$$w_I^i = \sup_{v>0: u^i(v)\mathbb{P}(\mathcal{V}\leq v) - p\mathbb{P}(\mathcal{V}>v) > -p} \Big(u^i(v)\mathbb{P}(\mathcal{V}\leq v) - p\mathbb{P}(\mathcal{V}>v)\Big).$$

Let $v>0$ be such that $u^i(v)\mathbb{P}(\mathcal{V}\leq v) - p\mathbb{P}(\mathcal{V}>v) > -p$ and $\hat{c}^i(v)$ be an optimal control for $u^i(v)$, so that $u^i(v) = \tilde{I}^i(\hat{c}^i(v))$. Since $-\tilde{J}(v,\hat{c}^i(v)) \leq 1$, by Lemma 14 one has $\{\mathcal{V}\leq v\} \subset \mathcal{A}^i(\hat{c}^i(v))$ and when $p<+\infty$,

$$\begin{aligned} u^i(v)\mathbb{P}(\mathcal{V}\leq v) - p\mathbb{P}(\mathcal{V}>v) &= (u^i(v)+p)\mathbb{P}(\mathcal{V}\leq v) - p \\ &\leq (u^i(v)+p)\mathbb{P}\left(\mathcal{A}^i(\hat{c}^i(v))\right) - p \\ &= \tilde{I}^i(\hat{c}^i(v))\mathbb{P}(\mathcal{A}^i(\hat{c}^i(v)) - p(1-\mathbb{P}(\mathcal{A}^i(\hat{c}^i(v)) \leq u_I^i. \end{aligned}$$

When $p=+\infty$, the left-most side of the above inequalities is still not greater than the right-most side as $1 = \mathbb{P}(\mathcal{V}\leq v) = \mathbb{P}(\mathcal{A}^i(\hat{c}^i(v)))$.
Since $v>0$ such that $u^i(v)\mathbb{P}(\mathcal{V}\leq v) - p\mathbb{P}(\mathcal{V}>v) > -p$ is arbitrary, we get $u_I^i \geq w_I^i$.
Finally we prove that $w_I^i \geq u_I^i$ in case $u_I^i > -p$. Let $\varepsilon \in (0, \frac{u_I^i+p}{2})$ and c^i be an ε-optimal control for u_I^i. Since

$$\tilde{I}^i(c^i)\mathbb{P}\left(\mathcal{A}^i(c^i)\right) - p(1-\mathbb{P}\left(\mathcal{A}^i(c^i)\right)) \geq u_I^i - \varepsilon > \frac{u_I^i - p}{2} \geq -p,$$

one has $\mathbb{P}\left(\mathcal{A}^i(c^i)\right) > 0$. Since $\mathcal{V}>0$ and $\mathcal{A}^i(c^i) = \{\mathcal{V}\leq \bar{v}(c^i)\}$ for $\bar{v}(c^i) = \sup\{v\geq 0 : -\tilde{J}^i(v,c^i)\leq 1\}$, one deduces that $\bar{v}(c^i) \in (0,\infty)\cup\{\infty\}$.
For any $v\in(0,+\infty)$ such that $-\tilde{J}^i(v,c^i)\leq 1$, one has $u^i(v)\geq \tilde{I}^i(c^i)$.
• If $\bar{v}(c^i)\in(0,+\infty)$ then, by Lemma 14, $-\tilde{J}^i(\bar{v}(c^i),c^i) = 1$ so that

$$\begin{aligned} w_I^i \geq u^i(\bar{v}(c^i))\mathbb{P}(\mathcal{V}\leq \bar{v}(c^i)) - p\mathbb{P}(\mathcal{V}>\bar{v}(c^i)) &\geq \tilde{I}^i(c^i)\mathbb{P}(\mathcal{V}\leq \bar{v}(c^i)) - p\mathbb{P}(\mathcal{V}>\bar{v}(c^i)) \\ &= \tilde{I}^i(c^i)\mathbb{P}(\mathcal{A}^i(c^i)) - p(1-\mathbb{P}(\mathcal{A}^i(c^i)) \geq u_I^i - \varepsilon. \end{aligned}$$

• If $\bar{v}(c^i) = +\infty$, then $\mathbb{P}\left(\mathcal{A}^i(c^i)\right) = 1$ and for all $v>0$, $w_I^i \geq \tilde{I}^i(c^i)\mathbb{P}(\mathcal{V}\leq v) - p\mathbb{P}(\mathcal{V}>v)$ and the same conclusion as before holds by taking the limit $v\to\infty$ in this inequality under the assumption that either $p<+\infty$ or $\mathbb{P}(\mathcal{V}>v)=0$ for v large enough. Since $\varepsilon>0$ is arbitrarily small, we get $w_I^i \geq u_I^i$, which ends the proof. □

To prove Theorem 8, we need the following properties of the value functions of the problems with complete information that firm J's risk aversion is v.

