Fast construction of irreducible polynomials over finite fields

Jean-Marc Couveignes, Reynald Lercier

FAST CONSTRUCTION OF IRREDUCIBLE POLYNOMIALS OVER FINITE FIELDS

JEAN-MARC COUVEIGNES AND REYNALD LERCIER

Abstract. We present a randomized algorithm that on input a finite field $\mathbf{K}$ with q elements and a positive integer d outputs a degree d irreducible polynomial in $\mathbf{K}[x]$. The running time is $d^{1+\varepsilon(d)} \times (\log q)^{5+\varepsilon(q)}$ elementary operations. The function ε in this expression is a real positive function belonging to the class $o(1)$, especially, the complexity is quasi-linear in the degree d. Once given such an irreducible polynomial of degree d, we can compute random irreducible polynomials of degree d at the expense of $d^{1+\varepsilon(d)} \times (\log q)^{1+\varepsilon(q)}$ elementary operations only.

1. Introduction

This article deals with the following problem: given a prime integer p, a power $q = p^w$ of p, a finite field $\mathbf{K}$ with q elements, and a positive integer d, find a degree d irreducible polynomial in $\mathbf{K}[x]$. We assume that the finite field $\mathbf{K}$ is given as a quotient $(\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ where $m(z)$ is a degree w monic irreducible polynomial in $(\mathbb{Z}/p\mathbb{Z})[z]$. We assume furthermore that polynomials are given in a dense representation. The complexity of algorithms will be evaluated in terms of the number of necessary elementary operations. Additions, subtractions and comparisons in $\mathbf{K}$ require a constant times $\log q$ elementary operations. Multiplication and division require $(\log q) \times (\log \log q)^{1+\varepsilon(q)}$ elementary operations. Note that in this paper, the notation $\varepsilon(x)$ stands for a real positive function of the parameter x alone, belonging to the class $o(1)$.

It has been proven by Adleman and Lenstra [1] that under the generalized Riemann hypothesis there exists an algorithm that constructs a degree d irreducible polynomial over $\mathbf{K}$ in *deterministic* polynomial time in d and $\log q$. There is no known unconditional proof of this result. The main algorithms in this paper are *Las Vegas probabilistic*. The behavior of a Las Vegas algorithm depends on the input of course, but also on the result of some random choices. One has to flip coins. A Las Vegas algorithm either stops with the correct result or informs that it failed. The running time of the algorithm is bounded from above in terms of the size of the input only (this upper bound should not depend on the random choices). For *each* input, one asks that the probability that the algorithm succeeds is $\geqslant 1/2$. See Papadimitriou's book [19] for a formal definition of the main complexity classes.

A classical probabilistic approach to finding irreducible polynomials consists in first choosing a random polynomial of degree d and then testing for its irreducibility. The probability that a polynomial of degree d be irreducible is greater than or equal to $1/(2d)$. See Lidl and Niederreiter [16, Ex. 3.26 and 3.27, page 142] and Lemma 9 of Section 6 below. In order to check whether a polynomial $f(x)$ is irreducible, we may use Ben-Or's irreducibility test [2]. This test has maximal complexity $(\log q)^{2+\varepsilon(q)} \times d^{2+\varepsilon(d)}$ elementary operations while its average complexity is $(\log q)^{2+\varepsilon(q)} \times d^{1+\varepsilon(d)}$ elementary operations according to Panario

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and Richmond [18]. The average complexity of finding an irreducible polynomial with this method is thus $d^{2+\varepsilon(d)} \times (\log q)^{2+\varepsilon(q)}$ elementary operations. All the known algorithms have a quadratic factor at least in d. A survey can be found in the work of Lenstra [11, 12] and Shoup [22, section 1.2]. It seems difficult to improve on these existing methods as long as we use an irreducibility test.

So we are driven to consider very particular polynomials. For example, Adleman and Lenstra [1] construct irreducible polynomials in this way. Their method uses Gauss periods that are relative traces of roots of unity. In Section 2, we recall how efficient known methods can be for very special values of the degree d. We reach quasi-linear complexity in d when $d = \ell^\delta$ is a power of a prime divisor ℓ of $p(q-1)$. Section 3.1.2 explains how to construct a degree $d_1 d_2$ irreducible polynomial once given two irreducible polynomials of coprime degrees d_1 and d_2. We explain in Sections 4 and 5 how to construct irreducible polynomials using isogenies between elliptic curves. Thanks to this new construction, we reach quasi-linear complexity in d when $d = \ell^\delta$ is a power of a prime ℓ and ℓ does not divide $p(q-1)$. Putting everything together, we obtain a probabilistic algorithm that finds degree d irreducible polynomials in $\mathbf{K}[x]$ in quasi-linear time in d, without any restriction on d nor q. Our constructions are summarized in Section 6. In Section 3, we state several useful preliminary results about finite fields, polynomials and elliptic curves.

Remark. One may wonder if the algorithms and complexity estimates in this paper are still valid when the base field is not presented as a quotient of the form $(\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$. Following [12, Section 1], one may assume for example that elements in $\mathbf{K}$ are represented as vectors in $(\mathbb{Z}/p\mathbb{Z})^w$. Assume we are given the vector corresponding to the unit element 1. Assume also we are given a black box or an algorithm that computes multiplications and divisions of elements in $\mathbf{K}$. In this situation, before applying the algorithms presented in this paper, we should first construct an isomorphism between the given $\mathbf{K}$ and a quotient ring of the form $(\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$. To this end, we first look for a generator τ of the $(\mathbb{Z}/p\mathbb{Z})$-algebra $\mathbf{K}$. We pick a random element τ in $\mathbf{K}$. The probability that τ generates $\mathbf{K}$ over $\mathbb{Z}/p\mathbb{Z}$ is at least $1/2$ according to Lemma 9 of Section 6. We compute the powers τ^k for $0 \leqslant k \leqslant w$. These are $w+1$ vectors of length w. We compute the kernel of the corresponding matrix in $\mathcal{M}_{w\times(w+1)}(\mathbb{Z}/p\mathbb{Z})$. If the dimension of this kernel is bigger than 1, then τ is not a generator, so we pick a different τ and start again. If the kernel has dimension 1, we obtain the minimal polynomial $m(z) \in (\mathbb{Z}/p\mathbb{Z})[z]$ of τ, and an explicit isomorphism κ from $\tilde{\mathbf{K}} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ onto $\mathbf{K}$. All this requires $O(w)$ operations in $\mathbf{K}$ and $O(w^3)$ operations in $\mathbb{Z}/p\mathbb{Z}$. Given any degree d irreducible polynomial $\tilde{f}(x)$ in $\tilde{\mathbf{K}}[x]$, we deduce an irreducible polynomial in $\mathbf{K}[x]$ by applying the isomorphism κ to every coefficient in $\tilde{f}(x)$. This requires dw^2 operations in $\mathbb{Z}/p\mathbb{Z}$. So our algorithms and complexity estimates remain valid in that case, as long as elementary operations in $\mathbf{K}$ can be computed in time $(\log q)^{4+\varepsilon(q)}$ elementary operations. This includes all the reasonable known models for finite fields, including normal bases, explicit data [12] and towers of extensions.

Notations. If $\mathbf{K}$ is a field with characteristic p and q is a power of p, we denote by $\Phi_q : \mathbf{K} \to \mathbf{K}$ the morphism which raises to the q-th power. If $\mathbf{G}$ is an algebraic group over a field with q elements, we denote by $\varphi_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}^{(q)}$, the Frobenius morphism.

The book [17] by Liu provides a nice introduction to abstract algebraic geometry. A good monograph on elliptic curve is [23].

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2. Basic constructions

In this section, $\mathbf{K}$ is a finite field with $q = p^w$ elements and Ω is an algebraic closure of $\mathbf{K}$. For every positive integer k, we denote by $\mathbb{F}_{p^k}$ the unique subfield of Ω with p^k elements. We explain how to quickly construct a degree d irreducible polynomial when d is a prime power ℓ^δ and ℓ divides $p(q-1)$. All the constructions in this section are known, but deserve to be quickly surveyed. Section 2.1 deals with the case $\ell = p$. Section 2.2 deals with the case when ℓ is a prime divisor of $(q-1)$. Section 2.3 is concerned with the special case $\ell = 2$ and q odd. In Section 2.4, we illustrate on a simple example how to use Kummer theory together with descent when the roots of unity lie in a non-trivial extension of the base field. Although the results in Section 2.4 are not necessary to prove our main theorems in Section 6, several ideas at work in this section play a decisive role later in the slightly more advanced context of Section 5.

2.1. **Artin-Schreier towers.** In this section, we are given a p-th power $d = p^\delta$ and we want to construct a degree d irreducible polynomial in $\mathbf{K}[x]$. We use a construction of Lenstra and de Smit [13] in that case. If k and l are two positive integers such that l divides k, we define the polynomial $T_{l,k}(x) = x + x^{p^l} + x^{p^{2l}} + \cdots + x^{p^{(\frac{k}{l}-1)l}}$. For every positive integer k, we denote by $\mathcal{A}_k \subset \Omega$ the subset consisting of all scalars $a \in \Omega$ such that the three following conditions hold true:

- a generates $\mathbb{F}_{p^k}$ over $\mathbb{F}_p$, *i.e.* $\mathbb{F}_p(a) = \mathbb{F}_{p^k}$,
- a has non-zero absolute trace, *i.e.* $T_{1,k}(a) \neq 0$,
- a^{-1} has non-zero absolute trace, *i.e.* $T_{1,k}(a^{-1}) \neq 0$.

We set $I(X) = (X^p - 1)/(\sum_{1 \leqslant i \leqslant p-1} X^i)$. This rational fraction induces a p to 1 surjective map

$$I : \Omega - \mathbb{F}_p \to \Omega - \{0\}.$$

We check that $I^{-1}(\mathcal{A}_k) \subset \mathcal{A}_{pk}$ for every $k \geqslant 1$. Indeed, if $a \in \mathcal{A}_k$ and if $I(b) = a$ then $b \neq 1$ and

$$\frac{1}{(1-b)^p} - \frac{1}{1-b} = \frac{b^p - b}{(b-1)^{p+1}} = \frac{b + \cdots + b^{p-1}}{b^p - 1} = a^{-1}.$$

So $1/(1-b)$ is a root of the separable polynomial $x^p - x = a^{-1}$. This polynomial is irreducible over $\mathbb{F}_{p^k}[x]$ because the absolute trace of a^{-1} is non-zero. So $\mathbb{F}_p(b) = \mathbb{F}_{p^{pk}}$. Further, b is a root of the polynomial $x^p - a(x^{p-1} + \cdots + x) - 1$. So the trace $T_{k,pk}(b)$ of b relative to the extension $\mathbb{F}_{p^{pk}}/\mathbb{F}_{p^k}$ is a. As a consequence the absolute trace of b is $T_{1,pk}(b) = T_{1,k}(T_{k,pk}(b)) = T_{1,k}(a)$, the absolute trace of a, and it is non-zero. Now b^{-1} is a root of the reversed polynomial $x^p + a(x^{p-1} + \cdots + x) - 1$. So the trace of b^{-1} relative to the extension $\mathbb{F}_{p^{pk}}/\mathbb{F}_{p^k}$ is $-a$. As a consequence, the absolute trace of b^{-1} is the opposite of the absolute trace of a, and it is non-zero.

