The legacy of 50 years of fuzzy sets: A discussion

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Abstract: This note provides a brief overview of the main ideas and notions underlying fifty years of research in fuzzy set and possibility theory, two important settings introduced by L. A. Zadeh for representing sets with unsharp boundaries and uncertainty induced by granules of information expressed with words. The discussion is organized on the basis of three potential understanding of the grades of membership to a fuzzy set, depending on what the fuzzy set intends to represent: a group of elements with borderline members, a plausibility distribution, or a preference profile. It also questions the motivations for some existing generalized fuzzy sets. This note clearly reflects the shared personal views of its authors.
keywords: fuzzy set, possibility theory, similarity, uncertainty, preference.

1 Introduction

The founding paper on fuzzy sets [72], written by Lotfi Zadeh, is 50 years old. This seminal paper, sometimes ill-regarded at the beginning, has given rise to a huge literature, several dedicated journals, and many conferences each year for several decades now. It has affected many areas of scientific research (sometimes marginally, sometimes significantly) ranging from mathematics (especially many-valued logics, topology, algebra and category theory) to engineering practice especially in modeling, control, optimization, and data processing, but also with some clear impact on techniques devoted to pattern recognition and image processing, operations research, artificial intelligence, databases and information systems. Besides, fuzzy sets have influenced uncertainty analysis through the introduction of possibility theory [80] based on the use of membership functions for representing incomplete information, a bit more than a decade after the publication of the founding paper. The latter issue has contributed to a clarification of the confusion, pervading early years, between fuzzy sets and probability.

This discussion paper tries to organize the legacy of fuzzy sets in an orderly way, highlighting the main ideas, sometimes misunderstood, and pointing out what seem to be promising trends and barren areas as well as indicating some neglected views of interest. The paper will briefly situate various subfields of fuzzy sets in the light of the various interpretations of membership functions in terms of distance, preference or uncertainty [20], and suggest potentially fruitful fuzzy set-inspired topics for future research and applications.

2 Basic ideas behind fuzzy sets

The introduction of the notion of a fuzzy set by L. A. Zadeh was motivated by the fact that, quoting the founding paper [72]:

> "imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction".

This seems to have been a continuous concern in all Zadeh's papers since the beginning, as well as the need to develop a sound mathematical framework for handling these "classes". This purpose required an effort to go beyond classical binary-valued logic, the usual setting for classes. Although many-valued logics had been there for a while, what is really remarkable is that due to this concern, Zadeh started to think in terms of *sets* rather than only in terms of degrees of truth. Since a set is a very basic notion, it was opening the road to the introduction of the fuzzification of any set-based notions such as relations, events, or intervals, while sticking with the many-valued logic point of view only does not lead you to consider such generalized notions. In other words, while Boolean algebras are underlying both propositional logic and naive set theory, the set point of view may be found richer in terms of mathematical modeling, and the same thing takes place when moving from many-valued logics to fuzzy sets. Moreover, the study of set operations on fuzzy sets has in return strongly contributed to a renewal of many-valued logics (see [14] for an introductory overview).

According to the founding paper [72], a fuzzy set represents "a class of objects with a continuum of grades of membership",

but in a footnote, Zadeh acknowledges that "the range of the membership function can be taken to be a suitable partially ordered set". This is an important remark, which opens the road to to more abstract constructs and to type-n fuzzy sets as well. Zadeh also observes that the case where the unit interval is used as a membership scale "corresponds to a multivalued logic with a continuum of truth values in the interval $[0, 1]$", acknowledging the link with many-valued logics.

So, a fuzzy set can be understood as a class equipped with an ordering of elements expressing that some objects are more in the class than others. However, in order to generalise the Boolean connectives, we need more than a mere relation between elements if one is to extend intersection, union and complement of sets, let alone implication to fuzzy sets. The set of possible membership grades has to be a complete lattice [33] so as to capture union and intersection, and either the concept of residuation or an order-reversing function are needed in order to express some kind of negation and implication.

Moreover, from the beginning, it was made clear that fuzzy sets were not meant as probabilities in disguise, since one can read [72] that

"the notion of a fuzzy set is completely non-statistical in nature"

and that it provides

"a natural way of dealing with problems where the source of imprecision is the absence of sharply defined criteria of membership rather than the presence of random variables."

