SOME GEOMETRIC PROPERTIES OF THE SUBORDINATION FUNCTION ASSOCIATED TO AN OPERATOR-VALUED FREE CONVOLUTION SEMIGROUP

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ABSTRACT. In his article *On the free convolution with a semicircular distribution*, Biane found very useful characterizations of the boundary values of the imaginary part of the Cauchy-Stieltjes transform of the free additive convolution of a probability measure on $\mathbb{R}$ with a Wigner (semicircular) distribution. Biane's methods were recently extended by Huang to measures which belong to the partial free convolution semigroups introduced by Nica and Speicher. This note further extends some of Biane's methods and results to free convolution powers of operator-valued distributions and to free convolutions with operator-valued semicirculars. In addition, it investigates properties of the Julia-Carathéodory derivative of the subordination functions associated to such semigroups, extending results from [7].

1. INTRODUCTION

Free probability, introduced by Voiculescu in order to study free group factors, gained considerable importance after the discovery in [26] of the connection between freeness and the asymptotic behaviour of large random matrices. One of the most significant consequences of the main result of [26] is the fact that two independent selfadjoint random matrices H_N, A_N - H_N being a gaussian matrix - are asymptotically free as $N \to \infty$. Thanks to Wigner's work, it is known since the '50s that the asymptotic distribution of H_N as $N \to \infty$ is the semicircle law. In particular, the distribution of $A_N + H_N$ is modeled by Voiculescu's free additive convolution [30] of a standard semicircular distribution with the limiting distribution of A_N. In [12], this convolution is analyzed in great detail: among others, a formula for the density of the corresponding distribution is provided, and it is shown that this density is bounded, continuous and analytic wherever positive. However, in order to study the asymptotic eigenvalue distribution of more general selfadjoint polynomials $P(A_N, H_N)$ it is necessary to consider the more general framework of free convolutions of *operator-valued distributions* [27, 22, 23, 17, 5]. In the present note, we find certain operator-valued counterparts of Biane's results from [12]; necessarily, several of the conclusions of [12] do not hold in this more general setting.

As it is shown in [3], there exists an intimate connection between the free additive convolution with an operator-valued semicircular distribution and the free convolution powers of operator-valued distributions. In particular, it turns out that the analytic tools used in the study of free convolution powers of operator-valued distributions are generalizations of the analytic tools used in the study of the free convolution with an operator-valued semicircular distribution. Thus, we write our proofs in the more general context. This has the advantage of allowing us to draw conclusions about more general free additive convolutions of operator-valued distributions.

The second section is dedicated to introducing the main concepts and tools we require. We state and prove our main results in the third and fourth section.

2. Noncommutative functions, distributions and convolutions

2.1. **Noncommutative probability spaces and distributions.** Following D. Voiculescu [30, 27], by a noncommutative probability space we understand a pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a unital $*$-algebra over $\mathbb{C}$ and $\tau\colon \mathcal{A} \to \mathbb{C}$ is a positive linear functional with $\tau(1) = 1$. Let $\mathcal{B}$ be a unital C*-algebra. A $\mathcal{B}$-valued non-commutative probability space is a triple $(\mathcal{A}, \mathbb{E}_\mathcal{B}, \mathcal{B})$, where $\mathcal{A}$ is a unital $*$-algebra containing $\mathcal{B}$ as a $*$-subalgebra and $\mathbb{E}_\mathcal{B}$ is a unit-preserving positive conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ (in particular, the units of $\mathcal{A}$ and $\mathcal{B}$ coincide). If $\mathcal{B} \subset \mathcal{A}$ is an inclusion of unital C*-algebras, then we call $(\mathcal{A}, \mathbb{E}_\mathcal{B}, \mathcal{B})$ a $\mathcal{B}$-valued noncommutative C*-probability space. For simplicity, we will suppress the subscript of $\mathbb{E}_\mathcal{B}$ whenever there is no risk of confusion, and denote our conditional expectation by $\mathbb{E}$. Elements $X \in \mathcal{A}$ are called *random variables* or (in the second context) *$\mathcal{B}$-valued* (or *operator-valued*) *random variables.*

We use the notation $\mathcal{B}\langle \mathcal{X}_1, \mathcal{X}_2, \dots \mathcal{X}_n\rangle$ for the $*$-algebra freely generated by $\mathcal{B}$ and the noncommuting selfadjoint symbols $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$. If $X \in \mathcal{A}$ is a selfadjoint element, then we will also use the notation $\mathcal{B}\langle X\rangle$ for the $*$-algebra generated by X and $\mathcal{B}$. Following [3] we denote the set of all positive, unit preserving, conditional expectations from $\mathcal{B}\langle \mathcal{X}\rangle$ to $\mathcal{B}$ by $\Sigma(\mathcal{B})$. Given $\mu \in \Sigma(\mathcal{B})$, its n^{th} moment is the $n-1$-linear map $\mu_n\colon \mathcal{B}\times\cdots\times\mathcal{B} \to \mathcal{B}$ defined by $\mu_n(b_1, b_2, \dots, b_{n-1}) := \mu\left[\mathcal{X}b_1\mathcal{X}b_2\cdots\mathcal{X}b_{n-1}\mathcal{X}\right]$. We define the zeroth moment to be $1 \in \mathcal{B}$ and the first moment to be $\mu[\mathcal{X}] \in \mathcal{B}$. We also denote $\Sigma_0(\mathcal{B})$ the set of all $\mu \in \Sigma(\mathcal{B})$ whose moments do not grow faster than exponentially, that is, all $\mu \in \Sigma(\mathcal{B})$ for which there exists some $M > 0$ such that, for all positive integers m, all $b_1, \dots, b_n \in M_m(\mathcal{B})$ and $\mathcal{X}_m = \mathcal{X} \otimes 1_m$ we have that

$$\|(\mu \otimes \mathrm{Id}_m)(\mathcal{X}_m b_1 \mathcal{X}_m b_2 \cdots \mathcal{X}_m b_n \mathcal{X}_m)\| < M^{n+1}\|b_1\|\cdots\|b_n\|.$$

If $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ is a $\mathcal{B}$-valued noncommutative probability space and $X = X^* \in \mathcal{A}$, we define its distribution with respect to $\mathbb{E}$ to be the element $\mu_X \in \Sigma_0(\mathcal{B})$ satisfying

$$\mu_X(P(\mathcal{X})) = \mathbb{E}(P(X)) \text{ for all } P(\mathcal{X}) \in \mathcal{B}\langle\mathcal{X}\rangle.$$

If X belongs to a $\mathcal{B}$-valued noncommutative C*-probability space, then $\mu_X \in \Sigma_0(\mathcal{B})$. Conversely, as shown by Voiculescu in [27], if $\mu \in \Sigma_0(\mathcal{B})$, then there exist a $\mathcal{B}$-valued C*-noncommutative probability space containing an element $X = X^*$ such that $\mu_X = \mu$. In the simpler case $\mathcal{B} = \mathbb{C}$, μ_X can be identified with a Borel probability measure supported on the compact set $\sigma(X)$, the spectrum of X (see [2]).

2.2. **Free independence and relevant transforms.** We present next the free independence, and define the relevant analytic transforms, in a C*-algebraic context, as this is the context that is considered most often in this paper.

Definition 2.1. *Let $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ be a $\mathcal{B}$-valued noncommutative C*-probability space and $\{X_i\}_{i\in I}$ be a family of selfadjoint elements from $\mathcal{A}$. The family $\{X_i\}_{i\in I}$ is said to be* freely independent *(or just* free*) over $\mathcal{B}$ with respect to $\mathbb{E}$ if for any $n \in \mathbb{N}$, $\mathbb{E}[A_1 \cdots A_n] = 0$ whenever $A_j \in \mathcal{B}\langle X_{\iota(j)}\rangle \cap \ker(\mathbb{E})$, $\iota(j) \in I$, $\iota(k) \neq \iota(k+1)$ for all $k \in \{1, \dots, n-1\}$.*

If X, Y are two freely independent $\mathcal{B}$-valued noncommutative random variables, then μ_{X+Y} depends only on μ_X and μ_Y and is called the free additive convolution of μ_X and μ_Y. We denote μ_{X+Y} by $\mu_X \boxplus \mu_Y$.

It is natural to denote $\underbrace{\mu \boxplus \cdots \boxplus \mu}_{n \text{ times}}$ by $\mu^{\boxplus n}$. Obviously, $\{\mu^{\boxplus n} | n \in \mathbb{N}\}$ forms a discrete semigroup. A remarkable result of Nica and Speicher [19] states that for any Borel probability measure μ on $\mathbb{R}$, there exists a partial semigroup, i.e. a family $\{\mu^{\boxplus t} \colon t \geq 1\}$ such that $\mu^{\boxplus 1} = \mu$ and $\mu^{\boxplus s+t} = \mu^{\boxplus s} \boxplus \mu^{\boxplus t}$, $s, t \geq 1$ (see also [11]). This result has been extended by Curran [14] to certain operator-valued distributions. However, as it will be seen below, in the operator-valued context, analytic transforms indicate that it should be possible - or rather natural - to consider convolution powers indexed by a subset of the set of completely positive self-maps of $\mathcal{B}$. The main result of [3] states precisely that: given $\mu \in \Sigma_0(\mathcal{B})$, there exists a family

$$\left\{\mu^{\boxplus \alpha} | \alpha \colon \mathcal{B} \to \mathcal{B} \text{ completely positive, } \alpha - \mathrm{Id}_{\mathcal{B}} \text{ completely positive}\right\} \subset \Sigma_0(\mathcal{B})$$

such that $\mu^{\boxplus \mathrm{Id}_{\mathcal{B}}} = \mu$ and $\mu^{\boxplus \alpha+\beta} = \mu^{\boxplus \alpha} \boxplus \mu^{\boxplus \beta}$. Moreover, as shown in [3, Corollary 7.6], whenever $\mu \in \Sigma_0(\mathcal{B})$ is $\boxplus$-infinitely divisible, $\mu^{\boxplus \alpha} \in \Sigma_0(\mathcal{B})$ for *any* completely positive $\alpha \colon \mathcal{B} \to \mathcal{B}$.