Since $\mathcal{A}_1 = \mathbb{F}_p - \{0\}$, we deduce that $\#\mathcal{A}_{p^k} \geqslant (p-1)p^k$. In particular the fiber above 1 of the iterated rational fraction $I^{(\delta)}$ is irreducible over $\mathbb{F}_p$. See [17, Section 3.1] for the definition of the fiber of a morphism over a point, and [17, Section 2.4] for the definition of an irreducible scheme. If w is prime to p, then this fiber remains irreducible over $\mathbf{K} = \mathbb{F}_q$. In general, we factor the degree w of $\mathbb{F}_q/\mathbb{F}_p$ as $w = p^e w'$ where w' is prime to p. We first look for an element $a \in \mathcal{A}_{p^e} \subset \mathbb{F}_q$. Using the remarks above we can find such an a by solving e Artin-Schreier equations with coefficients in $\mathbb{F}_q$. To this end, we write down the matrix of the $\mathbb{F}_p$-linear map $x \mapsto x^p - x$ in the $\mathbb{F}_p$-basis $(1, z, \dots, z^{w-1})$ of $\mathbf{K} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$. We then solve the e corresponding $\mathbb{F}_p$-linear systems of dimension w. Altogether, finding a requires a constant times $w \times \log p$ operations in $\mathbf{K}$ and a constant times ew^3 operations in $\mathbb{F}_p$. Since $w = O(\log q)$ and $e = O(\log w) = O(\log\log q)$ we end up with a complexity of $(\log q)^{4+\varepsilon(q)}$ elementary operations.

The fiber $I^{-\delta}(a)$ is a degree p^δ irreducible divisor over $\mathbb{F}_{p^{p^e}}$. It remains irreducible over $\mathbf{K} = \mathbb{F}_q$. It remains to compute the annihilating polynomial of this fiber. We compute the iterated rational fraction $I^\delta(x) = N(x)/D(x)$. Composition of polynomials and power series can be computed in quasi-linear time i.e. $d^{1+\varepsilon(d)} \times (\log q)^{1+\varepsilon(q)}$ elementary operations, using recent results by Umans and Kedlaya [24, 10] (see Corollary 1 in Section 3.1.3). An older algorithm due to Brent and Kung has exponent $(\omega+1)/2 + \varepsilon(d)$ where ω is the exponent in matrix multiplication. So we can compute $N(x)$ and $D(x)$ at the expense of $p^{\delta(1+\varepsilon(p^\delta))} \times (\log q)^{1+\varepsilon(q)}$, that is $d^{1+\varepsilon(d)} \times (\log q)^{1+\varepsilon(q)}$ elementary operations. The polynomial $f(x) = N(x) - aD(x)$ is an irreducible degree d polynomial in $\mathbf{K}[x]$.

We thus have proven the following lemma.

Lemma 1 (Artin-Schreier extensions)**.** *There exists a deterministic algorithm that on input a finite field $\mathbf{K} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ with cardinality $q = p^w$ and a positive integer δ computes an irreducible degree $d = p^\delta$ polynomial in $\mathbf{K}[x]$ at the expense of $(\log q)^{4+\varepsilon(q)} + d^{1+\varepsilon(d)} \times (\log q)^{1+\varepsilon(q)}$ elementary operations.*

Example. We take $p = 2$, $q = 4$, $\delta = 2$ and $d = 4$. We assume $\mathbf{K} = \mathbb{F}_2[z]/(z^2 + z + 1)$, so $e = 1$. We know that $1 \in \mathcal{A}_1$. We set $a = z \bmod z^2 + z + 1$ and check that $I(a) = 1$, so $a \in \mathcal{A}_2$. We compute $I(I(x)) = (x^4 + x^2 + 1)/(x^3 + x)$ and set $f(x) = x^4 + x^2 + 1 - a(x^3 + x)$. This is an irreducible polynomial in $\mathbf{K}[x]$.

2.2. **Radicial extensions.** In this section, ℓ is a prime dividing $q - 1$. Let $d = \ell^\delta$ for some positive integer δ. In the special case $\ell = 2$ we further ask that $\ell^2 = 4$ divide $q - 1$. We want to construct a degree d irreducible polynomial in $\mathbf{K}[x]$. This is a very classical case.

We write $q - 1 = \ell^e \ell'$ where ℓ' is prime to ℓ. We first look for a generator a of the ℓ-Sylow subgroup of $\mathbb{F}_q^*$. To find such a generator, we pick a random element in $\mathbb{F}_q^*$ and raise it to the power ℓ'. Call a the result. Check that $a^{\ell^{e-1}} \neq 1$. If this is not the case, start again. The probability of success is $1 - 1/\ell$. The average complexity of finding such an a is $O(\log q)$ operations in $\mathbb{F}_q$. The polynomial $f(x) = x^d - a$ is irreducible in $\mathbb{F}_q[x]$. This is well known but we try to prove it in a way that will be easily adapted to a more general context later.

The $\ell^{\delta+e}$-torsion $\mathbf{G}_m[\ell^{\delta+e}]$ of the multiplicative group $\mathbf{G}_m/\mathbb{F}_q$ is isomorphic to $(\mathbb{Z}/\ell^{\delta+e}\mathbb{Z}, +)$ and the Frobenius endomorphism $\varphi_{\mathbf{G}_m} : \mathbf{G}_m \to \mathbf{G}_m$ acts on it as multiplication by q. The order of $q = 1 + \ell'\ell^e$ in $(\mathbb{Z}/\ell^{e+\delta}\mathbb{Z})^*$ is $\ell^\delta = d$. So the Frobenius Φ_q acts transitively on the roots of $f(x)$.

Example. We take $p = 5$, $q = 5$, $\ell = 2$, $\delta = 3$ and $d = 8$. We check that 4 divides $p - 1$. In particular $e = 2$ and $\ell' = 1$. The class $a = 2 \bmod 5$ generates the 2-Sylow subgroup of $(\mathbb{Z}/5\mathbb{Z})^*$. Indeed $2^4 = 1 \bmod 5$ and $2^2 = -1 \bmod 5$. We set $f(x) = x^8 - 2$.

2.3. **A special case.** In this section, we assume that p is odd, $\ell = 2$ and $d = 2^\delta$ for some positive δ. We need to adapt the methods of Section 2.2 in that special case because the group of units in $\mathbb{Z}/d\mathbb{Z}$ that are congruent to 1 modulo ℓ is no longer cyclic when $\ell = 2$ and $\delta > 2$. We want to construct a degree d irreducible polynomial in $\mathbf{K}[x]$. This time we assume that 2^2 does not divide $q - 1$. So q is congruent to 3 modulo 4. We set $Q = q^2$ and observe that 4 divides $Q - 1$.

We first look for a generator c of $\mathbb{F}_Q$ over $\mathbf{K} = \mathbb{F}_q$. For example we take c a root of the polynomial $y^2 - r$ where r is not a square in $\mathbf{K}$ (we can take $r = -1$). If $\delta = 1$ we are done. Assume now $\delta \geqslant 2$. We write $Q - 1 = 2^e \ell'$ where ℓ' is prime to 2. We find a generator a of the 2-Sylow subgroup of $\mathbb{F}_Q^*$. The polynomial $F(x) = x^{d/2} - a$ is irreducible in $\mathbb{F}_Q[x]$. It remains to derive from $F(x)$ an irreducible polynomial $f(x)$ of degree d in $\mathbf{K}[x]$. We call $\bar{a} = \Phi_q(a) = a^q$, the conjugate of a over $\mathbb{F}_q$. We can compute it at the expense of $O(\log q)$ operations in $\mathbf{K}$. It is clear that $\bar{a} \neq a$ because the order of a is divisible by 4 and there is no point of order 4 in $\mathbf{G}_m(\mathbb{F}_q)$. The polynomial $f(x) = (x^{d/2} - a)(x^{d/2} - \bar{a})$ has coefficients in $\mathbf{K}$. It is irreducible over $\mathbf{K}$. Indeed, any root b of $x^{d/2} - a$ is also a root of $f(x)$. The field $\mathbb{F}_q(b)$ generated by b over $\mathbb{F}_q$ contains a and it has degree $d/2$ over $\mathbb{F}_q(a) = \mathbb{F}_Q$ because $F(x)$ is irreducible in $\mathbb{F}_Q[x]$. So $f(x)$ is irreducible in $\mathbf{K}[x]$.

Sections 2.2 and 2.3 prove the following lemma.

Lemma 2 (Kummer extensions)**.** *There exists a probabilistic (Las Vegas) algorithm that on input a finite field $\mathbf{K} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ with cardinality $q = p^w$, a prime integer ℓ dividing $q - 1$, and a positive integer δ, computes an irreducible degree $d = \ell^\delta$ polynomial in $\mathbf{K}[x]$ at the expense of $(\log q)^{2+\varepsilon(q)} + d \log q$ elementary operations.*

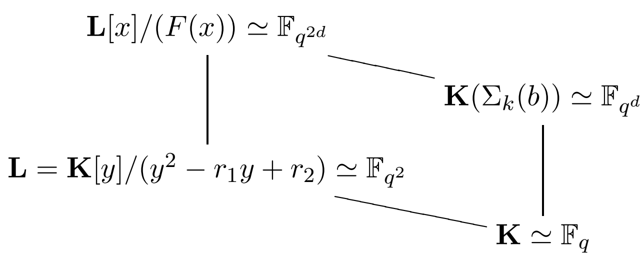
Example. We take $p = 7$, $q = 7$, $\ell = 2$, $\delta = 3$ and $d = 8$. Since 4 does not divide $q - 1$, we set $Q = q^2 = 49$. We factor $49 - 1 = 2^4 \times 3$ so $e = 4$ and $\ell' = 3$. We check that $r = 3 \bmod 7$ is not a square in $\mathbb{F}_7$. So we set $c = y \bmod y^2 - 3 \in \mathbb{F}_7[y]/(y^2 - 3)$. We set $a = (1 + c)^3 = 3 - c$ and check $a^{16} = 1$ and $a^8 = -1$. We set $F(x) = x^4 - a$. We compute $\bar{a} = a^7 = 3 + c$. We set $f(x) = (x^4 - a)(x^4 - \bar{a}) = x^8 + x^4 - 1$. This is an irreducible polynomial in $\mathbb{F}_7[x]$.

2.4. **Descent.** In this section, we recall how to use Kummer theory when roots of unity are missing. We do not hope to find a quasi-optimal algorithm that way. But several important algorithmic questions arise naturally in this context.

