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Lévy Processes—From Probability to Finance and Quantum Groups

David Applebaum

The theory of stochastic processes was one of the most important mathematical developments of the twentieth century. Intuitively, it aims to model the interaction of "chance" with "time". The tools with which this is made precise were provided by the great Russian mathematician A. N. Kolmogorov in the 1930s. He realized that probability can be rigorously founded on measure theory, and then a stochastic process is a family of random variables $(X(t), t \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a measurable space $(E, \mathcal{E})$. Here Ω is a set (the sample space of possible outcomes), $\mathcal{F}$ is a σ-algebra of subsets of Ω (the events), and P is a positive measure of total mass 1 on $(\Omega, \mathcal{F})$ (the probability). E is sometimes called the state space. Each $X(t)$ is a $(\mathcal{F}, \mathcal{E})$ measurable mapping from Ω to E and should be thought of as a random observation made on E at time t. For many developments, both theoretical and applied, E is Euclidean space $\mathbb{R}^d$ (often with $d = 1$); however, there is also considerable interest in the case where E is an infinite dimensional Hilbert or Banach space, or a finite-dimensional Lie group or manifold. In all of these cases $\mathcal{E}$ can be taken to be the Borel σ-algebra generated by the open sets. To model probabilities arising within quantum theory, the scheme described above is insufficiently general and must be embedded into a suitable noncommutative structure.

Stochastic processes are not only mathematically rich objects. They also have an extensive range of applications in, e.g., physics, engineering, ecology, and economics—indeed, it is difficult to conceive of a quantitative discipline in which they do not feature. There is a limited amount that can be said about the general concept, and much of both theory and applications focusses on the properties of specific classes of process that possess additional structure. Many of these, such as random walks and Markov chains, will be well known to readers. Others, such as semimartingales and measure-valued diffusions, are more esoteric. In this article, I will give an introduction to a class of stochastic processes called *Lévy processes*, in honor of the great French probabilist Paul Lévy, who first studied them in the 1930s. Their basic structure was understood during the "heroic age" of probability in the 1930s and 1940s and much of this was due to Paul Lévy himself, the Russian mathematician A. N. Khintchine, and to K. Itô in Japan. During the past ten years, there has been a great revival of interest in these processes, due to new theoretical developments and also a wealth of novel applications—particularly to option pricing in mathematical finance. As well as a vast number of research papers, a number of books on the subject have been published ([3], [11], [1], [2], [12]) and there have been annual international conferences devoted to these processes since 1998. Before we begin the main part of the article, it is worth

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listing some of the reasons why Lévy processes are so important:

- There are many important examples, such as Brownian motion, the Poisson process, stable processes, and subordinators.
- They are generalizations of random walks to continuous time.
- They are the simplest class of processes whose paths consist of continuous motion interspersed with jump discontinuities of random size appearing at random times.
- Their structure contains many features, within a relatively simple context, that generalize naturally to much wider classes of processes, such as semimartingales, Feller-Markov processes, processes associated to Dirichlet forms, and (generalizing the strictly stable Lévy processes) self-similar processes.
- They are a natural model of noise that can be used to build stochastic integrals and to drive stochastic differential equations.
- Their structure is mathematically robust and generalizes from Euclidean space to Banach and Hilbert spaces, Lie groups, and symmetric spaces, and algebraically to quantum groups.

The Structure of Lévy Processes

We will take $E = \mathbb{R}^d$ throughout the first part of this article.

Definition. A Lévy process $X = (X(t), t \geq 0)$ is a stochastic process satisfying the following:

(L1) X has independent and stationary increments,

(L2) Each $X(0) = 0$ (with probability one[1]),

(L3) X is *stochastically continuous*, i.e., for all $a > 0$ and for all $s \geq 0, \lim_{t \to s} P(|X(t) - X(s)| > a) = 0$.

