Instantaneous shrinking and single point extinction for viscous Hamilton-Jacobi equations with fast diffusion

Razvan Gabriel Iagar, Philippe Laurençot, Christian Stinner

Instantaneous shrinking and single point extinction for viscous Hamilton-Jacobi equations with fast diffusion

Razvan Gabriel Iagar [*,†]
Philippe Laurençot [‡]
Christian Stinner [§]

October 1, 2015

Abstract

For a large class of non-negative initial data, the solutions to the quasilinear viscous Hamilton-Jacobi equation $\partial_t u - \Delta_p u + |\nabla u|^q = 0$ in $(0,\infty)\times\mathbb{R}^N$ are known to vanish identically after a finite time when $2N/(N+1) < p \le 2$ and $q \in (0, p-1)$. Further properties of this extinction phenomenon are established herein: *instantaneous shrinking* of the support is shown to take place if the initial condition u_0 decays sufficiently rapidly as $|x|\to\infty$, that is, for each $t>0$, the positivity set of $u(t)$ is a bounded subset of $\mathbb{R}^N$ even if $u_0>0$ in $\mathbb{R}^N$. This decay condition on u_0 is also shown to be optimal by proving that the positivity set of any solution emanating from a positive initial condition decaying at a slower rate as $|x|\to\infty$ is the whole $\mathbb{R}^N$ for all times. The time evolution of the positivity set is also studied: on the one hand, it is included in a fixed ball for all times if it is initially bounded (*localization*). On the other hand, it converges to a single point at the extinction time for a class of radially symmetric initial data, a phenomenon referred to as *single point extinction*. This behavior is in sharp contrast with what happens when q ranges in $[p-1, p/2)$ and $p \in (2N/(N+1), 2]$ for which we show *complete extinction*. Instantaneous shrinking and single point extinction take place in particular for the semilinear viscous Hamilton-Jacobi equation when $p=2$ and $q\in(0,1)$ and seem to have remained unnoticed.

AMS Subject Classification: 35K59, 35K67, 35K92, 35B33, 35B40.

Keywords: finite time extinction, singular diffusion, viscous Hamilton-Jacobi equation, gradient absorption, instantaneous shrinking, single point extinction.

[*] Instituto de Ciencias Matemáticas (ICMAT), Nicolas Cabrera 13-15, Campus de Cantoblanco, Madrid, Spain, *e-mail:* razvan.iagar@icmat.es

[†] Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700, Bucharest, Romania.

[‡] Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, F–31062 Toulouse Cedex 9, France. *e-mail:* Philippe.Laurencot@math.univ-toulouse.fr

[§] Technische Universität Kaiserslautern, Felix-Klein-Zentrum für Mathematik, Paul-Ehrlich-Str. 31, D-67663 Kaiserslautern, Germany, *e-mail:* stinner@mathematik.uni-kl.de

1 Introduction and results

We perform a detailed study of the finite time extinction phenomenon for a class of diffusion equations with a gradient absorption term, of the form

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0, \quad u = u(t,x), \ (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

where, as usual,

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u) \ ,$$

supplemented with the initial condition

$$u(0) = u_0, \qquad x \in \mathbb{R}^N. \tag{1.2}$$

Throughout the paper we assume that

$$u_0 \in W^{1,\infty}(\mathbb{R}^N) \ , \qquad u_0 \geq 0 \ , \ u_0 \not\equiv 0 \ . \tag{1.3}$$

The range of exponents under consideration is

$$p_c := \frac{2N}{N+1} < p \leq 2, \qquad 0 < q < p-1, \tag{1.4}$$

in which it is already known that extinction in finite time takes place for initial data decaying sufficiently rapidly as $|x| \to \infty$, that is: there exists $T_e \in (0,\infty)$ such that $u(T_e, x) = 0$ for any $x \in \mathbb{R}^N$, but $\|u(t)\|_\infty > 0$ for any $t \in (0, T_e)$. The time T_e is usually referred to as *the extinction time* of the solution u. Let us notice at this point that the range of exponents includes both the *semilinear case* $p = 2$ (with $q \in (0,1)$), and the *singular diffusion case* $p \in (p_c, 2)$ (with $q \in (0, p-1)$). These two diffusion operators usually depart strongly in their qualitative properties, but in the range (1.4), the gradient absorption term is dominating the evolution, thus explaining the similarity of the results for the linear and singular diffusions.

The main feature concerning equations such as (1.1) is the *competition* between the two terms in the equation: a diffusion one and an absorption one, in form of a gradient term. As the properties of the diffusion equation and of the Hamilton-Jacobi equation (without diffusion) are very different, it is of interest to study the effects of their merging in the equation, depending on the relative positions of the exponents p and q.

For the semilinear case $p = 2$, a number of results are available by now, due to the possibility of using semigroup theory or linear techniques. Thus, it has been shown that there appear two critical values for the exponent q, namely $q = q_* := (N+2)/(N+1)$ and $q = 1$. The qualitative theory, including the large time behavior, is now well understood for exponents $q > 1$ after the series of works [2, 3, 4, 5, 9, 10, 19, 20, 22]. In this range $q > 1$, the diffusion has an important influence on the evolution: either completely dominating, when $q > (N+2)/(N+1)$, leading to asymptotic simplification, or having a similar effect to the Hamilton-Jacobi part for $q \in (1, q_*]$, leading to a resonant, logarithmic-type behavior for $q = q_*$ [19], or a behavior driven by *very singular solutions* for $q \in (1, q_*)$ [4]. Much less is known for the complementary range of q, that is, $0 < q \leq 1$, where the Hamilton-Jacobi term starts to have a very strong influence on the dynamics. The limit exponent $q = 1$ is highly critical and not yet fully understood though optimal temporal decay estimates are established in [8], while extinction in finite time has been shown for $q \in (0,1)$, see [6, 7, 22]. Still, a complete understanding of the extinction phenomenon is missing and requires a deeper study. So far, the class of initial data for which finite time extinction takes place has not yet been identified and more qualitative information on the behavior of the support of the solution and on the rate and shape of the extinction are still lacking. The purpose of the present work is to shed some further light on the extinction phenomenon for (1.1) and provide more detailed information on the above mentioned issues, not only in the semilinear case $p = 2$ but also when the diffusion is nonlinear and singular corresponding to $p \in (p_c, 2)$.

Indeed, considering the quasilinear diffusion operator Δ_p is a natural nonlinear generalization, but due to the fact that linear techniques are not available anymore, its study is more involved and results were obtained only recently. We are interested in the fast diffusion case $p \in (p_c, 2)$, for which the qualitative theory is developed starting from [25], where all exponents $q > 0$ are considered. In particular, two critical exponents are identified in [25]: $q = q_* := p - N/(N+1)$ and $q = p/2$. These critical values limit ranges of parameters with different behaviors: diffusion dominates for $q > q_*$, while there is a balance between diffusion and absorption for $q \in [p/2, q_*]$ leading to logarithmic decay for $q = q_*$, algebraic decay for $q \in (p/2, q_*)$, and exponential decay for $q = p/2$. Finally, finite time extinction occurs for $0 < q < p/2$ and this is the range of the parameter q we are interested in. We actually perform a deeper study of the extinction range $0 < q < p/2$ with $p \in (p_c, 2)$ which reveals very interesting and surprising features having not been observed before, as far as we know. We mention at this point that we restrict the analysis to $p > p_c$ as there is a competition in this range between the diffusion term which aims at positivity and the gradient absorption term which is the driving mechanism of extinction. We left aside the critical case $p = p_c$ which is trickier to handle, as well as the case $p \in (1, p_c)$, for which a different competition takes place. Indeed, finite time extinction is also known to take place for the diffusion equation without the gradient absorption term when $p \in (1, p_c)$, and there is then a competition between two extinction mechanisms stemming from the diffusion and the gradient absorption, respectively.

Before describing more precisely our results, let us recall that the finite time extinction phenomenon has already been observed as the outcome of a competition between diffusion and absorption effects, in particular for another important diffusive model, the porous medium equation with zero order absorption

$$\partial_t u - \Delta u^m + u^q = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{1.5}$$

with $m > 0$ and $q \in (0,1)$. A striking feature is the *instantaneous shrinking of the support* for non-negative solutions, that is, the solution $u(t)$ to (1.5) at time t is compactly supported for all $t > 0$ even though the initial condition $u(0)$ is positive in $\mathbb{R}^N$. This phenomenon was first noticed in [14] for $m = 1$ and later extended to (1.5) and its variants (including variable coefficients in front of the absorption term and/or an additional convection term) for $q \in (0,1)$ and $m > q$ in [1, 11, 14, 17, 21, 23, 26, 27, 28], as well as to other equations such as

$$\partial_t u - \Delta_p u + |u|^{q-1} u = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N,$$

for suitable ranges of the parameters p and q, see [12, 29] and the references therein. More precise information on the behavior near the extinction time are available for (1.5) in one space dimension $N = 1$ when $0 < m \le 1$ and $0 < q < 1$, [15, 16, 24]. It is shown that the case $q = m$ is critical and the mechanism of extinction is different whether $m < q < 1$ or $0 < q < m$: in the former, *simultaneous or complete extinction* occurs, that is, the solution is positive everywhere in $\mathbb{R}^N$ prior to the extinction time and vanishes identically at the extinction time. In the latter, *single point extinction* takes place as shown in [16], that is, the positivity set of $u(t)$ shrinks to a point as t approaches the extinction time. The limit case $q = m$ is simpler and explicit, being the only case studied also for $N > 1$ [13].

The purpose of this paper is to investigate the occurrence of the above mentioned phenomena for Eq. (1.1). Concerning instantaneous shrinking, we show that it takes place in the range $q \in (0, p-1)$, but only for initial conditions which decay sufficiently fast as $|x| \to \infty$. On the contrary, for positive initial data with a slow decay as $|x| \to \infty$, the solution is positive everywhere in $\mathbb{R}^N$ for all times. This is in sharp contrast with the situation for (1.5), where it is sufficient that the initial condition decays to zero as $|x| \to \infty$ for instantaneous shrinking to take place [1, 11, 14]. The occurrence of instantaneous shrinking in (1.1) thus not only depends on the parameters p and q but also on the shape of the initial condition. Coming back to complete extinction, we also show that this is the generic behavior when $q \in [p-1, p/2)$, while we identify a class of initial data for which single point extinction occurs when $q \in (0, p-1)$.

Main results. We denote in the sequel the (spatial) positivity set $\mathcal{P}(t)$ at time $t \geq 0$ of a solution u to (1.1)-(1.2) by

$$\mathcal{P}(t) := \{x \in \mathbb{R}^N : u(t,x) > 0\}. \tag{1.6}$$

We begin with some features of the time evolution of the positivity set according to the decay of u_0 at infinity.

Theorem 1.1 (Instantaneous shrinking and localization)**.** *Let u be a solution to the Cauchy problem* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.3) *and*

$$u_0(x) \leq C(1+|x|)^{-\theta}, \qquad x \in \mathbb{R}^N, \qquad \theta > \frac{q}{1-q} \tag{1.7}$$

for some $C > 0$, and exponents p, q as in (1.4)*. Then:*

(i) Instantaneous shrinking: for any $t > 0$, $\mathcal{P}(t)$ is a bounded subset of $\mathbb{R}^N$.

(ii) Localization: for any $\tau > 0$ there exists $\varrho_\tau > 0$ such that $\mathcal{P}(t) \subseteq B(0, \varrho_\tau)$ for all $t \geq \tau$.

(iii) Extinction: there is $T_e > 0$ such that $u(t) \equiv 0$ for all $t \geq T_e$.

In other words, the dynamics forces the support of the solution to become compact immediately (at any time $t > 0$) even if u_0 is positive in $\mathbb{R}^N$, that is, $\mathcal{P}(0) = \mathbb{R}^N$. It then remains confined inside a ball for $t \geq \tau > 0$, the radius of the ball depending only on τ. Two steps are needed to prove Theorem 1.1: we first construct a supersolution to (1.1) on $(0, t_0) \times \mathbb{R}^N$ for a sufficiently small time t_0 which is positive for $t = 0$ but has compact support for $t \in (0, t_0)$. Due to the gradient absorption term, it does not seem to be possible to adapt the approach used for (1.5) and the supersolution has to be constructed in a different way. The second step is to establish that solutions to (1.1)-(1.2) emanating from compactly supported initial data enjoy the localization property, that is, their support stays forever in a fixed ball of $\mathbb{R}^N$. Combining these two steps provides the first two assertions of Theorem 1.1, the last assertion being a straightforward consequence of the compactness of the support for positive times, [22, Corollary 9.1], and [25, Theorem 1.2(iii)].

Theorem 1.1 turns out to be false under a sole condition of decay to zero at infinity of u_0 and the following result shows a strikingly different behavior for initial data with a sufficiently slow spatial decay at infinity.

Theorem 1.2 (Non-extinction and non-localization)**.** *Let u be a solution to the Cauchy problem* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.3) *and*

$$\lim_{|x|\to\infty} |x|^{q/(1-q)} u_0(x) = \infty, \qquad u_0(x) > 0 \text{ for any } x \in \mathbb{R}^N, \tag{1.8}$$

and exponents p, q as in (1.4)*. Then*

$$\mathcal{P}(t) = \mathbb{R}^N \qquad \text{for any } t > 0.$$

Theorem 1.2 means that, when the initial condition has a sufficiently fat tail at infinity, it has enough mass to remain positive everywhere for all times in spite of the dominating absorption effect. When $p = 2$ and $q \in (0, 1)$ it is established in [7] with the help of suitable subsolutions and we extend here this approach to the whole range (1.4).

A consequence of Theorem 1.2 is that the decay condition (1.7) on u_0 is optimal for instantaneous shrinking to take place when the parameters p and q satisfy (1.4). The optimality of the range (1.4) of the exponents p and q requires a different argument: we show in Proposition 4.4 that, when $p \in (p_c, 2)$ and $q \in [p-1, p/2)$, only complete extinction takes place, that is,

$$\mathcal{P}(t) = \mathbb{R}^N \qquad \text{for any } t \in (0, T_e).$$

the finiteness of the extinction time T_e for that range being proved in [25] for initial data decaying suffciently rapidly at infinity.

Returning to finite time extinction, we are able to improve Theorem 1.1 as well as [25, Theorem 1.2(iii)], showing that solutions vanish after a finite time even in the limit case for the decay $\theta = q/(1-q)$ in (1.7), which is excluded in Theorem 1.1. However, no information on the evolution of the positivity set is provided.

Theorem 1.3 (Improved finite time extinction)**.** *Let u be a solution to the Cauchy problem* (1.1)-(1.2)*, with an initial condition u_0 satisfying* (1.3) *and*

$$u_0(x) \le C_0(1 + |x|)^{-q/(1-q)}, \qquad x \in \mathbb{R}^N, \tag{1.9}$$

for some $C_0 > 0$ and p, q as in (1.4)*. Then extinction in finite time takes place: there exists $T_e \in (0,\infty)$ such that $u(T_e, x) = 0$ for any $x \in \mathbb{R}^N$, but $\|u(t)\|_\infty > 0$ for any $t \in (0, T_e)$.*

Theorem 1.3 is proved in [7] when $p = 2$ and $q \in (0, 1)$ and we extend it here to the whole range (1.4). Its proof relies on the construction of self-similar supersolutions vanishing identically after a finite time.

We now delve deeper in the extinction mechanism and aim at studying how extinction takes place. Let u be a solution to (1.1) and $T_e \in (0,\infty)$ its extinction time, assuming that u vanishes in finite time. Recalling the definition (1.6) of the positivity set $\mathcal{P}(t)$, we define *the extinction set* of u by

$$\mathcal{E}(u) := \left\{ \begin{array}{l} x \in \mathbb{R}^N : \text{there exist } \{x_n \ : \ n \ge 1\} \subset \mathbb{R}^N, \ \{t_n \ : \ n \ge 1\} \subset (0, T_e) \\ \text{such that } x_n \to x, \ t_n \to T_e \text{ as } n \to \infty, \ u(t_n, x_n) > 0 \text{ for all } n \end{array} \right\}.$$

We say that u presents *simultaneous or complete extinction* if $\mathcal{E}(u) = \mathbb{R}^N$ while it presents *single point extinction* when $\mathcal{E}(u)$ is a singleton. Simultaneous extinction is the most common phenomenon; for example, it occurs for the standard subcritical fast diffusion equation (without absorption terms). Here, it happens that the opposite and less standard phenomenon occurs. More precisely:

Theorem 1.4 (Single point extinction)**.** *Let u be a solution to the Cauchy problem* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.3)*, and exponents p, q as in* (1.4)*. Assume further that:*

(a) $u_0 \in C^1(\mathbb{R}^N)$ is radially symmetric and radially non-increasing,

(b) u_0 is compactly supported in $B(0, R_0)$ for some $R_0 > 0$ and satisfies the following condition

$$u_0(x) \le \kappa_{p,q} |x - x_0|^{\omega} , \qquad x \in \mathbb{R}^N \tag{1.10}$$

for all $x_0 \in \partial B(0, R_0)$, with

$$\kappa_{p,q} := \frac{p-1-q}{p-q} \left(\frac{p-1}{p-1-q} + N - 1 \right)^{-1/(p-1-q)} \qquad \text{and} \qquad \omega := \frac{p-q}{p-1-q} , \tag{1.11}$$

(c) and there exists $\delta_0 > 0$ such that

$$\left| \nabla u_0^{(p-q-1)/(p-q)}(x) \right| \ge \delta_0 |x|^{1/(p-1-q)} , \qquad x \in B(0, R_0) . \tag{1.12}$$

Let $T_e \in (0, \infty)$ be the extinction time of u, which is finite according to Theorem 1.1. Then, there exist $\varrho_1 > 0$ and $\varrho_2 > 0$ such that

$$B\left(0, \varrho_1 (T_e - t)^{\sigma}\right) \subseteq \mathcal{P}(t) \subseteq B\left(0, \varrho_2 (T_e - t)^{\nu}\right) \qquad \text{for any } t \in (T_e/2, T_e), \tag{1.13}$$

where

$$\sigma := \frac{p-q-1}{(p-q)(1-q)}, \qquad \nu := \frac{p(p-q-1)^2}{2(p-q)(p-2q)}.$$

Consequently, u presents single point extinction *at the origin: $\mathcal{E}(u) = \{0\}$.*

As we shall see below the first inclusion in (1.13) holds true for any radially symmetric and radially non-increasing initial condition u_0 with compact support. The second inclusion requires the more restrictive conditions (1.10) and (1.12) on u_0, the former guaranteeing that the positivity set of u stays inside the ball $B(0, R_0)$.