We assume $\ell = 3$ and $d = 3^\delta$ and $p = q \neq 3$. We assume that 3 does not divide $q - 1$. So q is congruent to 2 modulo 3, and we cannot apply the method in Section 2.2. We experiment in this simple context an idea that will be decisive in Section 5. We base change to a small auxiliary extension. We set $Q = q^2$ and observe that 3 divides $Q - 1$. We shall deal with the field $\mathbb{F}_Q$ with Q elements. We note that this idea is valid for any prime ℓ, but the degree of the auxiliary extension $\mathbb{F}_Q/\mathbb{F}_q$ might be quite large (up to $\ell - 1$) for a general ℓ.

We first need to build a computational model for the field $\mathbb{F}_Q$. For example we pick a degree 2 irreducible polynomial $y^2 - r_1 y + r_2$ in $\mathbf{K}[x]$ and set $\mathbf{L} = \mathbf{K}[y]/(y^2 - r_1 y + r_2)$. We set $c = y \bmod y^2 - r_1 y + r_2$. We write $Q - 1 = 3^e \ell'$ where ℓ' is prime to 3. We find a generator a of the 3-Sylow subgroup of $\mathbf{L}^*$. The polynomial $F(x) = x^d - a$ is irreducible in $\mathbf{L}[x]$. It remains to derive from $F(x)$ an irreducible polynomial $f(x)$ of degree d in $\mathbf{K}[x]$.

Let $b = x \bmod F(x)$. This is a root of $F(x)$ in $\mathbf{L}[x]/(F(x))$. The latter field has q^{2d} elements. Recall Φ_q is the map which raises to the q-th power. We have $\Phi_Q = \Phi_q^2$. For any α in $\mathbf{L}[x]/(F(x))$, we set $\Sigma_1(\alpha) = \alpha + \Phi_q^d(\alpha)$ and $\Sigma_2(\alpha) = \alpha \times \Phi_q^d(\alpha)$.



Since d is a *prime power*, at least one among $\Sigma_1(b)$ and $\Sigma_2(b)$ generates an extension of degree d of $\mathbb{F}_q$ (see Lemma 3 of Section 3.1.1). In other words, there exists a $k \in \{1, 2\}$ such that the polynomial

$$f(x) = \prod_{0 \leqslant l < d} (x - \Phi_q^l(\Sigma_k(b)))$$

is irreducible of degree d in $\mathbf{K}[x]$. Three questions now worry us.

(1) How to compute $\Sigma_k(b)$ for $k \in \{1, 2\}$?
(2) How to find the good integer k ?
(3) How to compute $f(x)$ starting from $F(x)$?

Question 1 boils down to asking how to compute $\Phi_q^d(b)$. A first method would be to compute $\Phi_q^d(b)$ as b^{q^d} at the expense of a constant times $d \log q$ operations in $\mathbf{L}[x]/(F(x))$. This would require a constant times $\log q \times d^{2+\varepsilon(d)}$ operations in $\mathbf{K}$. This is too much for us.

Instead of that, we should remind ourselves of the geometric origin of the polynomial $F(x)$. Indeed, b lies in $\mathbf{G}_m[3^{e+\delta}]$. We write $q^d = R \bmod 3^{e+\delta}$ where $0 \leqslant R < 3^{e+\delta} \leqslant Qd$. Then $\Phi_q^d(b) = b^R$ can be computed at the expense of a constant times $\log R \leq \log Q + \log d$ operations in $\mathbf{L}[x]/(F(x))$. This requires a constant times $\log q \times d^{1+\varepsilon(d)}$ operations in $\mathbf{K}$.

Question 2 can be solved by comparing $\Sigma_1(b)$ and its conjugate by $\Phi_q^{3^{\delta-1}}$. We have

$$\Phi_q^{3^{\delta-1}}(\Sigma_1(b)) = \Sigma_1(\Phi_q^{3^{\delta-1}}(b)) = \Phi_q^{3^{\delta-1}}(b) + \Phi_q^{3^\delta + 3^{\delta-1}}(b).$$

Each of the two terms in the above sum can be computed as explained in the answer of Question 1. Since $\Sigma_2(b) = 1$ here, we already know that $\Sigma_1(b)$ is the good candidate. But we keep the more naive approach in mind.

Question 3 is related to the following problem: we are given $\Sigma_k(b)$ for some $k \in \{1, 2\}$. We know that $\Sigma_k(b)$ generates the degree d extension of $\mathbf{K}$ inside $\mathbf{L}[x]/(F(x))$. Therefore its minimal polynomial $f(x)$ in the latter extension has coefficients in $\mathbf{K}$. We want to compute this degree d polynomial in $\mathbf{K}[x]$. One can apply a general algorithm for this task, such as the one given by Kedlaya and Umans ([24, 10] and Theorem 2 below). They show that it is possible to compute this minimal polynomial at the expense of $d^{1+\varepsilon(d)} \times (\log Q)^{1+\varepsilon(q)}$ elementary operations. Thus, the complexity is quasi-linear in the degree d.

Example. We take $p = q = 5$, $\ell = 3$, $\delta = 2$, $d = 9$. So $Q = 25$, $Q - 1 = 3 \times 8$, $e = 1$ and $\ell' = 8$. We check that $r = 2 \bmod 5$ is not a square. We set $c = y \bmod y^2 - 2 \in \mathbb{F}_5[y]/(y^2 - 2)$. We compute $a = (1 + c)^8 = 2 + 3c$. We check $a^3 = 1$ and $a \neq 1$. We set $F(x) = x^9 - a$ and $b = x \bmod F(x)$. We need to compute the conjugate of b above $\mathbb{F}_{5^9}$. This is b^{5^9}.

Recall b lies in $\mathbf{G}_m[27]$. So we don't raise b to the power 5^9 brutally. We rather compute $5^9 = 1953125 = -1 \bmod 27$. So $\Phi_{5^9}(b) = 1/b = 2(y+1)x^8 \bmod (x^9 - 2 - 3y, y^2 - 2, 5)$. The product $\Sigma_2(b) = 1$ is not the good candidate. So we compute the minimal polynomial of $\Sigma_1(b) = b + 1/b$ and find $f(x) = x^9 + x^7 + 2x^5 + 4x + 1 \in \mathbb{F}_5[x]$.

3. Preliminary algebraic and geometric results

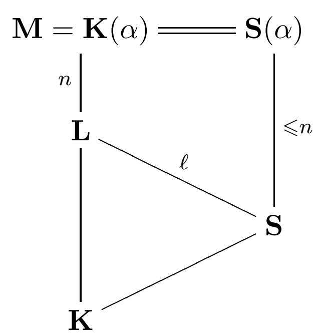
We introduce several algebraic and algorithmic results about finite field extensions and elliptic curves over finite fields

3.1. Finite field extensions.

In this section, we collect algebraic and algorithmic results about finite fields.

3.1.1. *Generator of a subextension.* We prove the following lemma.

Lemma 3 (Subfield generated by a symmetric function)**.** *Let $\mathbf{M}$ be a finite field and let $\mathbf{K}$ be a subfield of $\mathbf{M}$. We assume that the degree of $\mathbf{M}$ over $\mathbf{K}$ is a power of a prime integer ℓ. Let α be a generator of $\mathbf{M}$ over $\mathbf{K}$. Let $\mathbf{L}$ be a subfield of $\mathbf{M}$ containing $\mathbf{K}$. Let n be the degree of $\mathbf{M}$ over $\mathbf{L}$. Let $(\Sigma_k(\alpha))_{0<k\leqslant n}$ be the n symmetric functions of α above $\mathbf{L}$. Then at least one among these n symmetric functions generates $\mathbf{L}$ over $\mathbf{K}$.*



Proof. If $\mathbf{L} = \mathbf{K}$, there is nothing to prove. When $\mathbf{L}$ is a non trivial extension of $\mathbf{K}$, the degree of this extension is a power of ℓ. Let $\mathbf{S}$ be the unique maximal proper subfield of $\mathbf{L}$ containing $\mathbf{K}$. The degree of $\mathbf{L}$ over $\mathbf{S}$ is ℓ. The extension $\mathbf{M}/\mathbf{L}$ is cyclic of finite degree n, a power of ℓ.

The n elementary symmetric functions of α over $\mathbf{L}$ are the coefficients of the minimal polynomial of α, seen as an element in the $\mathbf{L}$-algebra $\mathbf{M}$. We claim that at least one of these symmetric functions $(\Sigma_k(\alpha))_{0<k\leqslant n}$ generates $\mathbf{L}$ over $\mathbf{K}$. Otherwise, all these functions would be contained in $\mathbf{S}$. The field $\mathbf{S}(\alpha)$ would then be a degree $\leqslant n$ algebraic extension of $\mathbf{S}$. Since $\mathbf{S}(\alpha)$ contains $\mathbf{K}(\alpha)$, $\mathbf{S}(\alpha)$ is $\mathbf{M}$. But the degree of $\mathbf{M}$ over $\mathbf{L}$ is n, and this is greater than or equal to the degree of $\mathbf{M}$ over $\mathbf{S}$. So $\mathbf{L} = \mathbf{S}$. A contradiction. $\square$

3.1.2. *Compositum.* In this section, $\mathbf{K}$ is a finite field with $q = p^w$ elements and Ω is an algebraic closure of $\mathbf{K}$. For every positive integer k, we denote by $\mathbb{F}_{p^k}$ the unique subfield of Ω with p^k elements. We have seen in Section 2 how to construct an irreducible polynomial of degree d in $\mathbf{K}[x]$ when d is a prime power ℓ^δ and ℓ divides $p(q-1)$. In Sections 4 and 5, we shall treat the case when d is a prime power ℓ^δ and ℓ is prime to $p(q-1)$. The last problem to be considered is thus the following one: given two irreducible polynomials $f_1(x)$ and $f_2(x)$ in $\mathbf{K}[x]$ with coprime degrees d_1 and d_2, construct a degree $d_1 d_2$ irreducible polynomial.

Lemma 4 (Composed sum of two polynomials)**.** *There exists a deterministic algorithm that on input a finite field* $\mathbf{K} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ *with* q *elements, two irreducible polynomials* f_1 *and* f_2 *in* $\mathbf{K}[x]$ *of coprime degree* d_1 *and* d_2*, computes a degree* $d_1 d_2$ *irreducible polynomial in* $\mathbf{K}[x]$ *at the expense of* $(d_1d_2)^{1+\varepsilon(d_1d_2)} \times (\log q)^{1+\varepsilon(q)}$ *elementary operations.*

Proof. Let $\alpha_1 \in \Omega$ be a root of $f_1(x)$. Let $\alpha_2 \in \Omega$ be a root of $f_2(x)$. We first show that $\alpha_1+\alpha_2$ generates an extension of degree d_1d_2 of $\mathbb{F}_q$. Indeed, let $\Phi \in \mathrm{Gal}(\Omega/\mathbb{F}_q)$ be an automorphism that fixes $\alpha_1 + \alpha_2$,

$$\Phi(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2 \,.$$

One deduces that $\Phi(\alpha_1) - \alpha_1 = \alpha_2 - \Phi(\alpha_2)$ is an element γ of the intersection $\mathbb{F}_q$ of $\mathbb{F}_{q^{d_1}}$ and $\mathbb{F}_{q^{d_2}}$. The order of Φ acting on $\mathbb{F}_{q^{d_1}}$ divides d_1. So $\Phi^{d_1}(\alpha_1) - \alpha_1 = d_1\gamma = 0$. We prove in the same way that $d_2\gamma = 0$. Since d_1 and d_2 are coprime we deduce that $\gamma = 0$. Thus Φ acts trivially on $\mathbb{F}_{q^{d_1}} = \mathbb{F}_q(\alpha_1)$ and on $\mathbb{F}_{q^{d_2}} = \mathbb{F}_q(\alpha_2)$, therefore also on their compositum $\mathbb{F}_{q^{d_1d_2}}$. So $\alpha_1 + \alpha_2$ generates this compositum. Note that the same argument proves that $\alpha_1\alpha_2$ generates $\mathbb{F}_{q^{d_1d_2}}$.