Of these three axioms, (L1) is the most important, and we begin by explaining what it means. It focusses on the increments $\{X(t) - X(s); 0 \leq s \leq t < \infty\}$. Stationarity of these means that $P(X(t) - X(s) \in A) = P(X(t-s) - X(0) \in A)$ for all Borel sets A, i.e., the distribution of $X(t) - X(s)$ is invariant under shifts $(s, t) \to (s + h, t + h)$. Independence means that given any finite ordered sequence of times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$, the random variables $X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are (statistically) independent. We emphasize again that (L1) is the key defining axiom for Lévy processes; indeed, for many years they were known as "processes with stationary and independent increments". Of the other axioms, (L2) is a convenient normalization and (L3) is a technical (but important) assumption that enables us to do serious analysis.

The Lévy-Khintchine Formula

To understand the structure of a generic Lévy process, we employ Fourier analysis. The *characteristic function* of $X(t)$ is the mapping $\phi_t : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\phi_t(u) = \mathbb{E}(e^{iu \cdot X(t)}) = \int_{\mathbb{R}^d} e^{iu \cdot y} p_t(dy),$$

where p_t is the law (or distribution) of $X(t)$, i.e., $p_t = P \circ X(t)^{-1}$, and $\mathbb{E}$ denotes expectation. ϕ_t is continuous and positive definite; indeed, a famous theorem of Bochner asserts that all continuous positive definite mappings from $\mathbb{R}^d$ to $\mathbb{C}$ are Fourier transforms of finite measures on $\mathbb{R}^d$.

It follows from the axiom (L1) that each $X(t)$ is infinitely divisible, i.e., for each $n \in \mathbb{N}$, there exists a probability measure $p_{t,n}$ on $\mathbb{R}^d$ with characteristic function $\phi_{t,n}$ such that $\phi_t(u) = (\phi_{t,n}(u))^n$, for each $u \in \mathbb{R}^d$. The characteristic functions of infinitely divisible probability measures were completely characterized by Lévy and Khintchine in the 1930s. Their result, which we now state, is fundamental for all that follows:

Theorem 0.1 [The Lévy-Khintchine Formula]. *If $X = (X(t), t \geq 0)$ is a Lévy process, then $\phi_t(u) = e^{t\eta(u)}$, for each $t \geq 0, u \in \mathbb{R}^d$, where*

$$\eta(u) = ib \cdot u - \frac{1}{2} u \cdot au + \int_{\mathbb{R}^d - \{0\}} [e^{iu \cdot y} - 1 - iu \cdot y 1_{||y||<1}(y)] \nu(dy), \tag{0.1}$$

for some $b \in \mathbb{R}^d$, a non-negative definite symmetric $d \times d$ matrix a and a Borel measure ν on $\mathbb{R}^d - \{0\}$ for which $\int_{\mathbb{R}^d - \{0\}} (||y||^2 \wedge 1) \nu(dy) < \infty$. Conversely, given a mapping of the form (0.1) we can always construct a Lévy process for which $\phi_t(u) = e^{t\eta(u)}$.

One of our goals is to give a probabilistic interpretation to the Lévy-Khintchine formula. The mapping $\eta : \mathbb{R}^d \to \mathbb{C}$ is called the *characteristic exponent* of X. It is conditionally positive definite in that $\sum_{i,j=1}^n c_i \overline{c_j} \eta(u_i - u_j) \geq 0$, for all $n \in \mathbb{N}, c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$. A theorem due to Schoenberg asserts that all continuous, hermitian (i.e., $\eta(u) = \overline{\eta(-u)}$, for all $u \in \mathbb{R}^d$), conditionally positive maps from $\mathbb{R}^d$ to $\mathbb{C}$ that satisfy $\eta(0) = 0$ must take the form (0.1). The triple (b, a, ν) is called the *characteristics* of X. It determines the law p_t. The measures ν that can appear in (0.1) are called *Lévy measures*.

We begin the task of interpreting (0.1) by examining some examples. The first two that we consider are very well known in probability theory—indeed, each has an extensive theoretical development in its own right with many applications.

Examples of Lévy Processes

1. Brownian Motion and Gaussian Processes

We define a Brownian motion $B_a = (B_a(t), t \geq 0)$ to be a Lévy process with characteristics $(0, a, 0)$.