As far as we know, Theorem 1.4 is the *first example* of single point extinction for equations with gradient absorption. Single point extinction was already observed for the heat equation with zero order absorption term in a bounded domain Ω of $\mathbb{R}^N$ with homogeneous Dirichlet boundary conditions

$$\partial_t u = \Delta u - u^q, \quad (t, x) \in (0, \infty) \times \Omega,$$

when $q \in (0, 1)$, the result being valid for a specific class of inital conditions [17]. In space dimension $N = 1$, similar results are available for (1.5) in the range $0 < q < m < 1$ [16].

The proof of Theorem 1.4 is technically involved, following the general strategy used by Friedman and Herrero [17], but with new and decisive contributions of the optimal gradient estimates established in [25, Theorem 1.3]. Notice also that we are able to drop any restriction of the type $\partial_t u(0,x) = \Delta u_0(x) - u_0(x)^q \geq 0$ on the initial data, as required in [17].

Organization of the paper. After recalling the well-posedness of (1.1)-(1.2) and a few properties of solutions in Section 2, we begin with proving the instantaneous shrinking phenomenon, as stated in Theorem 1.1, to which we devote Section 3. The proof of Theorem 1.1 is completed in Section 4, where we prove the localization of $\mathcal{P}(t)$, $t \geq 0$, for compactly supported initial data, as well as some side results showing that the range (1.4) is optimal for localization to take place. The localization property allows us to derive upper and lower bounds at the extinction time which are gathered in Section 5. We go on with the proof of Theorem 1.2, performed in Section 6, and the proof of Theorem 1.3, done in the subsequent Section 7. All these proofs have in common the fact that they rely on the maximum principle, used in suitable ways according to the case to be dealt with. In particular, subsolutions and supersolutions of different kinds with suitable behaviors are constructed along these sections. Finally, we devote Section 8 to the proof of Theorem 1.4, which is the most involved technically and is further divided into several subsections. The paper ends with a technical Appendix where we provide rigorous proofs for some estimates and calculations performed only at a formal level in Section 8 for the simplicity of the reading.

Notation. We introduce the parabolic operator $\mathcal{L}$ defined by

$$\mathcal{L}z := \partial_t z - \Delta_p z + |\nabla z|^q \quad \text{in} \quad (0,\infty) \times \mathbb{R}^N . \tag{1.14}$$

If z is radially symmetric with respect to the space variable then, setting $r := |x|$ and $z(t,r) = z(t,|x|)$, an alternative formula for $\mathcal{L}z$ is the following:

$$\mathcal{L}z = \partial_t z - (p-1)|\partial_r z|^{p-2}\partial_r^2 z - \frac{N-1}{r}|\partial_r z|^{p-2}\partial_r z + |\partial_r z|^q . \tag{1.15}$$

2 Well-posedness

We collect in this section some properties of the Cauchy problem (1.1)-(1.2) and its solutions. We first recall the well-posedness of (1.1)-(1.2), established in [20] for $p = 2$ and in [25] for $p \in (1,2)$.

Proposition 2.1. *Let $p \in (p_c, 2]$ and $q > 0$. Given an initial condition u_0 satisfying* (1.3)*, there is a unique nonnegative viscosity solution*

$$u \in \mathcal{BC}([0,\infty) \times \mathbb{R}^N) \cap L^\infty(0,\infty; W^{1,\infty}(\mathbb{R}^N))$$

to (1.1)-(1.2) *which is also a weak solution if $p \in (p_c, 2)$ and a classical solution if $p = 2$. In addition, it satisfies*

$$0 \leq u(t,x) \leq \|u_0\|_\infty , \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N . \tag{2.1}$$

We next show that radial symmetry and radial monotonicity are both preserved by (1.1).

Lemma 2.2. *Let u be a solution to* (1.1) *such that its initial condition u_0 satisfies* (1.3) *and is radially symmetric and non-increasing (that is, for $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$, there holds $u_0(y) \geq u_0(z)$ whenever $|y| \leq |z|$). Then $x \mapsto u(t,x)$ is radially symmetric and non-increasing for any $t > 0$.*

Proof. The radial symmetry is immediate from the rotational invariance of Eq. (1.1) and the uniqueness statement in Proposition 2.1.

Consider next $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$ such that $|y| < |z|$ and define $x_0 := (z-y)/2$, the hyperplane $\mathcal{H} := \{x \in \mathbb{R}^N \ : \ \langle x, x_0\rangle > 0\}$, and the functions $v_\pm(t,x) := u(t, x \pm x_0)$ for $(t,x) \in (0,\infty) \times \mathcal{H}$. On the one hand, for $x \in \partial\mathcal{H}$, there holds $\langle x, x_0\rangle = 0$, so that

$$|x+x_0|^2 = |x|^2 + |x_0|^2 = |x-x_0|^2,$$

and the radial symmetry of u entails that

$$v_+(t,x) = u(t,x+x_0) = u(t,x-x_0) = v_-(t,x) \qquad \text{for all } t>0 \quad \text{and} \quad x \in \partial\mathcal{H} \ .$$

On the other hand, if $x \in \mathcal{H}$, then

$$|x+x_0|^2 = |x|^2 + |x_0|^2 + 2\langle x, x_0\rangle \ge |x|^2 + |x_0|^2 - 2\langle x, x_0\rangle = |x-x_0|^2,$$

and the radial monotonicity of u_0 implies that

$$v_+(0,x) = u_0(x+x_0) \le u_0(x-x_0) = v_-(0,x).$$

Since v_+ and v_- both solve (1.1) in $(0,\infty)\times\mathcal{H}$, we infer from the comparison principle and the previous properties that $v_+ \le v_-$ in $(0,\infty)\times\mathcal{H}$. In particular, $(y+z)/2 \in \mathcal{H}$ and we obtain

$$u(t,y) = v_-\left(t, \frac{y+z}{2}\right) \ge v_+\left(t, \frac{y+z}{2}\right) = u(t,z)$$

as claimed. □

We finally recall that extinction in finite time occurs for $p \in (p_c, 2]$ and $q \in (0, p/2)$ when the initial condition u_0 is compactly supported.

Proposition 2.3. *Let $p \in (p_c, 2]$ and $q \in (0,p/2)$. Let u_0 be an initial condition satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Assume further that u_0 is compactly supported. There exists $T_e > 0$ depending on N, p, q, and u_0 such that*

$$\mathcal{P}(t) = \emptyset \text{ for } t \ge T_e, \qquad \mathcal{P}(t) \ne \emptyset \text{ for } t \in [0, T_e).$$

Proposition 2.3 is shown in [22, Corollary 9.1] for $p = 2$ and in [25, Theorem 1.2(iii)] for $p < 2$. It is actually proved in the latter that finite time extinction takes place for a broader class of initial data, namely, if there exist $C_0 > 0$ and $Q > 0$ such that

$$u_0(x) \le C_0|x|^{-(p-Q)/(Q-p+1)}, \qquad x \in \mathbb{R}^N, \tag{2.2}$$

where

$$Q = q \text{ if } q \in (q_1, p/2), \quad Q \in (q_1, p/2) \text{ if } q \in (0, q_1], \quad \text{and } q_1 := \max\left\{p-1, \frac{N}{N+1}\right\} \ .$$

Remark 2.4. *It is worth pointing out here that Theorem 1.3 includes [25, Theorem 1.2(iii)] when the range of (p,q) is* (1.4). *Indeed, if u_0 satisfies* (2.2) *then it satisfies* (1.9).

3 Instantaneous shrinking

In this section, we show that the phenomenon of instantaneous shrinking takes place, thus proving the first assertion in Theorem 1.1. Its proof will be completed in the next section, which deals with the localization part, once the support is known to be compact. More precisely, we show here the following result.

Proposition 3.1. *Let u be a solution to the Cauchy problem* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.3) *and* (1.7) *for some $C > 0$, and exponents p, q as in* (1.4). *There exists $t_0 > 0$ such that $\mathcal{P}(t)$ is bounded for $t \in (0, t_0)$.*

Proof. We look for a supersolution to Eq. (1.1) of the form

$$\Sigma(t,x) = \left[\frac{A}{1+|x|^\alpha} - \eta(t)\right]_+^\gamma, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{3.1}$$

where $z_+ := \max\{z, 0\}$ denotes the positive part of the real number z, the positive parameters $A > 0$, $\alpha \in (0,1)$, $\gamma > 1$, and the function η being to be determined. For further use, we set

$$r := |x|, \quad y := \frac{A}{1+r^\alpha} - \eta(t).$$

Owing to the radial symmetry of Σ, it follows from (1.15) that

$$\mathcal{L}\Sigma = \partial_t\Sigma - (p-1)|\partial_r\Sigma|^{p-2}\partial_r^2\Sigma - \frac{N-1}{r}|\partial_r\Sigma|^{p-2}\partial_r\Sigma + |\partial_r\Sigma|^q.$$

We further require that $\eta(0) = 0$ and that η is non-decreasing, that is, $\eta' \geq 0$. In the previous notation, we notice that

$$\partial_t\Sigma(t,x) = -\gamma y_+^{\gamma-1}\eta'(t),$$

$$\partial_r\Sigma(t,x) = -A\alpha\gamma y_+^{\gamma-1}\frac{r^{\alpha-1}}{(1+r^\alpha)^2},$$

and

$$\begin{aligned}\partial_r^2\Sigma(t,x) = {} & A^2\alpha^2\gamma(\gamma-1)y_+^{\gamma-2}\frac{r^{2\alpha-2}}{(1+r^\alpha)^4} \\ & - A\alpha\gamma y_+^{\gamma-1}\frac{r^{\alpha-2}}{(1+r^\alpha)^3}\left[\alpha - 1 - (\alpha+1)r^\alpha\right],\end{aligned}$$

whence, we get

$$
\begin{aligned}
\mathcal{L}\Sigma &= -\gamma y_+^{\gamma-1}\eta'(t) + (A\alpha\gamma)^q y_+^{q(\gamma-1)} \frac{r^{q(\alpha-1)}}{(1+r^\alpha)^{2q}} - (A\alpha\gamma)^{p-2} y_+^{(\gamma-1)(p-2)} \frac{r^{(\alpha-1)(p-2)}}{(1+r^\alpha)^{2(p-2)}} \\
&\times \left[(p-1)A^2\alpha^2\gamma(\gamma-1)y_+^{\gamma-2}\frac{r^{2\alpha-2}}{(1+r^\alpha)^4} - (p-1)A\alpha\gamma y_+^{\gamma-1}\frac{r^{\alpha-2}}{(1+r^\alpha)^3}\left(\alpha-1-(\alpha+1)r^\alpha\right)\right. \\
&\left. - A\alpha\gamma(N-1)y_+^{\gamma-1}\frac{r^{\alpha-2}}{(1+r^\alpha)^2}\right] \\
&= (A\alpha\gamma)^q y_+^{q(\gamma-1)} \frac{r^{q(\alpha-1)}}{(1+r^\alpha)^{2q}} - \gamma y_+^{\gamma-1}\eta'(t) + (A\alpha\gamma)^{p-1} y_+^{(\gamma-1)(p-1)-1} \frac{r^{(\alpha-1)(p-1)-1}}{(1+r^\alpha)^{2(p-1)}} \\
&\times \left[(N-1)y_+ + (p-1)y_+\frac{2\alpha-(1+\alpha)(1+r^\alpha)}{1+r^\alpha} - A\alpha(\gamma-1)(p-1)\frac{r^\alpha}{(1+r^\alpha)^2}\right] \\
&\geq (A\alpha\gamma)^q y_+^{q(\gamma-1)} \frac{r^{q(\alpha-1)}}{(1+r^\alpha)^{2q}} - \gamma y_+^{\gamma-1}\eta'(t) + (A\alpha\gamma)^{p-1} y_+^{(\gamma-1)(p-1)-1} \frac{r^{(\alpha-1)(p-1)-1}}{(1+r^\alpha)^{2(p-1)}} \\
&\times \left[-(1+\alpha)(p-1)y_+ - \frac{A\alpha(\gamma-1)(p-1)}{1+r^\alpha}\right],
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
\mathcal{L}\Sigma &\geq (A\alpha\gamma)^q y_+^{q(\gamma-1)} \frac{r^{q(\alpha-1)}}{(1+r^\alpha)^{2q}} - \gamma y_+^{\gamma-1}\eta'(t) \\
&- (A\alpha\gamma)^{p-1} A\alpha(\gamma-1)(p-1) y_+^{(\gamma-1)(p-1)-1} \frac{r^{(\alpha-1)(p-1)-1}}{(1+r^\alpha)^{2(p-1)+1}} \\
&- (1+\alpha)(p-1)(A\alpha\gamma)^{p-1} y_+^{(\gamma-1)(p-1)} \frac{r^{(\alpha-1)(p-1)-1}}{(1+r^\alpha)^{2(p-1)}}.
\end{aligned} \tag{3.2}
$$

We now choose

$$
\gamma = \frac{p-q}{p-q-1} > 1,
$$

so that

$$
\gamma - 1 = \frac{1}{p-1-q} \qquad \text{and} \qquad (\gamma-1)(p-1)-1 = q(\gamma-1).
$$

Since $\gamma > 1$ and $\eta \geq 0$, we notice that

$$
y_+ \leq \frac{A}{1+r^\alpha}, \qquad \text{and} \qquad y_+^{\gamma-1} = y_+^{q(\gamma-1)} y_+^{(1-q)(\gamma-1)} \leq y_+^{q(\gamma-1)} \frac{A^{(1-q)(\gamma-1)}}{(1+r^\alpha)^{(1-q)(\gamma-1)}}. \tag{3.3}
$$

Consequently, owing to the fact that $\eta'(t) \geq 0$ for all $t > 0$ and plugging (3.3) into (3.2), we deduce

$$\begin{aligned}
\mathcal{L}\Sigma \geq\ & y_+^{q(\gamma-1)} \left[(A\alpha\gamma)^q \frac{r^{q(\alpha-1)}}{(1+r^\alpha)^{2q}} - \frac{A^{(1-q)(\gamma-1)}\gamma}{(1+r^\alpha)^{(1-q)(\gamma-1)}} \eta'(t) \right. \\
& \left. -(p-1)(\alpha\gamma)^{p-1} A^p (1+\alpha\gamma) \frac{r^{(\alpha-1)(p-1)-1}}{(1+r^\alpha)^{2(p-1)+1}} \right] \\
\geq\ & y_+^{q(\gamma-1)} \left[\frac{(A\alpha\gamma)^q}{2} \frac{r^{q(\alpha-1)}}{(1+r^\alpha)^{2q}} - \frac{A^{(1-q)(\gamma-1)}\gamma}{(1+r^\alpha)^{(1-q)(\gamma-1)}} \eta'(t) \right. \\
& \left. + \frac{(A\alpha\gamma)^q}{2} \frac{r^{q(\alpha-1)}}{(1+r^\alpha)^{2q}} - (p-1)(\alpha\gamma)^{p-1} A^p (1+\alpha\gamma) \frac{r^{(\alpha-1)(p-1)-1}}{(1+r^\alpha)^{2(p-1)+1}} \right] \\
=\ & y_+^{q(\gamma-1)} \frac{A^{(1-q)(\gamma-1)}\gamma}{(1+r^\alpha)^{(1-q)(\gamma-1)}} S_1 + y_+^{q(\gamma-1)} \frac{(A\alpha\gamma)^q}{2} \frac{r^{(\alpha-1)(p-1)-1}}{(1+r^\alpha)^{2(p-1)+1}} S_2 \ ,
\end{aligned} \tag{3.4}$$

where

$$S_1 := \frac{(\alpha\gamma)^q}{2\gamma} \frac{A^{(q-1)\gamma+1} r^{q(\alpha-1)}}{(1+r^\alpha)^{q(\gamma+1)+1-\gamma}} - \eta'(t)$$

and

$$S_2 := \frac{r^{1+(\alpha-1)(q-p+1)}}{(1+r^\alpha)^{2(q-p+1)-1}} - 2(p-1)(\alpha\gamma)^{p-1-q} A^{p-q} (1+\alpha\gamma)$$