It is thus enough to compute the minimal polynomial of the sum, resp. the product, of α_1 and α_2. For this task, one may follow works by Bostan, Flajolet, Salvy and Schost [3], based on algorithms for symmetric power sums due to Kaltofen and Pan [9] and Schönhage [20]. The resulting polynomial is called the *composed sum*, resp. the *composed product*, of f_1 and f_2. See also [4]. This yields an algorithm with a quasi-linear time complexity in d_1d_2. □

3.1.3. *Fast composition.* The following theorems were recently proven by Umans and Kedlaya [10].

Theorem 1 (Kedlaya and Umans)**.** *There exists a deterministic algorithm that on input a finite field* $\mathbf{K} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ *with* q *elements and three polynomials* $f(x)$*,* $g(x)$ *and* $h(x)$ *in* $\mathbf{K}[x]$ *with degrees bounded by* d*, outputs the remainder* $f(g(x)) \bmod h(x)$ *at the expense of* $d^{1+\varepsilon(d)}(\log q)^{1+\varepsilon(q)}$ *elementary operations.*

Theorem 2 (Kedlaya and Umans)**.** *There exists a deterministic algorithm that on input a finite field* $\mathbf{K} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ *with* q *elements, a degree* d *irreducible monic polynomial* $f(x)$ *in* $\mathbf{K}[x]$*, and a degree* $\leqslant d-1$ *polynomial* $g(x)$ *in* $\mathbf{K}[x]$*, outputs the minimal polynomial* $h(x) \in \mathbf{K}[x]$ *of the class* $g(x) \bmod f(x)$ *at the expense of* $d^{1+\varepsilon(d)}(\log q)^{1+\varepsilon(q)}$ *elementary operations.*

The following corollary of Theorem 1 is particularly useful.

Corollary 1. *There exists a deterministic algorithm that on input a finite field* $\mathbf{K}$ *with* q *elements given by a quotient* $(\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ *and two rational fractions* $F(x)$ *and* $G(x)$ *in* $\mathbf{K}(x)$ *with respective degrees* d_F *and* d_G*, outputs the composition* $F(G(x)) = u(x)/v(x)$ *where* $u(x)$ *and* $v(x)$ *are coprime polynomials, at the expense of* $(d_Fd_G)^{1+\varepsilon(d_Fd_G)}(\log q)^{1+\varepsilon(q)}$ *elementary operations.*

We first notice that the problem is trivial if one of the two fractions has degree 1. Composing F and G with rational linear fractions we may assume that $F(0) = G(0) = 0$. We compute the Taylor expansions at 0 of either fractions and we compose them using the algorithm in Theorem 1. We recover the numerator $u(x)$ and denominator $v(x)$ of the corresponding fraction using the fast extended Euclid algorithm [5, Chapter 11].

3.2. **Elliptic curve over finite fields.** We now state several known and useful facts about elliptic curves over finite fields.

3.2.1. *Quotient isogenies.* Let $\mathbf{K}$ be a finite field of characteristic p and cardinality q. Let E be an elliptic curve over $\mathbf{K}$. We denote by $\varphi_E : E \to E$ the degree q Frobenius endomorphism of E. Let t be the trace of φ_E. Let $\mathcal{O}$ be the quotient ring $\mathbb{Z}[X]/(X^2 - tX + q)$ and let α be the class of X in $\mathcal{O}$. Let $\epsilon_E : \mathcal{O} \to \mathrm{End}(E)$ be the ring homomorphism that maps α onto φ_E. We say that ϵ_E is the *standard labeling* of E.

Let S be a subset of $\mathcal{O}$ containing an integer that is prime to p. We define the *kernel* of S in E to be the intersection of the kernels of all endomorphisms $\epsilon_E(s)$ for $s \in S$. This is a finite étale subgroup of E. So, it is characterized by its set of geometric points. We denote it $E[S]$.

Now let F be another elliptic curve over $\mathbf{K}$ and let $\iota : E \to F$ be an isogeny defined over $\mathbf{K}$. Let $\epsilon_F : \mathcal{O} \to \mathrm{End}(F)$ be the morphism of free $\mathbb{Z}$-modules that sends 1 onto the identity and α onto φ_F. For any element s in $\mathcal{O}$, we have

$$\iota \circ \epsilon_E(s) = \epsilon_F(s) \circ \iota. \tag{1}$$

Indeed, the identity above is true for $s = \alpha$ because ι is defined over $\mathbf{K}$. It is evidently true also for $s = 1$. Therefore it is true for all s in $\mathcal{O}$ by linearity. We deduce from Eq. (1) that ϵ_F is a ring homomorphism, just as ϵ_E.

Now let G be a third elliptic curve over $\mathbf{K}$. Let $\jmath : F \to G$ be an isogeny defined over $\mathbf{K}$. We define $\epsilon_G : \mathcal{O} \to \mathrm{End}(G)$ as before. Assume $\iota : E \to F$ is separable with kernel $E[S]$ where S is a subset of $\mathcal{O}$ containing a prime to p integer. Assume $\jmath : F \to G$ is separable with kernel $F[T]$ where T is a subset of $\mathcal{O}$ containing a prime to p integer. Then the kernel of

$$\jmath \circ \iota : \; E \xrightarrow{\iota} F \xrightarrow{\jmath} G$$

is $E[ST]$. Indeed, both the kernel of $\jmath \circ \iota$ and $E[ST]$ are étale, so they are characterized by their geometric points. Now let x be a point in the kernel of $\jmath \circ \iota$. Its image $\iota(x)$ by ι lies in the kernel of $\jmath$. Therefore it is killed by T: for any element t of T one has $\epsilon_F(t)(\iota(x)) = 0_F$. So $\iota(\epsilon_E(t)(x)) = 0_F$ and $\epsilon_E(t)(x)$ belongs in the kernel of ι. Thus it is killed by S: for any s in S we have $\epsilon_E(s)(\epsilon_E(t)(x)) = 0_E$ or equivalently $\epsilon_E(st)(x) = 0$. Therefore x lies in $E[ST]$.

Conversely, let x be a point in $E[ST]$. Let t be an element in T. We observe that $\epsilon_E(t)(x)$ is killed by S, so it belongs to the kernel of ι. Thus $\iota(\epsilon_E(t)(x)) = \epsilon_F(t)(\iota(x)) = 0_F$. So $\iota(x)$ is killed by T, therefore it belongs to the kernel of $\jmath$. Thus $\jmath(\iota(x)) = 0_G$.

Lemma 5 (Composition of quotient isogenies)**.** *Let $\mathbf{K}$ be a finite field with characteristic p. Let E be an elliptic curve over $\mathbf{K}$. Let t be the trace of the Frobenius endomorphism of E. Let $\mathcal{O}$ be the quotient ring $\mathbb{Z}[X]/(X^2 - tX + q)$ and let $\epsilon_E : \mathcal{O} \to \mathrm{End}(E)$ be the standard labeling. Let S be a subset of $\mathcal{O}$ containing a prime to p integer and let $\iota : E \to F$ be the quotient by $E[S]$ isogeny. Let T be a subset of $\mathcal{O}$ containing a prime to p integer and let $\jmath : F \to G$ be the quotient by $F[T]$ isogeny. Then the kernel of $\jmath \circ \iota$ is $E[ST]$.*

We note that a general and conceptual study of the action of rings over group schemes was initiated by Serre [21], Giraud [6], Waterhouse [26], and Lenstra [15]. In the case of elliptic curves one may use canonical lifts and reduce to complex multiplication theory in characteristic zero. We prefer a more self-contained and elementary approach.

3.2.2. *Density of elliptic curves with an ℓ-torsion point.* Let $\mathbf{K}$ be a finite field with q elements and let ℓ be a prime integer. Lenstra [14] and Howe [7] give estimates for the density of elliptic curves over $\mathbf{K}$ whose number of $\mathbf{K}$-rational points is divisible by ℓ. In this section, we recall what these authors mean by density and we explain why this density fits with the uniform density on Weierstrass curves.

We call $\mathcal{E}(\mathbf{K})$ the set of $\mathbf{K}$-isomorphism classes of elliptic curves over $\mathbf{K}$. The $\mathbf{K}$-isomorphism class of a curve $E/\mathbf{K}$ is denoted by $[E]$. One defines a measure on the finite set $\mathcal{E}(\mathbf{K})$ in the following way: the measure of a class $[E]$ is the inverse of the group of $\mathbf{K}$-automorphisms of E. So the measure of a subset S of $\mathcal{E}(\mathbf{K})$ is

$$\mu_{\mathcal{E}}(S) = \sum_{[E]\in S} \frac{1}{\#\operatorname{Aut}_{\mathbf{K}}(E)}. \tag{2}$$

Lenstra and Howe prove that the measure of the full set $\mathcal{E}(\mathbf{K})$ is q.

Now, let $\mathcal{W}(\mathbf{K})$ be the set of Weierstrass elliptic curves over $\mathbf{K}$,

$$E/\mathbf{K} \ : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3\,. \tag{3}$$

We denote by $\mu_{\mathcal{W}}$ the uniform measure on this set: the $\mu_{\mathcal{W}}$-measure of a subset of $\mathcal{W}(\mathbf{K})$ is defined to be its cardinality. This is a very convenient measure. In order to pick a random Weierstrass curve according to this measure, we just choose each coefficient a_1, a_2, a_3, a_4, a_6 at random with the uniform probability in $\mathbf{K}$ and we check that the discriminant is non-zero (if it is zero we start again).

Let $\gamma : \mathcal{W}(\mathbf{K}) \to \mathcal{E}(\mathbf{K})$ be the map that to every curve E associates its isomorphism class $[E]$. This is a surjection: every elliptic curve over $\mathbf{K}$ has a Weierstrass model over $\mathbf{K}$. Let $\mathbf{A}(\mathbf{K})$ be the group of projective transforms of the form

$$(X : Y : Z) \mapsto (u^2X + rZ : u^3Y + su^2X + tZ : Z)$$

where $u \in \mathbf{K}^*$ and $r, s, t \in \mathbf{K}$. This group acts on the set $\mathcal{W}(\mathbf{K})$ of Weierstrass elliptic curves over $\mathbf{K}$. Two Weierstrass elliptic curves over $\mathbf{K}$ are isomorphic over $\mathbf{K}$ if and only if they lie in the same orbit for the action of $\mathbf{A}(\mathbf{K})$. Further the group of $\mathbf{K}$-automorphisms of a Weierstrass elliptic curve is isomorphic to the stabilizer of E in $\mathbf{A}(\mathbf{K})$.