For the computation of free convolutions, Voiculescu [30, 27] introduced the R-transform. In order to define it, let

$$G_\mu(b) = \mu\left[(b - \mathcal{X})^{-1}\right], \quad \Im b > 0. \tag{1}$$

Here we denote $\Im b = (b - b^*)/2i$, $\Re b = (b + b^*)/2$, and we write $a > 0$ if $a = a^*$ and $\sigma(a) \subset (0, +\infty)$. The notation $a \geq 0$ is used when we require only that $a = a^*$ and $\sigma(a) \subset [0, +\infty)$. If $\mu \in \Sigma_0(\mathcal{B})$, then $M_\mu(b) = \mu\left[(1 - b\mathcal{X})^{-1} b\right] = G_\mu(b^{-1})$ has an analytic continuation to a neighbourhood of zero and maps 0 to itself. A simple computation shows that $M_\mu'(0) = \mathrm{Id}_{\mathcal{B}}$, so that, by the inverse function theorem for Banach spaces, M_μ has a unique compositional inverse, denoted by $M_\mu^{\langle -1 \rangle}$, around zero which maps zero to itself. Thus, both $b^{-1} M_\mu^{\langle -1 \rangle}(b)$ and $M_\mu^{\langle -1 \rangle}(b) b^{-1}$ are analytic around zero and map zero to one. The R-transform of μ is defined via the formula $bR_\mu(b) = (M_\mu^{\langle -1 \rangle}(b) b^{-1})^{-1} - 1$. We prefer a slightly different form of the definition of R_μ, namely

$$R_\mu(b) = G_\mu^{\langle -1 \rangle}(b) - b^{-1}. \tag{2}$$

This formula is well-defined on an open set which has zero in its closure, and thus determines R_μ. The essential property of the R-transform, found by Voiculescu, is that

$$R_\mu(b) + R_\nu(b) = R_{\mu \boxplus \nu}(b)$$

on a sufficiently small neighbourhood of zero in $\mathcal{B}$. Clearly then, for any linear completely positive map $\alpha \colon \mathcal{B} \to \mathcal{B}$ such that $\alpha - \mathrm{Id}_{\mathcal{B}}$ is still completely positive, $\mu^{\boxplus \alpha}$ will be given by

$$R_{\mu^{\boxplus \alpha}}(b) = \alpha(R_\mu(b)), \tag{3}$$

on a neighbourhood of zero. It has been shown in [3] that such a $\mu^{\boxplus \alpha} \in \Sigma_0(\mathcal{B})$ exists for any $\mu \in \Sigma_0(\mathcal{B})$. A different, simpler proof of this result is given in [24], where it is also shown that the requirement that $\alpha - \mathrm{Id}_{\mathcal{B}}$ is itself completely positive cannot be generally omitted.

It is quite obvious from (1) that G_μ plays a role similar to that of the Cauchy-Stieltjes transform in classical probability. However, unlike the classical Cauchy-Stieltjes transform, G_μ alone does not generally encode all of the distribution μ, but only its symmetric moments. It has been a crucial insight of Voiculescu that G_μ is just the first level of a noncommutative function that *does* encode all of μ: this will be outlined in the next subsection.

2.3. **Noncommutative functions and transforms.** In this subsection we largely follow [6, 21] in describing the noncommutative extensions of the analytic transforms introduced in the previous subsection, and [18] in the definition of noncommutative sets and functions. We refer to these three articles and [28, 29] for details on, and proofs of, the statements below.

If S is a nonempty set, we denote by $M_{m\times n}(S)$ the set of all matrices with m rows and n columns having entries from S. For simplicity, we let $M_n(S) := M_{n\times n}(S)$. Given C*-algebra $\mathcal{B}$, a *noncommutative set* is a family $\Omega := (\Omega_n)_{n\in\mathbb{N}}$ such that

(a) for each $n \in \mathbb{N}$, $\Omega_n \subseteq M_n(\mathcal{B})$;
(b) for each $m, n \in \mathbb{N}$, we have $\Omega_m \oplus \Omega_n \subseteq \Omega_{m+n}$.

The noncommutative set Ω is called *right admissible* if in addition the condition (c) below is satisfied:

(c) for each $m, n \in \mathbb{N}$ and $a \in \Omega_m, b \in \Omega_n, w \in M_{m\times n}(\mathcal{B})$, there is an $\epsilon > 0$ such that $\begin{pmatrix} a & zw \\ 0 & b \end{pmatrix} \in \Omega_{m+n}$ for all $z \in \mathbb{C}, |z| < \epsilon$.

Given C^*-algebras $\mathcal{B}, \mathcal{C}$ and a noncommutative set $\Omega \subseteq \coprod_{n=1}^\infty M_n(\mathcal{B})$, a *noncommutative function* is a family $f := (f_n)_{n\in\mathbb{N}}$ such that $f_n \colon \Omega_n \to M_n(\mathcal{C})$ and

(1) $f_m(a) \oplus f_n(b) = f_{m+n}(a \oplus b)$ for all $m, n \in \mathbb{N}$, $a \in \Omega_m, b \in \Omega_n$;
(2) for all $n \in \mathbb{N}$, $f_n(T^{-1}aT) = T^{-1}f_n(a)T$ whenever $a \in \Omega_n$ and $T \in GL_n(\mathbb{C})$ are such that $T^{-1}aT$ belongs to the domain of definition of f_n.

A remarkable result (see [18, Theorem 7.2]) states that, under very mild conditions on Ω, local boundedness for f implies each f_n is analytic as a map between Banach spaces.

As mentioned in the previous section, the function G_μ encodes only the symmetric part of the distribution μ. It was an extremely important remark of Voiculescu that G_μ has a noncommutative extension:

$$G_\mu^{[n]}(b) = (\mu \otimes \mathrm{Id}_n)\left[(b - \mathcal{X} \otimes 1_n)^{-1}\right], \quad n \in \mathbb{N}. \tag{4}$$

There are two noncommutative sets which are natural domains of definition for $(G_\mu^{[n]}(b))_{n\in\mathbb{N}}$ and for $(G_\mu^{[n]}(b^{-1}))_{n\in\mathbb{N}}$, respectively: the noncommutative operator upper half-plane $(\mathbb{H}^+(M_n(\mathcal{B})))_{n\in\mathbb{N}}$, where $\mathbb{H}^+(M_n(\mathcal{B})) = \{b \in M_n(\mathcal{B}) \colon \Im b > 0\}$, and the set of nilpotent matrices with entries from $\mathcal{B}$, respectively. Remarkably, as shown in [28], $G_\mu^{[n]}$ maps $\mathbb{H}^+(M_n(\mathcal{B}))$ into $\mathbb{H}^-(M_n(\mathcal{B})) := -\mathbb{H}^+(M_n(\mathcal{B}))$ and $G_\mu^{[n]}(b^*) = G_\mu^{[n]}(b)^*$. It is clear that the restriction of $(G_\mu^{[n]})_{n\in\mathbb{N}}$ to either of these two noncommutative sets determines $(G_\mu^{[n]})_{n\in\mathbb{N}}$. For a description of how to explicitly recover μ from $(G_\mu^{[n]})_{n\in\mathbb{N}}$ via its moments, we refer to [6, 21].

It follows from its definition that the R-transform has itself a noncommutative extension, which determines μ uniquely. The level-one relation (3) extends to $R_{\mu^{\boxplus\alpha}}^{[n]}(b) = (\alpha \otimes \mathrm{Id}_n)(R_\mu^{[n]}(b))$ for $b \in M_n(\mathcal{B})$ of small enough norm. From this formula and the noncommutative extension of (2) we obtain, by adding b^{-1}, the relation $\left(G^{[n]}_{\mu^{\boxplus\alpha}}\right)^{\langle -1\rangle}(b) = (\alpha\otimes \mathrm{Id}_n)\left(\left(G^{[n]}_{\mu}\right)^{\langle -1\rangle}(b)\right) - (\alpha\otimes\mathrm{Id}_n - \mathrm{Id}_{\mathcal{B}}\otimes\mathrm{Id}_n)(b^{-1})$. Replacing b by $G^{[n]}_{\mu^{\boxplus\alpha}}(b)$ provides $b = (\alpha\otimes\mathrm{Id}_n)\left(G^{[n]}_{\mu}\right)^{\langle -1\rangle}\left(G^{[n]}_{\mu^{\boxplus\alpha}}(b)\right) - (\alpha\otimes\mathrm{Id}_n - \mathrm{Id}_{\mathcal{B}}\otimes\mathrm{Id}_n)\left(G^{[n]}_{\mu^{\boxplus\alpha}}(b)^{-1}\right)$. With the notations

$$F^{[n]}_{\mu}(b) = G^{[n]}_{\mu}(b)^{-1}, \quad b\in\mathbb{H}^+(M_n(\mathcal{B})),\ n\in\mathbb{N}, \tag{5}$$

$$h^{[n]}_{\mu}(b) = F^{[n]}_{\mu}(b) - b, \quad b\in\mathbb{H}^+(M_n(\mathcal{B})),\ n\in\mathbb{N}, \tag{6}$$

and

$$\omega^{[n]}_{\alpha}(b) = \left(G^{[n]}_{\mu}\right)^{\langle -1\rangle}\left(G^{[n]}_{\mu^{\boxplus\alpha}}(b)\right), \tag{7}$$

we re-write equation (3) as

$$\omega^{[n]}_{\alpha}(b) = b + [(\alpha - \mathrm{Id}_{\mathcal{B}})\otimes\mathrm{Id}_n]\, h^{[n]}_{\mu}\left(\omega^{[n]}_{\alpha}(b)\right), \quad b\in\mathbb{H}^+(M_n(\mathcal{B})),\ n\in\mathbb{N}, \tag{8}$$

with $\omega^{[n]}_{\alpha}\colon \mathbb{H}^+(M_n(\mathcal{B}))\to\mathbb{H}^+(M_n(\mathcal{B}))$. The above argument for the existence of $(\omega^{[n]}_{\alpha})_{n\in\mathbb{N}}$ is obviously not complete: for the rigorous proof, we refer to [3, Theorem 8.4]. This same theorem also states that for any $b\in\mathbb{H}^+(M_n(\mathcal{B}))$, $\omega^{[n]}_{\alpha}(b)\in\mathbb{H}^+(M_n(\mathcal{B}))$ is the unique attracting fixed point of the map $f^{[n]}_b\colon\mathbb{H}^+(M_n(\mathcal{B}))\to\mathbb{H}^+(M_n(\mathcal{B}))$, $f^{[n]}_b(w) = b + [(\alpha-\mathrm{Id}_{\mathcal{B}})\otimes\mathrm{Id}_n]\,h^{[n]}_{\mu}(w)$, and the right inverse of the map $H^{[n]}\colon\mathbb{H}^+(M_n(\mathcal{B}))\to M_n(\mathcal{B})$, $H^{[n]}(w) = w - [(\alpha-\mathrm{Id}_{\mathcal{B}})\otimes\mathrm{Id}_n]\,h^{[n]}_{\mu}(w)$.