Our goal is now to show that Σ is a supersolution to the Cauchy problem (1.1)-(1.2) in $(0,t_0) \times \left(\mathbb{R}^N \setminus B(0,R)\right)$ for some $t_0 > 0$ small enough and $R > 1$ sufficiently large. To this end, we estimate separately S_1 and S_2. On the one hand, since $2(p-1-q) > 0$ by (1.4),

$$\begin{aligned}
S_2 &= r^{1+(\alpha-1)(q-p+1)} (1+r^\alpha)^{1+2(p-1-q)} - 2(p-1)(\alpha\gamma)^{p-1-q} A^{p-q} (1+\alpha\gamma) \\
&\geq r^{(\alpha+1)(p-q)} - 2(1+\alpha\gamma)(p-1)(\alpha\gamma)^{p-1-q} A^{p-q} \\
&\geq R^{(\alpha+1)(p-q)} - 2(1+\alpha\gamma)(p-1)(\alpha\gamma)^{p-1-q} A^{p-q},
\end{aligned} \tag{3.5}$$

provided $r > R$. On the other hand, for $r > R$ such that $y > 0$,

$$\left(\frac{r^\alpha}{1+r^\alpha} \right)^{q(\alpha-1)/\alpha} \geq 1 \quad \text{and} \quad \frac{1}{1+r^\alpha} \geq \frac{\eta(t)}{A},$$

since $\alpha \in (0,1)$, hence

$$\begin{aligned}
S_1 &\geq \frac{(\alpha\gamma)^q}{2\gamma} A^{(q-1)\gamma+1} \left(\frac{1}{1+r^\alpha} \right)^{(\alpha(1+q\gamma-\gamma)+q)/\alpha} - \eta'(t) \\
&\geq \frac{(\alpha\gamma)^q}{2\gamma} A^{(q-1)\gamma+1} \left(\frac{\eta(t)}{A} \right)^{(\alpha(1+q\gamma-\gamma)+q)/\alpha} - \eta'(t),
\end{aligned}$$

provided that $\alpha(1+q\gamma-\gamma)+q > 0$. Since $q+1 < p \leq 2$,

$$1 + q\gamma - \gamma = \frac{q(p-q)-1}{p-q-1} \leq \frac{q(2-q)-1}{p-q-1} = -\frac{(1-q)^2}{p-q-1} < 0 \ ,$$

and the previous condition on α reads

$$\alpha < \alpha_2 := \min\left\{ \frac{q}{\gamma(1-q)-1}, 1 \right\} \ .$$

The lower bound on S_1 then becomes

$$S_1 \geq \frac{(\alpha\gamma)^q}{2\gamma} A^{-q/\alpha} \eta(t)^{(\alpha(1+q\gamma-\gamma)+q)/\alpha} - \eta'(t). \tag{3.6}$$

We thus choose η as the solution to the differential equation

$$\eta'(t) = \frac{(\alpha\gamma)^q}{2\gamma} A^{-q/\alpha} \eta(t)^{(\alpha(1+q\gamma-\gamma)+q)/\alpha}, \qquad \eta(0) = 0, \tag{3.7}$$

which is possible when $(\alpha(1+q\gamma-\gamma)+q)/\alpha < 1$. Taking into account the precise value of γ, this condition reads

$$\alpha > \alpha_1 := \frac{p-1-q}{p-q} \frac{q}{1-q} = \frac{q}{\gamma(1-q)}.$$

It is easy to check that $\alpha_1 < \alpha_2$, so that, any $\alpha \in (\alpha_1, \alpha_2)$ satisfies the conditions we assumed up to now. Since $S_1 \geq 0$ by (3.6) and (3.7), we infer from (3.4) and (3.5) that Σ is a supersolution to (1.1) in $(0,\infty) \times \mathbb{R}^N \setminus B(0,R)$, provided that

$$R^{(\alpha+1)(p-q)} \geq 2(1+\alpha\gamma)(p-1)(\alpha\gamma)^{p-1-q} A^{p-q}\ . \tag{3.8}$$

It remains to check that Σ is a supersolution also for the initial and boundary conditions, that is

$$\Sigma(0,x) = \left(\frac{A}{1+r^\alpha}\right)^\gamma \geq u_0(x) \quad \text{for any } x \in \mathbb{R}^N \setminus B(0,R), \tag{3.9}$$

and

$$\Sigma(t,x) \geq u(t,x), \quad \text{for any } t \in (0,t_0),\ |x| = R. \tag{3.10}$$

In order to check (3.9), we first readily notice that in the range given by (1.4)

$$\gamma = \frac{p-q}{p-1-q} > \frac{q}{1-q},$$

so that, owing to (1.7), there exists $\theta' > 0$ such that

$$u_0(x) \leq C(1+|x|)^{-\theta'}, \quad \frac{q}{1-q} = \gamma\alpha_1 < \theta' < \gamma\alpha_2 = \min\left\{\frac{\gamma q}{\gamma(1-q)-1}, \gamma\right\}\ .$$

Indeed, if the initial decay exponent θ satisfies $\theta < \gamma\alpha_2$, then we take $\theta' = \theta$, while, if $\theta \geq \gamma\alpha_2$, we anyway have

$$u_0(x) \leq C(1+|x|)^{-\theta} \leq C(1+|x|)^{-\theta'}, \qquad x \in \mathbb{R}^N,$$

for any $\theta' \in (q/(1-q), \gamma\alpha_2)$. Then, we define

$$\alpha := \frac{\theta'}{\gamma} \in (\alpha_1, \alpha_2), \tag{3.11}$$

and we have

$$\begin{aligned} u_0(x) &\leq C(1+|x|)^{-\theta'} = \frac{C}{(1+r)^{\alpha\gamma}} \leq \frac{C}{2^{\gamma(\alpha-1)}(1+r^\alpha)^\gamma} \\ &= \left(\frac{C^{1/\gamma}}{2^{\alpha-1}} \frac{1}{1+r^\alpha}\right)^\gamma, \end{aligned}$$

after using the elementary inequality

$$(1+r)^\alpha \geq 2^{\alpha-1}(1+r^\alpha).$$

We thus derive the inequality in (3.9) by requiring that

$$A \geq \frac{C^{1/\gamma}}{2^{\alpha-1}}. \tag{3.12}$$

In order to establish (3.10), we further prescribe the following condition:

$$A > (1+R^\alpha)\|u_0\|_\infty^{1/\gamma}, \tag{3.13}$$

with α already chosen in (3.11). By a direct integration of the differential equation (3.7), we obtain

$$\eta(t) = \left[\frac{(\alpha\gamma)^q(1-\beta)}{2\gamma}A^{-q/\alpha}\right]^{1/(1-\beta)} t^{1/(1-\beta)}, \qquad \beta := \frac{\alpha(1+q\gamma-\gamma)+q}{\alpha} \in (0,1),$$

for $t \geq 0$. Taking into account that $\eta(t) \to 0$ as $t \to 0$, there exists $t_0 > 0$ sufficiently small such that, for $t \in [0, t_0]$ and $x \in \mathbb{R}^N$, $|x| = R$,

$$\left(\frac{A}{1+R^\alpha} - \eta(t)\right)^\gamma \geq \left(\frac{A}{1+R^\alpha} - \eta(t_0)\right)^\gamma > \|u_0\|_\infty \geq u(t,x),$$

which implies (3.10). It only remains to show that the conditions (3.8), (3.12) and (3.13) are compatible. To this end, we first choose $R > 0$ sufficiently large such that it satisfies simultaneously the following estimates:

$$\begin{aligned} R^\alpha &> \frac{C^{1/\gamma}}{2^{\alpha-1}\|u_0\|_\infty^{1/\gamma}} - 1, \\ R^{\alpha+1} &> 2\left[2(p-1)(1+\alpha\gamma)(\alpha\gamma)^{p-1-q}\right]^{1/(p-q)}(1+R^\alpha)\|u_0\|_\infty^{1/\gamma}, \end{aligned} \tag{3.14}$$

then choose $A > 0$ such that

$$(1+R^\alpha)\|u_0\|_\infty^{1/\gamma} < A < 2(1+R^\alpha)\|u_0\|_\infty^{1/\gamma}. \tag{3.15}$$

It is immediate to check that these choices of R and A satisfy (3.8), (3.12), and (3.13). Thus, letting $\gamma = (p-q)/(p-q-1)$, α as in (3.11), R as in (3.14), and A as in (3.15), the function Σ introduced in (3.1) is a supersolution to the Cauchy problem (1.1)-(1.2) in $(0, t_0) \times \mathbb{R}^N \setminus B(0,R)$. By the comparison principle, we get

$$u(t,x) \leq \Sigma(t,x), \qquad t \in (0,t_0), \ |x| \geq R.$$

Consequently, for $t \in (0, t_0]$, we infer from the definition (3.1) of Σ that $\Sigma(t,x) = 0$ for $|x|^\alpha > A/\eta(t) - 1$, so that $\mathcal{P}(t)$ is bounded. □

4 Propagation of the support. Localization

In this section, we complete the proof of Theorem 1.1 by showing the second statement, observing that it readily implies the third one by Proposition 2.3. To this end we actually establish that, if u_0 vanishes sufficiently rapidly at $x_0 \in \mathbb{R}^N$, then $u(t,x_0) = 0$ for all $t \geq 0$.

Proposition 4.1. *Let u_0 be an initial condition satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Assume further that (p,q) satisfies* (1.4) *and that there is $x_0 \in \mathbb{R}^N$ such that* (1.10) *holds true, with constants $\kappa_{p,q}$ and ω given in* (1.11)*. Then $u(t,x_0) = 0$ for all $t \geq 0$.*

Proof. Owing to the invariance by translation of (1.1) we may assume without loss of generality that $x_0 = 0$. Setting $\Sigma(x) := \kappa_{p,q} r^\omega$ and $r := |x|$ for $x \in \mathbb{R}^N$, we infer from the radial symmetry of Σ and (1.15) that

$$\mathcal{L}\Sigma(x) = (\omega\kappa_{p,q})^q r^{q(\omega-1)} - (\omega\kappa_{p,q})^{p-1}\left[(p-1)(\omega-1)+N-1\right] r^{(\omega-1)(p-1)-1} .$$

Since

$$q(\omega-1) = \frac{q}{p-1-q} = (\omega-1)(p-1)-1 \quad \text{ and } \quad (p-1)(\omega-1)+N-1 = (\omega\kappa_{p,q})^{q+1-p} ,$$

we end up with

$$\mathcal{L}\Sigma(x) = (\omega\kappa_{p,q})^q r^{q(\omega-1)}\left[1-(\omega\kappa_{p,q})^{p-1-q}(\omega\kappa_{p,q})^{q+1-p}\right] = 0 .$$

Consequently Σ is a solution to (1.1) and we infer from (1.10) and the comparison principle that $u(t,x) \leq \Sigma(x)$ for $(t,x) \in [0,\infty)\times\mathbb{R}^N$. In particular, $u(t,0) \leq \Sigma(0) = 0$ for $t \geq 0$ as claimed □

The localization property is then a straightforward consequence of Proposition 4.1.

Proposition 4.2. *Let u_0 be an initial condition satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Assume further that (p,q) satisfies* (1.4) *and that u_0 is compactly supported. There exists $R > 0$ such that*

$$\mathcal{P}(t) \subseteq B(0,R), \qquad \text{for any } t \geq 0 .$$

Proof. Let $R_0 > 0$ be such that $\mathcal{P}(0) \subseteq B(0,R_0)$ and set $R := R_0 + \left(\|u_0\|_\infty \kappa_{p,q}^{-1}\right)^{1/\omega}$ with $\kappa_{p,q}$ and ω defined in (1.11). Consider $x_0 \in \mathbb{R}^N$ such that $|x_0| \geq R$. On the one hand, for $x \in B(x_0, R-R_0)$,

$$|x| \geq |x_0| - |x-x_0| \geq R-(R-R_0) = R_0$$

and $u_0(x) = 0 \leq \kappa_{p,q}|x-x_0|^\omega$. On the other hand, for $x \notin B(x_0,R-R_0)$,

$$u_0(x) \leq \|u_0\|_\infty = \kappa_{p,q}(R-R_0)^\omega \leq \kappa_{p,q}|x-x_0|^\omega .$$

We are then in a position to apply Proposition 4.1 and conclude that $u(t,x_0) = 0$ for all $t \geq 0$. Since x_0 is arbitrary in $\mathbb{R}^N \setminus B(0,R)$ we have shown that $\mathcal{P}(t) \subseteq B(0,R)$ for all $t \geq 0$ and the proof is complete. □

Theorem 1.1 is now an immediate corollary of Propositions 2.3, 3.1 and 4.2. Indeed, there is $t_0 > 0$ such that $\mathcal{P}(t)$ is bounded for $t \in (0,t_0]$ by Proposition 3.1. Proposition 2.3 applied to $u(\cdot + t_0)$ then implies the finite time extinction of u while Proposition 4.2 guarantees the localization property for $t \geq \tau$ and $\tau > 0$.

Another consequence of Proposition 4.1 is the *infinite waiting time* phenomenon, that is, the fact that the positivity set might not expand with time.

Proposition 4.3. *Let u_0 be an initial condition satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Assume further that (p,q) satisfies* (1.4) *and that u_0 is compactly supported in $B(0,R_0)$ for some $R_0>0$ and satisfies* (1.10) *for all $x_0\in\partial B(0,R_0)$. Then*

$$\mathcal{P}(t)\subseteq\mathcal{P}(0)\ , \qquad t\ge 0\ .$$

Proof. In view of Proposition 4.1 it is sufficient to check that u_0 satisfies (1.10) for all $x_0\in\mathbb{R}^N$ such that $|x_0|>R_0$. Indeed, consider $x_0\in\mathbb{R}^N$ with $|x_0|>R_0$. Set $\xi:=x_0/|x_0|$. Since (1.10) is satisfied for $R_0\xi$, there holds

$$u_0(x)\le\kappa_{p,q}|x-R_0\xi|^\omega\ , \qquad x\in\mathbb{R}^N\ .$$

Consider now $x\in B(0,R_0)$. Then $x=\langle x,\xi\rangle\xi+y$ and, since $|\langle x,\xi\rangle|\le|x|<R_0$,

$$|x-R_0\xi|^2=(R_0-\langle x,\xi\rangle)^2+|y|^2\le(|x_0|-\langle x,\xi\rangle)^2+|y|^2=|x-x_0|^2\ .$$

Consequently, $u_0(x)\le\kappa_{p,q}|x-x_0|^\omega$ for all $x\in B(0,R_0)$. This inequality being obviously true for $x\notin B(0,R_0)$ since $u_0(x)=0$, we have thus shown that u_0 satisfies (1.10) for all $x_0\notin B(0,R_0)$ and Proposition 4.3 readily follows from Proposition 4.1. □

The last result of this section is devoted to the optimality of the range (1.4) of the parameters p and q for instantaneous shrinking to take place.

Proposition 4.4. *Consider an initial condition u_0 satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Let $p\in(p_c,2)$ and assume that u_0 satisfies* (2.2)*. If $q\in[p-1,p/2)$ then u vanishes identically after a finite time T_e and*

$$\mathcal{P}(t)=\mathbb{R}^N \quad \text{for} \quad t\in(0,T_e)\ .$$

Observe that Proposition 4.4 does not apply to $p=2$ as the range $[p-1,p/2)$ is empty in that case.

Proof. The occurrence of finite time extinction is a consequence of the discussion after Proposition 2.3. Fix $t_0\in(0,T_e)$. By Proposition 2.3, there exists $x_0\in\mathbb{R}^N$ such that $u(t_0,x_0)>0$. Let $B(x_0,\varrho)$ be the largest ball centered at x_0 and included in $\mathcal{P}(t_0)$. We split the analysis into two cases:

- $q\in(p-1,p/2)$: it follows from the gradient estimates [25, Theorem 1.3(iii)] that for any $x\in B(x_0,\varrho)$, we have

$$\left|\nabla u^{-(q-p+1)/(p-q)}(t_0,x)\right|\le K_0:=C\left(1+\|u_0\|_\infty^{(p-2q)/p(p-q)}t_0^{-1/p}\right),$$

hence

$$u^{-(q-p+1)/(p-q)}(t_0,x)\le u^{-(q-p+1)/(p-q)}(t_0,x_0)+K_0|x-x_0|,$$

or equivalently

$$\left[K_0|x-x_0|+u^{-(q-p+1)/(p-q)}(t_0,x_0)\right]^{-(p-q)/(q-p+1)}\le u(t_0,x).$$

Assuming now for contradiction that $\varrho<\infty$, there exists $y\in\partial B(x_0,\varrho)$ such that $u(t_0,y)=0$, contradicting the previous lower bound. Consequently, $\varrho=\infty$ and thus $\mathcal{P}(t_0)=\mathbb{R}^N$.

- $q = p-1$: it follows from [25, Theorem 1.3(iv)] that, for $x \in B(x_0, \varrho)$,

$$|\nabla \log u(t_0,x)| \le K_0 := C\left(1 + \|u_0\|_\infty^{(2-p)/p} t_0^{-1/p}\right),$$

whence

$$\log u(t_0,x_0) \le \log u(t_0,x) + K_0|x-x_0|,$$

which implies

$$0 < u(t_0,x_0) e^{-K_0|x-x_0|} \le u(t_0,x),$$

for any $x \in B(x_0,\varrho)$. Arguing as in the previous case, this readily implies that $\varrho = \infty$ and thus $\mathcal{P}(t_0) = \mathbb{R}^N$. □

5 Upper and lower bounds at the extinction time

An interesting consequence of the localization property established in the previous section is the derivation of temporal upper and lower bounds for the L^∞-norm of solutions to (1.1)-(1.2) with compactly supported initial data.