So the orbit of a Weierstrass curve $E/\mathbf{K}$ under the action of $\mathbf{A}(\mathbf{K})$ is the fiber $\gamma^{-1}([E])$ and the cardinality of this fiber is the quotient $\#\mathbf{A}(\mathbf{K}) \,/\, \#\operatorname{Aut}_{\mathbf{K}}(E)$. Therefore, if S is a subset of $\mathcal{E}(\mathbf{K})$ and if T is its preimage by γ, then the measures of S and T are proportional,

$$\mu_{\mathcal{W}}(T) = \#\mathbf{A}(\mathbf{K}) \times \mu_{\mathcal{E}}(S) \text{ where } \#\mathbf{A}(\mathbf{K}) = (q-1)q^3.$$

In particular, if we want to pick a random $\mathbf{K}$-isomorphism class of elliptic curve according to the measure $\mu_{\mathcal{E}}$, it suffices to pick a random Weierstrass elliptic curve according to the uniform measure $\mu_{\mathcal{W}}$.

We now can state a special case of the main result in Howe's paper [7].

Theorem 3 (Howe)**.** *Let q be a prime power and let $\mathbf{K}$ a field with q elements. Let $\mathcal{E}(\mathbf{K})$ be the set of $\mathbf{K}$-isomorphism classes of elliptic curves over $\mathbf{K}$. Let $\mu_{\mathcal{E}}$ be the measure on this set defined by Eq. (2). Let ℓ be a prime integer not dividing $q-1$. The isomorphism classes in $\mathcal{E}(\mathbf{K})$ of elliptic curves having a $\mathbf{K}$-rational point of order ℓ form a subset of density $r(\ell, q)$ where*

$$\left| r(\ell, q) - \frac{1}{\ell - 1} \right| \leq \frac{4\ell(\ell+1)}{(\ell-1)\sqrt{q}}.$$

We deduce the following corollary.

Corollary 2 (Density of elliptic curves with an ℓ-torsion point)**.** *Let q be a prime power and let $\mathbf{K}$ be a field with q elements. Let $\mathcal{W}(\mathbf{K})$ be the set of Weierstrass elliptic curves over $\mathbf{K}$. Let $\mu_{\mathcal{W}}$ be the uniform measure on this set. Let ℓ be a prime integer not dividing $q-1$. The density $r(\ell, q)$ of Weierstrass curves having a $\mathbf{K}$-rational point of order ℓ satisfies*

$$\left| r(\ell, q) - \frac{1}{\ell - 1} \right| \leq \frac{4\ell(\ell+1)}{(\ell-1)\sqrt{q}}.$$

4. Isogeny fibers

In this section, we show how to construct irreducible polynomials using elliptic curves.

Let $\mathbf{K}$ be a field and let Ω be an algebraic closure of $\mathbf{K}$. Let $E/\mathbf{K}$ be an elliptic curve given by the Weierstrass equation (3). We denote by $O_E = [0:1:0]$ the origin of E and by $x = X/Z$, $y = Y/Z$ the affine coordinates associated with the projective coordinates $[X : Y : Z]$.

Let $E'/\mathbf{K}$ be another elliptic curve in Weierstrass form. We define X', Y', Z', a'_1, a'_2, a'_3, a'_4, a'_6, x', y', O' similarly. Let $\iota/\mathbf{K} : E/\mathbf{K} \to E'/\mathbf{K}$ be a degree d separable isogeny. We assume that d is a positive odd number and the kernel $\operatorname{Ker}\iota$ is cyclic. Let $T \in E(\Omega)$ be a generator of $\operatorname{Ker}\iota$. Let $\psi_\iota(x) \in \mathbf{K}[x]$ be the degree $(d-1)/2$ polynomial

$$\psi_\iota(x) = \prod_{1 \leqslant k \leqslant (d-1)/2} (x - x(kT)) \,. \tag{4}$$

There exists a degree d polynomial $\phi_\iota(x) \in \mathbf{K}[x]$ and a polynomial $\omega_\iota(x, y) = \omega_0(x) + y\omega_1(x)$ in $\mathbf{K}[x, y]$ with degree 1 in y such that the image of the point (x, y) by ι is (x', y') where $x' = \phi_\iota(x)/\psi_\iota^2(x)$ and $y' = \omega_\iota(x, y)/\psi_\iota^3(x)$. We call $I(x)$ the rational fraction $I(x) = \phi_\iota(x)/\psi_\iota^2(x)$.

Now let A be a $\mathbf{K}$-rational point on E' such that $2A \neq O'$ and let $B \in E(\Omega)$ be a point on E such that $\iota(B) = A$. We define the degree d polynomial

$$f_{\iota,A}(x) = \phi_\iota(x) - x'(A)\psi_\iota^2(x) \in \mathbf{K}[x].$$

Its roots are the $x(B + kT)$ for $0 \leqslant k < d$, and they are pairwise distinct because $2A \neq O'$. So $f_{\iota,A}(x)$ is a degree d separable polynomial. The coordinate x lies in the field $\mathbf{K}(E)$ of $\mathbf{K}$-rational functions on E. So the map $x : E(\Omega) - O \to \Omega$ induces a Galois equivariant bijection between the fiber $\iota^{-1}(A)$ and the roots of $f_{\iota,A}(x)$. In particular, $f_{\iota,A}(x)$ is irreducible if and only if the fiber $\iota^{-1}(A)$ is. The fiber $\iota^{-1}(A)$ over A is an affine scheme with ring $\mathbf{K}[x]/(f_{\iota,A}(x))$ and the class of y in this ring is given by

$$y = \frac{y'(A)\psi_\iota^3(x) - \omega_0(x)}{\omega_1(x)} \bmod f_{\iota,A}(x) \,. \tag{5}$$

Then, the two questions that worry us are the following ones.

(1) Can we compute $f_{\iota,A}(x)$ quickly, *e.g.* in quasi-linear time in d ?
(2) Under which conditions is $f_{\iota,A}(x)$ irreducible ?

These two questions are successively addressed in Sections 4.1 and 4.2. In Section 4.3 we deduce a fast algorithm that constructs a degree d irreducible polynomial in $\mathbf{K}[x]$ when $\mathbf{K}$ is a finite field with $q = p^w$ elements and $d = \ell^\delta$ is a power of a prime ℓ such that ℓ is prime to $p(q-1)$ and $4\ell \leqslant q^{\frac{1}{4}}$. Larger primes ℓ are considered in Section 5.

4.1. **Calculation of the polynomial** $f_{\iota,A}(x)$**.** For any geometric point $P \in E(\Omega)$, we denote by $\tau_P : E \to E$ the translation by P. Let x_P be the function $x \circ \tau_{-P}$ and similarly let y_P be the function $y \circ \tau_{-P}$. If $P = kT$, we moreover define $x_k = x_{kT}$ and $y_k = y_{kT}$. Recall d is assumed to be odd.

In this section, we present methods for fast construction of isogenies. Section 4.1.1 concerns isogenies with split cyclic kernel and Vélu's formulae. Section 4.1.2 recalls how one can take advantage of the decomposition of an isogeny into several ones with smaller degrees. This is particularly useful when $E/\mathbf{K}$ has complex multiplication and the isogeny in question is the quotient isogeny associated to some power of an invertible prime ideal in the endomorphism ring of E. This idea is detailed in Section 4.1.3.

4.1.1. *Vélu's isogenies.* Let T be a $\mathbf{K}$-rational point and let ι be the isogeny given by Vélu's formulae [25],

$$\left\{ \begin{array}{rcl} x' & = & x + \sum_{0<k<d} (x_k - x(kT)) \ , \\ y' & = & y + \sum_{0<k<d} (y_k - y(kT)) \ . \end{array} \right. \tag{6}$$

We put some order in Eq. (6). Using the addition law on E, we first express x_k in terms of x and y,

$$\begin{aligned} x_{kT} \cdot (x - x(kT))^2 = {} & x(kT)x^2 + (a_3 + 2y(kT) + a_1 x(kT))\, y \\ & + \left(a_4 + a_1^2 x(kT) + a_1 a_3 + 2a_2 x(kT) + a_1 y(kT) + x(kT)^2\right) x \\ & + a_3^2 + a_1 a_3 x(kT) + a_3 y(kT) + a_4 x(kT) + 2a_6. \end{aligned} \tag{7}$$

We deduce that $(x_{kT} + x_{-kT} - 2x(kT)) \cdot (x - x(kT))^2$ is equal to

$$\begin{aligned} & (6x(kT)^2 + (a_1^2 + 4a_2)x(kT) + a_1 a_3 + 2a_4)\, x - 2x(kT)^3 \\ & \qquad + (a_1 a_3 + 2a_4)x(kT) + a_3^2 + 4a_6 \,. \end{aligned} \tag{8}$$

One computes the rational fraction $x' = \phi_\iota(x)/\psi_\iota^2(x)$ using Eqs. (6) and (8) by gathering the terms relative to k and $-k$, with the help of a "divide and conquer" strategy [5, Chapter 10]. Complexity is quasi-linear in d. A similar calculation gives us the explicit form of $y' = \omega_\iota(x, y)/\psi_\iota^3(x)$.

Example. We take $p = 7$, $q = 7$ and $d = 5$. The elliptic curve $E/\mathbb{F}_7$ with equation $y^2 = x^3 + x + 4$ has got ten $\mathbb{F}_7$-rational points. The point $T = (6, 4)$ has order $\ell = 5$. The group generated by T is

$$\langle T \rangle = \{O_E,\ (6, 4),\ (4, 4),\ (4, 3),\ (6, 3)\} \ .$$

The corresponding isogenous curve E' is given by Vélu's formulae, $E' : y^2 = x^3 + 3\,x + 4$. Moreover, Eq. (6) yields

$$\begin{aligned} x' = x + \frac{y + 6\,x^2 + 2\,x}{(x+1)^2} - 6 + \frac{y + 4\,x^2 + 3\,x + 5}{(x+3)^2} - 4 + \\ \frac{6\,y + 4\,x^2 + 3\,x + 5}{(x+3)^2} - 4 + \frac{6\,y + 6\,x^2 + 2\,x}{(x+1)^2} - 6\,. \end{aligned}$$

Using Eq. (8), we find an expression for x' in terms of x alone,

$$x' = x + \frac{x+2}{(x+1)^2} + \frac{1}{(x+3)^2} = \frac{x^5 + x^4 + 2\,x^3 + 5\,x^2 + 4\,x + 5}{(x+3)^2\,(x+1)^2}\,.$$

It remains to choose a point A in $E'(\mathbb{F}_7)$. We set $A = (1,1)$, a point of order 5, and we finally obtain,

$$f_{\iota,A}(x) = x^5 + x^4 + 2\,x^3 + 5\,x^2 + 4\,x + 5 - (x+3)^2\,(x+1)^2 = x^5 + x^3 + 4\,x^2 + x + 3\,.$$

4.1.2. *Composition of isogenies.* Assume d factors as $d_1 d_2$. Then the degree d isogeny $\iota : E \to E'$ decomposes as $\iota = \iota_2 \circ \iota_1$ where $\iota_1 : E \to F$ is a degree d_1 isogeny and $\iota_2 : F \to E_2$ is a degree d_2 isogeny. The kernel of ι_1 is generated by $d_2 T$ and the kernel of ι_2 is generated by $\iota_1(T)$. Let $I(x)$ be the degree d rational fraction associated with ι. Define similarly $I_1(x)$ and $I_2(x)$. Then $I(x) = I_2(I_1(x))$. We may then compute $I(x)$ in three steps: first compute $I_1(x)$, then compute $I_2(x)$, and finally compute the composition $I = I_2 \circ I_1$ using work by Umans and Kedlaya [24, 10] (see Corollary 1 in Section 3.1.3).