Lemma 15 *The function $v \mapsto u^i(v)$ is non-increasing and continuous on $\mathbb{R}_+$.*

Proof We do the proof for Situation 2. The same holds (with the control (α, β, γ) instead of $(a, \alpha, \beta, \gamma)$) for Situation 1. For $v \geq 0$, let $\mathcal{A}_v = \{(a, \alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ : -\tilde{J}^2(v, a, \alpha, \beta, \gamma) \leq 1\}$. By Lemma 14, one has $\mathcal{A}_v \subset \mathcal{A}_{v'}$ when $v' \leq v$. Therefore $v \mapsto u^2(v)$ is non-increasing.
• Let us check the right-continuity of u^2 i.e. that $\liminf_{v' \to v^+} u^2(v') \geq u^2(v)$. According to Proposition 6, there exists $(a, \alpha, \beta, \gamma) \in \mathcal{A}_v$ such that $u^2(v) = \tilde{I}^2(a, \alpha, \beta, \gamma)$.
Either $-\tilde{J}^2(v, a, \alpha, \beta, \gamma) < 1$ and by continuity of $v' \mapsto -\tilde{J}^2(v', a, \alpha, \beta, \gamma)$, $(a, \alpha, \beta, \gamma) \in \mathcal{A}_{v'}$ for v' close enough to v so that the conclusion holds.
Or $-\tilde{J}^2(v, a, \alpha, \beta, \gamma) = 1$ so that for $v' > v$, $-\tilde{J}^2(v', a, \alpha, \beta, \gamma) > 1$, and $\alpha_{v'} = \alpha + \frac{1}{v'} \ln(-\tilde{J}^2(v', a, \alpha, \beta, \gamma)) > 0$ is such that $-\tilde{J}^2(v', a, \alpha_{v'}, \beta, \gamma)) = 1$ and $\lim_{v' \to v^+} \tilde{I}^2(a, \alpha_{v'}, \beta, \gamma) = \tilde{I}^2(a, \alpha, \beta, \gamma)$.
• For the left-continuity, we consider a sequence $(v_n)_n$ of positive numbers increasing to a finite limit v_∞. According to Theorem 6, there exists $(a_n, \alpha_n, \beta_n, \gamma_n) \in \mathcal{A}_{v_n}$ such that $u^2(v_n) = \tilde{I}^2(a_n, \alpha_n, \beta_n, \gamma_n)$. By Lemma 11 and the proof of Lemma 13, $(a_n, \alpha_n, \beta_n, \gamma_n)$ stays in a compact subset of $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ so one may extract a subsequence that we still index by n for simplicity such that $(a_n, \alpha_n, \beta_n, \gamma_n)$ tends to $(a_\infty, \alpha_\infty, \beta_\infty, \gamma_\infty)$. By continuity of $\tilde{I}^2$ and $\tilde{J}^2$, one has $\tilde{J}^2(v_\infty, a_\infty, \alpha_\infty, \beta_\infty, \gamma_\infty) = \lim_{n \to \infty} \tilde{J}^2(v_n, a_n, \alpha_n, \beta_n, \gamma_n)$ so that $(a_\infty, \alpha_\infty, \beta_\infty, \gamma_\infty) \in \mathcal{A}_{v_\infty}$ and therefore $\lim_{n \to \infty} \tilde{I}^2(a_n, \alpha_n, \beta_n, \gamma_n) = \tilde{I}^2(a_\infty, \alpha_\infty, \beta_\infty, \gamma_\infty) \leq u^2(v_\infty)$.
With the monotonicity of u^2, we conclude that this function is continuous. □

We prove the existence of a Stackelberg equilibrium with incomplete information, firm I leader.
Proof of Theorem 8 Let $v_0 := \inf\{v > 0 : \mathbb{P}(\mathcal{V} \leq v) > 0\}$.