Proposition 5.1. *Let u_0 be an initial condition satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Assume further that the exponents p and q satisfy* (1.4) *and that u_0 is compactly supported. Then u vanishes identically at a finite time $T_e > 0$ and there is $C_1 > 0$ such that*

$$\|u(t)\|_\infty \ge C_1 (T_e - t)^{1/(1-q)}\ , \qquad t \in [0,T_e]\ . \tag{5.1}$$

In addition, given $\vartheta \in (0,1)$, there is $C_2(\vartheta) > 0$ such that

$$\|u(t)\|_\infty \le C_2(\vartheta) (T_e - t)^{\vartheta(p-q)/(p-2q)}\ , \qquad t \in [T_e/2,T_e]\ . \tag{5.2}$$

Proof. We first infer from the compactness of the support of u_0, Proposition 2.3, and Proposition 4.2 that the extinction time T_e of u is positive and finite and that there exists $R > 0$ such that

$$\mathcal{P}(t) \subseteq B(0,R)\ , \qquad t \in [0,T_e]\ . \tag{5.3}$$

We next recall the following gradient estimates, the first one being proved in [25, Theorem 1.3(v)] for $p \in (p_c,2)$ and in Proposition A.1 for $p=2$:

$$\left|\nabla u^{(p-q-1)/(p-q)}(t,x)\right| \le C_1 \left(1 + \|u(s)\|_\infty^{(p-2q)/p(p-q)} (t-s)^{-1/p}\right), \tag{5.4}$$

and the next one proved in [25, Theorem 1.7] for $p \in (p_c,2)$ and in [20, Theorem 2] for $p=2$:

$$|\nabla u(t,x)| \le C_2 \|u(s)\|_\infty^{1/q} (t-s)^{-1/q} \tag{5.5}$$

for $(t,x) \in (0,\infty)\times\mathbb{R}^N$ and $s \in [0,t)$, the constants C_1 and C_2 depending only on N, p, and q. Thanks to (5.3) and (5.5), we may argue as in [30, Proposition 3.5] and establish that

$$\|u(t)\|_\infty \ge C (T_e - t)^{1/(1-q)}, \qquad t \in (0,T_e)\ .$$

We next deduce from (5.4) that

$$\left|\nabla u^{(p-q-1)/(p-q)}(t,x)\right| \le C\ , \qquad (t,x) \in [T_e/2,T_e]\times\mathbb{R}^N\ ,$$

so that

$$|\nabla u(t,x)| \le C u(t,x)^{1/(p-q)}, \qquad x \in \mathcal{P}(t)\ , \quad t \in [T_e/2, T_e]\ . \tag{5.6}$$

Consider $\varrho \ge 1$ and $t \in [T_e/2, T_e]$. We infer from (1.1) that

$$\begin{aligned} \frac{1}{\varrho+1} \frac{d}{dt} \|u(t)\|_{\varrho+1}^{\varrho+1} &= - \int_{\mathbb{R}^N} \left(\varrho u^{\varrho-1} |\nabla u|^p + u^\varrho |\nabla u|^q \right)\ dx \\ &= - \int_{\mathcal{P}(t)} \left(\varrho u^{\varrho-1} |\nabla u|^p + u^\varrho |\nabla u|^q \right)\ dx\ . \end{aligned}$$

Integrating with respect to time over (t, T_e) and using (5.6), we obtain

$$\begin{aligned} \|u(t)\|_{\varrho+1}^{\varrho+1} &= (\varrho+1) \int_t^{T_e} \int_{\mathcal{P}(s)} \left(\varrho u^{\varrho-1} |\nabla u|^p + u^\varrho |\nabla u|^q \right)\ dxds \\ &\le C(1+\varrho)^2 \int_t^{T_e} \int_{\mathcal{P}(s)} u^{(\varrho(p-q)+q)/(p-q)}\ dxds\ . \end{aligned}$$

It then follows from (5.3), the Hölder inequality, and the time monotonicity of $s \mapsto \|u(s)\|_{\varrho+1}$ that

$$\begin{aligned} \|u(t)\|_{\varrho+1}^{\varrho+1} &\le C(1+\varrho)^2 |B(0,R)|^{(p-2q)/(\varrho+1)(p-q)} \int_t^{T_e} \|u(s)\|_{\varrho+1}^{(\varrho(p-q)+q)/(p-q)}\ ds \\ &\le C(\varrho)(T_e - t) \|u(t)\|_{\varrho+1}^{(\varrho(p-q)+q)/(p-q)}\ . \end{aligned}$$

Consequently,

$$\|u(t)\|_{\varrho+1} \le C(\varrho)(T_e - t)^{(p-q)/(p-2q)}\ , \qquad t \in [T_e/2, T_e]\ . \tag{5.7}$$

We finally combine (5.6) and the Gagliardo-Nirenberg inequality to obtain, for $t \in [T_e/2, T_e]$,

$$\begin{aligned} \|u(t)\|_\infty &\le C(\varrho) \|\nabla u(t)\|_\infty^{N/(\varrho+1+N)} \|u(t)\|_{\varrho+1}^{(\varrho+1)/(\varrho+1+N)} \\ &\le C(\varrho) \|u(t)\|_\infty^{N/(\varrho+1+N)(p-q)} \|u(t)\|_{\varrho+1}^{(\varrho+1)/(\varrho+1+N)}\ . \end{aligned}$$

Consequently

$$\|u(t)\|_\infty \le C(\varrho) \|u(t)\|_{\varrho+1}^{(\varrho+1)(p-q)/((\varrho+1)(p-q)+N(p-1-q))}\ ,$$

which, together with (5.7), gives (5.2) as ϱ can be chosen arbitrarily large. □

When $p = 2$ the upper bound (5.2) can be improved, allowing the value $\vartheta = 1$ in (5.2).

Proposition 5.2. *Let u_0 be an initial condition satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Assume further that $p = 2$, $q \in (0,1)$, and that u has a finite extinction time $T_e > 0$. Then there is $C_3 > 0$ depending on q, T_e, and u_0 such that*

$$\|u(t)\|_\infty \le C_3 (T_e - t)^{(2-q)/(2-2q)}\ , \qquad t \in [0, T_e]\ .$$

Proof. We adapt the proof of [24, Proposition 2.2]. Assume for contradiction that, for each $n \ge 1$, there are $x_n \in \mathbb{R}^N$ and $t_n \in (0, T_e)$ such that

$$u(t_n, x_n) \ge n (T_e - t_n)^{(2-q)/(2-2q)}\ . \tag{5.8}$$

Since $u \le \|u_0\|_\infty$ and $q < 1$, it readily follows from (5.8) that

$$\lim_{n\to\infty} t_n = T_e \ . \tag{5.9}$$

As $u(T_e) \equiv 0$, we infer from the variation of constants formula for (1.1) that for $t \in (0, T_e)$

$$0 = u(T_e) = e^{(T_e-t)\Delta} u(t) - \int_t^{T_e} e^{(T_e-s)\Delta} |\nabla u(s)|^q ds \ . \tag{5.10}$$

Now, for $t \in (T_e/2, T_e)$, it follows from Proposition A.1 and the properties of the fundamental solution of the heat equation that there is $c_1 > 0$ such that

$$\begin{aligned}
I(t,x) \ &:= \left(\int_t^{T_e} e^{(T_e-s)\Delta} |\nabla u(s)|^q ds \right)(x) \\
&= \int_t^{T_e} \frac{1}{(4\pi(T_e-s))^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4(T_e-s)} \right) |\nabla u(s,y)|^q dy ds \\
&= \left(\frac{2-q}{1-q} \right)^q \int_t^{T_e} \frac{1}{(4\pi(T_e-s))^{N/2}} \\
&\quad \times \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4(T_e-s)} \right) u^{q/(2-q)}(s,y) |\nabla u^{(1-q)/(2-q)}(s,y)|^q dy ds \\
&\le c_1 \int_t^{T_e} \frac{1}{(4\pi(T_e-s))^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4(T_e-s)} \right) u^{q/(2-q)}(s,y) dy ds \\
&\le c_1 \int_t^{T_e} \left[\frac{1}{(4\pi(T_e-s))^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4(T_e-s)} \right) u(s,y) dy \right]^{q/(2-q)} ds,
\end{aligned} \tag{5.11}$$

where we use Jensen's inequality for concave functions in the last step in view of $q < 2-q$. Introducing

$$\begin{aligned}
h(t,x) &:= \left(e^{(T_e-t)\Delta} u(t) \right)(x) = \frac{1}{(4\pi(T_e-t))^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4(T_e-t)} \right) u(t,y) dy \ , \\
H(t,x) &:= \int_t^{T_e} h^{q/(2-q)}(s,x) ds
\end{aligned}$$

for $(t,x) \in (T_e/2, T_e) \times \mathbb{R}^N$, we infer from (5.10) and (5.11) that

$$h(t,x) = I(t,x) \le c_1 H(t,x), \qquad (t,x) \in (T_e/2, T_e) \times \mathbb{R}^N \ . \tag{5.12}$$

This implies that

$$-\partial_t H(t,x) = h^{q/(2-q)}(t,x) \le c_2 H^{q/(2-q)}(t,x)$$

with some $c_2 > 0$, whence an integration shows that

$$-\frac{2-q}{2-2q} \left(H^{(2-2q)/(2-q)}(T_e,x) - H^{(2-2q)/(2-q)}(t,x) \right) \le c_2 (T_e - t) \ ,$$

for $(t,x) \in (T_e/2, T_e) \times \mathbb{R}^N$. Consequently, using $H(T_e) \equiv 0$ and (5.12), we obtain $c_3 > 0$ such that

$$h(t,x) \le c_1 H(t,x) \le c_3 (T_e - t)^{(2-q)/(2-2q)} \ , \qquad (t,x) \in (T_e/2, T_e) \times \mathbb{R}^N \ . \tag{5.13}$$

Now in view of (5.9) there is $n_0 \in \mathbb{N}$ such that $t_n \in (T_e/2, T_e)$ for all $n \geq n_0$. We use once more Proposition A.1 along with (5.8) to obtain $c_4 > 0$ such that, for all $n \geq n_0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned} u^{(1-q)/(2-q)}(t_n, x) &\geq u^{(1-q)/(2-q)}(t_n, x_n) - c_4|x - x_n| \\ &\geq n^{(1-q)/(2-q)}(T_e - t_n)^{1/2} - c_4|x - x_n| \geq \frac{n^{(1-q)/(2-q)}}{2}(T_e - t_n)^{1/2} , \end{aligned}$$

provided that $x \in B\left(x_n, \frac{n^{(1-q)/(2-q)}}{2c_4}(T_e - t_n)^{1/2}\right)$. We then infer from the latter estimate, (5.13), the definition of h, and the nonnegativity of u that

$$\begin{aligned} c_3(T_e - t_n)^{(2-q)/(2-2q)} &\geq h(t_n, x_n) \\ &\geq \frac{1}{(4\pi(T_e - t_n))^{N/2}} \int_{B\left(x_n, \frac{n^{(1-q)/(2-q)}}{2c_4}(T_e - t_n)^{1/2}\right)} \exp\left(-\frac{|x_n - y|^2}{4(T_e - t_n)}\right) \\ &\qquad \times \frac{n}{2^{(2-q)/(1-q)}}(T_e - t_n)^{(2-q)/(2-2q)} dy \end{aligned}$$

for all $n \geq n_0$. We conclude that

$$c_3 \geq \frac{n}{2^{(2-q)/(1-q)}(4\pi)^{N/2}} \int_{B\left(0, \frac{n^{(1-q)/(2-q)}}{2c_4}\right)} \exp\left(-\frac{|z|^2}{4}\right) dz$$

for all $n \geq n_0$, and a contradiction. □

6 Non-extinction

This section is devoted to the proof of Theorem 1.2. As in [7] (which only deals with the case $p = 2$), it relies on the comparison principle, this time with suitable subsolutions which we construct now.

Lemma 6.1. *Assume that* (p, q) *satisfies* (1.4). *There exists* $b_0 > 0$ *depending only on* p *and* q *such that, given* $T > 0$, $a > 0$, *and* $b \in (0, b_0)$, *the function*

$$w(t, x) := (T - t)^{1/(1-q)} \left(a + b|x|^\theta\right)^{-\gamma}, \qquad \theta = \frac{p}{p-1}, \ \gamma = \frac{q(p-1)}{p(1-q)}, \tag{6.1}$$

is a subsolution to (1.1) *in* $(0, T) \times \mathbb{R}^N$ *provided* a *is large enough.*

Proof. Since w is radially symmetric and letting $r = |x|$ as usual, we note that

$$\mathcal{L}w = \partial_t w - |\partial_r w|^{p-2} \left[(p-1)\partial_r^2 w + \frac{N-1}{r}\partial_r w\right] + |\partial_r w|^q.$$

Setting $y := a + b|x|^\theta$, we easily have:

$$\partial_t w(t, x) = -\frac{1}{1-q}(T - t)^{q/(1-q)} y^{-\gamma}, \qquad \partial_r w(t, x) = -\gamma\theta b(T - t)^{1/(1-q)} r^{\theta-1} y^{-\gamma-1},$$

and

$$\partial_r^2 w(t, x) = -\gamma\theta b(T - t)^{1/(1-q)} y^{-\gamma-2} r^{\theta-2} \left[(\theta - 1)y - (1 + \gamma)\theta b r^\theta\right],$$

whence, after easy manipulations,

$$\begin{aligned}\mathcal{L}w = (T-t)^{q/(1-q)} y^{-\gamma} &\left[(\gamma\theta b)^q r^{q(\theta-1)} y^{\gamma(1-q)-q} - \frac{1}{1-q} \right] \\ &+ (\gamma\theta b)^{p-1} (T-t)^{(p-1)/(1-q)} r^{\theta(p-1)-p} y^{-(p-1)\gamma-p} \\ &\times \left[((p-1)(\theta-1)+N-1)y - (1+\gamma)\theta b(p-1) r^\theta \right],\end{aligned}$$

or equivalently,

$$\begin{aligned}(T-t)^{-q/(1-q)} y^\gamma \mathcal{L}w &= (\gamma\theta b)^q r^{q(\theta-1)} y^{\gamma(1-q)-q} - \frac{1}{1-q} \\ &+ (\gamma\theta b)^{p-1} (T-t)^{(p-1-q)/(1-q)} r^{\theta(p-1)-p} y^{(2-p)\gamma-p} \\ &\times \left[((p-1)(\theta-1)+N-1-(p-1)\theta(1+\gamma))y + (1+\gamma)\theta(p-1)a \right] \\ &= (\gamma\theta b)^q r^{q(\theta-1)} y^{\gamma(1-q)-q} - \frac{1}{1-q} \\ &+ (\gamma\theta b)^{p-1} (T-t)^{(p-1-q)/(1-q)} r^{\theta(p-1)-p} y^{(2-p)\gamma-p} \\ &\times \left[(1+\gamma)\theta(p-1)a + (N-1-(p-1)(1+\theta\gamma))y \right].\end{aligned}$$

Recalling now the values of θ and γ from (6.1) and that $q < p-1$, we obtain

$$\begin{aligned}(T-t)^{-q/(1-q)} y^\gamma \mathcal{L}w &= (\gamma\theta)^q b^{q/\theta} (br^\theta)^{q/p} y^{\gamma(1-q)-q} - \frac{1}{1-q} \\ &+ (\gamma\theta b)^{p-1} (T-t)^{(p-1-q)/(1-q)} y^{(2-p)\gamma-p} \\ &\times \left[(1+\gamma)pa + \left(N-1-\frac{p-1}{1-q} \right) y \right] \\ &\le (\gamma\theta)^q b^{q/\theta} - \frac{1}{1-q} \\ &+ (\gamma\theta b)^{p-1} T^{(p-1-q)/(1-q)} y^{(2-p)\gamma-p+1} [(1+\gamma)p+N-1] \\ &\le (\gamma\theta)^q b^{q/\theta} - \frac{1}{1-q} \\ &+ (\gamma\theta b)^{p-1} T^{(p-1-q)/(1-q)} a^{(2-p)\gamma-p+1} [(1+\gamma)p+N-1],\end{aligned} \tag{6.2}$$

where we have used in the last inequality the fact that

$$(2-p)\gamma - p + 1 = \frac{(p-1)(2q-p)}{p(1-q)} < 0 \tag{6.3}$$

due to $q < p-1 \le p/2$. Setting

$$b_0 := \left(2(1-q)(\gamma\theta)^q\right)^{-\theta/q} ,$$

it follows from (6.2) that, for $b \in (0, b_0)$,

$$\begin{aligned}(T-t)^{-q/(1-q)} y^\gamma \mathcal{L}w \le (\gamma\theta b)^{p-1} T^{(p-1-q)/(1-q)} a^{(2-p)\gamma-p+1} [(1+\gamma)p+N-1] \\ - \frac{1}{2(1-q)}.\end{aligned}$$

We use once more (6.3) to conclude that the right hand side of the previous inequality is negative for a large enough, that is, w is a subsolution to (1.1) in $(0,T) \times \mathbb{R}^N$. $\square$