In a nutshell, the main idea behind fuzzy sets is to make membership to sets gradual rather than abrupt. For instance, in the case of totally ordered universes, changing sharp membership thresholds into soft ones. It leads to extending the usual notions from set theory, logic, and inference, replacing Boolean algebra by many-valued ones, as well as all forms of set-valued mathematics to fuzzy set-valued mathematics. Presented as such, note that this extension is prima facie not related to the idea of uncertainty.

Fuzziness should also not be confused with vagueness [79], which is exclusively a concept pertaining to natural language. Indeed, the representation of gradual properties is not the unique information processing scenario that gives rise to borderline cases, one of the features of vagueness [13]. Vagueness refers to uncertainty of meaning (the membership function is ill-known), which is distinct from gradualness (membership is a matter of degree) [12]. The idea of typicality underlying linguistic terms is more connected to the one of similarity than to uncertainty [58].

As a consequence,

- originally, fuzzy sets were designed to formalize the idea of soft classification, which is more in agreement with the way people use categories in natural language.
- fuzziness is just implementing the idea of gradation in all forms of reasoning and problem-solving, as for Zadeh, everything is a matter of degree.
- a degree of membership is an abstract notion to be interpreted in practice.

According to the area of application, several interpretations can be found such as degree of similarity (to a prototype in a class), degree of plausibility, or degree of preference [20]. We now survey the use of fuzzy sets with respect to these three semantics.

3 Membership grades related to distance

The idea of representing a class by a fuzzy set [2], and later [79, 83] the fuzzy set representation of linguistic terms naming classes, is underlain by the idea of gradual transition between a set of elements that fully belong to the class, or that are fully representative of the term, and a set of elements that do not belong at all to the class, or that are definitely excluded by the meaning of the term. The first set, whose elements have membership 1, may be understood as the typical elements of the fuzzy set. More generally, the membership degree of an element to a fuzzy set is a degree of typicality of this element with respect to the class or the term represented, which is all the greater as the element is closer to the set of typical elements. Thus, in this view, membership grades can be naturally related to the idea of distance.

The most popular part of the fuzzy set literature deals with clustering, modeling and control, where gradual transitions between classes and their use in interpolation are the basic contribution of fuzzy sets. Intuitively speaking, a cluster gathers elements that are rather close to each other (or close to some core element(s)), while they are well-separated from the elements in the other cluster(s). Thus, the notions of graded proximity, similarity (dissimilarity) are at work in fuzzy clustering. With gradual clusters, the key issue is to define fuzzy partitions. The most widely used definition of a fuzzy partition, originally due to Ruspini [59], where the sum of membership grades of one element to the various classes is 1, suggests a connection (present in Ruspini's paper) between membership grades and probabilities, according to which a degree of membership of an element in a class can be identified with the probability that the element will be assigned to the class. Even though this view seems to be at odds with Zadeh's non-statistical intuitions, it is not surprising at all, as the closer an object to the prototypes of a class, the more often it will be assigned to this class. Measuring probability by distance is already present in the early times of statistics, when Gauss discovered the normal distribution as the only error function compatible with the least squares method (minimizing the Euclidean distance to observations), as explained by Stigler [63]. The statistical point of view on clustering is just a reversal of perspective with respect to the one of fuzzy sets, whereby the more often an object is assigned to a class, the closer is this object to the prototypes of the class. The question is then whether we measure strength of membership by observing frequencies in a training set or by computing distances to class prototypes. Using Gaussian-shaped distributions the two points of view are formally equivalent. However, fuzzy sets theory offers a more flexible mathematical framework for error-minimizing estimation methods, that cover distances other than Euclidean, as in the case of data reconciliation methods [15].