If $\lim_{v \to v_0^+} u^i(v) \leq -p$, , then $v_I^i = w_I^i = -p$.
If not, $\lim_{v \to v_0^+} u^i(v) > -p$ and we assume that

$$v_1 := \sup\{v > 0 : \mathbb{P}(\mathcal{V} > v) > 0\} < +\infty.$$

If $p = +\infty$, then the optimization problem (34) clearly admits the solution $v^\star = v_1$.
If $p < +\infty$, then there exists $v > v_0$ close enough to v_0 such that

$$(u^i(v) + p)\mathbb{P}(\mathcal{V} \leq v) > 0.$$

Let us deduce existence of a solution to the optimization problem (34). Since u^1 (resp. u^2) is bounded from above by $\sup_{a \in \mathbb{R}_+} b^a(a) < +\infty$ (resp. $\sup_{a \in \mathbb{R}_+} F(a) < +\infty$) and $\mathcal{V}$ takes its values in $(0, +\infty)$, one has

$$\lim_{v \to 0} (u^i(v) + p)\mathbb{P}(\mathcal{V} \leq v) = 0.$$

The function $v \mapsto (u^i(v) + p)\mathbb{P}(\mathcal{V} \leq v)$ being upper-semicontinuous on the closed set $\{v \in [\varepsilon, v_1] : u_i(v) + p \geq 0\}$ for each $\varepsilon > 0$, we conclude that the optimization problem (34) has a solution $v^\star \in (0, v_1]$ if $v_0 = 0$ and in $[v_0, v_1]$ otherwise.
Moreover $u^i(v^\star) > -p$. Let $\hat{c}^i(v^\star)$ be an optimal control for $u^i(v^\star)$ such

that $u^i(v^\star) = \tilde{I}^i(\hat{c}^i(v^\star))$. Since $-\tilde{J}^i(v^\star, \hat{c}^i(v^\star)) \leq 1$, by Lemma 14 one has $\{\mathcal{V} \leq v^\star\} \subset \mathcal{A}^i(\hat{c}^i(v^\star))$ and

$$\begin{aligned} w_I^i &= u^i(v^\star)\mathbb{P}(\mathcal{V} \leq v^\star) - p\mathbb{P}(\mathcal{V} > v^\star) = (u^i(v^\star) + p)\mathbb{P}(\mathcal{V} \leq v^\star) - p \\ &\leq (u^i(v^\star) + p)\mathbb{P}\left(\mathcal{A}^i(\hat{c}^i(v^\star))\right) - p \\ &= \tilde{I}^i(\hat{c}^i(v^\star))\mathbb{P}(\mathcal{A}^i(\hat{c}^i(v^\star))) - p(1 - \mathbb{P}(\mathcal{A}^i(\hat{c}^i(v^\star))) \\ &\leq u_I^i. \end{aligned}$$

With Theorem 7, we conclude that $(\hat{c}^i(v^\star))$ solves problem (30)-(31).

□

8.2 Nash equilibrium in incomplete information

We consider both situations, the proofs are quite similar. Let $I(a, \alpha, \beta, \gamma, e, m) :=$

$$\mathbb{E}\left[b^a(a) - e^{-u\left(b^m(m) + b^e(e) - \alpha - \beta(\mu\ - \varphi(e) - \delta\psi(a)) - \gamma g(m)\right)}\right]$$

respectively $I(a, \alpha, \beta, \gamma, e, m) :=$

$$\mathbb{E}\left[F(a) - e^{-u\left(b^m(m) + b^e(e) - \alpha - \beta(\mu\ - \varphi(e) - \delta\psi(a)) - \gamma g(m)\right)}\right],$$

and $J(v, a, \alpha, \beta, \gamma, e, m) :=$

$$-\mathbb{E}\left[e^{-v\left(\alpha + (\beta - 1)(\mu\ - \varphi(e) - \delta\psi(a)) - e + \gamma g(m) - m - (1 + r_J)(D + a)\right)}\right]$$

respectively

$$J(v, a, \alpha, \beta, \gamma, e, m) := -\mathbb{E}\left[e^{-v\left(\alpha + (\beta - 1)(\mu\ - \varphi(e) - \delta\psi(a)) - e + \gamma g(m) - m\right)}\right].$$