With this technical lemma, we are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let u_0 be an initial condition satisfying (1.3) and (1.8). Fix $b \in (0, b_0)$ and $T > 0$. Since

$$\lim_{|x|\to\infty} u_0(x)|x|^{q/(1-q)} = \infty,$$

there exists $R > 0$ sufficiently large such that

$$\begin{aligned} u_0(x) &\geq T^{1/(1-q)} b^{-q(p-1)/p(1-q)} |x|^{-q/(1-q)} \\ &\geq T^{1/(1-q)} \left(a + b|x|^{p/(p-1)}\right)^{-q(p-1)/p(1-q)}, \end{aligned} \tag{6.4}$$

for any $a > 0$ and $x \in \mathbb{R}^N \setminus B(0, R)$. It remains to show that it is possible to choose $a > 0$ such that (6.4) also holds true inside $B(0, R)$. Since $u_0 > 0$ in $\mathbb{R}^N$ and $\overline{B(0,R)}$ is a compact set, we infer from the continuity of u_0 that

$$\delta := \inf\{u_0(x) : x \in B(0, R)\} > 0.$$

Choose then $a > 0$ sufficiently large such that

$$T^{1/(1-q)} a^{-q(p-1)/p(1-q)} < \delta \ ,$$

and set

$$w(t, x) := (T - t)^{1/(1-q)} \left(a + b|x|^{p/(p-1)}\right)^{-q(p-1)/p(1-q)}, \qquad (t, x) \in (0, T) \times \mathbb{R}^N.$$

Then

$$u_0(x) \geq \delta > T^{1/(1-q)} a^{-q(p-1)/p(1-q)} \geq w(0, x), \tag{6.5}$$

for any $x \in B(0, R)$ and we combine (6.4) and (6.5) to conclude that $u_0(x) \geq w(0, x)$ for any $x \in \mathbb{R}^N$. Owing to the choice $b \in (0, b_0)$, the function w is a subsolution to (1.1) according to Lemma 6.1, and the comparison principle ensures that

$$u(t, x) \geq w(t, x) > 0, \qquad \text{for any } t \in (0, T), \ x \in \mathbb{R}^N.$$

Consequently, $\mathcal{P}(t) = \mathbb{R}^N$ for all $t \in (0, T)$ and, since $T > 0$ is arbitrarily chosen, we conclude that $\mathcal{P}(t) = \mathbb{R}^N$ for any $t > 0$, as stated. $\square$

7 Improved finite time extinction property

This section is devoted to the proof of Theorem 1.3, which improves the range of initial data for which finite time extinction takes place. As in the previous section, the main argument is again the comparison principle, this time with suitable supersolutions. The idea to construct them is adapted from [7, Lemma 7], where similar supersolutions are built in the semilinear case $p = 2$ and $q \in (0, 1)$. To this end we define

$$\alpha := \frac{p - q}{p - 2q} > 0, \qquad \beta := \frac{q - p + 1}{p - 2q} < 0, \tag{7.1}$$

which are the usual self-similar exponents associated to Eq. (1.1) and look for self-similar supersolutions of (1.1).

Lemma 7.1. *Assume that (p,q) satisfies (1.4) and that $p<2$. There exists $A_0>0$ depending only on N, p, and q such that, for all $T>0$ and $A\in(0,A_0)$, the function*

$$W(t,x) := (T-t)^\alpha f(|x|(T-t)^\beta), \qquad (t,x)\in(0,T)\times\mathbb{R}^N, \tag{7.2}$$

with

$$f(y) := A(1+y^2)^{-\gamma}, \quad y\in\mathbb{R}, \qquad \gamma := \frac{q}{2(1-q)},$$

is a supersolution to (1.1) in $(0,T)\times\mathbb{R}^N$.

Proof. Since W is radially symmetric, we again have (with $r=|x|$)

$$\mathcal{L}W = \partial_t W - (p-1)|\partial_r W|^{p-2}\partial_r^2 W - \frac{N-1}{r}|\partial_r W|^{p-2}\partial_r W + |\partial_r W|^q.$$

Setting $y := |x|(T-t)^\beta = r(T-t)^\beta$ and taking into account that

$$\alpha - 1 = (p-1)(\alpha+\beta) + \beta = q(\alpha+\beta) = \frac{q}{p-2q} > 0,$$

we readily find

$$\begin{aligned}\mathcal{L}W = (T-t)^{\alpha-1}\Big[&-\alpha f(y) - \beta y\partial_y f(y) - (p-1)|\partial_y f(y)|^{p-2}\partial_y^2 f(y)\\ &-\frac{N-1}{y}|\partial_y f(y)|^{p-2}\partial_y f(y) + |\partial_y f(y)|^q\Big].\end{aligned}$$

Since

$$\partial_y f(y) = -2A\gamma y(1+y^2)^{-\gamma-1}$$

and

$$\partial_y^2 f(y) = -2A\gamma(1+y^2)^{-\gamma-1}\left[1 - \frac{2(\gamma+1)y^2}{1+y^2}\right],$$

we further obtain

$$\begin{aligned}\mathcal{L}W &= (T-t)^{\alpha-1}(1+y^2)^{-(\gamma+1)}\Big[-\alpha A - (\alpha-2\beta\gamma)Ay^2\\ &\quad +(p-1)(2A\gamma)^{p-1}\frac{(1+y^2)^{(2-p)(\gamma+1)}}{y^{2-p}}\left(1-\frac{2(\gamma+1)y^2}{1+y^2}\right)\\ &\quad +(N-1)(2A\gamma)^{p-1}\frac{(1+y^2)^{(2-p)(\gamma+1)}}{y^{2-p}} + (2A\gamma)^q y^q(1+y^2)^{(1-q)(\gamma+1)}\Big]\\ &= (T-t)^{\alpha-1}(1+y^2)^{-(\gamma+1)}H(y),\end{aligned}$$

where

$$\begin{aligned}H(y) &:= (2A\gamma)^{p-1}\frac{(1+y^2)^{(2-p)(\gamma+1)}}{y^{2-p}}\left[p+N-2-\frac{2(p-1)(\gamma+1)y^2}{1+y^2}\right]\\ &\quad + (2A\gamma)^q y^q(1+y^2)^{(1-q)(\gamma+1)} - A\alpha - (\alpha-2\beta\gamma)Ay^2.\end{aligned}$$

We first note that

$$\begin{aligned}&\frac{1}{2}(2A\gamma)^q y^q(1+y^2)^{(1-q)(\gamma+1)} - (\alpha-2\beta\gamma)Ay^2\\ &\qquad \ge \frac{(2A\gamma)^q}{2}y^q y^{2(1-q)(\gamma+1)} - (\alpha-2\beta\gamma)Ay^2\\ &\qquad \ge Ay^2\left[\frac{(2\gamma)^q}{2}A^{q-1} - (\alpha-2\beta\gamma)\right].\end{aligned} \tag{7.3}$$

In order to estimate the remaining terms in the expression of $H(y)$, we have to split the analysis into two cases according to the values of y. More precisely, set

$$y_0 := \frac{1}{\sqrt{4(\gamma+1)}} .$$

- If $y \in [0, y_0]$, we find

$$\begin{aligned}
(2A\gamma)^{p-1} & \frac{(1+y^2)^{(2-p)(\gamma+1)}}{y^{2-p}} \left[p+N-2-\frac{2(p-1)(\gamma+1)y^2}{1+y^2} \right] - A\alpha \\
& \ge (2A\gamma)^{p-1} y^{p-2} \left[p-1-\frac{2(p-1)(\gamma+1)y^2}{1+y^2} \right] - A\alpha \\
& \ge (2A\gamma)^{p-1} (p-1) y^{p-2} \left[1-2(\gamma+1)y_0^2 \right] - A\alpha \\
& \ge (2A\gamma)^{p-1} \frac{p-1}{2} y_0^{p-2} - A\alpha \\
& \ge A \left[\frac{(p-1)(2\gamma)^{p-1}}{2y_0^{2-p}} A^{p-2} - \alpha \right] .
\end{aligned} \tag{7.4}$$

- If $y \ge y_0$, we have

$$\begin{aligned}
(2A\gamma)^{p-1} & \frac{(1+y^2)^{(2-p)(\gamma+1)}}{y^{2-p}} \left[p+N-2-\frac{2(p-1)(\gamma+1)y^2}{1+y^2} \right] \\
& + \frac{1}{2} (2A\gamma)^q y^q (1+y^2)^{(1-q)(\gamma+1)} - A\alpha \\
\ge & -2(p-1)(1+\gamma)(2A\gamma)^{p-1} \frac{(1+y^2)^{(2-p)(\gamma+1)}}{y^{2-p}} \frac{y^2}{1+y^2} \\
& + \frac{(2A\gamma)^q}{2} y^q y^{2(1-q)(\gamma+1)} - A\alpha \\
\ge & -2(p-1)(1+\gamma)(2A\gamma)^{p-1} \frac{(1+y^2)^{(2-p)(\gamma+1)}}{y^{2-p}} + \frac{(2A\gamma)^q}{2} y^2 - A\alpha \\
\ge & \frac{(2A\gamma)^q}{4} y_0^2 - A\alpha + \frac{(2A\gamma)^q}{4} y^2 \\
& - 2(p-1)(\gamma+1)(2A\gamma)^{p-1} \left(1+\frac{1}{y_0^2} \right)^{(2-p)(\gamma+1)} y^{(2-p)(2(\gamma+1)-1)} \\
\ge & A \left[\frac{(2\gamma)^q}{4} y_0^2 A^{q-1} - \alpha \right] + (2A\gamma)^{p-1} y^{(2-p)/(1-q)} \\
& \times \left[\frac{(2\gamma)^{q-p+1}}{4} y_0^{(p-2q)/(1-q)} A^{q-p+1} - 2(p-1)(\gamma+1) \left(\frac{1+y_0^2}{y_0^2} \right)^{(2-p)(\gamma+1)} \right] .
\end{aligned} \tag{7.5}$$

Since $0 < q < p-1 < 1$ and $p < 2$, we realize that the right-hand sides of (7.3), (7.4), and (7.5) are simultaneously positive provided A is sufficiently small. It then follows that W defined in (7.2) is a supersolution to (1.1) in $(0,T) \times \mathbb{R}^N$ for A small enough. □

Completing the proof of Theorem 1.3 becomes now an easy task. A noteworthy fact to be mentioned is that the choice of $A > 0$ in Lemma 7.1 is independent of the fixed extinction time $T > 0$.

Proof of Theorem 1.3. For $p = 2$ and $q \in (0,1)$ Theorem 1.3 is proved in [7, Theorem 2]. Consider now $p < 2$. Let u_0 be an initial condition satisfying (1.3) and (1.9) and $T > 1$. We fix $A \in (0, A_0)$ so that the function

$$W(t,x) = (T-t)^\alpha A \left(1 + (T-t)^{2\beta}|x|^2\right)^{-q/2(1-q)}, \qquad (t,x) \in (0,T) \times \mathbb{R}^N,$$

is a supersolution to (1.1) in $(0,T) \times \mathbb{R}^N$ according to Lemma 7.1. We notice that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned} W(0,x) &= T^\alpha A \left(1 + T^{2\beta}|x|^2\right)^{-q/2(1-q)} \\ &\geq T^\alpha A \left(T^{2\beta} + T^{2\beta}|x|^2\right)^{-q/2(1-q)} \\ &= T^{\alpha - \beta q/(1-q)} A \left(1 + |x|^2\right)^{-q/2(1-q)} \\ &= T^{1/(1-q)} A \left(1 + |x|^2\right)^{-q/2(1-q)}. \end{aligned}$$

Since $1 + |x|^2 \leq (1 + |x|)^2$, for any $x \in \mathbb{R}^N$, we further obtain that

$$\begin{aligned} W(0,x) &\geq T^{1/(1-q)} A (1 + |x|)^{-q/(1-q)} \\ &= \frac{T^{1/(1-q)} A}{C_0} C_0 (1 + |x|)^{-q/(1-q)} \geq \frac{T^{1/(1-q)} A}{C_0} u_0(x), \end{aligned}$$

for any $x \in \mathbb{R}^N$. Letting $T > 0$ to be sufficiently large such that

$$\frac{T^{1/(1-q)} A}{C_0} > 1,$$

we have $W(0,x) \geq u_0(x)$ for any $x \in \mathbb{R}^N$. Consequently, by the comparison principle,

$$W(t,x) \geq u(t,x), \qquad (t,x) \in (0,T) \times \mathbb{R}^N .$$

Noting that $W(t,0) = A(T-t)^\alpha$ and $W(t,x) \leq A(T-t)^{1/(1-q)}|x|^{-q/(1-q)}$ for $x \neq 0$, we realize that $W(t,x) \longrightarrow 0$ as $t \to T$ for all $x \in \mathbb{R}^N$, which in particular implies that u vanishes in finite time. Moreover, its extinction time satisfies $T_e \leq T$. □

8 Single point extinction

This rather long section is devoted to the proof of Theorem 1.4. The single point extinction is an immediate consequence of the double inclusion (1.13), which is the main result to prove. This is divided into several steps, corresponding to subsections in the sequel.

8.1 Lower bound for the positivity set

We begin with the first inclusion in (1.13).

Lemma 8.1. *Let u be a solution to the Cauchy problem* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.3) *which is radially symmetric and non-increasing. Assume further that*

u_0 is compactly supported and denote its finite extinction time by T_e. There exists $\varrho_1 > 0$ such that

$$B(0,\varrho_1(T_e-t)^\sigma) \subseteq \mathcal{P}(t) \qquad \text{for any } t\in(T_e/2,T_e) \quad \textit{with} \quad \sigma := \frac{p-q-1}{(p-q)(1-q)},$$

where $\mathcal{P}(t)$ denotes the positivity set of u at time t, see (1.6).

Proof. Since $\|u(t)\|_\infty = u(t,0)$ due to the radial symmetry and monotonicity of $u(t)$ provided by Lemma 2.2, we infer from Proposition 5.1 that

$$u(t,0) \ge C_1(T_e-t)^{1/(1-q)}, \qquad t\in(0,T_e).$$

Next, it readily follows from the mean-value theorem and the gradient estimate (5.4) that

$$u^{(p-q-1)/(p-q)}(t,x) - u^{(p-q-1)/(p-q)}(t,0) \ge -C|x|, \qquad t\in[T_e/2,T_e],$$

whence, using the above lower bound for $u(t,0)$,

$$u^{(p-q-1)/(p-q)}(t,x) \ge C_1^{(p-q-1)/(p-q)}(T_e-t)^\sigma - C|x| > 0,$$

provided

$$|x| < \frac{C_1^{\sigma(1-q)}}{C}(T_e-t)^\sigma, \quad t\in(T_e/2,T_e),$$

ending the proof. □

As already pointed out in the Introduction, Lemma 8.1 holds true without requiring the assumptions (1.10) and (1.12) on u_0. It is in the next subsection where they will be needed.