The view of a fuzzy set as a fuzzy cluster of elements, clusters forming a partition, has led Zadeh to emphasize the idea of granulation as a core concept supporting fuzzy logic [88], while in the crisp case, granulation and the notion of partition are basic in the theory of rough sets [54]. Interestingly enough, some bridges can be established between, (fuzzy) clusters, extensional fuzzy sets [40, 42], granulation, graded indistinguishability, and formal concept analysis [30] [25]. One application of fuzzy granulation is the notion of fuzzy transform [55] of a real-valued function with respect to a Ruspini's fuzzy partition [59] where the coefficients representing the transform are obtained in terms of the integral of the product of the function with each of the basic functions defining the partition (which evaluate the degree of adequacy of the value of the variable with the corresponding element of the fuzzy partition). From these numbers, an approximation of the function can be recovered. The same concept of fuzzy granulation is at work in density estimation methods based on fuzzy histograms [65], where the analogy with kernel-based methods is striking: again a kernel expresses similarity and plays the same interpolation role as a membership function but it is couched in the phraseology of, and formalised inside, probability theory.

The role of fuzzy sets in modeling and control originated in the idea of fuzzy algorithms and programs [73], where fuzzy instructions are instructions involving fuzzy labels. In that respect, the role of fuzzy if-then rules was soon recognized [76], while their interpolation power was emphasized later [86]. Basically, it offers a reconciliation between logical notions such as Boolean categories and inference, and numerical modeling techniques in engineering, that extensively exploit the notion of (linear) interpolation. A fuzzy model (e. g., fuzzy rules in the sense of Takagi-Sugeno [67]) is typically a collection of local usual mathematical models, each defined on gradual overlapping domains forming a partition of the input space. These mathematical models are related via an interpolation scheme taking advantage of soft boundaries and membership grades to neighboring domains. Such interpolation schemes are similar to the ones of neural nets. The bridge between neural nets and fuzzy sets leads to a useful trade-off between model accuracy (thanks to universal approximation capabilities) and model interpretability, provided that the fuzzy sets appearing in the rules remain meaningful for the expert, as in the first fuzzy controllers (e.g., [48]). It is also worth mentioning that the inference mechanism underlying Takagi-Sugeno fuzzy rules is close in spirit to case-based decision theory, later axiomatized by Gilboa and Schmeidler [32]: in the latter, the decision to apply should maximize a counterpart of expected utility where probabilities are replaced by similarities to previous cases where the decision led to results whose utility is known, while in the former, since the potential decisions belong to a continuum, the similarity based weighting is directly applied to the (linear) models of actions given in the conclusion part of the fuzzy rules.

4 Membership grades related to uncertainty

Fuzziness is also often interpreted as a form of uncertainty. However, this view is sometimes based on a misunderstanding. In its original understanding a grade of membership is not considered as a degree of (un)certainty [22]: asserting that a man is almost bald (implying he has almost no hairs) differs from saying that this man is almost certainly bald (leaving the possibility that finally he is not bald at all).

Fuzzy sets can represent uncertainty (in a gradual way) because crisp sets are often used to represent an ill-known value (in a crisp way like in interval analysis or in propositional logic). Viewed as representing uncertainty a set just distinguishes between values that are considered possible and values that are not, and fuzzy sets just introduce grades to soften boundaries of an uncertainty set. So in a fuzzy set it is the set that captures uncertainty [23]. This point of view echoes an important distinction made by Zadeh himself [79] between

- conjunctive (fuzzy) sets, where the set is viewed as the conjunction of its elements. This is the case for clusters discussed in the previous section. But also with set-valued attributes like the languages spoken more or less fluently by an individual.

- and disjunctive fuzzy sets which corresponds to mutually exclusive possible values of an ill-known single-valued attribute, like the ill-known birth nationality of an individual.

In the latter case, fuzzy sets are called possibility distributions [80] and act as elastic constraints on a precise value [77, 78]. Possibility distributions have an epistemic flavor, since they represent the information we have at our disposal about what values remain more or less possible for the variable under consideration, and what values are (already) known as impossible. This epistemic view is in complete contrast with the ontic view underlying conjunctive fuzzy sets [26].