Of importance in our analysis will be the following result of Popa and Vinnikov [21, Theorem 6.6], re-phrased in terms of the noncommutative function h:

Theorem 2.2. *Let $\mu\in\Sigma_0(\mathcal{B})$ be given. Then there exists a linear map $\eta_\mu\colon\mathcal{B}\langle\mathcal{X}\rangle\to\mathcal{B}$ and $M\in(0,+\infty)$ such that for any $k\in\mathbb{N}$, $x_1,\dots,x_k\in\mathcal{B}\langle\mathcal{X}\rangle$, we have*

$$\left(\eta_\mu\left[x_j^* x_i\right]\right)_{i,j=1}^{k}\geq 0 \quad \textit{in } M_k(\mathcal{B}),$$

$$\|\eta_\mu\left[\mathcal{X}b_1\mathcal{X}b_2\cdots\mathcal{X}b_k\mathcal{X}\right]\| < M^{n+1}\|b_1\|\|b_2\|\cdots\|b_n\| \quad \textit{for all } b_1,\dots,b_n\in\mathcal{B},$$

and

$$h^{[n]}_{\mu}(b) = (\eta_\mu\otimes\mathrm{Id}_n)\left[(\mathcal{X}\otimes 1_n - b)^{-1}\right] - (\mu\otimes\mathrm{Id}_n)(\mathcal{X}\otimes 1_n), \quad b\in\mathbb{H}^+(M_n(\mathcal{B})), n\in\mathbb{N}.$$

In [21] it is shown that under the assumption that $\mu\in\Sigma_0(\mathcal{B})$, $\mathcal{B}\langle\mathcal{X}\rangle$ has a natural C*-algebra completion, and then the first statement of the theorem about the norm-bounded η_μ becomes equivalent to its complete positivity. This will be important in our proofs. We finally write (8) as

$$\begin{aligned}\omega^{[n]}_{\alpha}(b) &= b - [(\alpha-\mathrm{Id}_{\mathcal{B}})\otimes\mathrm{Id}_n]\,(\mu\otimes\mathrm{Id}_n)(\mathcal{X}\otimes 1_n)\\ &\quad + [(\alpha-\mathrm{Id}_{\mathcal{B}})\otimes\mathrm{Id}_n]\,(\eta_\mu\otimes\mathrm{Id}_n)\left[\left(\mathcal{X}\otimes 1_n - \omega^{[n]}_{\alpha}(b)\right)^{-1}\right],\end{aligned} \tag{9}$$

for $b\in\mathbb{H}^+(M_n(\mathcal{B}))$, $n\in\mathbb{N}$. This equation determines $(\omega^{[n]}_{\alpha})_{n\in\mathbb{N}}$ and thus, via the relation $G^{[n]}_{\mu}\circ\omega^{[n]}_{\alpha} = G^{[n]}_{\mu^{\boxplus\alpha}}$,equivalent to (7), determines $\mu^{\boxplus\alpha}$ in terms of μ and α.

We conclude this section with a simple remark in light of [3, 27, 25]: assume that in equation (9) above, $\mu(\mathcal{X}) = 0$ and $\nu := \eta_\mu$ is a conditional expectation.

Denote $\beta := \alpha - \mathrm{Id}_{\mathcal{B}}$, and assume that β is still completely positive. Then the above equation becomes

$$\omega_\beta^{[n]}(b) = b + (\beta \otimes \mathrm{Id}_n)(\nu \otimes \mathrm{Id}_n)\left[(\mathcal{X} \otimes 1_n - \omega_\beta^{[n]}(b))^{-1}\right], \quad b \in \mathbb{H}^+(M_n(\mathcal{B})), n \in \mathbb{N}.$$

This is precisely the subordination equation generalizing the results of [12, Lemma 4] to the operator-valued context: if γ_β is the centered operator-valued semicircular distribution of variance β, then

$$G_{\nu \boxplus \gamma_\beta}^{[n]} = G_\nu^{[n]} \circ \omega_\beta^{[n]}, \quad n \in \mathbb{N}. \tag{10}$$

There are deeper reasons for the similarity between the above formula and (9), reasons evidentiated in the case $\mathcal{B} = \mathbb{C}$ in [8, 9] and which are explored in [3] for arbitrary $\mathcal{B}$.

For the purposes of our present study, we specify the object of interest: the solution in $\mathbb{H}^+(\mathcal{B})$ of the functional equation

$$\omega(b) = b + \mathbf{a} + \eta\left[(\mathcal{X} - \omega(b))^{-1}\right], \quad b \in \mathbb{H}^+(\mathcal{B}), \tag{11}$$

and its noncommutative extension to the noncommutative operator upper half-plane, where $\mathcal{B}$ is an arbitrary unital C*-algebra, $\mathbf{a} = \mathbf{a}^* \in \mathcal{B}$, $\mathcal{B}\langle\mathcal{X}\rangle$ has a C*-algebra completion in which $\mathcal{X} = \mathcal{X}^*$, and $\eta \colon \mathcal{B}\langle\mathcal{X}\rangle \to \mathcal{B}$ is bounded, completely positive. The function ω is necessarily the right inverse of

$$H(w) = w - \mathbf{a} - \eta\left[(\mathcal{X} - w)^{-1}\right], \quad \in \mathbb{H}^+(\mathcal{B}). \tag{12}$$

These facts were proved in [3] and from now on we will take them for granted.

3. $(\Re\omega(\cdot + iq), \Im\omega(\cdot + iq))$ is the graph of a function

Let γ_t be the semicircular (Wigner) law of variance $t \in (0, +\infty)$ and let μ be an arbitrary Borel probability measure on $\mathbb{R}$. In [12, Lemma 2] it is shown that the imaginary part of the Cauchy-Stieltjes transform of $\mu \boxplus \gamma_t$ is, up to a factor of $-\pi^{-1}$, equal to the function $v_t(u)$ given as

$$v_t(u) := \inf\left\{v \geq 0 | \int_{\mathbb{R}} \frac{t d\mu(x)}{(u-x)^2 + v^2} \leq 1\right\},$$

and moreover, that this infimum is reached (i.e. $t\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_t(u)^2} = 1$) whenever $v_t(u) > 0$. Our next proposition establishes a slightly weaker (and necessarily so) operator-valued counterpart of this result. We denote by $\mathcal{B}^{sa}$ the set of all selfadjoint elements of the C*-algebra $\mathcal{B}$, by $\mathcal{B}^+$ its subset of nonnegative elements, and by $\mathcal{B}^{++}$ the (open) subset of $\mathcal{B}^{sa}$ of strictly positive (i.e. nonnegative and invertible in $\mathcal{B}$) elements.

Proposition 3.1. *Let $\mathcal{B}$ be a C*-algebra, η be a completely positive map on the C*-completion of $\mathcal{B}\langle\mathcal{X}\rangle$ and $\mathbf{a}$ be a selfadjoint element of $\mathcal{B}$. For any fixed $q \in \mathcal{B}$, $q > 0$, there exists a function $v_q \colon \mathcal{B}^{sa} \to \mathcal{B}^{++}$ such that*

$$v_q(u) = q + \eta\left[((\mathcal{X} - u)v_q(u)^{-1}(\mathcal{X} - u) + v_q(u))^{-1}\right],$$

for all $u \in \mathcal{B}^{sa}$. Moreover, the correspondence $u \mapsto v_q(u)$ is uniformly bounded (with a bound depending on q, η) and the restriction to $\mathcal{B}^{sa}$ of an analytic map.

Proof. It is useful to clarify first the relation between our proposition and Equation (11): taking imaginary part in this equation and recalling that (i) $\mathcal{B}\langle\mathcal{X}\rangle$ has a C*-algebra structure, (ii) $\mathcal{X}=\mathcal{X}^*$, and (iii) η is positive, provides us with

$$\Im\omega(b)=\Im b+\eta\left[\left((\mathcal{X}-\Re\omega(b))(\Im\omega(b))^{-1}(\mathcal{X}-\Re\omega(b))+\Im\omega(b)\right)^{-1}\right].$$

We fix $\Im b=q>0$: then our proposition states that the imaginary part of $\omega(b)$ is a continuous function of the real part of $\omega(b)$. Here, of course, $\Re\omega(b)$ is viewed as an independent variable.

Thus, let us fix $q>0$. Define

$$g_q\colon\mathcal{B}^{sa}\times\mathcal{B}^{++}\to\mathcal{B}^{++},\quad g_q(u,v)=q+\eta\left[((\mathcal{X}-u)v^{-1}(\mathcal{X}-u)+v)^{-1}\right].$$

For any $\epsilon=\epsilon^*\in\mathcal{B}$ and $v>0$, the relation $(v+i\epsilon)^{-1}=(v+\epsilon v^{-1}\epsilon)^{-1}-i(v+\epsilon v^{-1}\epsilon)^{-1}\epsilon v^{-1}$ implies that

$$\begin{aligned}&(\mathcal{X}-u)(v+i\epsilon)^{-1}(\mathcal{X}-u)+v+i\epsilon=\\&\quad(\mathcal{X}-u)(v+\epsilon v^{-1}\epsilon)^{-1}(\mathcal{X}-u)+v+i\left(\epsilon-(\mathcal{X}-u)(v+\epsilon v^{-1}\epsilon)^{-1}\epsilon v^{-1}(\mathcal{X}-u)\right)\end{aligned}$$

which guarantees that the real part (in the C*-algebra completion of $\mathcal{B}\langle\mathcal{X}\rangle$) of $(\mathcal{X}-u)(v+i\epsilon)^{-1}(\mathcal{X}-u)+v+i\epsilon$ is greater than v. This makes it invertible for *any* $\epsilon=\epsilon^*\in\mathcal{B}$, allowing the extension of g_q to $\mathcal{B}^{sa}\times(-i)\mathbb{H}^+(\mathcal{B})$, and, moreover, guarantees that $\Re g_q(u,v+i\epsilon)\geq q$ for any $(u,v+i\epsilon)\in\mathcal{B}^{sa}\times(-i)\mathbb{H}^+(\mathcal{B})$. We have thus re-written $g_q(u,\cdot)$ as a self-map of the noncommutative operator right half-plane. Observe that $v>q/2$ implies $\|v^{-1}\|<2\|q^{-1}\|$. Since for any selfadjoint V the relation $\|(v+iV)^{-1}\|\leq\|v^{-1}\|$ holds, the above relation implies

$$\left\|\eta\left[\left((\mathcal{X}-u)(v+i\epsilon)^{-1}(\mathcal{X}-u)+v+i\epsilon\right)^{-1}\right]\right\|\leq\|\eta\|\|v^{-1}\|<2\|\eta\|\|q^{-1}\|.$$

Precisely the same argument as the one from the proof of [3, Theorem 8.4] shows that $g_q(u,\cdot)$ maps a bounded subdomain $\mathcal{D}$ of $\{w\in(-i)\mathbb{H}^+(\mathcal{B})\colon\Re w\geq q/2\}$, depending on u and q, strictly inside itself. The Earle-Hamilton theorem [15, Section 11.1] guarantees that $g_q(u,\cdot)$ has precisely one attracting fixed point in $(-i)\mathbb{H}^+(\mathcal{B})+q$ for any $u\in\mathcal{B}^{sa}$, point which we call $v_q(u)$. Moreover, the function $w\mapsto g_q(u,w)$ is shown in the same reference to be a *strict* contraction in the Kobayashi metric, with the contraction coefficient depending continuously on the distance from $g_q(u,\mathcal{D})$ to the complement of $\mathcal{D}$. Thus, the dependence of the fixed point on u,q is necessarily sequentially continuous (recall that the dependence $u\mapsto g_q(u,v)$ is smooth - in fact analytic). Since on any (norm)-bounded subset of $\mathbb{H}^+(\mathcal{B})$ which is at a strictly positive (norm)-distance from $\mathcal{B}\setminus\mathbb{H}^+(\mathcal{B})$, the topology generated by the Kobayashi metric coincides with the norm topology, this makes the correspondence $u\mapsto v_q(u)$ norm-continuous. As $g_q(u,v)>0$ for all $u=u^*,v>0$, the uniqueness of the attracting fixed point of $g_q(u,\cdot)$ implies that it necessarily belongs to $\mathcal{B}^{++}$.