8.2 Upper bound for the positivity set

We move now to the more involved part, that is, to prove the second inclusion in (1.13) which turns out to be rather technical. Let u_0 be a radially symmetric and non-increasing initial condition satisfying (1.3). Assume further that u_0 is compactly supported in $B(0,R_0)$ for some $R_0>0$ and satisfies (1.10) for all $x_0\in\partial B(0,R_0)$ as well as (1.12). Denoting the corresponding solution to the Cauchy problem (1.1)-(1.2) by u we infer from Theorem 1.1 and Proposition 4.3 that there is $T_e>0$ such that

$$\mathcal{P}(t)\subseteq B(0,R_0), \quad t\in(0,T_e), \qquad \text{and} \quad \mathcal{P}(t)=\emptyset, \quad t\ge T_e. \tag{8.1}$$

Furthermore, Lemma 2.2 and the assumptions on u_0 guarantee that

$$\partial_r u(t,r)\le 0\ , \qquad (t,r)\in(0,T_e)\times(0,R_0)\ . \tag{8.2}$$

Lemma 8.2. *There exists $\delta>0$ such that*

$$|\nabla u(t,x)| \ge \delta^{1/(p-1)}|x|^{1/(p-1-q)}u(t,x)^{1/(p-q)}, \qquad (t,x)\in(0,T_e)\times B(0,R_0)\ . \tag{8.3}$$

Proof. We adapt an idea from [17, Lemma 2.2] which takes its origin in the study of blow-up problems, see [18, Lemma 2.2]. For $z>0$, we set

$$a(z) := z^{(p-2)/2}\ , \qquad b(z) := z^{q/2}\ ,$$

so that, using the radial variable $r := |x|$, Eq. (1.1) reads

$$\partial_t(r^{N-1}u) = \partial_r \left(r^{N-1}a(|\partial_r u|^2)\partial_r u\right) - r^{N-1}b(|\partial_r u|^2) \;, \qquad (t,r) \in (0,T_e)\times(0,R_0) \;. \quad (8.4)$$

We define the auxiliary function

$$J(t,r) := r^{N-1}a(|\partial_r u(t,r)|^2)\partial_r u(t,r) + c(r)F(u(t,r)), \qquad (t,r)\in(0,T_e)\times(0,R_0),$$

where the functions $c \geq 0$ and $F \geq 0$ are to be determined later and assumed to satisfy

$$c(0) = 0 \;, \quad F(0) = 0 \;, \quad F' \geq 0 \;, \quad F'' \leq 0 \;. \qquad (8.5)$$

We aim at finding c and F such that $J \leq 0$ in $(0,T_e)\times(0,R_0)$. Since $F(0) = 0$, we first note that (8.1) and (8.2) imply that

$$J(t,0) = 0 \;, \qquad J(t,R_0) = R_0^{N-1}|\partial_r u(t,R_0)|^{p-2}\partial_r u(t,R_0) \leq 0, \qquad t\in(0,T_e) \;, \quad (8.6)$$

as well as

$$J(t,r) = -r^{N-1}|\partial_r u(t,r)|^{p-1} + c(r)F(u(t,r)), \qquad (t,r)\in(0,T_e)\times(0,R_0) \;.$$

The following calculations are performed at a formal level. A rigorous justification requires some approximating procedures and will be completed in the Appendix, a detailed account of the formal calculations being given for the easiness of the reading. Introducing

$$g := -\partial_r u \geq 0 \quad \text{and} \quad a_1(z) := 2za'(z) + a(z) \;, \quad z > 0 \;,$$

we infer from (8.4) that

$$\begin{aligned}
\partial_t J &= a_1(g^2)\partial_t\left(r^{N-1}\partial_r u\right) + c(r)F'(u)\partial_t u \\
&= a_1(g^2)\partial_t\left[\partial_r\left(r^{N-1}u\right) - (N-1)r^{N-2}u\right] + \frac{c(r)}{r^{N-1}}F'(u)\partial_t(r^{N-1}u) \\
&= a_1(g^2)\partial_r\left[\partial_r\left(r^{N-1}a(g^2)\partial_r u\right) - r^{N-1}b(g^2)\right] \\
&\quad + \left(\frac{c(r)}{r^{N-1}}F'(u) - \frac{N-1}{r}a_1(g^2)\right)\left[\partial_r\left(r^{N-1}a(g^2)\partial_r u\right) - r^{N-1}b(g^2)\right] \\
&= a_1(g^2)\partial_r^2\left(r^{N-1}a(g^2)\partial_r u\right) - 2r^{N-1}(a_1 b')(g^2)\partial_r u\partial_r^2 u \\
&\quad + \left(\frac{c(r)}{r^{N-1}}F'(u) - \frac{N-1}{r}a_1(g^2)\right)\partial_r\left(r^{N-1}a(g^2)\partial_r u\right) - c(r)F'(u)b(g^2) \;.
\end{aligned}$$

Next

$$\begin{aligned}
\partial_r J &= \partial_r\left(r^{N-1}a(g^2)\partial_r u\right) + c'(r)F(u) + c(r)F'(u)\partial_r u \;, \\
\partial_r^2 J &= \partial_r^2\left(r^{N-1}a(g^2)\partial_r u\right) + c''(r)F(u) + 2c'(r)F'(u)\partial_r u \\
&\qquad + c(r)F''(u)g^2 + c(r)F'(u)\partial_r^2 u \;,
\end{aligned}$$

and it follows from the formulas for $\partial_t J$ and $\partial_r^2 J$ that

$$\begin{aligned}
\partial_t J - a_1(g^2)\partial_r^2 J &= 2r^{N-1}(a_1 b')(g^2)g\partial_r^2 u \\
&\quad + \left(\frac{c(r)}{r^{N-1}}F'(u) - \frac{N-1}{r}a_1(g^2)\right)\partial_r\left(r^{N-1}a(g^2)\partial_r u\right) \\
&\quad - c(r)F'(u)b(g^2) - a_1(g^2)c''(r)F(u) + 2a_1(g^2)c'(r)F'(u)g \\
&\quad - c(r)F''(u)a_1(g^2)g^2 - a_1(g^2)c(r)F'(u)\partial_r^2 u \;.
\end{aligned}$$

We now use the formula for $\partial_r J$ to replace the terms involving $\partial_r^2 u$ in the above identity and obtain

$$\begin{aligned}
\partial_t J - a_1(g^2)\partial_r^2 J = & \left(2b'(g^2)g - \frac{c(r)}{r^{N-1}}F'(u)\right) r^{N-1}\partial_r \left(a(g^2)\partial_r u\right) \\
& + \left(\frac{c(r)}{r^{N-1}}F'(u) - \frac{N-1}{r}a_1(g^2)\right) \partial_r \left(r^{N-1}a(g^2)\partial_r u\right) \\
& - c(r)F'(u)b(g^2) - a_1(g^2)c''(r)F(u) + 2a_1(g^2)c'(r)F'(u)g \\
& - c(r)F''(u)a_1(g^2)g^2 \\
= & \, 2b'(g^2)g\left[\partial_r J - c'(r)F(u) + c(r)F'(u)g\right] + 2(N-1)r^{N-2}(ab')(g^2)g^2 \\
& - \frac{N-1}{r}c(r)F'(u)a(g^2)g - \frac{N-1}{r}a_1(g^2)\left[\partial_r J - c'(r)F(u) + c(r)F'(u)g\right] \\
& - c(r)F'(u)b(g^2) - c''(r)F(u)a_1(g^2) + 2c'(r)F'(u)a_1(g^2)g \\
& - c(r)F''(u)a_1(g^2)g^2 \ .
\end{aligned}$$

Consequently

$$\begin{aligned}
& \partial_t J - a_1(g^2)\partial_r^2 J + \left(\frac{N-1}{r}a_1(g^2) - 2b'(g^2)g\right)\partial_r J \\
& = 2\left[(N-1)r^{N-2}a(g^2)g - c'(r)F(u)\right] b'(g^2)g + \left[\frac{N-1}{r}c'(r) - c''(r)\right] F(u)a_1(g^2) \\
& \quad + \left[2c'(r)a_1(g^2) - \frac{N-1}{r}c(r)(a+a_1)(g^2)\right] F'(u)g - c(r)F''(u)a_1(g^2)g^2 \\
& \quad - \left[b(g^2) - 2b'(g^2)g^2\right] c(r)F'(u) \ .
\end{aligned}$$

Since

$$a_1(z) = (p-1)a(z) \le a(z) \quad \text{and} \quad b(z) - 2zb'(z) = (1-q)b(z) \ , \qquad z > 0 \ ,$$

we infer from the non-negativity of c and g, and the monotonicity (8.5) of F, that

$$\partial_t J - a_1(g^2)\partial_r^2 J + \left(\frac{N-1}{r}a_1(g^2) - 2b'(g^2)g\right)\partial_r J \le \sum_{i=1}^{3} \mathcal{R}_i \ , \tag{8.7}$$

where

$$\begin{aligned}
\mathcal{R}_1 & := 2\left[(N-1)r^{N-2}a(g^2)g - c'(r)F(u)\right] b'(g^2)g \ , \\
\mathcal{R}_2 & := \left[\frac{N-1}{r}c'(r) - c''(r)\right] F(u)a_1(g^2) \ , \\
\mathcal{R}_3 & := 2\left[c'(r) - \frac{N-1}{r}c(r)\right] F'(u)a_1(g^2)g - c(r)F''(u)a_1(g^2)g^2 \\
& \qquad - (1-q)c(r)F'(u)b(g^2) \ .
\end{aligned}$$

We now choose

$$c(r) = r^\lambda \ , \quad r \ge 0 \ , \qquad F(z) = \delta z^\beta \ , \quad z \ge 0 \ ,$$

where $\lambda > N$, $\delta \in (0,1)$, and $\beta \in (0,1)$ are yet to be determined. Note that the latter constraint on β complies with (8.5). With this choice,

$$\mathcal{R}_2 = -\lambda(\lambda - N)\delta(p-1)r^{\lambda-2}u^\beta g^{p-2} \ , \tag{8.8}$$

while

$$\mathcal{R}_1 = 2r^{\lambda-1}b'(g^2)g\left[(N-1)r^{N-1-\lambda}a(g^2)g - \delta\lambda u^\beta\right] . \tag{8.9}$$

Moreover, since $\lambda > N$ and $\beta \in (0,1)$,

$$\begin{aligned}
\mathcal{R}_3 &= \delta\Big[2(\lambda-N+1)\beta(p-1)r^{\lambda-1}u^{\beta-1}g^{p-1} \\
&\qquad +\beta(1-\beta)(p-1)r^\lambda u^{\beta-2}g^p - (1-q)\beta r^\lambda u^{\beta-1}g^q\Big] \\
&\le \beta\delta\left[\frac{2(\lambda-N+1)}{r}g^{p-1-q} + \frac{1-\beta}{u}g^{p-q} - (1-q)\right]r^\lambda u^{\beta-1}g^q .
\end{aligned} \tag{8.10}$$

Now let $\kappa > 0$ and define

$$\mathcal{J}_\kappa := \{(t,r) \in (0,T_e)\times(0,R_0) \ : \ J(t,r) \ge \kappa\} .$$

Owing to the definition of J there holds

$$r^{N-1}g^{p-1} = r^{N-1}a(g^2)g \le \kappa + r^{N-1}g^{p-1} \le c(r)F(u) = \delta r^\lambda u^\beta \quad \text{in} \quad \mathcal{J}_\kappa . \tag{8.11}$$

A first consequence of (8.11) and the positivity of κ is that $r > 0$ and $u > 0$ in $\mathcal{J}_\kappa$, so that (8.8) yields

$$\mathcal{R}_2 = -\lambda(\lambda-N)\delta(p-1)r^{\lambda-2}u^\beta g^{p-2} < 0 \quad \text{in} \quad \mathcal{J}_\kappa \tag{8.12}$$

in view of $\lambda > N$ and $p \in (1,2]$. Moreover, we deduce from (8.9), (8.10), and (8.11) that, in $\mathcal{J}_\kappa$,

$$\begin{aligned}
\mathcal{R}_1 &\le 2r^{\lambda-1}b'(g^2)g\left[(N-1)r^{-\lambda}\delta r^\lambda u^\beta - \delta\lambda u^\beta\right] \\
&\le -2(\lambda-N+1)\delta r^{\lambda-1}u^\beta b'(g^2)g \le 0
\end{aligned} \tag{8.13}$$

and

$$\begin{aligned}
\mathcal{R}_3 &\le \beta\delta\Bigg[\frac{2(\lambda-N+1)}{r}\left(\delta r^{\lambda-N+1}u^\beta\right)^{(p-1-q)/(p-1)} \\
&\qquad +\frac{1-\beta}{u}\left[\delta r^{\lambda-N+1}u^\beta\right]^{(p-q)/(p-1)} - (1-q)\Bigg]r^\lambda u^{\beta-1}g^q \\
&\le \beta\delta\Big[2(\lambda-N+1)\delta^{(p-1-q)/(p-1)}r^{((\lambda-N)(p-1-q)-q)/(p-1)}u^{\beta(p-1-q)/(p-1)} \\
&\qquad + (1-\beta)\delta^{(p-q)/(p-1)}r^{(\lambda-N+1)(p-q)/(p-1)}u^{(\beta(p-q)-(p-1))/(p-1)} \\
&\qquad - (1-q)\Big]r^\lambda u^{\beta-1}g^q .
\end{aligned}$$

In order to ensure $\mathcal{R}_3 \le 0$ in $\mathcal{J}_\kappa$, we choose

$$\beta := \frac{p-1}{p-q} \in (0,1) \quad \text{and} \quad \lambda := N + \frac{q}{p-1-q} > N .$$

Recalling that $q < p-1$, we use (2.1) and (8.1) to obtain that, in $\mathcal{J}_\kappa$,

$$\begin{aligned}
\mathcal{R}_3 &\le \beta\delta\Big[2(\lambda-N+1)\delta^{(p-1-q)/(p-1)}\|u_0\|_\infty^{(p-1-q)/(p-q)} \\
&\qquad +(1-\beta)\delta^{(p-q)/(p-1)}R_0^{(p-q)/(p-q-1)} - (1-q)\Big]r^\lambda u^{\beta-1}g^q \\
&\le -\frac{\beta\delta(1-q)}{2}r^\lambda u^{\beta-1}b(g^2) \le 0 ,
\end{aligned} \tag{8.14}$$

provided δ is chosen suitably small (depending on $\|u_0\|_\infty$ and R_0). Collecting (8.7), (8.12), (8.13), and (8.14), we have shown that

$$\partial_t J - a_1(g^2)\partial_r^2 J + \left(\frac{N-1}{r}a_1(g^2) - 2b'(g^2)g\right)\partial_r J < 0 \ , \qquad (t,r)\in\mathcal{J}_\kappa \ , \tag{8.15}$$

as soon as δ is suitably small.

Assume now for contradiction that the maximum of J in $[0,T_e]\times[0,R_0]$ exceeds κ. It is attained at some point $(t_0,r_0)\in[0,T_e]\times[0,R_0]$ and it readily follows from (8.6) that $r_0\in(0,R_0)$. In addition, (1.12) ensures that, for $r\in(0,R_0)$,

$$\begin{aligned}
J(0,r) &= r^\lambda u_0(r)^{(p-1)/(p-q)}\left[\delta - \left(\frac{1}{r^{1/(p-1-q)}}\frac{|\partial_r u_0(r)|}{u_0(r)^{1/(p-q)}}\right)^{p-1}\right] \\
&= r^\lambda u_0(r)^{(p-1)/(p-q)}\left[\delta - \left(\frac{p-q}{p-1-q}\frac{1}{r^{1/(p-1-q)}}|\partial_r u_0^{(p-1-q)/(p-q)}(r)|\right)^{p-1}\right] \\
&\le r^\lambda u_0(r)^{(p-1)/(p-q)}\left[\delta - \left(\frac{p-q}{p-1-q}\delta_0\right)^{p-1}\right] \le 0
\end{aligned}$$

by choosing $\delta>0$ even smaller. Consequently, $t_0>0$ and we conclude that

$$\partial_t J(t_0,r_0)\ge 0 \ , \quad \partial_r J(t_0,r_0) = 0 \ , \quad \partial_r^2 J(t_0,r_0)\le 0 \ ,$$

which contradicts (8.15), since $(t_0,r_0)\in\mathcal{J}_\kappa$. Therefore, $J\le\kappa$ in $[0,T_e]\times[0,R_0]$. Since κ is an arbitrary positive number, we have shown that $J\le 0$ in $[0,T_e]\times[0,R_0]$ from which (8.3) follows.

We stress once more that this proof holds only at formal level as the coefficients in the equation solved by J may be singular. A complete justification is postponed to the Appendix. □

The proof of Theorem 1.4 is now an easy consequence of the previous analysis as shown below.

Proof of Theorem 1.4. We wish to prove (1.13), from which the single point extinction follows readily. The left inclusion in (1.13) has been proved in Lemma 8.1 and we are thus left with the proof of the right inclusion. Fix $t\in(T_e/2,T_e)$. Taking into account the radial symmetry and monotonicity of $u(t)$, we deduce from Lemma 8.2 that

$$|\nabla u(t,x)| \ge \delta^{1/(p-1)}|x|^{1/(p-1-q)}\ u(t,x)^{1/(p-q)} \ , \qquad x\in B(0,R_0) \ ,$$

hence, recalling (8.1),

$$\frac{p-1-q}{p-q}\delta^{1/(p-1)}|x|^{1/(p-1-q)} \le \left|\nabla u^{(p-1-q)/(p-q)}(t,x)\right| \ , \qquad x\in\mathcal{P}(t) \ . \tag{8.16}$$

For $x\in\mathcal{P}(t)$, the monotonicity properties of $u(t)$ guarantee that $\varrho x\in\mathcal{P}(t)$ for all $\varrho\in[0,1]$

and, for any $\varrho \in (0,1)$, we infer from (1.4) and (8.16) that

$$\begin{aligned}
u^{(p-1-q)/(p-q)}(t,x) &= u^{(p-1-q)/(p-q)}(t,\varrho x) + \int_\varrho^1 \left\langle \nabla u^{(p-1-q)/(p-q)}(t,\sigma x), x \right\rangle \, d\sigma \\
&= u^{(p-1-q)/(p-q)}(t,\varrho x) - \int_\varrho^1 \left| \nabla u^{(p-1-q)/(p-q)}(t,\sigma x) \right| |x| \, d\sigma \\
&\le \|u(t)\|_\infty^{(p-1-q)/(p-q)} - C|x| \int_\varrho^1 |\sigma x|^{1/(p-1-q)} \, d\sigma \\
&\le \|u(t)\|_\infty^{(p-1-q)/(p-q)} - C|x|^{(p-q)/(p-1-q)} \left(1 - \varrho^{(p-q)/(p-1-q)} \right) .
\end{aligned}$$

We now apply Proposition 5.1 and let $\varrho \to 0$ to obtain that, given $\vartheta \in (0,1)$,

$$u^{(p-q-1)/(p-q)}(t,x) \le C_2(\vartheta)^{(p-q-1)/(p-q)} (T_e - t)^{\vartheta(p-q-1)/(p-2q)} - C|x|^{(p-q)/(p-1-q)} .$$

Since $x \in \mathcal{P}(t)$, this implies in particular that the right-hand side of the previous inequality is positive, that is,

$$|x| < C(\vartheta)(T_e - t)^{\vartheta(p-q-1)^2/(p-q)(p-2q)} .$$

Consequently, choosing $\vartheta = p/2$,

$$\mathcal{P}(t) \subseteq B(0, \varrho_2 (T_e - t)^\nu), \qquad \nu = \frac{p(p-q-1)^2}{2(p-q)(p-2q)} ,$$

for some $\varrho_2 > 0$. Since $t \in (T_e/2, T_e)$ is arbitrary, we have established the right inclusion in (1.13) and thereby completed the proof. □

A Proofs of Lemma 8.2 and gradient estimates for $p = 2$

In this technical section we provide a fully rigorous proof of Lemma 8.2, as well as a gradient estimate for solutions to (1.1) for $p = 2$. The latter, besides of its interest as an independent result, provides an essential technical tool in the proofs of our main results, and complements the gradient estimates in [25, Theorem 1.3], valid for $p \in (p_c, 2)$.