4.1.3. *A special simple case.* We now assume that $\mathbf{K}$ is a finite field with $q = p^w$ elements. Let $\varphi_E : E \to E$ be the Frobenius endomorphism of E and t be its trace. Let $\mathcal{O}$ be the quotient ring $\mathbb{Z}[X]/(X^2 - tX + q)$ and let α be the class of X in $\mathcal{O}$. We call $\epsilon : \mathcal{O} \to \mathrm{End}(E)$ the ring monomorphism that sends α onto φ_E. For every subset S of $\mathcal{O}$, we define the *kernel* of S in E to be the intersection of all the kernels of the endomorphisms $\epsilon(s)$ for $s \in S$. We denote it by $E[S]$. Let ℓ be a prime not dividing $p(q-1)$. We assume that ℓ divides the order $q + 1 - t$ of $E(\mathbf{K})$. As a consequence ℓ is coprime to the discriminant $t^2 - 4q$ of $\mathcal{O}$.

We have

$$X^2 - tX + q = (X-1)(X-q) \bmod \ell\,,$$

because $1 - t + q$ is divisible by ℓ and the product of the roots of $X^2 - tX + q$ equals q. Furthermore, the roots $(1 \bmod \ell)$ and $(q \bmod \ell)$ are distinct because ℓ does not divide $q - 1$. Let $\mathfrak{l} = (\ell, \alpha - 1)$ be the prime ideal in $\mathcal{O}$ above ℓ and containing $\alpha - 1$. The kernel of $\mathfrak{l}$ in E is $E[\ell](\mathbf{K})$, the rational part of the ℓ-torsion of E. This is a cyclic group of order ℓ because ℓ divides $q + 1 - t$ and ℓ is coprime to $p(q-1)$.

Let k be a positive integer. According to Hensel's lemma, there exist two integers λ_k and μ_k in $[0, \ell^k[$ such that $\lambda_k = 1 \bmod \ell$, $\mu_k = q \bmod \ell$ and

$$X^2 - tX + q = (X - \lambda_k)(X - \mu_k) \bmod \ell^k.$$

The ideal $\mathfrak{l}^k$ of $\mathcal{O}$ is generated by ℓ^k and $\alpha - \lambda_k$. We show that the kernel $E[\mathfrak{l}^k]$ of $\mathfrak{l}^k$ in E (in the sense of section 3.2.1) is a cyclic group of order ℓ^k inside $E(\Omega)$. Let $\iota_k : E \to E_k$ be the quotient isogeny by $E[\mathfrak{l}^k]$. The elliptic curve E_k is defined over $\mathbf{K}$, a finite field with q elements. Let $\epsilon_k : \mathcal{O} \to \mathrm{End}(E_k)$ be the ring homomorphism that sends α onto the q-Frobenius endomorphism of E_k. The two homomorphisms ϵ and ϵ_k are compatible with the isogeny ι_k in the sense that for every s in $\mathcal{O}$ one has $\epsilon_k(s) = \iota_k \circ \epsilon(s) \circ \iota_k^{-1}$. Using Lemma 5 of Section 3.2.1, we see that $\iota_{k+1} : E \to E_{k+1}$ decomposes as $\jmath_{k+1} \circ \iota_k$ where $\jmath_{k+1} : E_k \to E_{k+1}$ is the degree ℓ isogeny with kernel $E_k[\mathfrak{l}] = E_k[\ell](\mathbf{K})$. In particular, ι_k has degree ℓ^k, so the order of the kernel $E[\mathfrak{l}^k]$ of ι_k is ℓ^k. This kernel is a subgroup of $E[\ell^k]$ that does not contain the full ℓ torsion, therefore it is cyclic. We obtain in this way a chain of degree ℓ isogenies

$$E \xrightarrow{\jmath_1} E_1 \longrightarrow \cdots \longrightarrow E_k \xrightarrow{\jmath_{k+1}} E_{k+1} \longrightarrow \cdots$$

We denote by $I_k(x) \in \mathbf{K}(x)$ the degree ℓ^k rational fraction associated with ι_k. We denote by $J_k \in \mathbf{K}(x)$ the degree ℓ rational fraction associated with $\jmath_k$. In order to compute $\jmath_1 = \iota_1$, we pick a random point in $E(\mathbf{K})$ and multiply it by $(q+1-t)/\ell$. If the result is non-zero, we are done, otherwise we start again. We then compute $I_1 = J_1$ using Vélu's formulae in Section 4.1.1. Every rational fraction J_k can be computed the same way. The composition $I_k = J_k \circ \cdots J_2 \circ J_1$ can be computed using the method in Paragraph 4.1.2.

4.2. **Irreducibility conditions.** We assume that we still are in the situation of Paragraph 4.1.3. Let ℓ be a prime not dividing $p(q-1)$. In particular ℓ is odd. We assume that ℓ divides the order $q+1-t$ of some elliptic curve $E/\mathbf{K}$. As a consequence ℓ is coprime to t^2-4q. We want to construct an irreducible polynomial $f(x) \in \mathbf{K}[x]$ with degree $d = \ell^\delta$. We factor $q+1-t$ as $q+1-t = \ell^e \ell'$ where ℓ' is prime to ℓ.

There exist two integers $\lambda_{e+\delta}$ and $\mu_{e+\delta}$ such that

$$\lambda_{e+\delta} = 1 \bmod \ell^e \quad , \quad \mu_{e+\delta} = q \bmod \ell^e \,,$$
$$X^2 - tX + q \ = \ (X-\lambda_{e+\delta})(X-\mu_{e+\delta}) \bmod \ell^{e+\delta}\,.$$

We write $\lambda_{e+\delta} = 1 + \ell^e\ell''$ with ℓ'' prime to ℓ. In the sequel, we set $\lambda = \lambda_{e+\delta}$ and $\mu = \mu_{e+\delta}$. Let now

$$\mathfrak{d} = (d, \alpha - \lambda) = (\ell, \alpha-\lambda)^\delta = \mathfrak{l}^\delta.$$

This is an invertible ideal. Its kernel $E[\mathfrak{d}]$ in E is the kernel of the isogeny $\iota_\delta : E \to E_\delta$. The ℓ-Sylow subgroup of $E_\delta(\mathbf{K})$ is the kernel of $\mathfrak{l}^e = (\ell^e, \alpha-1)$ in E_δ and it is cyclic. Let A be a generator of it. Let $B \in E(\Omega)$ such that $\iota_\delta(B) = A$. Then, B generates the kernel of $\mathfrak{l}^{e+\delta} = (\ell^{e+\delta}, \varphi_E - \lambda)$ in E. Especially,

$$\varphi_E(B) = \lambda B, \tag{9}$$

and the order of $\lambda = 1 + \ell^e\ell''$ in $(\mathbb{Z}/\ell^{e+\delta}\mathbb{Z})^*$ is $d = \ell^\delta$. Thus, the Galois orbit of B has cardinality d and the polynomial $f_{\iota,A}(X)$ is irreducible.

4.3. **Existence conditions.** Assume we are given a finite field $\mathbf{K}$ with characteristic p and cardinality q and an integer $d = \ell^\delta$ such that ℓ is prime to $p(q-1)$. We look for a degree d irreducible polynomial in $\mathbf{K}[x]$. The construction in Section 4.2 requires an elliptic curve over $\mathbf{K}$ such that ℓ divides the cardinality $q+1-t$ of $E(\mathbf{K})$. Is there any such elliptic curve ? How can we find it ?

If $\ell \leqslant 2\sqrt{q}$, then there are at least two consecutive integer multiples of ℓ in the interval $[q+1-2\sqrt{q}, q+1+2\sqrt{q}]$. At least one of them is not congruent to 1 modulo p. So there exists at least one elliptic curve with cardinality divisible by the prime ℓ. We want to bound from below the number of such elliptic curves. We use the results of Lenstra [14] extended by Howe [7].

From Theorem 3 and Corollary 2 of Section 3.2.2, we deduce that the probability that a Weierstrass elliptic curves over a finite field $\mathbf{K}$ with q element has an order divisible by ℓ is $1/(\ell-1)$, up to an error term bounded in absolute value by $8\ell/\sqrt{q}$. We deduce that if

$$4\ell \leqslant q^{\frac{1}{4}}, \tag{10}$$

then this proportion is at least $1/(2\ell)$.

In that case, we can find such an elliptic curve in the following way: we pick a random Weierstrass elliptic curve over $\mathbf{K}$. We compute its cardinality using Schoof's algorithm at the expense of $(\log q)^{5+\varepsilon(q)}$ elementary operations. If this cardinality is divisible by ℓ we are done.

Otherwise, we try again. The average number of trials is $O(\ell)$. The expected time to find the needed curve E is $\ell(\log q)^{5+\varepsilon(q)}$ elementary operations provided condition (10) holds true.

All in all, we need $\ell \times (\log q)^{5+\varepsilon(q)}$ elementary operations to find the first elliptic curve, then $\delta^{1+\epsilon(\delta)} \times \ell^{1+\varepsilon(\ell)} \times (\log q)^{2+\varepsilon(q)}$ elementary operations to compute the δ isogenies of degree ℓ, and $d^{1+\varepsilon(d)} \times (\log q)^{1+\varepsilon(q)}$ elementary operations to compose these isogenies. The conclusion of this section is the following.

Lemma 6 (Isogeny fiber)**.** *There exists a probabilistic (Las Vegas) algorithm that on input a finite field $\mathbf{K}$ with characteristic p and cardinality $q = p^w$, a prime integer ℓ not dividing $p(q-1)$ such that $4\ell \leqslant q^{\frac{1}{4}}$, and a positive integer δ, computes an irreducible polynomial in $\mathbf{K}[x]$ of degree $d = \ell^\delta$, at the expense of $\ell \times (\log q)^{5+\varepsilon(q)} + d^{1+\varepsilon(d)} \times (\log q)^{2+\varepsilon(q)}$ elementary operations.*

5. Base change

In this section, $\mathbf{K} = (\mathbb{Z}/p\mathbb{Z})[z]/(m(z))$ is a finite field with $q = p^w$ elements. We still assume here that $d = \ell^\delta$ is a power of a prime ℓ where ℓ is prime to $p(q-1)$. We look for a degree d irreducible polynomial in $\mathbf{K}[x]$. However, we no longer assume that $4\ell \leqslant q^{\frac{1}{4}}$.