The first illustration of the power of possibility theory proposed by Zadeh was an original theory of approximate reasoning [81, 84, 85], later reworked for emphasizing new points [87, 90], where pieces of knowledge are represented by possibility distributions that fuzzily restrict the possible values of variables, or tuples of variables. These possibility distributions are combined conjunctively, and then projected in order to compute fuzzy restrictions acting on the variables of interest. This view is the one at work in constraint satisfaction problems (CSP), and anticipates weighted CSP [4] by many years, although without any algorithmic concerns. One research direction, quite in the spirit of the objective of "computing with words" [87], would be to further explore the possibility of a syntactic (or symbolic) computation of the inference step (at least for some noticeable fragments of this general approximate reasoning theory), where the obtained results are parameterized by fuzzy set membership functions that would be used only for the final interpretation of the results. An illustration of this idea can be found in an approach to reasoning with relative orders of magnitude [39]. It is also at work in possibilistic logic [27], a very elementary formalism handling pairs made of a Boolean formula and a certainty weight. Such pairs encode simple possibility distributions.

Associated with a possibility distribution, is a possibility measure [80], which is a max-decomposable set function. Thus, one can evaluate the possibility of a crisp, or fuzzy, statement of interest, given the available information supposed to be represented by a possibility distribution. It is also important to notice that the introduction of possibility theory by Zadeh was closely related to the modeling of information expressed in natural language. This view contrasts with the motivations of the English economist Schackle [61, 62], interested in a non-probabilistic view of expectation, who designed a formally similar theory, but rather based on the idea of degree of impossibility understood as a degree of surprise.

However, apart from a brief mention in [82], Zadeh does not explicitly use of the notion of necessity (the natural dual of the modal notion of possibility) in his work on possibility theory and approximate reasoning. Still, it is important to distinguish between statements that are *necessarily* true (to some extent), i.e. whose negation is almost impossible, from the statements that are only *possibly* true (to some extent) depending on the way the fuzzy knowledge would be made precise. The simultaneous use of the two notions is often required in applications of possibility theory. Besides, in possibility theory, apart from a (strong) necessity measure reflecting, by complementation, what is known as being more or less impossible, and the dual (weak) possibility measure reflecting consistency with the available information, there exist two other set functions of interest (see, e.g., [21]): a strong possibility measure which guarantees that *all* interpretations in a subset are possible to some extent, and a dual weak necessity measure. The joint use of these four set functions provides the proper setting for representing bipolar information, when one distinguishes between positive information, e.g., what has been observed, and negative information, corresponding, e.g., to what is not ruled out by generic knowledge [24].

Interestingly, necessity (resp. possibility) measures are formally special case of belief (resp. plausibility) functions (already pointed out in Shafer's book [64]) and special cases of coherent lower probabilities in the sense of Walley [70]. This direction is at odds with Zadeh's motivations for possibility theory but it opens the way to the systematic use of membership functions (as possibility distributions) to represent incomplete information in statistics, from likelihood functions to confidence intervals and probabilistic inequalities [10] as well as dispersion measures and parameter estimation methods [50].

Necessity degrees are also the building blocks of possibilistic logic (see [27] for an up-to-date overview), where classical logic formulas are associated with lower bounds of a necessity measure for assessing their certainty. The semantics of a (conjunctive) set of possibilistic logic formulas is expressed by a possibility distribution over a set of interpretations (or possible worlds). This is an example of possibility distribution defined on an abstract, non-ordered referential, where the possibility distribution is no longer the precisiation of a linguistic term on an ordered often continuous universe of discourse as it is the case in Zadeh's approximate reasoning approach. Possibilistic logic has found a number of developments in artificial intelligence including the handling of inconsistency, the modeling of exception-tolerant reasoning, the fusion of logical knowledge bases, and the design of possibilistic counterparts to Bayesian networks, which are semantically equivalent to possibilistic logic bases, but exhibit a graphical representation. See [27] for an introduction to these issues and for references. The definition of possibilistic networks requires the introduction of the notion of conditioning in possibility theory. It turns out that two forms of conditioning make sense, one defined with minimum, the other with product. These two forms of conditioning differentiate qualitative and quantitative possibility theory [21]. Let us also mention a recent application of a possibilistic logic-like handling of uncertain functional dependencies to the design of databases containing dubious tuples [44].

The links and differences between modal logic and possibility theory have been a matter of debates, not always well-focused, for a long time; see [91] for a recent position paper by Zadeh. Still, the recent development of generalized possibilistic logic (GPL) (see [27] for a brief account and references), where one can reason both in terms of possibility and necessity, and whose axiomatics is as the one of a (graded) epistemic modal logic, sheds some light on the question: the semantics of GPL is in terms of *sets* of possibility distributions (rather than a unique possibility distribution as in basic possibilistic logic), while the semantics of general modal logics require accessibility relations.