Lemma 8.2 was proved in Section 8 at a formal level, presenting the essential calculations that give the ideas and essence of the proof, but allowing us at that point, for the simplicity of the exposition, to use results such as the maximum principle (or comparison principle) that are not automatically granted when we deal with singular coefficients. This is why we include the rigorous proof of this lemma here. To this end, we introduce a regularization of (1.1), already successfully used by the authors in [25, Section 6], in order to avoid the difficulties coming from the singularity at points where $\nabla u = 0$. Let then u be a solution to the Cauchy problem (1.1)-(1.2) associated to an initial condition u_0 satisfying the assumptions (a)-(c) of Theorem 1.4. We recall that u vanishes identically after a finite time T_e and that its positivity set $\mathcal{P}(t)$ is included in $B(0, R_0)$ for all $t \ge 0$, see (8.1).

For $\varepsilon \in (0, 1/2)$, we define:

$$a_\varepsilon(z) := (z + \varepsilon^2)^{(p-2)/2}, \qquad b_\varepsilon(z) := (z + \varepsilon^2)^{q/2}, \qquad z \ge 0, \tag{A.1}$$

and consider the following Cauchy problem:

$$\left\{ \begin{array}{l} \partial_t \tilde{u}_\varepsilon - \mathrm{div}\left(a_\varepsilon(|\nabla \tilde{u}_\varepsilon|^2) \nabla \tilde{u}_\varepsilon \right) + b_\varepsilon\left(|\nabla \tilde{u}_\varepsilon|^2 \right) - \varepsilon^q = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ \tilde{u}_\varepsilon(0) = u_{0,\varepsilon} + \varepsilon^\gamma, \qquad x \in \mathbb{R}^N, \end{array} \right. \tag{A.2}$$

where $\gamma \in (0,p/4) \cap (0,q/2)$ is a small positive parameter such that $\gamma < \min\{p-1, 1-q\}$ and $u_{0,\varepsilon} \in C^\infty(\mathbb{R}^N)$ is a non-negative smooth approximation of the initial condition u_0, in the sense that it converges to u_0 uniformly on compact sets in $\mathbb{R}^N$ and satisfies

$$0 \le u_{0,\varepsilon} \le \|u_0\|_\infty, \qquad \|\nabla u_{0,\varepsilon}\|_\infty \le (1 + C(u_0)\varepsilon)\|\nabla u_0\|_\infty. \tag{A.3}$$

It is proved in [5] and [25, Section 6] that (A.2) has a unique classical solution $\tilde{u}_\varepsilon$ and that, as $\varepsilon \to 0$, it is an approximation of the solution u to (1.1)-(1.2) with initial condition u_0 in the following sense:

$$u(t,x) = \lim_{\varepsilon\to 0} \tilde{u}_\varepsilon(t,x), \qquad \nabla u(t,x) = \lim_{\varepsilon\to 0} \nabla\tilde{u}_\varepsilon(t,x), \tag{A.4}$$

for almost every $(t,x) \in (0,\infty) \times \mathbb{R}^N$, the first convergence being actually uniform in compact sets of $(0,\infty) \times \mathbb{R}^N$. In addition, if u_0 is radially symmetric and non-increasing, then $u_{0,\varepsilon}$ can be chosen to be radially symmetric and non-increasing as well, so that $x \mapsto \tilde{u}_\varepsilon(t,x)$ is radially symmetric and non-increasing for any $t \ge 0$ and $\varepsilon \in (0,1/2)$. We next define

$$u_\varepsilon(t,x) := \tilde{u}_\varepsilon(t,x) - \varepsilon^q t \ , \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N \ ,$$

and observe that the comparison principle and (A.2) imply that

$$u_\varepsilon(t,x) \ge \varepsilon^\gamma - \varepsilon^q t \ge \frac{\varepsilon^\gamma}{2} \ , \qquad (t,x) \in (0,\tau_\varepsilon) \times \mathbb{R}^N \ , \tag{A.5}$$

with $\tau_\varepsilon := \varepsilon^{\gamma-q}/2$ and that u_ε solves

$$\left\{ \begin{array}{l} \partial_t u_\varepsilon - \mathrm{div}\left(a_\varepsilon(|\nabla u_\varepsilon|^2)\nabla u_\varepsilon\right) + b_\varepsilon\left(|\nabla u_\varepsilon|^2\right) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ u_\varepsilon(0) = u_{0,\varepsilon} + \varepsilon^\gamma, \qquad x \in \mathbb{R}^N \ . \end{array} \right. \tag{A.6}$$

Since $\tau_\varepsilon \to \infty$ as $\varepsilon \to 0$, we may assume that ε is taken sufficiently small to ensure $T_e \le \tau_\varepsilon$. With these approximations in mind, we are ready to give the complete proof of Lemma 8.2 as well as that of the gradient estimate (5.4) for $p = 2$.

A.1 Proof of Lemma 8.2

Owing to the C^1-smoothness of u_0, the gradient convergence in (A.4) is also uniform on compact subsets of $\mathbb{R}^N$. Consequently,

$$m_\varepsilon := \varepsilon + \|u_{0,\varepsilon} - u_0\|_{C^1(B(0,R_0))} \underset{\varepsilon\to 0}{\longrightarrow} 0 \ . \tag{A.7}$$

Introducing

$$r_\varepsilon := \min\{r \in [0,R_0) \ : \ u_0(r) \le m_\varepsilon^{1/4}\} \ ,$$

the properties of u_0 assumed in Theorem 1.4 and (A.7) ensure that there is $\varepsilon_0 \in (0,1/2)$ such that

$$r_\varepsilon > s_\varepsilon := m_\varepsilon^{(p-1-q)/4(p-q)} \quad \text{and} \quad m_\varepsilon \in (0,1) \quad \text{for any} \quad \varepsilon \in (0,\varepsilon_0) \ . \tag{A.8}$$

We then infer from the radial monotonicity of u_0, (1.12), (A.7), and (A.8) that, for $r \in [s_\varepsilon, r_\varepsilon]$,

$$\begin{aligned}
|\partial_r u_{0,\varepsilon}(r)| &\geq |\partial_r u_0(r)| - m_\varepsilon \\
&\geq \delta_0 r^{1/(p-1-q)} u_0(r)^{1/(p-q)} - m_\varepsilon \\
&\geq \frac{\delta_0}{2} r^{1/(p-1-q)} u_{0,\varepsilon}^{1/(p-q)}(r) + \frac{\delta_0}{2} r^{1/(p-1-q)} \left(u_0^{1/(p-q)} - u_{0,\varepsilon}^{1/(p-q)} \right)(r) \\
&\quad + \frac{\delta_0}{2} m_\varepsilon^{1/2(p-q)} - m_\varepsilon \\
&\geq \frac{\delta_0}{2} r^{1/(p-1-q)} u_{0,\varepsilon}^{1/(p-q)}(r) - \frac{\delta_0}{2} R_0^{1/(p-1-q)} m_\varepsilon^{1/(p-q)} + \frac{\delta_0}{2} m_\varepsilon^{1/2(p-q)} - m_\varepsilon \\
&\geq \frac{\delta_0}{2} r^{1/(p-1-q)} u_{0,\varepsilon}^{1/(p-q)}(r) \\
&\quad + m_\varepsilon^{1/2(p-q)} \left(\frac{\delta_0}{2} - \frac{\delta_0}{2} R_0^{1/(p-1-q)} m_\varepsilon^{1/2(p-q)} - m_\varepsilon^{(2(p-q)-1)/2(p-q)} \right) .
\end{aligned}$$

Since $p - q > 1$, we infer from (A.7) and the above inequality that, taking ε_0 smaller if necessary, there holds

$$|\partial_r u_{0,\varepsilon}(r)| \geq \frac{\delta_0}{2} r^{1/(p-q-1)} u_{0,\varepsilon}(r)^{1/(p-q)} , \qquad r \in [s_\varepsilon, r_\varepsilon] , \qquad \varepsilon \in (0, \varepsilon_0) . \tag{A.9}$$

Now fix $\varepsilon \in (0, \varepsilon_0)$. Recalling that a_ε and b_ε are given by (A.1), we define

$$a_{1,\varepsilon}(z) := 2z a_\varepsilon'(z) + a_\varepsilon(z) , \qquad z \geq 0 ,$$

and the auxiliary function

$$J_\varepsilon(t,r) := r^{N-1} a_\varepsilon(|\partial_r u_\varepsilon(t,r)|^2) \partial_r u_\varepsilon(t,r) + c(r) F(u_\varepsilon(t,r)) , \qquad (t,r) \in (0,T_e) \times (0,R_0) .$$

Since u_ε solves (A.6) and

$$\begin{aligned}
(p-1) a_\varepsilon(z) \leq \ a_{1,\varepsilon}(z) &= (z+\varepsilon^2)^{(p-4)/2} [(p-1)z + \varepsilon^2] \leq a_\varepsilon(z) , \\
b_\varepsilon(z) - 2z b_\varepsilon'(z) &= (z+\varepsilon^2)^{(q-2)/2} [(1-q)z + \varepsilon^2] \geq (1-q) b_\varepsilon(z)
\end{aligned}$$

for $z \geq 0$, we may perform the same computations as in the proof of Lemma 8.2 with $(a_\varepsilon, b_\varepsilon)$ instead of (a,b) and derive the analogue of (8.7):

$$\partial_t J_\varepsilon - a_{1,\varepsilon}(g_\varepsilon^2) \partial_r^2 J_\varepsilon + \left(\frac{N-1}{r} a_{1,\varepsilon}(g_\varepsilon^2) - 2 b_\varepsilon'(g_\varepsilon^2) g_\varepsilon \right) \partial_r J_\varepsilon \leq \sum_{i=1}^3 \mathcal{R}_{i,\varepsilon} , \tag{A.10}$$

where $g_\varepsilon := -\partial_r u_\varepsilon \geq 0$ and

$$\begin{aligned}
\mathcal{R}_{1,\varepsilon} &:= 2 \left[(N-1) r^{N-2} a_\varepsilon(g_\varepsilon^2) g_\varepsilon - c'(r) F(u_\varepsilon) \right] b_\varepsilon'(g_\varepsilon^2) g_\varepsilon , \\
\mathcal{R}_{2,\varepsilon} &:= \left[\frac{N-1}{r} c'(r) - c''(r) \right] F(u_\varepsilon) a_{1,\varepsilon}(g_\varepsilon^2) , \\
\mathcal{R}_{3,\varepsilon} &:= 2 \left[c'(r) - \frac{N-1}{r} c(r) \right] F'(u_\varepsilon) a_{1,\varepsilon}(g_\varepsilon^2) g_\varepsilon - c(r) F''(u_\varepsilon) a_{1,\varepsilon}(g_\varepsilon^2) g_\varepsilon^2 \\
&\qquad - (1-q) c(r) F'(u_\varepsilon) b_\varepsilon(g_\varepsilon^2) .
\end{aligned}$$

Observe that the positivity (A.5) of u_ε guarantees that $F'(u_\varepsilon)$ and $F''(u_\varepsilon)$ are well-defined, even if F is not twice differentiable at zero. As in the proof of Lemma 8.2 we choose

$$c(r) = r^\lambda \ , \quad r \ge 0 \ , \qquad F(z) = \delta z^\beta \ , \quad z \ge 0 \ ,$$

where $\delta > 0$ is to be determined and

$$\lambda = N + \frac{q}{p-1-q} > N \ , \qquad \beta = \frac{p-1}{p-q} \in (0,1) \ .$$

With this choice,

$$\mathcal{R}_{2,\varepsilon} \le -\lambda(\lambda-N)\delta(p-1) r^{\lambda-2} u_\varepsilon^\beta a_\varepsilon(g_\varepsilon^2) \ , \tag{A.11}$$

while

$$\mathcal{R}_{1,\varepsilon} = 2r^{\lambda-1} b_\varepsilon'(g_\varepsilon^2) g_\varepsilon \left[(N-1) r^{N-1-\lambda} a_\varepsilon(g_\varepsilon^2) g_\varepsilon - \delta\lambda u_\varepsilon^\beta \right] \tag{A.12}$$

and

$$\mathcal{R}_{3,\varepsilon} \le \beta\delta \left[\frac{2(\lambda-N+1)}{r} \left(g_\varepsilon^2+\varepsilon^2\right)^{(p-1-q)/2} + \frac{1-\beta}{u_\varepsilon} \left(g_\varepsilon^2+\varepsilon^2\right)^{(p-q)/2} \right.$$
$$\left. - (1-q) \right] r^\lambda u_\varepsilon^{\beta-1} b_\varepsilon(g_\varepsilon^2) \ . \tag{A.13}$$

Now let $\kappa > 0$ and define

$$\mathcal{J}_{\kappa,\varepsilon} := \{(t,r) \in (0,T_e) \times (0,R_0) \ : \ J_\varepsilon(t,r) \ge \kappa\} \ .$$

Then

$$\kappa + r^{N-1} a_\varepsilon(g_\varepsilon^2) g_\varepsilon \le c(r) F(u_\varepsilon) = \delta r^\lambda u_\varepsilon^\beta \quad \text{in} \quad \mathcal{J}_{\kappa,\varepsilon} \ ,$$

from which we deduce that, if $\kappa \ge R_0^{N-1} \varepsilon^{p-1}$, then

$$r^{N-1} (g_\varepsilon^2+\varepsilon^2)^{(p-1)/2} \le r^{N-1} a_\varepsilon(g_\varepsilon^2) g_\varepsilon + R_0^{N-1}\varepsilon^{p-1} \le \delta r^\lambda u_\varepsilon^\beta \quad \text{in} \quad \mathcal{J}_{\kappa,\varepsilon} \ . \tag{A.14}$$

This inequality implies in particular that $r > 0$ and $u_\varepsilon > 0$ in $\mathcal{J}_{\kappa,\varepsilon}$ and (A.11) yields

$$\mathcal{R}_2 \le -\lambda(\lambda-N)\delta(p-1) r^{\lambda-2} u_\varepsilon^\beta \left(g_\varepsilon^2+\varepsilon^2\right)^{(p-2)/2} < 0 \quad \text{in} \quad \mathcal{J}_{\kappa,\varepsilon} \tag{A.15}$$

in view of $\lambda > N$ and $p \in (1,2]$. We then infer from (A.12), (A.13), and (A.14) that, in $\mathcal{J}_{\kappa,\varepsilon}$,

$$\mathcal{R}_{1,\varepsilon} \le -2(\lambda-N+1)\delta r^{\lambda-1} u_\varepsilon^\beta b_\varepsilon'(g_\varepsilon^2) g_\varepsilon \le 0 \tag{A.16}$$

and

$$\begin{aligned} \mathcal{R}_{3,\varepsilon} &\le \beta\delta \Big[2(\lambda-N+1)\delta^{(p-1-q)/(p-1)} \left(\|u_0\|_\infty + \varepsilon^\gamma\right)^{(p-1-q)/(p-q)} \\ &\qquad + (1-\beta)\delta^{(p-q)/(p-1)} R_0^{(p-q)/(p-1-q)} - (1-q) \Big] r^\lambda u_\varepsilon^{\beta-1} b_\varepsilon(g_\varepsilon^2) \\ &\le -\frac{\beta\delta(1-q)}{2} r^\lambda u_\varepsilon^{\beta-1} b_\varepsilon(g_\varepsilon^2) \le 0 \ , \end{aligned} \tag{A.17}$$

provided δ is chosen suitably small (depending on $\|u_0\|_\infty$ and R_0) and independent of $\varepsilon \in (0,\varepsilon_0)$ as $\|u_0\|_\infty + \varepsilon^\gamma \le \|u_0\|_\infty + 1$. Collecting (A.10), (A.15), (A.16), and (A.17), we end up with

$$\partial_t J_\varepsilon - a_{1,\varepsilon}(g_\varepsilon^2) \partial_r^2 J_\varepsilon + \left(\frac{N-1}{r} a_{1,\varepsilon}(g_\varepsilon^2) - 2b_\varepsilon'(g_\varepsilon^2) g_\varepsilon \right) \partial_r J_\varepsilon < 0 \qquad \text{for } (t,r) \in \mathcal{J}_{\kappa,\varepsilon} \ , \tag{A.18}$$

this inequality being true only for $\kappa \geq R_0^{N-1}\varepsilon^{p-1}$.
Next, introducing

$$M_\varepsilon := \sup_{t\in[0,T_e]} u_\varepsilon(t,R_0) ,$$

we infer from the monotonicity of $r \mapsto u_\varepsilon(t,r)$ that

$$J_\varepsilon(t,0) = 0 , \qquad J_\varepsilon(t,R_0) \leq \delta R_0^\lambda M_\varepsilon^\beta , \quad t \in [0,T_e] . \tag{A.19}$$

In addition, given $r \in (0,R_0)$,

$$J_\varepsilon(0,r) \leq r^{N-1} a_\varepsilon(|\partial_r u_{0,\varepsilon}(r)|^2)\partial_r u_{0,\varepsilon}(r) + \delta r^\lambda u_{0,\varepsilon}^\beta(r) + \delta R_0^\lambda \varepsilon^{\gamma\beta} .$$