In order to conclude, we must show that the correspondence $u\mapsto v(u)$ extends analytically to a neighbourhood of $\mathcal{B}^{sa}$. Direct computations show that $g_{q\otimes 1_2}^{[2]}$ can be extended to the set $\mathcal{D}_2$ of elements

$$\left\{\left(\begin{pmatrix}u_1&c\\0&u_2\end{pmatrix},\begin{pmatrix}w_1&d\\0&w_2\end{pmatrix}\right)\colon u_1,u_2\in\mathcal{B}^{sa},w_1,w_2\in(-i)\mathbb{H}^+(\mathcal{B}),c,d\in\mathcal{B}\right\}:$$

the expressions of the $(1,1)$ and $(2,2)$ entries are $g_q(u_1,w_1)$ and $g_q(u_2,w_2)$, respectively, the $(2,1)$ entry is zero, and the $(1,2)$ entry is

$$\eta\Big[((\mathcal{X}-u_1)w_1^{-1}(\mathcal{X}-u_1)+w_1)^{-1}\ \big[(\mathcal{X}-u_1)w_1^{-1}c+cw_2^{-1}(\mathcal{X}-u_2)$$
$$-d+\ (\mathcal{X}-u_1)w_1^{-1}dw_2^{-1}(\mathcal{X}-u_2)\big]\,((\mathcal{X}-u_2)w_2^{-1}(\mathcal{X}-u_2)+w_2)^{-1}\Big]. \tag{13}$$

This makes $g^{[2]}_{q\otimes 1_2}$ into a self-map of $\mathcal{D}_2$. For u_1,u_2,c fixed, $\begin{pmatrix} w_1 & d \\ 0 & w_2\end{pmatrix}\mapsto g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u_1 & c \\ 0 & u_2\end{pmatrix},\begin{pmatrix} w_1 & d \\ 0 & w_2\end{pmatrix}\right)$ maps the set

$$\mathcal{D}_1=\left\{\begin{pmatrix} w_1 & d \\ 0 & w_2\end{pmatrix}: w_1,w_2\in(-i)\mathbb{H}^+(\mathcal{B}), d\in\mathcal{B}\right\}$$

into itself. We have noted that for fixed u_1,u_2,c, the relations $\Re w_1,\Re w_2>q/2$ imply uniform norm boundedness for the factors $((\mathcal{X}-u_j)w_j^{-1}(\mathcal{X}-u_j)+w_j)^{-1}, j\in\{1,2\}$ in the C*-algebra completion of $\mathcal{B}\langle\mathcal{X}\rangle$, as well as of $(\mathcal{X}-u_j)w_j^{-1}$ etc. However, this bound might be quite large, making the estimates on (13) uniform, but useless in terms of mapping a bounded subset of $\mathcal{D}_1$ into itself, thus precluding another direct application of the Earle-Hamilton Theorem. We shall go around this inconvenient fact.

It is clear that if $\begin{pmatrix} w_1 & d \\ 0 & w_2\end{pmatrix}\mapsto g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u_1 & c \\ 0 & u_2\end{pmatrix},\begin{pmatrix} w_1 & d \\ 0 & w_2\end{pmatrix}\right)$ has a fixed point in $\mathcal{D}_1$, then the $(1,1)$ and $(2,2)$ entries of this fixed point must be $v_q(u_1)$ and $v_q(u_2)$, respectively. This puts a very strong restriction on the $(1,2)$ entry of the fixed point: it must be of the form

$$\begin{aligned}\eta\Big[&((\mathcal{X}-u_1)v_q(u_1)^{-1}(\mathcal{X}-u_1)+v_q(u_1))^{-1}\\ &\times\big[(\mathcal{X}-u_1)v_q(u_1)^{-1}c+cv_q(u_2)^{-1}(\mathcal{X}-u_2)\\ &-d+(\mathcal{X}-u_1)v_q(u_1)^{-1}dv_q(u_2)^{-1}(\mathcal{X}-u_2)\big]\\ &\times((\mathcal{X}-u_2)v_q(u_2)^{-1}(\mathcal{X}-u_2)+v_q(u_2))^{-1}\Big],\end{aligned}$$

for some $d\in\mathcal{B}$. This fixed point, if existing, must depend linearly on c. Thus, we are allowed to re-scale c as small (in norm) as we desire. However, we are still inconvenienced by the (implicit) requirement that the norm of the remaining part of the expression above (the terms not containing c) is *strictly* less than $\|d\|$. In order to address this problem, we need to enlarge the domain of definition of $g^{[2]}_{q\otimes 1_2}$. For u_1,u_2,c fixed as above, with the possible proviso that c might be re-scaled (see equation (14) below), and $\varepsilon>0$, we consider the set

$$\tilde{\mathcal{D}}_1^{\varepsilon}=\left\{\begin{pmatrix} w_1 & d \\ m & w_2\end{pmatrix}\in M_2(\mathcal{B})\colon \Re\begin{pmatrix} w_1 & d \\ m & w_2\end{pmatrix}>\varepsilon 1\otimes 1_2\right\}.$$

The defining inequality of $\tilde{\mathcal{D}}_1^{\varepsilon}$ requires $\Re w_j>\varepsilon 1$ and $\frac{d^*+m}{2}(\Re w_1-\varepsilon 1)^{-1}\frac{d+m^*}{2}<(\Re w_2-\varepsilon 1)$. In order to study $g^{[2]}_{q\otimes 1_2}$, we consider the expression

$$\left[\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2\end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2\end{pmatrix}^{-1}\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2\end{pmatrix}+\begin{pmatrix} w_1 & d \\ m & w_2\end{pmatrix}\right]^{-1}$$

which appears under $\eta\otimes\mathrm{Id}_2$ in the formula of $g^{[2]}_{q\otimes 1_2}$. Under the assumption that the argument belongs to $\tilde{\mathcal{D}}^{\varepsilon}_1$, we determine under what conditions the element under the inverse has positive real part, and hence the whole expression above has positive real part (recall that $(-i)\mathbb{H}^+(\mathcal{B})$ is invariant under taking inverse). We write

$$\begin{aligned}
&\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\\
&= \begin{pmatrix} \mathcal{X}-u_1 & 0 \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{X}-u_1 & 0 \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\\
&\quad -\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{X}-u_1 & 0 \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\\
&\quad -\begin{pmatrix} \mathcal{X}-u_1 & 0 \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\\
&\quad +\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

for $\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix} \in \tilde{\mathcal{D}}^{\varepsilon}_1$. It is clear that the first term on the right hand side above has real part greater than or equal to zero. Since the real part of $\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}$ is greater than ε times the unit of $M_2(\mathcal{B})$, it follows that the norm of its inverse is no greater than ε^{-1}. Thus, for all $c\in\mathcal{B}$ with

$$\|c\| < \min\left\{\frac{1}{2}, \frac{\varepsilon^2}{4+16\|\mathcal{X}\|+8(\|u_1\|+\|u_2\|)}\right\}, \tag{14}$$

the norm of the sum of the real parts of the seconf and third terms is strictly less than $\varepsilon/2$. We conclude that

$$\Re\left[\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\right] > -\frac{\varepsilon}{2}1\otimes 1_2.$$

This guarantees that the real part of

$$\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2 \end{pmatrix}\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{X}-u_1 & -c \\ 0 & \mathcal{X}-u_2 \end{pmatrix}+\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}$$

is strictly greater than $\frac{\varepsilon}{2}1\otimes 1_2$. If we choose $\varepsilon\in(0,1)$ sufficiently small so that $q>2\varepsilon 1$ in $\mathcal{B}\langle\mathcal{X}\rangle$, then $\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}\mapsto g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix},\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}\right)$ maps $\tilde{\mathcal{D}}^{\varepsilon}_1$ in a bounded subset of itself which is at strictly positive distance from the complement of $\tilde{\mathcal{D}}^{\varepsilon}_1$, as shown above. The Earle-Hamilton Theorem [15, Section 11.1] applies to provide a unique attracting fixed point in $\tilde{\mathcal{D}}^{\varepsilon}_1$ for this correspondence. As noted above, upper diagonal matrices are mapped to upper diagonal matrices. Thus, any iteration of $g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix}, \quad\cdot\quad\right)$ that starts at an upper diagonal matrix cannot converge to a matrix that is not upper diagonal. Thus, we obtain a $d=d(u_1,u_2,c)$ such that $\begin{pmatrix} v_q(u_1) & d \\ 0 & v_q(u_2) \end{pmatrix}$ is the attracting fixed point of the correspondence given just above. As argued above, the dependence of the fixed point on the initial data (u_1, u_2, c) is norm-continuous. With the notation $d(u_1, u_2, c) = \Delta v_q(u_1, u_2)(c)$, justified by [18, Section 2], we obtain

$$\begin{aligned}
&\eta\Big[((\mathcal{X} - u_1)v_q(u_1)^{-1}(\mathcal{X} - u_1) + v_q(u_1))^{-1} \\
&\quad \times \big[(\mathcal{X} - u_1)v_q(u_1)^{-1}c + cv_q(u_2)^{-1}(\mathcal{X} - u_2) \\
&\quad - \Delta v_q(u_1, u_2)(c) + (\mathcal{X} - u_1)v_q(u_1)^{-1}\Delta v_q(u_1, u_2)(c)v_q(u_2)^{-1}(\mathcal{X} - u_2)\big] \\
&\quad \times ((\mathcal{X} - u_2)v_q(u_2)^{-1}(\mathcal{X} - u_2) + v_q(u_2))^{-1}\Big] = \Delta v_q(u_1, u_2)(c), \qquad (15)
\end{aligned}$$

for all $c \in \mathcal{B}$ of sufficiently small norm (estimated in (14)), and, by linearity, for all $c \in \mathcal{B}$. Moreover, this same norm estimate (14) is seen to be uniform for u_1, u_2 uniformly bounded. We conclude that the correspondence $(u_1, u_2, c) \mapsto \Delta v_q(u_1, u_2)(c)$ is not only continuous, but also locally uniformly bounded when the norm topologies are considered on $\mathcal{B}^{sa} \times \mathcal{B}^{sa} \times \mathcal{B}$ and $\mathcal{B}$. As shown in the same [18, Section 2], $\Delta v_q(u, u)(c) = \partial_u v_q(c)$. We conclude that the correspondence $u \mapsto v_q(u)$ is in fact C^1 in the Fréchet sense on $\mathcal{B}$.