We adapt the main idea in Section 2.4 to the context of elliptic curves: we base change to a small auxiliary extension. Let n be the smallest integer coprime with $\ell(\ell-1)$ such that $Q = q^n$ satisfies $4\ell \leqslant Q^{\frac{1}{4}}$. According to Iwaniec's result about Jacobsthal's problem [8] we have $n = (\log \ell)^{2+\varepsilon(\ell)}$. Let us remark that d is then coprime with $Q-1$ too.

Using e.g. the methods in Shoup [22], we find a degree n irreducible polynomial $m' \in \mathbf{K}[z']$. We set $\mathbf{L} = \mathbf{K}[z']/(m'(z'))$. A basis of this $(\mathbb{Z}/p\mathbb{Z})$-vector space is given by the $z^j z'^i$ for $0 \leqslant i < n$ and $0 \leqslant j < w$. Using the method given in the introduction, we find a generator τ of the $(\mathbb{Z}/p\mathbb{Z})$-algebra $\mathbf{L}$. We compute also the minimal polynomial $h(u) \in (\mathbb{Z}/p\mathbb{Z})[u]$ of τ. We set $\tilde{\mathbf{L}} = (\mathbb{Z}/p\mathbb{Z})[u]/(h(u))$. A basis of this $(\mathbb{Z}/p\mathbb{Z})$-vector space is given by the u^k for $0 \leqslant k < nw$. We compute and store the matrix of the isomorphism $\kappa : \tilde{\mathbf{L}} \to \mathbf{L}$ that sends $u \bmod h(u)$ onto τ. This is a $nw \times nw$ matrix with entries in $\mathbb{Z}/p\mathbb{Z}$. We also compute and store the inverse of this matrix. The image $\tilde{\mathbf{K}} = \kappa^{-1}(\mathbf{K})$ of $\mathbf{K}$ by κ^{-1} is the unique subfield with q elements inside $\tilde{\mathbf{L}}$.

The reason for introducing these two different models of the field with q^n elements is that, on the one hand, this field should be constructed as an extension of $\mathbf{K}$ because we shall have to descend to $\mathbf{K}$ later on; but on the other hand, the field with q^n elements should be also presented as a monogenic extension of $\mathbb{Z}/p\mathbb{Z}$, because all the algorithms described and used so far (and in particular the algorithms due to Umans and Kedlaya) require that the base field be presented as a monogenic extension of $\mathbb{Z}/p\mathbb{Z}$. One can now apply the construction of Section 4 to $\tilde{\mathbf{L}}$ and obtain an irreducible polynomial $F_{\iota,A}(x)$ of degree d in $\tilde{\mathbf{L}}[x]$, in time $(\log Q)^{5+\varepsilon(Q)} d^{1+\varepsilon(d)}$, that is

$$(\log q)^{5+\varepsilon(q)} d^{1+\varepsilon(d)}$$

elementary operations.

It remains to derive from $F_{\iota,A}(x)$ an irreducible polynomial $f(x)$ of degree d over $\mathbf{K}$. Recall $F_{\iota,A}(x)$ is the minimal polynomial of $x(B)$ where B is a geometric point of order $\ell^{e+\delta} \leqslant 4Qd$ on an elliptic curve E over $\tilde{\mathbf{L}}$. We also are given an integer λ such that $0 \leqslant \lambda < \ell^{e+\delta}$ and

$$\varphi_E(B) = \lambda B \tag{11}$$

where φ_E is the degree Q Frobenius endomorphism of $E/\tilde{\mathbf{L}}$. We set $\alpha = x(B) \in \tilde{\mathbf{L}}[x]/(F_{\iota,A}(x))$. Recall Φ_q is the map which raises to the q-th power. We have $\Phi_Q = \Phi_q^n$. The field $\tilde{\mathbf{L}}[x]/(F_{\iota,A}(x)) = \tilde{\mathbf{L}}(\alpha)$ is a degree d extension of $\tilde{\mathbf{L}}$. For any integer k between 1 and n, one denotes by $\Sigma_k(\alpha)$ the k-th symmetric function of the conjugates of α over the subfield with q^d elements. These conjugates are $\alpha,\ \Phi_q^d(\alpha),\ \Phi_q^{2d}(\alpha),\ \ldots,\ \Phi_q^{(n-1)d}(\alpha)$. Since d is a *prime power*, we deduce from Lemma 3 of Section 3.1.1 that at least one among these n symmetric functions generates the degree d extension of $\tilde{\mathbf{K}}$. In other words, there exists a k between 1 and n such that the polynomial

$$\tilde{f}(x) = \prod_{0 \leqslant l < d} (x - \Phi_q^l(\Sigma_k(\alpha)))$$

is irreducible of degree d in $\tilde{\mathbf{K}}[x] \subset \tilde{\mathbf{L}}[x]$.

Three questions now worry us, that we consider in turn in Sections 5.1, 5.2 and 5.3.

(1) How to compute $\Sigma_k(\alpha)$ and its conjugates ?
(2) How to find the good integer k ?
(3) How to compute $\tilde{f}(x) \in \tilde{\mathbf{K}}[x]$ starting from $F_{\iota,A}(x) \in \tilde{\mathbf{L}}[x]$?

5.1. **Computing symmetric functions.** First, we compute $\beta = y(B) \in \tilde{\mathbf{L}}[x]/(F_{\iota,A}(x))$ using Eq. (5). Let now l be an integer between 0 and $dn-1$. We explain how to compute $\alpha_l = \Phi_q^l(\alpha)$. We write $l = r + ns$ with $0 \leqslant r < n$ and $0 \leqslant s < d$. Then,

$$\alpha_l = \Phi_q^l(\alpha) = \Phi_q^r(\Phi_Q^s(\alpha)).$$

We first compute $\Phi_Q^s(\alpha) = x(\varphi_E^s(B)) = x(\lambda^s B)$ using Eq. (11). To this end, we write $\lambda^s = R \bmod \ell^{e+\delta}$ where $0 \leqslant R < \ell^{e+\delta}$ and we multiply the $\ell^{\delta+e}$-torsion point $B \in E(\tilde{\mathbf{L}}[x]/(F_{\iota,A}(x)))$ by R using fast exponentiation. This is done at the expense of a constant times $(\log Q + \log d)$ operations in $\tilde{\mathbf{L}}[x]/(F_{\iota,A}(x))$. One then raises $\Phi_Q^s(\alpha)$ to the q^r-th power at the expense of at most $n \log q$ operations modulo $F_{\iota,A}(x)$. Thus, each conjugate is computed at the expense of $d^{1+\varepsilon(d)}(\log q)^{2+\varepsilon(q)}$ elementary operations.

To compute all the $(\Sigma_k(\alpha))_{0<k\leqslant n}$, one computes the n conjugates α, $\Phi_q^d(\alpha), \ldots, \Phi_q^{(n-1)d}(\alpha)$ and one forms the corresponding polynomial of degree n. Altogether, the computation of the symmetric functions $(\Sigma_k(\alpha))_{0<k\leqslant n}$ requires

$$d^{1+\varepsilon(d)}(\log q)^{2+\varepsilon(q)}$$

elementary operations.

5.2. **Finding a generating symmetric function.** One seeks an integer k between 1 and n such that $\Sigma_k(\alpha)$ generates an extension of degree d of $\tilde{\mathbf{K}}$ (there is at least one such integer). So we successively test all the k between 1 and n. As n is small, this is not a problem. We know that $\Sigma_k(\alpha)$ generates the degree d extension of $\tilde{\mathbf{K}}$ if and only if

$$\Phi_q^{\ell^{\delta-1}}(\Sigma_k(\alpha)) \neq \Sigma_k(\alpha),$$

where $\ell^{\delta-1}$ is the unique maximal divisor of d. This condition is equivalent to

$$\Sigma_k(\Phi_q^{\ell^{\delta-1}}(\alpha)) \neq \Sigma_k(\alpha), \text{ or } \Sigma_k(\alpha_{\ell^{\delta-1}}) \neq \Sigma_k(\alpha).$$

One computes the $\Sigma_k(\alpha_{\ell^{\delta-1}})$'s in the same way as the $\Sigma_k(\alpha)$'s, following Section 5.1. It is then easy to compare $\Sigma_k(\alpha_{\ell^{\delta-1}})$ and $\Sigma_k(\alpha)$. One can thus find k in

$$d^{1+\varepsilon(d)}(\log q)^{2+\varepsilon(q)}$$

elementary operations.

5.3. **Computing minimal polynomials.** We now have an element $\Sigma_k(\alpha)$ of $\tilde{\mathbf{L}}[x]/(F_{\iota,A}(x))$ and we know that it actually belongs to the degree d extension of $\tilde{\mathbf{K}}$. But this is not really visible because $\Sigma_k(\alpha)$ is given in the basis $1, x, \ldots, x^{d-1}$ of $\tilde{\mathbf{L}}[x]/(F_{\iota,A}(x))$. Still, the minimal polynomial $\tilde{f}(x)$ of $\Sigma_k(\alpha)$ has coefficients in $\tilde{\mathbf{K}} \subset \tilde{\mathbf{L}}$. We compute this minimal polynomial. We use a general algorithm for this task, such as the one appearing in recent work by Umans and Kedlaya [24, 10]. See Theorem 2 in Section 3.1.3. This algorithm requires $d^{1+\varepsilon(d)} \times (\log Q)^{1+\varepsilon(Q)}$ elementary operations. Finally, we apply the isomorphism $\kappa : \tilde{\mathbf{L}} \to \mathbf{L}$ to every coefficient in $\tilde{f}(x)$ and we find a polynomial $f(x)$ with coefficients in $\mathbf{K} \subset \mathbf{L}$. This polynomial is irreducible in $\mathbf{K}[x]$.

All in all, the conclusion of this section is the following.

Lemma 7 (Base change)**.** *There exists a probabilistic (Las Vegas) algorithm that on input a finite field $\mathbf{K}$ with characteristic p and cardinality $q = p^w$, a prime integer ℓ not dividing $p(q-1)$, and a positive integer δ, computes an irreducible polynomial in $\mathbf{K}[x]$ of degree $d = \ell^\delta$, at the expense of $d^{1+\varepsilon(d)} \times (\log q)^{5+\varepsilon(q)}$ elementary operations.*

6. Construction of irreducible polynomials

6.1. **Finding one irreducible polynomial.** Given that we represent the finite field $\mathbf{K}$ in a reasonable way, as explained in the introduction, we can now state our main result.