Apart from the reasoning side, another important area of the application of the possibility theory-based understanding of fuzzy sets is the computation with ill-known quantities represented by fuzzy intervals. The possibility of performing arithmetic operations on fuzzy numbers was also pointed out by Zadeh [78], then developed by other scholars (see [19] for a survey in the XXth century). The calculus of fuzzy intervals is a gradual extension of *set-valued* mathematics and the extension principle underlying it can be expressed in possibility theory. The calculus of fuzzy intervals is instrumental in various areas including:

- systems of linear equations with fuzzy coefficients (see the paper by Lodwick and Dubois in this special issue) and differential equations with fuzzy initial values, and fuzzy set functions [45];

- fuzzy random variables for the handling of linguistic or imprecise statistical data [31, 8];

- fuzzy regression methods [52];

- operations research and optimisation under uncertainty [93, 16, 46].

This research trend contrasts with the mainstream fuzzy modeling approach, discussed in the previous section, which promotes mathematical models in the usual sense, albeit constructed by means of fuzzy sets, and does not express uncertainty. Systems analysis based on fuzzy intervals and fuzzy differential equations has been less developed than fuzzy modeling partly because it leads to complex calculations, but also because the epistemic nature of the fuzzy approach under the uncertainty interpretation is sometimes ill-understood at the practical level, running the risk of posing mathematical problems that are not always reflecting their intended meaning. For instance, the equality of membership functions on each side of a fuzzy linear equation with fuzzy numbers is very demanding and is not equivalent to the identity between actual values of the terms on each side of the equality. So there is sometimes a gap between mathematical results and the actual problem they are supposed to model in this area.

5 Membership grades related to preference

Another natural semantics for the grades of membership of a fuzzy set is in terms of degrees of satisfaction, when the fuzzy set represents a value function. For instance, the probability of a fuzzy event [74] has exactly the same form as the expected utility of an act after Savage, interpreting the utility function as the membership function of the fuzzy set of good consequences of this act. Likewise, in multicriteria decision evaluation, the rating profile of an object according to various criteria is easily viewed as the fuzzy set of satisfied criteria.

In this respect the invention of fuzzy set connectives by Zadeh has triggered a large literature on

- aggregation operations both from a mathematical point of view (see [43]) and for multicriteria evaluation (see [36] [1] [69] for recent books).
- maxmin approaches to optimization, in multicriteria linear programming, initiated in [68, 93] and more recently seen as a special case of valued constraint satisfaction [4].

This framework for multicriteria evaluation, constraint-based reasoning, and optimization is clearly part of the legacy of Zadeh's 1965 paper, as well as the one he published with Richard Bellman in 1970 [3]. Originally rather elementary (using max and min), it is characterized by:

- the assumption of a common value scale for the various factors, constraints or criteria. This is a strong assumption that nevertheless includes both quantitative and qualitative scales.

- A unified view of possible aggregation modes ranging from conjunctions and disjunctions to generalized means.

- Sophisticated criteria weighting schemes that allow for dependent criteria. On this issue, the fuzzy set literature has met the economic literature on Choquet integrals [35]. In the qualitative framework, the counterpart to Choquet integral is Sugeno integral [66, 35], also called fuzzy integral by Sugeno, because it uses maximum and minimum, respectively the basic disjunction and conjunction in fuzzy set theory. From its inception, it was construed as a tool for multiple criteria evaluation. Sugeno integral acted as a bridge between fuzzy set theory and Choquet integrals.

- Extensions to bipolar decision analysis methods that measure pros and cons of decisions, losses and gains in a separate way [34].