We use next the property of $\Delta v_q(u_1, u_2)(c)$ to be an attracting fixed point for the map on the left hand side of (15). More precisely, we write the left hand side of (15) as the sum of two linear maps (one of them, (16), is applied in (15) to c, the other, (17), to $\Delta v_q(u_1, u_2)(c)$):

$$\begin{aligned}
&\Delta_1 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(c) = \\
&\quad \eta\big[((\mathcal{X} - u_1)v_q(u_1)^{-1}(\mathcal{X} - u_1) + v_q(u_1))^{-1} \\
&\quad \times \big[(\mathcal{X} - u_1)v_q(u_1)^{-1}c + cv_q(u_2)^{-1}(\mathcal{X} - u_2)\big] \\
&\quad \times ((\mathcal{X} - u_2)v_q(u_2)^{-1}(\mathcal{X} - u_2) + v_q(u_2))^{-1}\big],
\end{aligned} \qquad (16)$$

and

$$\begin{aligned}
&\Delta_2 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(d) = \\
&\quad \eta\big[((\mathcal{X} - u_1)v_q(u_1)^{-1}(\mathcal{X} - u_1) + v_q(u_1))^{-1} \\
&\quad \big[(\mathcal{X} - u_1)v_q(u_1)^{-1}dv_q(u_2)^{-1}(\mathcal{X} - u_2) - d\big] \\
&\quad \times ((\mathcal{X} - u_2)v_q(u_2)^{-1}(\mathcal{X} - u_2) + v_q(u_2))^{-1}\big].
\end{aligned} \qquad (17)$$

The correspondence that we iterate is

$$\begin{pmatrix} v_q(u_1) & d \\ 0 & v_q(u_2) \end{pmatrix} \mapsto g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix}, \begin{pmatrix} v_q(u_1) & d \\ 0 & v_q(u_2) \end{pmatrix}\right). \qquad (18)$$

The right-hand side of this correspondence is

$$\begin{pmatrix} v_q(u_1) & \Delta_1 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(c) + \Delta_2 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(d) \\ 0 & v_q(u_2) \end{pmatrix}$$

(recall that $g_q(u_j, v_q(u_j)) = v_q(u_j)$ by the first part of the proof of Proposition 3.1). In order to save space, we denote just here $S(\cdot) = \Delta_1 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(\cdot)$, $T(\cdot) = \Delta_2 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(\cdot)$. This expression makes clear that the n^{th} iteration of $g^{[2]}$ provides to its $(1, 2)$ entry $T^n(d) + \sum_{j=0}^{n-1} T^j(S(c))$. We conclude from the existence of the limit as $n \to \infty$ of this expression for any c, d in a ball of small enough diameter (see Eq. (14)) that $\left\|T^n(d) + \sum_{j=0}^{n-1} T^j(S(c)) - \Delta v_q(u_1, u_2)(c)\right\| \to 0$ as $n \to \infty$. In particular, $\|T^n(d)\| \to 0$ as $n \to \infty$.

Now we use again the essential property of the map (18) to be a *strict* contraction in the Kobayashi metric, with the contraction coefficient uniformly bounded away from one when u_1, u_2, c vary very little in norm. This makes the above convergence to zero uniform in c and d for c, d in small enough norm-balls around zero in $\mathcal{B}$. In particular, $\|T^n\| \to 0$ as $n \to \infty$, imposing $\sigma(T) \subset \mathbb{D}$. The claimed result follows by a direct application of the analytic implicit function theorem. □

Remark 3.2.

(1) It should be noted that in the proof of the above proposition, the noncommutative structure of the functions involved has not come up in the proof of the existence and the norm-continuity of $u \mapsto v(u)$. In particular, for the proof of the existence, boundedness and norm-continuity of $u \mapsto v(u)$, the requirement of complete positivity of the linear map η can be relaxed to simple positivity. Moreover, the proof of the analyticity of this correspondence involves only the 2-positivity of η. However, the hypothesis of *complete* positivity is necessary, and sufficient, in order for the conclusion of Proposition 3.1 to hold at all levels $n \in \mathbb{N}$, and thus, in the light of the motivation for our investigation, it makes sense to keep it.

(2) The existence of $\omega(r+iq)$ for any $r = r^* \in \mathcal{B}$, proved in [3, Theorem 8.4], as an attracting fixed point of $f_{r+iq}(w) = r+iq+\mathbf{a}+\eta\left[(\mathcal{X}-w)^{-1}\right]$ guarantees that there are pairs of points $(\Re\omega(r+iq), \Im\omega(r+iq)) \in \mathcal{B}^{sa} \times \mathcal{B}^{++}$ such that $g_q(\Re\omega(r+iq), \Im\omega(r+iq)) = \Im\omega(r+iq)$. The uniqueness of the fixed point of $g_q(u,\cdot)$ guarantees that $v_q(\Re\omega(r+iq)) = \Im\omega(r+iq)$ whenever u is of the form $\Re\omega(r+iq)$; in particular, the set $\{(\Re\omega(r+iq), \Im\omega(r+iq)) : r \in \mathcal{B}^{sa}\}$ is the graph of a function defined on $\mathcal{B}^{sa}$ with values in $\mathcal{B}^{++}$.

(3) It is remarkable in this context that g is the first level of a noncommutative map having the properties described in Proposition 3.1 at each level n (in our proof, only levels $n = 1$ and $n = 2$ appear). Indeed, the noncommutative extension of g is written as $g^{[n]}_{q\otimes 1_n}(u,v) = q \otimes 1_n + (\eta \otimes \mathrm{Id}_n)\left[((\mathcal{X} \otimes 1_n - u)v^{-1}(\mathcal{X} \otimes 1_n - u) + v)^{-1}\right]$, for $u \in M_n(\mathcal{B})^{sa}$, $v \in (-i)\mathbb{H}^+(M_n(\mathcal{B}))$, $n \in \mathbb{N}$ ($M_n(\mathcal{B})^{sa}, n \in \mathbb{N}$, is a noncommutative set, but not an admissible one). This necessarily implies that the fixed point is itself noncommutative: if $u_n = u \otimes 1_n$, then $v^{[n]}_{q\otimes 1_n}(u \otimes 1_n) = v_q(u) \otimes 1_n$ (see [1] for this fact, and for more properties of noncommutative fixed points).

(4) When $\mathcal{B}$ is finite dimensional (a C*-algebra of complex matrices) - the most important case in the study of distributions of polynomials and rational functions in free random variables - it is much easier to show that $u \mapsto v_q(u)$ is analytic in the sense of Proposition 3.1. Indeed, the fact that $v_q(u)$ is an attracting fixed point for $v \mapsto g_q(u,v)$ which is in the interior of the domain of $g_q(u,\cdot)$ implies that all eigenvalues of $\partial_v g_q(u, v_q(u))$ are of absolute value strictly less than one. In order to see this, we write

$$g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} v & c \\ 0 & v \end{pmatrix}\right) = \begin{pmatrix} g_q(u,v) & \partial_v g_q(u,v)(c) \\ 0 & g_q(u,v) \end{pmatrix},$$

and observe that for $c \in \mathcal{B}$ satisfying $(v^{-1/2}cv^{-1/2})(v^{-1/2}cv^{-1/2})^* < 4$ (hence for any $c \in \mathcal{B}$ of sufficiently small norm), the real part of $\begin{pmatrix} v & c \\ 0 & v \end{pmatrix}$

is strictly positive in $M_2(\mathcal{B})$. Iterating the map $\begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix} \mapsto g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix}\right)$ provides convergence in the norm of $M_2(\mathcal{B})$ to the fixed point $\begin{pmatrix} v_q(u) & 0 \\ 0 & v_q(u) \end{pmatrix}$ as $n \to \infty$. Direct computation yields the formula $\begin{pmatrix} v_q(u) & \partial_v g_q(u, v_q(u))^{\circ n}(c) \\ 0 & v_q(u) \end{pmatrix}$ for the n^{th} iterate of the map $\begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix} \mapsto g^{[2]}_{q\otimes 1_2}\left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix}\right)$, so we must have $\lim_{n\to\infty} \partial_v g_q(u, v_q(u))^{\circ n}(c) = 0$ for all $c \in \mathcal{B}$. This requires the spectrum $\sigma(\partial_v g_q(u, v_q(u))) \subset \mathbb{D}$ (we have denoted by $\mathbb{D}$ the open unit disc in the complex plane). An application of the implicit function theorem for analytic functions provides the desired result, with a formula for the derivative of v_q given by

$$\partial_u v_q(u) = [\mathrm{Id}_{\mathcal{B}} - \partial_v g_q(u, v_q(u))]^{-1} \circ \partial_u g_q(u, v_q(u)). \tag{19}$$

Note that this argument also required 2-positivity for η. As a side benefit, note that $\sigma\left([\mathrm{Id}_{\mathcal{B}} - \partial_v g_q(u, v_q(u))]^{-1}\right) \subseteq \frac{1}{2} - i\mathbb{C}^+$.

A further, rather straightforward, corollary of the above proposition and remarks is recorded here.