As $\partial_r u_{0,\varepsilon}(r) \leq 0$, we obtain from (A.3) and (A.7) that

$$J_\varepsilon(0,r) \leq \delta s_\varepsilon^\lambda \|u_0\|_\infty^\beta + \delta R_0^\lambda \varepsilon^{\gamma\beta}, \qquad r \in (0,s_\varepsilon) , \tag{A.20}$$

as well as

$$J_\varepsilon(0,r) \leq \delta R_0^\lambda \left(m_\varepsilon + m_\varepsilon^{1/4}\right)^\beta + \delta R_0^\lambda \varepsilon^{\gamma\beta}, \qquad r \in (r_\varepsilon, R_0) . \tag{A.21}$$

For $r \in [s_\varepsilon, r_\varepsilon]$, we now divide the analysis into two regions with respect to the magnitude of $|\partial_r u_{0,\varepsilon}(r)|$. Either $|\partial_r u_{0,\varepsilon}(r)| \leq \varepsilon$ and we deduce from (A.9) that

$$u_{0,\varepsilon}(r)^{1/(p-q)} \leq \frac{2\varepsilon}{\delta_0} r^{-1/(p-q-1)} .$$

Thus, taking also into account that $\partial_r u_{0,\varepsilon}(r) \leq 0$ for any $r \geq 0$, we realize that

$$\begin{aligned} J_\varepsilon(0,r) &\leq \delta r^\lambda (2\varepsilon)^{p-1} \delta_0^{1-p} r^{-(p-1)/(p-1-q)} + \delta R_0^\lambda \varepsilon^{\gamma\beta} \\ &\leq \delta\delta_0^{1-p} R_0^{N-1} (2\varepsilon)^{p-1} + \delta R_0^\lambda \varepsilon^{\gamma\beta} . \end{aligned} \tag{A.22}$$

Or $|\partial_r u_{0,\varepsilon}(r)| > \varepsilon$ which implies that

$$a_\varepsilon(|\partial_r u_{0,\varepsilon}(r)|^2) \geq 2^{(p-2)/2} |\partial_r u_{0,\varepsilon}(r)|^{p-2} .$$

Therefore, using again (A.9),

$$\begin{aligned} J_\varepsilon(0,r) &\leq \delta r^\lambda u_{0,\varepsilon}(r)^\beta - 2^{(p-2)/2} r^{N-1} |\partial_r u_{0,\varepsilon}(r)|^{p-1} + \delta R_0^\lambda \varepsilon^{\gamma\beta} \\ &\leq \left(\delta - 2^{-p/2}\delta_0^{p-1}\right) r^\lambda u_{0,\varepsilon}(r)^\beta + \delta R_0^\lambda \varepsilon^{\gamma\beta} \\ &\leq \delta R_0^\lambda \varepsilon^{\gamma\beta} , \end{aligned} \tag{A.23}$$

provided $\delta < 2^{-p/2}\delta_0^{p-1}$. In view of (A.8) and (A.20)-(A.23) we have thus shown that, if δ is sufficiently small,

$$J_\varepsilon(0,r) \leq \delta\delta_0^{1-p} R_0^{N-1}(2\varepsilon)^{p-1} + \delta\|u_0\|_\infty^\beta s_\varepsilon^\lambda + \delta R_0^\lambda \left((2m_\varepsilon)^{\beta/4} + \varepsilon^{\gamma\beta}\right) , \qquad r \in (0,R_0) .$$

Consequently, if δ is sufficiently small and

$$\begin{aligned} \kappa = \kappa_\varepsilon := &\ \delta\delta_0^{1-p} R_0^{N-1}(2\varepsilon)^{p-1} + \delta\|u_0\|_\infty^\beta s_\varepsilon^\lambda + \delta R_0^\lambda \left((2m_\varepsilon)^{\beta/4} + \varepsilon^{\gamma\beta}\right) \\ &+ R_0^{N-1}\varepsilon^{p-1} + \delta R_0^\lambda M_\varepsilon^\beta , \end{aligned}$$

then the parabolic boundary $\{0\} \times (0, R_0)$ and $[0, T_e) \times \{0, R_0\}$ of $(0, T_e) \times (0, R_0)$ contains no point in $\mathcal{J}_{\kappa_\varepsilon,\varepsilon}$. Recalling (A.18) we may then argue as in the proof of Lemma 8.2 to conclude that

$$J_\varepsilon \le \kappa_\varepsilon \quad \text{in} \quad (0, T_e) \times (0, R_0) \ . \tag{A.24}$$

To complete the proof, we observe that $M_\varepsilon \to 0$ as $\varepsilon \to 0$ due to the uniform convergence (A.4) and the vanishing of u on $(0, T_e) \times \partial B(0, R_0)$. Combining this fact with (A.7) and (A.8) yields

$$\lim_{\varepsilon \to 0} \kappa_\varepsilon = 0 \ ,$$

and we may let $\varepsilon \to 0$ in (A.24) and use (A.4) and (A.7) to obtain the expected result.

A.2 Proof of (5.4) for $p = 2$.

Finally, we prove the gradient estimate (5.4) for $p = 2$.

Proposition A.1. *Consider an initial condition u_0 satisfying* (1.3) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Assume further that $p = 2$ and $q \in (0, 1)$. Then there is $C_1 > 0$ depending only on q such that*

$$\left| \nabla u^{(1-q)/(2-q)}(t, x) \right| \le C_1 \left(1 + \|u_0\|_\infty^{(1-q)/(2-q)} t^{-1/2} \right) , \tag{A.25}$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^N$.

Proof. We fix $\varepsilon \in (0, 1/2)$ and denote the classical solution to (A.2) by $\tilde{u}_\varepsilon$. Observe that $a_\varepsilon \equiv 1$ due to $p = 2$. In view of (A.3), the comparison principle implies

$$\varepsilon^\gamma \le \tilde{u}_\varepsilon(t, x) \le \|u_0\|_\infty + \varepsilon^\gamma, \qquad (t, x) \in [0, \infty) \times \mathbb{R}^N \ . \tag{A.26}$$

We further set

$$f(\xi) := \frac{1-q}{2-q} \xi^{(2-q)/(1-q)}, \ \xi \ge 0, \qquad v_\varepsilon := f^{-1}(\tilde{u}_\varepsilon), \qquad w_\varepsilon := |\nabla v_\varepsilon|^2,$$

and note that $f \in C^2([0, \infty)) \cap C^\infty((0, \infty))$ is strictly increasing. Hence, according to [5, formula (10)], we have

$$\mathcal{P}_\varepsilon w_\varepsilon \le 2 \left(\frac{f''}{f'} \right)' (v_\varepsilon) w_\varepsilon^2 - 2 \left(\frac{f''}{(f')^2} \right) (v_\varepsilon) \Theta_\varepsilon \left((f')^2 (v_\varepsilon) w_\varepsilon \right) w_\varepsilon \tag{A.27}$$

in $(0, \infty) \times \mathbb{R}^N$, where

$$\mathcal{P}_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - \Delta w_\varepsilon + 2 \left(f'(v_\varepsilon) b_\varepsilon' \left((f')^2 (v_\varepsilon) w_\varepsilon \right) - \left(\frac{f''}{f'} \right) (v_\varepsilon) \right) \nabla v_\varepsilon \cdot \nabla w_\varepsilon \ ,$$

$$\Theta_\varepsilon(\xi) := 2\xi b_\varepsilon'(\xi) - b_\varepsilon(\xi) + \varepsilon^q \ , \qquad \xi \ge 0 \ .$$

Since $q \in (0, 1)$, we have

$$\begin{aligned} \Theta_\varepsilon(\xi) &= q\xi(\xi + \varepsilon^2)^{(q-2)/2} - (\xi + \varepsilon^2)^{q/2} + \varepsilon^q \\ &= -(1-q)(\xi + \varepsilon^2)^{q/2} - q\varepsilon^2 (\xi + \varepsilon^2)^{(q-2)/2} + \varepsilon^q \\ &\ge -(1-q)(\xi + \varepsilon^2)^{q/2} + (1-q)\varepsilon^q \\ &\ge -(1-q)\xi^{q/2} \ , \qquad \xi \ge 0 \ . \end{aligned}$$

Hence, (A.27), the choice of f, the nonnegativity of w_ε, and Young's inequality imply

$$\begin{aligned}
\mathcal{P}_\varepsilon w_\varepsilon &\le -\frac{2}{(1-q)v_\varepsilon^2} w_\varepsilon^2 + 2\cdot\frac{1}{1-q} v_\varepsilon^{(q-2)/(1-q)} \cdot (1-q)\left(v_\varepsilon^{2/(1-q)} w_\varepsilon\right)^{q/2} w_\varepsilon \\
&\le -\frac{2}{(1-q)v_\varepsilon^2} w_\varepsilon^2 + \frac{2}{(1-q)v_\varepsilon^2} w_\varepsilon^{1+q/2} \\
&\le \frac{2}{(1-q)v_\varepsilon^2}\left(-w_\varepsilon^2 + \frac{2+q}{4} w_\varepsilon^2 + \frac{2-q}{4}\right) \\
&= -\frac{2-q}{2(1-q)v_\varepsilon^2}\left(w_\varepsilon^2 - 1\right) .
\end{aligned} \tag{A.28}$$

Considering next

$$W(t) := 1 + \frac{a}{t},\ t>0, \qquad \text{with } \ a := 2\left(\frac{1-q}{2-q}\right)^{q/(2-q)} (\|u_0\|_\infty + 1)^{2(1-q)/(2-q)},$$

noticing that (A.26) implies

$$v_\varepsilon = \left(\frac{2-q}{1-q}\tilde{u}_\varepsilon\right)^{(1-q)/(2-q)} \le \left(\frac{2-q}{1-q}(\|u_0\|_\infty + 1)\right)^{(1-q)/(2-q)},$$

and using the fact that $a>0$, we obtain

$$\begin{aligned}
\mathcal{P}_\varepsilon W + \frac{2-q}{2(1-q)v_\varepsilon^2}\left(W^2-1\right) &\ge -\frac{a}{t^2} + \frac{2-q}{2(1-q)v_\varepsilon^2}\cdot\frac{a^2}{t^2} \\
&\ge \frac{a}{t^2}\left[-1 + \frac{a}{2}\left(\frac{2-q}{1-q}\right)^{q/(2-q)} (\|u_0\|_\infty + 1)^{-2(1-q)/(2-q)}\right] \\
&\ge 0
\end{aligned}$$

for any $t>0$. Since $w_\varepsilon(0,x) < W(0) = \infty$ for $x \in \mathbb{R}^N$, we deduce from (A.28) and the comparison principle that

$$\left(\frac{2-q}{1-q}\right)^{(1-q)/(2-q)} \left|\nabla \tilde{u}_\varepsilon^{(1-q)/(2-q)}(t,x)\right| = |\nabla v_\varepsilon(t,x)| = w_\varepsilon^{1/2}(t,x) \le \left(1+\frac{a}{t}\right)^{1/2}$$

in $(0,\infty)\times\mathbb{R}^N$. Letting $\varepsilon \searrow 0$ and recalling (A.4), we end up with (A.25). □

Acknowledgments

R. I. is partially supported by the Spanish project MTM2012-31103 and the Severo Ochoa Excellence project SEV-2011-0087. C. S. acknowledges the support of the Carl Zeiss foundation. Part of this work was done while C. S. held a one month invited position at the Institut de Mathématiques de Toulouse, and during visits of R. I. to the Institut de Mathématiques de Toulouse. Both authors would like to express their gratitude for the support and hospitality.

References

[1] U. G. Abdullaev, *Instantaneous shrinking of the support of solutions to a nonlinear degenerate parabolic equation*, Math. Notes, **63** (1998), no. 3, 285–292.

[2] D. Andreucci, A. F. Tedeev, and M. Ughi, *The Cauchy problem for degenerate parabolic equations with source and damping*, Ukrainian Math. Bull., **1** (2004), 1–23.

[3] M. Ben-Artzi, Ph. Souplet, and F.B. Weissler, *The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces*, J. Math. Pures Appl., **81** (2002), 343–378.

[4] S. Benachour, G. Karch, and Ph. Laurençot, *Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations*, J. Math. Pures Appl., **83** (2004), 1275–1308.

[5] S. Benachour and Ph. Laurençot, *Global solutions to viscous Hamilton-Jacobi equations with irregular initial data*, Comm. Partial Differential Equations, **24** (1999), no. 11-12, 1999–2021.

[6] S. Benachour, Ph. Laurençot, and D. Schmitt, *Extinction and decay estimates for viscous Hamilton-Jacobi equations in* $\mathbb{R}^N$, Proc. Amer. Math. Soc., **130** (2001), no. 4, 1103–1111.

[7] S. Benachour, Ph. Laurençot, D. Schmitt, and Ph. Souplet, *Extinction and non-extinction for viscous Hamilton-Jacobi equations in* $\mathbb{R}^N$, Asympt. Anal., **31** (2002), 229–246.

[8] S. Benachour, B. Roynette, and P. Vallois, *Asymptotic estimates of solutions of* $u_t - \frac{1}{2}\Delta u = -|\nabla u|$ *in* $\mathbb{R}_+ \times \mathbb{R}^d$, $d \geq 2$, J. Funct. Anal., **144** (1997), 301–324.

[9] M.-F. Bidaut-Véron and N.A. Dao, L^∞ *estimates and uniqueness results for nonlinear parabolic equations with gradient absorption terms*, Nonlinear Anal., **91** (2013), 121–152.

[10] P. Biler, M. Guedda, and G. Karch, *Asymptotic properties of solutions of the viscous Hamilton-Jacobi equation*, J. Evolution Equations, **4** (2004), 75–97.

[11] M. Borelli and M. Ughi, *The fast diffusion equation with strong absorption: the instantaneous shrinking phenomenon*, Rend. Ist. Mat. Univ. Trieste, **26** (1994), 109–140.

[12] S.P. Degtyarev, *Conditions for instantaneous support shrinking and sharp estimates for the support of the solution of the Cauchy problem for a doubly non-linear parabolic equation with absorption*, Sb. Math., **199** (2008), no. 4, 511–538.

[13] M. del Pino and M. Saéz, *Asymptotic description of vanishing in a fast-diffusion equation with absorption*, Differential Integral Equations, **15** (2002), no. 8, 1009–1023.

[14] L. C. Evans and B. F. Knerr, *Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities*, Illinois J. Math., **23** (1) (1979), 153–166.

[15] R. Ferreira, V. A. Galaktionov, and J. L. Vázquez, *Uniqueness of asymptotic profiles for an extinction problem*, Nonlinear Anal., **50** (2002), no. 4, 495–507.

[16] R. Ferreira and J. L. Vázquez, *Extinction behaviour for fast diffusion equations with absorption*, Nonlinear Anal., **43** (2001), no. 8, 943–985.

[17] A. Friedman and M. A. Herrero, *Extinction properties of semilinear heat equations with strong absorption*, J. Math. Anal. Appl., **124** (1987), no. 2, 530–546.

[18] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J., **34** (1985), no. 2, 425–447.

[19] Th. Gallay and Ph. Laurençot, *Asymptotic behavior for a viscous Hamilton-Jacobi equation with critical exponent*, Indiana Univ. Math. J., **56** (2007), 459–479.

[20] B. H. Gilding, M. Guedda, and R. Kersner, *The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$*, J. Math. Anal. Appl., **284** (2003), 733–755.

[21] B. H. Gilding and R. Kersner, *Instantaneous shrinking in nonlinear diffusion-convection*, Proc. Amer. Math. Soc, **109** (1990), no. 2, 385–394.

[22] B. H. Gilding, *The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$, large-time behaviour*, J. Math. Pures Appl., **84** (2005), 753–785.

[23] M. A. Herrero and J. J. L. Velázquez, *On the dynamics of a semilinear heat equation with strong absorption*, Comm. Partial Differential Equations, **14** (1989), no. 12, 1653–1715.

[24] M. A. Herrero and J. J. L. Velázquez, *Approaching an extinction point in one-dimensional semilinear heat equations with strong absorption*, J. Math. Anal. Appl. **170** (1992), 353–381.

[25] R. Iagar and Ph. Laurençot, *Positivity, decay and extinction for a singular diffusion equation with gradient absorption*, J. Funct. Anal., **262** (2012), no. 7, 3186–3239.

[26] A. S. Kalashnikov, *Conditions for the instantaneous compactifications of carriers of solutions of semilinear parabolic equations and systems*, Math. Notes, **47** (1990), no. 1-2, 49–53.

[27] A. S. Kalashnikov, *Quasilinear degenerate parabolic equations with singular lowest terms and growing initial data*, Differential Equations, **29** (1993), no. 6, 857–866.

[28] R. Kersner and F. Nicolosi, *The nonlinear heat equation with absorption: effects of variable coefficients*, J. Math. Anal. Appl., **170** (1992), no. 2, 551–566.

[29] R. Kersner and A. Shishkov, *Instantaneous shrinking of the support of energy solutions*, J. Math. Anal. Appl, **198** (1996), no. 3, 729–750.

[30] Ph. Laurençot, *Large time behavior for diffusive Hamilton-Jacobi equations*, in Topics in Mathematical Modeling, Lecture Notes, vol. 4, Jindrich Necas Center for Mathematical Modeling, Praha, 2008.