Theorem 4. *There exists a probabilistic (Las Vegas) algorithm that on input a finite field $\mathbf{K}$ with characteristic p and cardinality $q = p^w$, and a positive integer d, returns a degree d irreducible polynomial in $\mathbf{K}[x]$. The algorithm requires $d^{1+\varepsilon(d)} \times (\log q)^{5+\varepsilon(q)}$ elementary operations.*

Proof. The algorithms runs as follows.

We first factor the degree d as $d = \prod_i \ell_i^{\delta_i}$, this requires $O(d)$ elementary operations. Then, Lemma 4 shows that it suffices to find an irreducible polynomial of degree $\ell_i^{\delta_i}$ for every i. So we may assume that $d = \ell^\delta$ is a prime power.

Then, we use the construction of Lemma 1 if $\ell = p$, and Lemma 2 if ℓ divides $q - 1$. Otherwise Lemma 6 applies. □

Remark. One may use a faster algorithm to compute the cardinality of elliptic curves, for instance the SEA algorithm, and hope to gain a $\log q$ speedup. But, at the time of writing, it is not clear how the probability of failure of the SEA algorithm can be rigorously related to the Las Vegas behavior of our construction, and we finally prefer to state a complexity result based on Schoof's algorithm only.

6.2. **Constructing random irreducible polynomials.** Let $\mathbf{K}$ be a finite field with cardinality q and characteristic p. Let $d \geqslant 2$ be an integer. We just explained how to quickly compute a degree d irreducible polynomial in $\mathbf{K}[x]$. We stress that the polynomials generated by our algorithm have a very special form. This might be a problem for some applications. In this section we explain how to construct *random* polynomials. We need an estimate for the number of degree d irreducible monic polynomials in $\mathbf{K}[x]$. We recall and prove a very classical lower bound [16, Ex. 3.26 and 3.27, page 142].

Lemma 8. *Let $\mathbf{K}$ be a finite field with q elements. Let $d \geqslant 2$ be an integer. The density of irreducible polynomials among the degree d monic polynomials is greater than or equal to*

$$\frac{1}{d}(1 - \frac{q}{q-1}(q^{-\frac{d}{2}} - q^{-d})).$$

Let $\mathbf{L}$ be a degree d field extension of $\mathbf{K}$. The density of generators of the $\mathbf{K}$-algebra $\mathbf{L}$ is greater than or equal to

$$1 - \frac{q}{q-1}(q^{-\frac{d}{2}} - q^{-d}).$$

Proof. Let Ω be an algebraic closure of $\mathbf{K}$ and let $\mathbf{L}$ be the unique degree d extension of $\mathbf{K}$ inside Ω. Call $\mathcal{G}_d$ the set of generators of the $\mathbf{K}$-algebra $\mathbf{L}$. This is the set of all α in $\mathbf{L}$ such that $\mathbf{K}(\alpha) = \mathbf{L}$. Let $\mathcal{I}_d$ be the set of degree d monic irreducible polynomials in $\mathbf{K}[x]$. Let $\rho : \mathcal{G}_d \to \mathcal{I}_d$ be the map that to every generator α associates its minimal polynomial. Every polynomial $P(x)$ in $\mathcal{I}_d$ has exactly d preimages by ρ, namely its d roots.

To enumerate the degree d monic irreducible polynomials, we just count the generators of $\mathbf{L}$ over $\mathbf{K}$. Let α be an element in $\mathbf{L}$. If α does not generate $\mathbf{L}$, then it belongs to a smaller extension of $\mathbf{K}$ inside $\mathbf{L}$. Therefore the complementary set of $\mathcal{G}_d$ in $\mathbf{L}$ is the union of all proper subfields of $\mathbf{L}$ containing $\mathbf{K}$. These subfields are in correspondence with the strict divisors of d. To any such divisor D, we associate the unique extension of $\mathbf{K}$ with degree D. It has q^D elements. The set of strict divisors of d is a subset of $\{1, 2, 3, 4, \dots, \lfloor \frac{d}{2} \rfloor\}$. So the number of elements in $\mathbf{L}$ that do not generate it over $\mathbf{K}$ is upper bounded by

$$q + q^2 + q^3 + q^4 + \cdots + q^{\lfloor \frac{d}{2} \rfloor} = q\frac{q^{\lfloor \frac{d}{2} \rfloor} - 1}{q-1} \leqslant \frac{q}{q-1}(q^{d/2} - 1).$$

The cardinality of $\mathcal{G}_d$ is thus $\geqslant q^d - \frac{q}{q-1}(q^{d/2} - 1)$ and the cardinality of $\mathcal{I}_d$ is

$$\geqslant \frac{q^d}{d} - \frac{q}{d(q-1)}(q^{d/2} - 1).$$ □

If $d \geqslant 2$, we deduce from Lemma 8 that the density of generators is $\geqslant 1 - 1/(q-1) = (q-2)/(q-1)$. So $\geqslant 1/2$ if $q \geqslant 3$. If $q = 2$ and $d \geqslant 4$ then this density is $\geqslant 1 - 2 \times 2^{-2} = 1/2$. If $q = 2$ and d equals 2 (resp. 3) then this density is $1/2$ (resp. $3/4$). If $d = 1$ then this density is 1. We deduce the following lemma.

Lemma 9 (Density of generators)**.** *Let $\mathbf{K}$ be a finite field with q elements. Let $d \geqslant 1$ be an integer. The density of irreducible polynomials among the degree d monic polynomials is greater than or equal to $1/2d$.*

Let $\mathbf{L}$ be a degree d field extension of $\mathbf{K}$. The density of generators in the $\mathbf{K}$-algebra $\mathbf{L}$ is greater than or equal to $1/2$.

As a corollary of Lemma 9, given an irreducible polynomial $f(x)$ of degree d computed with Theorem 4, one can compute a new completely *random* irreducible polynomial $g(x)$ at the expense of only $d^{1+\varepsilon(d)} \times (\log q)^{1+\varepsilon(q)}$ elementary operations. Indeed, we choose a random element in $\mathbf{L}$, the degree d extension of $\mathbf{K}$ constructed from $f(x)$, and we use Theorem 2 to compute its minimal polynomial. We obtain an irreducible polynomial $g(x)$ that has degree d with probability greater than $1/2$. So, we have a Las Vegas quasi-linear algorithm.

References

[1] L. M. Adleman and H. W. Lenstra, Jr. Finding irreducible polynomials over finite fields. *Proceedings of the eighteenth annual ACM Symposium on Theory of Computing*, pages 350–355, 1986.

[2] M. Ben-Or. Probabilistic algorithms in finite fields. *22nd Annual Symposium on Foundations of Computer Science*, 11:394–398, 1981.

[3] A. Bostan, Ph. Flajolet, B. Salvy, and É. Schost. Fast computation of special resultants. *Journal of Symbolic Computation*, 41(1):1–29, January 2006.

[4] A. Bostan, L. González-Vega, H. Perdry and É. Schost. From Newton sums to coefficients: complexity issues in characteristic p. *Proceedings of MEGA'05*, 2005.

[5] J. von zur Gathen and J. Gerhard. Modern computer algebra. Cambridge University Press, second edition, 2003.

[6] J. Giraud. Remarque sur une formule de Shimura-Taniyama. *Inventiones Mathematicae*, 5:231–236, 1968.

[7] E. W. Howe. On the group orders of elliptic curves over finite fields. *Compositio Mathematica*, 85:229–247, 1993.

[8] H. Iwaniec. On the problem of Jacobsthal. *Demonstratio Mathematica*, 11:225–231, 1978.

[9] E. Kaltofen and V. Y. Pan. Parallel solution of Toeplitz and Toeplitz-like linear systems over fields of small positive characteristic. In *Proceedings of PASCO'94. Lecture Notes Series on Computing*, World Scientific Publishing Company, 5:225–233, 1994.

[10] K. S. Kedlaya and C. Umans. Modular composition in any characteristic. *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science*, IEEE Computer Society, pp. 146–155, 2008.

[11] H. W. Lenstra, Jr. Algorithms for finite fields. *Number theory and cryptography (Sydney, 1989)*, Number Theory and Cryptography, London Mathematical Society, Lecture Note Series, Cambridge University Press, 154:76–85, 1990.

[12] H. W. Lenstra, Jr. Finding isomorphisms between finite fields. *Mathematics of Computation*, vol. 56, 193:329–347, 1991.

[13] H. W. Lenstra, Jr and B. de Smit. Standard models for finite fields: the definition. *http://www.math.leidenuniv.nl/∼desmit*, pages 1–4, 2008.

[14] H. W. Lenstra, Jr. Factoring integers with elliptic curves. *Annals of Mathematics*, 126:649–673, 1987.

[15] H. W Lenstra, Jr. Complex multiplication structure of elliptic curves. *Journal of Number Theory*, 56:227–241, 1996.

[16] R. Lidl and H. Niederreiter *Finite Fields*. Addison-Wesley, 1983.

[17] Q. Liu. Algebraic Geometry and Arithmetic Curves. *Oxford Graduate Texts in Mathematics*, 6, 2002.

[18] D. Panario and B. Richmond. Analysis of Ben-Or's polynomial irreducibility test. *Random Structures and Algorithms*, 13:439–456, 1998.

[19] C. H. Papadimitriou. *Computational complexity.* Addison Wesley, 1967.

[20] A. Schönhage. Fast parallel computation of characteristic polynomials by Leverrier's power sum method adapted to fields of finite characteristic. In *Automata, languages and programming (Lund, 1993).* Lecture Notes in Computer Science, Springer, 700:410–417.

[21] J.-P. Serre. Complex multiplication. in *Algebraic Number Theory*, edited by J. W. S. Cassels and A. Fröhlich, Academic Press, 1967.

[22] V. Shoup. Fast construction of irreducible polynomials over finite fields. In *Proceedings of the 4th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '93)*, pages 484–492, 1993.

[23] J. Silverman. *The Arithmetic of Elliptic Curves.* Springer-Verlag, Graduate Texts in Mathematics 106, 1986. Expanded 2nd Edition, 2009.

[24] C. Umans. Fast polynomial factorization and modular composition in small characteristic. In the *Proceedings of the 40th Annual ACM Symposium on Theory of Computing,*, pages 350–355, 1986.

[25] J. Vélu. Isogénies entre courbes elliptiques. *Comptes-Rendus de l'Académie des Sciences, Série I*, 273:238–241, juillet 1971.

[26] W. C. Waterhouse. Abelian varieties over finite fields. *Annales scientifiques de l'École Normale Supérieure, Série. 4, 2, 4:521–560. 1969.*

INRIA Bordeaux Sud-Ouest & Université de Toulouse.

Université de Toulouse le Mirail, Département de Mathématiques et Informatique, 5 allées Antonio Machado, F-31058 Toulouse cédex 9, France.
E-mail address: jean-marc.couveignes@math.univ-toulouse.fr

DGA, La Roche Marguerite, F-35174 Bruz Cedex, France.

Institut de Recherche Mathématique de Rennes, Université de Rennes 1, Campus de Beaulieu, F-35042 Rennes, France.
E-mail address: reynald.lercier@m4x.org