Corollary 3.3. *For any $q > 0$ in $\mathcal{B}$, the map $\mathcal{B}^{sa} \ni u \mapsto \Re\omega(u + iq) \in \mathcal{B}^{sa}$ is bijective. For any $u = u^*$, the map $\mathcal{B}^{++} \ni q \mapsto v_q(u) \in \mathcal{B}^{++}$ is injective.*

Proof. If $w = \Re\omega(u + iq) \in \Re\omega(\mathcal{B}^{sa} + iq)$, then the relation

$$\begin{aligned} u = {} & \Re\omega(u + iq) - \mathbf{a} - \\ & \eta\Big[v_q(\Re\omega(u + iq))^{-1}(\mathcal{X} - \Re\omega(u + iq)) \times \\ & \big((\mathcal{X} - \Re\omega(u + iq))v_q(\Re\omega(u + iq))^{-1}(\mathcal{X} - \Re\omega(u + iq)) + v_q(\Re\omega(u + iq))\big)^{-1}\Big] \end{aligned}$$

indicates that $u \mapsto \Re\omega(u + iq)$ is the right inverse of the map

$$w \mapsto \Phi_q(w) = w - \mathbf{a} - \eta\Big[v_q(w)^{-1}(\mathcal{X} - w)\big((\mathcal{X} - w)v_q(w)^{-1}(\mathcal{X} - w) + v_q(w)\big)^{-1}\Big].$$

This shows that $u \mapsto \Re\omega(u + iq)$ is injective. It follows from Proposition 3.1 that Φ_q has an analytic extension to a small enough norm-neighbourhood (depending on $q > 0$) of $\mathcal{B}^{sa}$. For elements w whose inverses w^{-1} are of small norm, it follows easily from the formula of Φ_q and the fixed-point equation satisfied by v_q (see Proposition 3.1) that $\|\Phi_q'(w) - \mathrm{Id}\|$ is small. Thus, by Equation (11), the classical inverse function theorem for Banach spaces applied to Φ_q in a point of the form $w_M = \Re\omega(M1 + iq)$ for $M \in (0, +\infty)$ sufficiently large provides a local inverse for Φ_q on a neighbourhood (in $\mathcal{B}$) of w_M and guarantees that $u \mapsto \Re\omega(u + iq)$ maps a neighbourhood of $1M$ onto a neighburhood of $w_M = \Re\omega(M1 + iq)$. Now

$$\begin{aligned} u = \Phi_q(\Re\omega(u + iq)) \quad & \Longrightarrow \quad \Re\omega(u + iq) = \Re\omega(\Phi_q(\Re\omega(u + iq)) + iq) \\ & \Longrightarrow \quad w = \Re\omega(\Phi_q(w) + iq), \end{aligned}$$

for all w in an open set containing w_M. On the other hand, formula (12) indicates that, as functions of u, both $\Re\omega(u + iq)$ and $\Im\omega(u + iq)$ have analytic extensions to small enough (depending on $q > 0$) norm-neighbourhoods in $\mathcal{B}$ of $\mathcal{B}^{sa}$. Thus, the fact that $\mathcal{B}^{sa}$ is a set of uniqueness for analytic maps in $\mathcal{B}$ implies that the relation $w = \Re\omega(\Phi_q(w) + iq)$ holds for all $w = w^* \in \mathcal{B}^{sa}$, so that $u \mapsto \omega(u + iq)$ is also surjective.

The second statement of the corollary is trivial. □

4. The derivative of ω

4.1. **Spectrum of the derivative.** For the case of $\mathcal{B} = \mathbb{C}$, it is shown in [7, Theorem 4.6] that the difference quotient of ω satisfies the inequality

$$\left|\frac{\omega(z_1) - \omega(z_2)}{z_1 - z_2}\right| \geq \frac{1}{2}, \quad z_1, z_2 \in \mathbb{C}^+ \cup \mathbb{R}.$$

This is shown by proving that $\Re\omega'(\alpha) > 1/2$ for all $\alpha \in \mathbb{C}^+$. The operator-valued counterpart of this statement has the following form:

Proposition 4.1. *Let $\mathcal{B}$ be a unital C*-algebra and let H and ω be defined as in* (12) *and* (11). *For any $b_1, b_2 \in \mathbb{H}^+(\mathcal{B})$, the spectrum of $\Delta\omega(b_1, b_2)$ as a linear operator from $\mathcal{B}$ to itself is included in $\{z \in \mathbb{C} \colon \Re z > 1/2\}$. In particular, $\Delta\omega(b_1, b_2) \colon \mathcal{B} \to \mathcal{B}$ is invertible for any $b_1, b_2 \in \mathbb{H}^+(\mathcal{B})$.*

Proof. The proof is very similar in spirit to the proof of [7, Theorem 4.6]. Consider $b_1, b_2 \in \mathbb{H}^+(\mathcal{B})$ and $c \in \mathcal{B}$ of sufficiently small norm so that $\begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} \in \mathbb{H}^+(M_2(\mathcal{B}))$. We evaluate ω on this matrix in order to obtain

$$\begin{aligned}\begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} &= H^{[2]}\left(\omega^{[2]}\begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} H(\omega(b_1)) & \Delta H(\omega(b_1), \omega(b_2))\Delta\omega(b_1, b_2)(c) \\ 0 & H(\omega(b_2)) \end{pmatrix}.\end{aligned}$$

This indicates that $\Delta H(\omega(b_1), \omega(b_2)) \circ \Delta\omega(b_1, b_2) = \mathrm{Id}_{\mathcal{B}}$. As shown in the proof of [3, Theorem 8.4], the point $\omega^{[2]}\begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} \in \mathbb{H}^+(M_2(\mathcal{B}))$ is the unique attracting fixed point of the self-map

$$f^{[2]} \colon w \mapsto \begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} + \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix} + (\eta \otimes \mathrm{Id}_{M_2(\mathbb{C})})\left[\left(\begin{pmatrix} \mathcal{X} & 0 \\ 0 & \mathcal{X} \end{pmatrix} - w\right)^{-1}\right]$$

of $\mathbb{H}^+(M_2(\mathcal{B}))$. The methods used in the proof of Proposition 3.1 apply to show that this map is a strict contraction in the Kobayashi metric. We conclude that the spectrum of $\Delta f(\omega(b_1), \omega(b_2))$ is included in the open unit disc $\mathbb{D}$. However, $\Delta H(\omega(b_1), \omega(b_2)) - \mathrm{Id}_{\mathcal{B}} = \Delta f(\omega(b_1), \omega(b_2))$, which implies by the definition of the spectrum that $\sigma(\Delta H(\omega(b_1), \omega(b_2))) \subset \mathbb{D} + 1$. Analytic functional calculus rules (see, for example, [13, Section II.1.5]) provide

$$\sigma(\Delta\omega(b_1, b_2)) = \sigma\left(\Delta H(\omega(b_1), \omega(b_2))^{-1}\right) \subset \{z \in \mathbb{C} \colon \Re z > 1/2\}.$$

□

As before, the proof of the above proposition only requires η to be 2-positive.

We note a significant element: if we consider a b_0 in the *boundary* of $\mathbb{H}^+(\mathcal{B})$ and we try to apply the implicit function theorem to the function $f(b, w) - w = b + \mathbf{a} + \eta\left[(\mathcal{X} - w)^{-1}\right] - w$ around a point (b_0, w_0), where w_0 is a point in $\mathbb{H}^+(\mathcal{B}) \cap C(\omega, b_0)$

(where $C(\omega, b_0)$ denotes the set of limit points of ω at b_0), it turns out that this is possible whenever $0 \notin \sigma(\Delta H(w_0, w_0))$. If $\mathcal{B}$ is finite dimensional, the set of points w_0 with positive imaginary part that satisfy such a condition is an analytic set. It turns out that this analytic set has several properties of interest, which become quite evident when one considers rather the map $(w_1, w_2) \mapsto \Delta H(w_1, w_2)$, and which will be investigated later.

4.2. **The Julia-Carathéodory derivative.** We discuss the Julia-Carathéodory derivatives at the distinguished boundary for the functions ω and h. We shall use results and methods from [4], especially Propositions 3.1, 3.2 and Theorem 2.2. Assume that the there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}^+$ converging nontangentially to zero so that the norm-limit $\lim_{n\to\infty} \omega(\alpha + z_n v) = \omega(\alpha)$ exists and is selfadjoint. This implies that

$$\lim_{n\to\infty} \omega(\alpha + z_n v) = \alpha + \mathbf{a} + \lim_{n\to\infty} \left(z_n v + \eta\left[(\mathcal{X} - \omega(\alpha + z_n v))^{-1}\right]\right),$$

In particular,

$$\lim_{n\to\infty} \eta\left[(\mathcal{X} - \omega(\alpha + z_n v))^{-1}\right] = \omega(\alpha) - \alpha - \mathbf{a}$$

in norm. If these statements are straightforward, providing more details on the properties of $\omega'(\alpha)$ requires a finer analysis.

Let us recall that for any $b \in \mathbb{H}^+(\mathcal{B})$, $\omega(b)$ is the attracting fixed point of the map $f_b(w) = b + \eta\left[(\mathcal{X} - w)^{-1}\right]$, $w \in \mathbb{H}^+(\mathcal{B})$. Recall from [4, Proposition 3.1] that

$$\left\|(\Im f_b(w_1))^{-\frac{1}{2}}(f_b(w_1) - f_b(w_2))(\Im f_b(w_2))^{-\frac{1}{2}}\right\| \le \left\|(\Im w_1)^{-\frac{1}{2}}(w_1 - w_2)(\Im w_2)^{-\frac{1}{2}}\right\|,$$

for all $w_1, w_2, b \in \mathbb{H}^+(\mathcal{B})$. In particular, if $w_1 = \omega(b) = f_b(\omega(b))$, then

$$\left\|(\Im \omega(b))^{-\frac{1}{2}}(\omega(b) - f_b(w_2))(\Im f_b(w_2))^{-\frac{1}{2}}\right\| \le \left\|(\Im \omega(b))^{-\frac{1}{2}}(\omega(b) - w_2)(\Im w_2)^{-\frac{1}{2}}\right\|, \tag{20}$$

for all $w_2, b \in \mathbb{H}^+(\mathcal{B})$. As in [4, Proposition 3.2], for $r \in (0, +\infty)$ and $c \in \mathbb{H}^+(\mathcal{B})$, define

$$B_n^+(c, r) = \left\{a \in \mathbb{H}^+(M_n(\mathcal{B}))\colon \left\|(\Im a)^{-\frac{1}{2}}(a - c \otimes 1_n)(\Im c \otimes 1_n)^{-\frac{1}{2}}\right\| \le r\right\},$$

and denote

$$\mathring{B}_n^+(c, r) = \left\{a \in \mathbb{H}^+(M_n(\mathcal{B}))\colon \left\|(\Im a)^{-\frac{1}{2}}(a - c \otimes 1_n)(\Im c \otimes 1_n)^{-\frac{1}{2}}\right\| < r\right\}.$$

It follows from (20) that $f_b^{[n]}(B_n^+(\omega(b), r)) \subseteq B_n^+(\omega(b), r)$ and $f_b^{[n]}(\mathring{B}_n^+(\omega(b), r)) \subseteq \mathring{B}_n^+(\omega(b), r)$ for any $r > 0$. Observe that, while by its definition $f_b^{[n]}(\mathring{B}_n^+(\omega(b), r)) \subset \mathbb{H}^+(M_n(\mathcal{B})) + b \otimes 1_n$, the inclusions above do not imply that $\mathring{B}_n^+(\omega(b), r) - b \otimes 1_n \subset \mathbb{H}^+(M_n(\mathcal{B}))$. As noted also in [4], the defining inequality of $B_n^+(c, r)$ can be re-writen as

$$(a - c \otimes 1_n)^*(\Im a)^{-1}(a - c \otimes 1_n) \le r^2 \Im c \otimes 1_n$$