Another important offspring of Zadeh's early papers is fuzzy preference modeling based on the gradual extension of equivalence relations and orderings proposed in [75], generalizing transitivity to maxmin transitivity. Preference modeling is the first step in multicriteria evaluation, whereby it is more natural for a person to represent his or her preference on a set of objects by an ordering relation than by a utility function. The basic tool for analyzing human-originated preference relations is their decomposition into strict preference, indifference and incomparability [6]. Using fuzzy relations, it is possible to express how much an object is preferred to another. A number of publications starting by an early paper of Orlowski [53] and later on triggered by the book by Fodor and Roubens [29] address the issue of decomposing a fuzzy relation into graded strict preference, indifference and incomparability. Bodenhofer [5] has shown fuzzy ordering relations should be envisaged conjointly with a fuzzy similarity relation expressing indifference.

Besides, here again the meaning of the values in a fuzzy relation matters, as it may either reflect intensity of preference, or uncertainty about all-or-nothing preference. Unless this is clarified, it is very difficult to apply fuzzy preference modeling in concrete decision-making problems. Indeed, one may argue that the measurement issue present in utility theory is for the most part obviated by the use of order relations, while this advantage is lost when making such relations fuzzy.

Note that the use of a fuzzy ordering relation implies that it makes sense to say: object 1 is preferred to object 2 to the same extent as object 3 is preferred to object 4, which suggests to consider preference difference measurement methods. A lot of work needs to be done to let such mathematical models of preference be used as a basis for multicriteria evaluation, in place of questionable methods based on triangular fuzzy numbers defined on arbitrary value scales [11].

Finally, it is not clear all valued relations can be put under the umbrella of fuzzy logic. In so-called reciprocal relations, the sum of the preference values of one object against the other and of the latter against the former is 1, so that it is more natural to interpret them in terms of probability of preference. Such reciprocal relations exist for a long time and differ from the valued relations introduced in Zadeh's paper. As a consequence, calling reciprocal relations fuzzy relations may be misleading [9].

In many practical applications, preference and uncertainty are conjointly present. For instance, fuzzy databases [56, 57] may be both a matter of

1. expressing preferences in the queries by specifying flexible restrictions on the desired attribute values or by handling priorities between attributes,
2. handling uncertainty when the database contains imprecise or fuzzy pieces of information.

When both issues are present, we are led to compute possibility and necessity measures for fuzzy events, in order to distinguish between answers that are certain (to some extent) to reach highly satisfactory values, and answers for which this is only possible (to some extent). At work here are pessimistic and optimistic decision criteria, which have been axiomatized [28], thus providing formal foundations for qualitative possibility theory.

Databases may be viewed as a repertory of cases. This leads to substitute similarity to uncertainty in possibilistic decision criteria, as in fuzzy case-based reasoning methods [18, 41], coming close to the idea of similarity-based possibility [92].

6 Higher-order membership grades

Ten years after the invention of fuzzy sets, Zadeh [78] considered the possibility of iterating the process of changing membership grades into (fuzzy) sets, giving birth to interval-valued fuzzy sets, fuzzy set-valued (type 2) fuzzy sets, and more generally type n fuzzy sets. While this idea is philosophically tempting and reflects the problematic issue of measuring membership grades of linguistic concepts, it has given birth to a high number of publications pertaining to variants of higher-order fuzzy sets, often reinventing the same notions under different names. To name a few:[1] intuitionistic fuzzy sets, vague sets, hesitant fuzzy sets, soft sets, arbitrary combinations of the above notions (for instance, interval-valued intuitionistic fuzzy sets, fuzzy soft sets, etc.).

There are several concerns to be pointed out with these complexifications (rather than generalizations) of fuzzy sets. They shed some doubt on the theoretical or applied merits of such developments of fuzzy set theory:

- Each new kind of fuzzy sets gives birth to a plethora of routine theoretical papers, redefining basic concepts of fuzzy sets in the new setting, irrespective of what the new membership grades mean, and presenting no motivations;

- Several of these constructions reinvent existing notions under different sometimes questionable names. For instance vague sets are the same as intuitionistic fuzzy sets, and formally they are just a different encoding of interval-valued fuzzy sets (a pair of nested sets, versus an orthopair of disjoint sets [7]). Moreover the link between intuitionistic fuzzy sets and intuitionism hardly exists[17]. Hesitant fuzzy sets were proposed already in 1975 under a different name [37]; soft sets are set-valued mappings, a notion that has been well-known for a long time in the theory of random sets.