For firm I, given the controls (a, e, m) of firm J (respectively (e, m)), the problem is to find (α, β, γ) (respectively $(a, \alpha, \beta, \gamma)$) maximizing

$$I(a, \alpha, \beta, \gamma, e, m)\mathbb{P}(J(\mathcal{V}, a, \alpha, \beta, \gamma, e, m) \geq -1) - p\mathbb{P}(J(\mathcal{V}, a, \alpha, \beta, \gamma, e, m) < -1).$$

As in Lemma 14, we have

Lemma 16 *For* $c = (a, \alpha, \beta, \gamma, e, m) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, $\bar{v}(c) := \sup\{v \geq 0 : J(v, c) \geq -1\}$ *belongs to* $[0, +\infty]$. *If* $\bar{v}(c) \in [0, +\infty)$, *then* $\{v \geq 0 : J(v, c) \geq -1\} = [0, \bar{v}(c)]$ *and* $J(\bar{v}(c), c) = -1$. *If* $\bar{v}(c) = +\infty$, *then* $\{v \geq 0 : J(v, c) \geq -1\} = [0, +\infty)$.

Proof of Theorem 9 Assume the existence of a Nash equilibrium $\hat{c} = (\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{e}, \hat{m})$ such that the value for firm I is greater than $-p$. This implies that $I(\hat{c}) > -p$ and $0 < \mathbb{P}(-J(\mathcal{V}, \hat{c}) \leq 1)$.
Since by Lemma 16, $\mathbb{P}(-J(\mathcal{V}, \hat{c}) \leq 1) = \mathbb{P}(\mathcal{V} \leq \bar{v}(\hat{c}))$, one has $\bar{v}(\hat{c}) > 0$.
We detail below the proof in Situation 1. The one in Situation 2 follows the same scheme, replacing the sets of control parameters (α, β, γ) and (a, e, m) respectively by $(a, \alpha, \beta, \gamma)$ and (e, m).
• Assume that $\bar{v}(\hat{c}) < +\infty$ and let $v \in (0, \bar{v}(\hat{c})]$ be such that $\mathbb{P}(\mathcal{V} \in (v, \bar{v}(\hat{c})]) = 0$.
For (α, β, γ) such that $-J(v, \hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) \leq 1$:
either $I(\hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) \leq -p$ and therefore $I(\hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) < I(\hat{c})$,
or $I(\hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) > -p$ and since
$\mathbb{P}(-J(\mathcal{V}, \hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) \leq 1) \geq \mathbb{P}(\mathcal{V} \leq v) = \mathbb{P}(\mathcal{V} \leq \bar{v}(\hat{c})) = \mathbb{P}(-J(\mathcal{V}, \hat{c}) \leq 1)$, then:

$$
\begin{aligned}
&I(\hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m})\mathbb{P}(\mathcal{V} \leq \bar{v}(\hat{c})) - p\mathbb{P}(\mathcal{V} > \bar{v}(\hat{c})) \\
&\leq I(\hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m})\mathbb{P}(-J(\mathcal{V}, \hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) \leq 1) - p\mathbb{P}(-J(\mathcal{V}, \hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) > 1) \\
&\leq I(\hat{c})\mathbb{P}(-J(\mathcal{V}, \hat{c}) \leq 1) - p\mathbb{P}(-J(\mathcal{V}, \hat{c}) > 1) = I(\hat{c})\mathbb{P}(\mathcal{V} \leq \bar{v}(\hat{c})) - p\mathbb{P}(\mathcal{V} > \bar{v}(\hat{c})),
\end{aligned}
$$

where the last inequality follows from the fact that $\hat{c}$ is a Nash equilibrium for the problem with incomplete information (cf. (67)).
This implies that $I(\hat{a}, \alpha, \beta, \gamma, \hat{e}, \hat{m}) \leq I(\hat{c})$. Since by Lemma 16, $J(v, \hat{c}) \geq -1$, we deduce that $\hat{c}$ is a Nash equilibrium for the problem with complete information and risk aversion v for firm J. By Theorem 2, we deduce that $\hat{\beta} = \frac{v}{u+v}$ so that the only $v \in (0, \bar{v}(\hat{c})]$ such that $\mathbb{P}(\mathcal{V} \in (v, \bar{v}(\hat{c})]) = 0$ is $\bar{v}(\hat{c}) = \hat{v}$.
• The same line of reasoning permits to conclude that in case $\bar{v}(\hat{c}) = +\infty$, $\forall v \in (0, +\infty)$, $\mathbb{P}(\mathcal{V} \in (v, +\infty)) > 0$. □

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