(and with $<$ for $\mathring{B}_n^+(c, r)$). Thus, under the additional assumption of the existence of a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}^+$ converging nontangentially to zero such that $\ell := \lim_{n\to\infty} \frac{\Im \omega(\alpha + z_n v)}{\|\Im \omega(\alpha + z_n v)\|}$ exists in the weak operator topology and is a strictly positive operator, we may take the limit of $\mathring{B}_1^+(\omega(\alpha+z_nv),\|\Im\omega(\alpha+z_nv)\|^{-1/2})$ as $n\to\infty$ in the sense that

$$\begin{aligned} H_1(\omega(\alpha),\ell) &\supseteq \bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} B_1^+(\omega(\alpha+z_kv),\|\Im\omega(\alpha+z_kv)\|^{-1/2}) \\ &\supseteq \bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} \mathring{B}_1^+(\omega(\alpha+z_kv),\|\Im\omega(\alpha+z_kv)\|^{-1/2}) \\ &\supseteq \bigcup_{n\in\mathbb{N}}\bigcap_{k\geq n} \mathring{B}_1^+(\omega(\alpha+z_kv),\|\Im\omega(\alpha+z_kv)\|^{-1/2}), \end{aligned}$$

where

$$H_n(\omega^{[n]}(\alpha\otimes 1_n),\ell\otimes 1_n) =$$
$$= \left\{w\in\mathbb{H}^+(M_n(\mathcal{B}))\colon (w-\omega^{[n]}(\alpha\otimes 1_n))^*(\Im w)^{-1}(w-\omega^{[n]}(\alpha\otimes 1_n))\leq \ell\otimes 1_n\right\} \tag{21}$$

plays the role of a noncommutative horodisc. (Note that under our current hypothesis ℓ might not belong to $\mathcal{B}$, but only to its enveloping W*-algebra.) Indeed, the second and third inclusion are obvious. We prove the first by contradiction. Assume that there is a point $x\in\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n}B_1^+(\omega(\alpha+z_kv),\|\Im\omega(\alpha+z_kv)\|^{-1/2})$ such that $x\notin H_1(\omega(\alpha),\ell)$, i.e. $(\Re x-\omega(\alpha))(\Im x)^{-1}(\Re x-\omega(\alpha))+\Im x\not\leq\ell$. This in turn means that there is a norm-one vector ξ in the Hilbert space on which the W*-envelope of $\mathcal{B}$ acts as a von Neumann algebra such that

$$\left\langle[(\Re x-\omega(\alpha))(\Im x)^{-1}(\Re x-\omega(\alpha))+\Im x]\xi,\xi\right\rangle>\langle\ell\xi,\xi\rangle.$$

By definition, we know that for any $n\in\mathbb{N}$ there exists some $k\geq n$ such that $x\in B_1^+(\omega(\alpha+z_kv),\|\Im\omega(\alpha+z_kv)\|^{-1/2})$, i.e. there exists a subsequence of $\{z_n\}_n$ (call it for simplicity $\{z_{n_k}\}_k$) such that

$$(x-\omega(\alpha+z_{n_k}v))^*(\Im x)^{-1}(x-\omega(\alpha+z_{n_k}v))\leq\frac{\Im\omega(\alpha+z_{n_k}v)\|}{\|\Im\omega(\alpha+z_{n_k}v)\|}.$$

Apply this inequality to ξ to obtain

$$\left\langle[(x-\omega(\alpha+z_{n_k}v))^*(\Im x)^{-1}(x-\omega(\alpha+z_{n_k}v))]\xi,\xi\right\rangle\leq\left\langle\frac{\Im\omega(\alpha+z_{n_k}v)\|}{\|\Im\omega(\alpha+z_{n_k}v)\|}\xi,\xi\right\rangle.$$

The assumption that $\frac{\Im\omega(\alpha+z_nv)}{\|\Im\omega(\alpha+z_nv)\|}$ converges to ℓ in the weak operator topology as $n\to\infty$ allows us to conclude that the right-hand term of the above converges to $\langle\ell\xi,\xi\rangle$ as $k\to\infty$. Recall that, according to our hypothesis, $\lim_{k\to\infty}\|\Im\omega(\alpha+z_{n_k}v)\|=0$ and $\lim_{k\to\infty}\|\Re\omega(\alpha+z_{n_k}v)-\omega(\alpha)\|=0$. For the fixed x, we then have

$$\lim_{k\to\infty}\|(x-\omega(\alpha+z_{n_k}v))^*(\Im x)^{-1}(x-\omega(\alpha+z_{n_k}v))-(x^*-\omega(\alpha))(\Im x)^{-1}(x-\omega(\alpha))\|=0,$$

so that the left-hand side of the above inequality converges to

$$\left\langle[(\Re x-\omega(\alpha))(\Im x)^{-1}(\Re x-\omega(\alpha))+\Im x]\xi,\xi\right\rangle,$$

providing an immediate contradiction. We now have all the necessary tools for proving the first main result of this subsection.

Theorem 4.2. *Assume that $\mathcal{B}$ is a von Neumann algebra, $\mathbf{a}=\mathbf{a}^*\in\mathcal{B}$, $\mathcal{X}=\mathcal{X}^*$ is a selfadjoint random variable in a von Neumann algebra which contains $\mathcal{B}$ as a von Neumann subalgebra, and η is a completely positive map on the* C**-algebra completion of $\mathcal{B}\langle\mathcal{X}\rangle$. Assume also that ω is the noncommutative function provided* *by* (11), $\alpha = \alpha^* \in \mathcal{B}$ *and* $\mathcal{B} \ni v > 0$ *are such that there exists a sequence* $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *converging to zero and*

$$\omega(\alpha) := \lim_{n\to\infty} \omega(\alpha + iy_n v)$$

exists in the norm topology and is a selfadjoint element of $\mathcal{B}$*. If*

$$\ell = \ell(v) := \lim_{n\to\infty} \frac{\Im\omega(\alpha + iy_n v)}{\|\Im\omega(\alpha + iy_n v)\|}$$

exists in the norm topology and is a strictly positive operator, then for any state φ *on* $\mathcal{B}$ *and any* $u > 0$,

$$\liminf_{z\to 0} \frac{\varphi\left(\Im\eta\left[(\mathcal{X} - \omega(\alpha) - zu)^{-1}\right]\right)}{\Im z} < +\infty.$$

The conclusion of the above theorem does require that the convergence to ℓ takes place in the norm topology, a considerably stronger requirement than before, but not surprising in the light of the results of, for example, [16].

Proof. By passing to a subsequence if necessary, we assume without loss of generality that $\{y_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $(0, 1)$. Since $f_b^{[n]}(B_n^+(\omega(b), r)) \subseteq B_n^+(\omega(b), r)$ for any $b \in \mathbb{H}^+(M_n(\mathcal{B}))$ and $r > 0$, it follows that $\eta\left[(\mathcal{X} - \cdot)^{-1}\right] = f_b^{[n]}(\cdot) - b$ maps $B_n^+(\omega(b), r)$ into $B_n^+(\omega(b), r) - b$, the shift of $B_n^+(\omega(b), r)$ by $-b$. In particular, as $w \mapsto \eta\left[(\mathcal{X} - w)^{-1}\right]$ sends the upper half-plane into its closure, $B_n^+(\omega(\alpha + iy_n v), \|\Im\omega(\alpha + iy_n v)\|^{-1/2})$ is sent inside $(B_n^+(\omega(\alpha + iy_n v), \|\Im\omega(\alpha + iy_n v)\|^{-1/2}) - \alpha - iy_n v) \cap \overline{\mathbb{H}^+(M_n(\mathcal{B}))}$. According to [4, Proposition 3.2], the sets $B_n^+(\omega(\alpha + iy_n v), \|\Im\omega(\alpha + iy_n v)\|^{-1/2})$ are bounded by

$$\|\Re\omega(\alpha + iy_n v)\| + \frac{1 + \|\Im\omega(\alpha + iy_n v)\| + \sqrt{1 + 4\|\Im\omega(\alpha + iy_n v)\|}}{2}$$
$$+ \sqrt{\frac{1 + \|\Im\omega(\alpha + iy_n v)\| + \sqrt{1 + 4\|\Im\omega(\alpha + iy_n v)\|}}{2}},$$

quantities that tend to $\|\omega(\alpha)\| + 2$ as $n \to \infty$. This makes the sets $H_n(\omega^{[n]}(\alpha \otimes 1_n), \ell \otimes 1_n)$, $n \in \mathbb{N}$, bounded in norm, uniformly in n (see also [1]).

It is for the next result that we need the strengthening of the convergence to ℓ to convergence in norm compared to the discussion before our present theorem. Under the hypothesis of weak operator convergence we were able to show that $H_1(\omega(\alpha), \ell)$ includes certain limsup and liminf of sets. Now we need to show in addition that

$$\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \mathring{B}_1^+(\omega(\alpha + iy_k v), \|\Im\omega(\alpha + iy_k v)\|^{-1/2}) \supseteq \mathring{H}_1(\omega(\alpha), \ell),$$

where $\mathring{H}_1(\omega(\alpha), \ell) = \{w \in \mathbb{H}^+(\mathcal{B})\colon (w - \omega(\alpha))^*(\Im w)^{-1}(w - \omega(\alpha)) < \ell\}$. This inclusion is equivalent to showing that for any fixed $w \in \mathring{H}_1(\omega(\alpha), \ell)$, there exists an $n_w \in \mathbb{N}$ such that

$$(w - \omega(\alpha + iy_n v))^*(\Im w)^{-1}(w - \omega(\alpha + iy_n v)) < \frac{\Im\omega(\alpha + iy_n v)}{\|\Im\omega(\alpha + iy_n v)\|}$$

for all $n \geq n_w$. This follows quite easily, however. Indeed, for w to satisfy the *strict* inequality $(w - \omega(\alpha))^*(\Im w)^{-1}(w - \omega(\alpha)) < \ell$ it follows that we can find an $\varepsilon \in (0, 1)$ such that $(w - \omega(\alpha))^*(\Im w)^{-1}(w - \omega(\alpha)) + \varepsilon 1 < \ell$. Then we only need to insure that our $n_w \in \mathbb{N}$ is sufficiently large in order to provide $\left\|\frac{\Im\omega(\alpha+iy_nv)}{\|\Im\omega(\alpha+iy_nv)\|}-\ell\right\|<\frac{\varepsilon}{8}$ and $\|\omega(\alpha)-\Re\omega(\alpha+iy_nv)\|+\|\Im\omega(\alpha+iy_nv)\|<\frac{\varepsilon}{8(1+\|w\|\|(\Im w)^{-1}\|)}$ for all $n\geq n_w$. This proves the claimed inclusion.