- Some generalizations of fuzzy sets often underly a misunderstanding [26]: are they special kinds of L-fuzzy sets or an approach to handling uncertainty about membership grades? The latter motivation is often put forward in the introduction of such papers, while the main text adopts an algebraic structure derived from L-fuzzy sets. For instance, many authors speak of "type-2-connectives". However, if a type 2 fuzzy set is understood as a fuzzy set of (a possibility distribution over) membership functions, it is enough to apply the extension principle to the type 1 fuzzy logic expression of interest, in order to compute the resulting fuzzy set-valued membership grade; there is no need to appeal to a specific algebraic structure on some set of higher-order membership grades different from the unit interval, all the more so as

[1] We omit references here, for the sake of conciseness; readers can find a lot of them by searching for the corresponding key-words.

compositionality of connectives is then lost [22]. There is no such thing as type-2 connectives in this case.

- Algebraic structures for complex membership grades stemming from higher-order fuzzy sets are in general special kind of lattices. Actually, these higher-order fuzzy sets are special cases of L-fuzzy sets, hence often redundant from a mathematical point of view [71, 38].

- The incurred complexity of higher-order fuzzy set-based approaches is sometimes not justified. Practical applications of type 2 fuzzy sets in the modeling area sometimes come down to models with more tuning parameters than usual fuzzy systems, but they are often standard input-output models after due defuzzification steps. So it is difficult to understand why they would outperform usual fuzzy systems without being subject to overfitting effects. In practice, many authors restrict to interval-valued fuzzy sets, under the strange name "interval type 2 fuzzy sets", to simplify calculations. Likewise applications of intuitionistic to multicriteria decision making seem to artificially increase the burden of collecting preference ratings in the form of pairs of numbers (or even of intervals) whose meaning may be more unclear to the user than mere membership values.

- Many methods using type 2 fuzzy sets revisit calculations with fuzzy intervals, without referring to the corresponding state of the art.

The point is not to claim that such variants of fuzzy sets are necessarily misleading or useless. They often try to capture convincing intuitions but are too often developed for their own sake, sometimes at odds with these intuitions. See [17] for a full-fledged discussion on intuitionistic fuzzy sets and interval-valued fuzzy sets, and [26] for the clash of intuitions between notions of bipolarity and uncertainty pervading intuitionistic fuzzy sets. Regarding soft sets, only outlined in the founding paper [51], they were originally meant as an extension of the alpha-cut mapping to non-nested sets, a concept more recently considered by several authors [23, 60, 49] in a more applied perspective. However, followers of the soft set trend often adopt the set-valued mapping point of view without reference to cuts of fuzzy sets. For instance, the highly cited paper of Maji *et al.* [47] seem to consider soft sets in the algebraic framework of formal concept analysis [30] only.

As a consequence, there is an effort to be pursued in terms of motivation, mathematical rigor, and convincing applications, in order to make this part of the fuzzy set legacy worth developing further.

7 Conclusion

The intention of this note was to overview research topics that stemmed from Zadeh's founding paper and early subsequent publications of his, and that seem to have a promising future. However, our discussion has no pretense to provide an exhaustive coverage of all the potential application fields of fuzzy set and possibility theory. Several noticeable ones have not been cited (e.g., information retrieval, machine learning), and those that have been mentioned are mainly there for illustrating various usages of fuzzy set notions, rather than for advocating their merits with respect to other approaches.

Note that fuzzy set research now reaches a point where the corpus of basic tools has been already considerably developed, and very few new basic concepts seem to have emerged in the last 10 years. These basic tools become more and more accepted in various established disciplines (for example, fuzzy systems in non-linear control engineering, fuzzy clustering in data analysis, fuzzy interval computations in risk analysis, etc.). In this sense, fuzzy set theory has come of age. These numerous achievements contrast with various attempts to fuzzify mathematical notions or complexify existing fuzzy set concepts, which can be called "fuzzification for its own sake", that seems to be driven, as in many fields nowadays, by the pressure to publish papers in the academic world. This increases the number of publications without always contributing much to science, while at a higher level in the society "the pursuit of knowledge for his own sake is increasingly being replaced by a quest for education as a ticket to a better-paying job", as denounced by Zadeh himself [89] and deplored by the scientific community.

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