Let $h(w)=\eta\left[(\mathcal{X}-w)^{-1}\right]$. As seen above, $h(B_1^+(\omega(\alpha+iy_kv),\|\Im\omega(\alpha+iy_kv)\|^{-1/2})\subseteq B_1^+(\omega(\alpha+iy_kv),\|\Im\omega(\alpha+iy_kv)\|^{-1/2})-\alpha-iy_kv$. We have

$$
\begin{aligned}
&h(\mathring{H}_1(\omega(\alpha),\ell))\\
&\quad\subseteq\ h\left(\bigcup_{n\in\mathbb{N}}\bigcap_{k\geq n}\mathring{B}_1^+(\omega(\alpha+iy_kv),\|\Im\omega(\alpha+iy_kv)\|^{-1/2})\right)\\
&\quad\subseteq\ h\left(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n}B_1^+(\omega(\alpha+iy_kv),\|\Im\omega(\alpha+iy_kv)\|^{-1/2})\right)\\
&\quad\subseteq\ \bigcap_{n\in\mathbb{N}}h\left(\bigcup_{k\geq n}B_1^+(\omega(\alpha+iy_kv),\|\Im\omega(\alpha+iy_kv)\|^{-1/2})\right)\\
&\quad=\ \bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n}h\left(B_1^+(\omega(\alpha+iy_kv),\|\Im\omega(\alpha+iy_kv)\|^{-1/2})\right)\\
&\quad\subseteq\ \bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n}\left[B_1^+(\omega(\alpha+iy_kv),\|\Im\omega(\alpha+iy_kv)\|^{-1/2})-\alpha-iy_kv\right]\cap\overline{\mathbb{H}^+(\mathcal{B})}\\
&\quad=\ \bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n}\left[B_1^+(\omega(\alpha+iy_kv)-\alpha,\|\Im\omega(\alpha+iy_kv)\|^{-1/2})-iy_kv\right]\cap\overline{\mathbb{H}^+(\mathcal{B})}\\
&\quad=\ \bigcap_{n>N}\bigcup_{k\geq n}\left[B_1^+(\omega(\alpha+iy_kv)-\alpha,\|\Im\omega(\alpha+iy_kv)\|^{-1/2})-iy_kv\right]\cap\overline{\mathbb{H}^+(\mathcal{B})},
\end{aligned}
$$

for all $N\in\mathbb{N}$. As $\{y_n\}_n$ is strictly decreasing, $-i[0,y_n]v\subsetneq -i[0,y_{n-1}]v$. Observe the inclusion $B_1^+(\omega(\alpha-iy_kv)-\alpha,\|\Im\omega(\alpha+iy_kv)\|^{-1/2})-iy_kv\subset B_1^+(\omega(\alpha+iy_kv)-\alpha,\|\Im\omega(\alpha+iy_kv)\|^{-1/2})-i[0,y_{k-j}]v$ for all $j\in\{1,\dots,k\}$. Since $(\cup_\iota A_\iota)+K=\cup_\iota(A_\iota+K)$ for any subsets $(A_\iota)_\iota,K$ of a topological vector space,

$$
\begin{aligned}
&\bigcup_{k\geq n}\left[B_1^+(\omega(\alpha+iy_kv)-\alpha,\|\Im\omega(\alpha+iy_kv)\|^{-1/2})-i[0,y_{n-1}]v\right]\\
&\qquad=\ -i[0,y_{n-1}]v+\left[\bigcup_{k\geq n}B_1^+(\omega(\alpha+iy_kv)-\alpha,\|\Im\omega(\alpha+iy_kv)\|^{-1/2})\right]\\
&\qquad\subseteq\ -i[0,y_N]v+\left[\bigcup_{k\geq n}B_1^+(\omega(\alpha+iy_kv)-\alpha,\|\Im\omega(\alpha+iy_kv)\|^{-1/2})\right],
\end{aligned}
$$

for all $n>N$. Unfortunately in general $(A\cap V)+K\subsetneq(A+K)\cap(V+K)$. However, if we have a decreasing sequence of norm-closed sets $A_1\supset A_2\supset A_3\supset\cdots$ and a norm-compact set K in some Banach space $\mathcal{Y}$, then $\bigcap_{n\in\mathbb{N}}(A_n+K)=(\bigcap_{n\in\mathbb{N}}A_n)+K$. Indeed, let $x\in\bigcap_{n\in\mathbb{N}}(A_n+K)$. This means that $x\in A_n+K$ for all $n\in\mathbb{N}$, so that there are $x_n\in A_n$ and $\kappa_n\in K$ such that $x=x_n+\kappa_n$. Since K is norm-compact, $\{\kappa_n\}_{n\in\mathbb{N}}$ has a norm-convergent subsequence $\{\kappa_{n_j}\}_{j\in\mathbb{N}}$, converging to $\kappa\in K$. Thus, $\lim_{j\to\infty}x_{n_j}=x-\lim_{j\to\infty}\kappa_{n_j}=x-\kappa\in A_n$ for all $n\in\mathbb{N}$. So

$x = (x - \kappa) + \kappa \in (\bigcap_{n\in\mathbb{N}} A_n) + K$, as claimed. We apply this to $K = -i[0, y_N]v$ and $A_n = \cup_{k\geq n} B_1^+(\omega(\alpha + iy_k v) - \alpha, \|\Im\omega(\alpha + iy_k v)\|^{-1/2})$ to conclude that

$$\begin{aligned} & h(\mathring{H}_1(\omega(\alpha), \ell)) \\ \subseteq & \bigcap_{n>N} \left\{ -i[0, y_N]v + \left[\bigcup_{k\geq n} B_1^+(\omega(\alpha + iy_k v) - \alpha, \|\Im\omega(\alpha + iy_k v)\|^{-1/2}) \right] \right\} \\ & \cap \overline{\mathbb{H}^+(\mathcal{B})} \\ = & \left[-i[0, y_N]v + \bigcap_{n>N} \bigcup_{k\geq n} B_1^+(\omega(\alpha + iy_k v) - \alpha, \|\Im\omega(\alpha + iy_k v)\|^{-1/2}) \right] \cap \overline{\mathbb{H}^+(\mathcal{B})} \\ \subseteq & [-i[0, y_N]v + H_1(\omega(\alpha) - \alpha, \ell)] \cap \overline{\mathbb{H}^+(\mathcal{B})}, \end{aligned}$$

for any $N \in \mathbb{N}$. As N is arbitrary and $y_N \to 0$ when $N \to \infty$, we obtain

$$h(\mathring{H}_1(\omega(\alpha), \ell) \subseteq \overline{H_1(\omega(\alpha) - \alpha, \ell)}.$$

Replacing the denominator in the definition $\ell = \lim_{n\to\infty} \frac{\Im\omega(\alpha+iy_n v)}{\|\Im\omega(\alpha+iy_n v)\|}$ by $\kappa\|\Im\omega(\alpha+iy_n v)\|$ for any $\kappa \in (0, +\infty)$ (which corresponds to changing radii in the definition of B^+ to $(\kappa\|\Im\omega(\alpha + iy_n v)\|)^{-1/2}$) yields

$$h(\mathring{H}_1(\omega(\alpha), \ell/\kappa)) \subseteq \overline{H_1(\omega(\alpha) - \alpha, \ell/\kappa)}. \tag{22}$$

Pick an arbitrary $u > 0$. There exists $\epsilon > 0$ such that $\omega(\alpha) + i\epsilon u \in \mathring{H}_1(\omega(\alpha), \ell)$. Indeed, this relation is equivalent to

$$(\omega(\alpha) + i\epsilon u - \omega(\alpha))^*(\epsilon u)^{-1}(\omega(\alpha) + i\epsilon u - \omega(\alpha)) < \ell,$$

i.e. $\epsilon u < \ell$. Since $\ell > 0$, the existence of such an ϵ is guaranteed (for ex. any $0 < \epsilon < (\|u\|\|\ell^{-1}\|)^{-1}$ will do). Fix such an ϵ and call it ϵ_0. It follows from the definition of $\mathring{H}_1$ that $(\epsilon_0/\kappa)u \in \mathring{H}_1(\omega(\alpha), \ell/\kappa)$. By applying [4, Relation (14)] with $r = (\kappa\|\Im\omega(\alpha+iy_n v)\|)^{-1/2}$ and letting $n \to \infty$ we obtain that $\|\Im w\| < \kappa^{-1}$ for any $w \in \mathring{H}_1(\omega(\alpha), \ell/\kappa)$ and $\|\Im w\| \leq \kappa^{-1}$ for any $w \in H_1(\omega(\alpha), \ell/\kappa)$. Now our theorem follows: $h(\omega(\alpha) + i(\epsilon_0/\kappa)u) \in \overline{H_1(\omega(\alpha), \ell/\kappa)}$ implies $\|\Im h(\omega(\alpha) + i(\epsilon_0/\kappa)u)\| \leq \kappa^{-1}$ for all $\kappa > 0$, so that

$$\frac{\varphi(\Im h(\omega(\alpha) + i(\epsilon_0/\kappa)u))}{\kappa^{-1}} \leq 1.$$

Taking $\epsilon_0\kappa^{-1} = y \downarrow 0$ provides us with

$$\frac{\varphi(\Im h(\omega(\alpha) + iyu))}{y} \leq \epsilon_0^{-1},$$

concluding the proof. We observe that, not surprisingly, the optimal bound ϵ_0^{-1} is given by the largest ϵ_0 for which $\omega(\alpha) + i\epsilon_0 u \in \mathring{H}_1(\omega(\alpha), \ell)$. □

Recall that $f_\alpha(w) = \alpha + \mathbf{a} + \eta\left[(\mathcal{X} - w)^{-1}\right] = \alpha + \mathbf{a} + h(w)$.

Corollary 4.3. *Assume that ω satisfies the hypotheses of Theorem 4.2 at $\alpha = \alpha^*$. Then for any $v > 0$, we have that*

$$\lim_{y\downarrow 0} f'_\alpha(\omega(\alpha) + iyv)$$

exists and is bounded, and

$$\left\| \lim_{y \downarrow 0} f'_\alpha \left(\omega(\alpha) + iy\ell\right)(\ell) \right\| \le 1.$$

Proof. We use the notations and definitions from Theorem 4.2 and its proof. The conclusion of Theorem 4.2 implies, according to [4, Theorem 2.2 (1)], that the limit $\lim_{y\downarrow 0} h'(\omega(\alpha) + iyv)$ exists and is bounded for all $v > 0$. Then

$$\lim_{y \downarrow 0} \frac{\Im h(\omega(\alpha) + iy\ell)}{y} = h'(\omega(\alpha))_\ell(\ell),$$

where $h'(\omega(\alpha))_\ell(b) = \lim_{y\downarrow 0} h'(\omega(\alpha) + iy\ell)(b)$, $b \in \mathcal{B}$. As it follows from the proof of Theorem 4.2, since the optimal ϵ_0 for $u = \ell$ is one, we have

$$\left\| \lim_{y \downarrow 0} h' \left(\omega(\alpha) + iy\ell\right)(\ell) \right\| \le 1.$$

Since $f'_\alpha = h'$, this proves the second statement. □

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