[1] *To be denoted w.p.1, henceforth.*

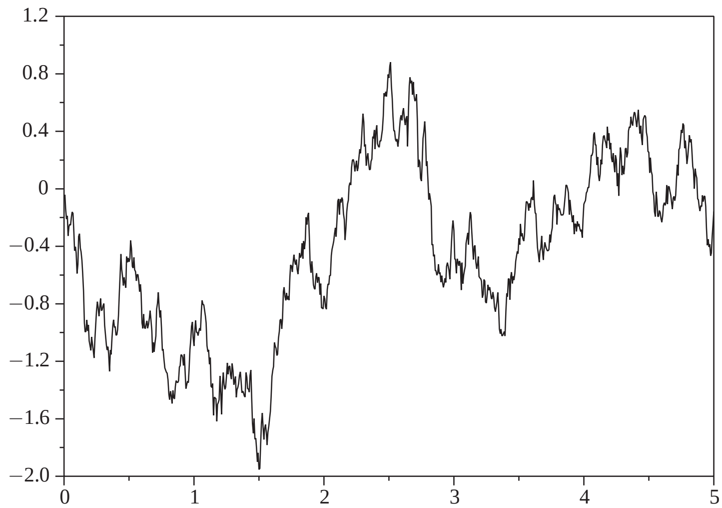

Figure 1. Simulation of standard Brownian motion. The path is continuous but nowhere differentiable. If you were to zoom in, the fractal nature of the path would become apparent and this reflects the self-similarity of the process.

It has mean zero and covariance $\mathbb{E}(B_a^i(s) B_a^j(t)) = a^{ij}(s \wedge t)$ (where $B_a^i(s)$ is the ith component of the vector $B_a(s)$). If a is positive definite, then each $B_a(t)$ has a normal distribution with density $f_{a,t}$ where

$$f_{a,t}(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}\sqrt{\det(a)}} \exp\left(-\frac{1}{2t}(x \cdot a^{-1}x)\right).$$

When $d = 1$, we write $B_1 = B$ and call it a standard Brownian motion. Brownian motion has a fascinating history. It is named after the botanist Robert Brown, who first observed, in the 1820s, the irregular motion of pollen grains immersed in water. By the end of the nineteenth century, the phenomenon was understood by means of kinetic theory as a result of molecular bombardment. Indeed, in 1905, Einstein, although ignorant of the discovery of the phenomenon and of previous work on it, predicted its existence from purely theoretical considerations. Five years earlier L. Bachelier had employed it to model the stock market, where the analogue of molecular bombardment is the interplay of the myriad of individual market decisions that determine the market price.

Standard Brownian motion was rigorously constructed by N. Wiener in the 1920s as a family of functionals on the space $C = C_0([0, \infty), \mathbb{R})$ of real-valued continuous functions on $[0, \infty)$ that vanish at zero. In so doing, he equipped the infinite-dimensional space C with a Gaussian measure that is now called *Wiener measure* in his honour. It follows that the paths $t \to B_a(t)(\omega)$, where $\omega \in C$, are continuous. In the 1930s Wiener, together with R. Paley and A. Zygmund, showed that the paths are nowhere differentiable (w.p.1).

Figure 1 presents a simulation of the paths of standard Brownian motion.

Brownian motion with drift is the Lévy process $C_{a,b} = (C_{a,b}(t), t \geq 0)$, with characteristics $(b, a, 0)$. Each $C_{a,b}(t)$ is a Gaussian random variable having mean vector tb and covariance matrix ta. In fact each $C_{a,b}(t) = bt + B_a(t)$. A Lévy process has continuous sample paths (w.p.1), or is Gaussian if and only if it is a Brownian motion with drift.

2. The Poisson Process

A Poisson process $N_\lambda = (N_\lambda(t), t \geq 0)$ with intensity $\lambda > 0$ is a Lévy process with characteristics $(0, 0, \lambda\delta_1)$, where δ_1 is a Dirac mass concentrated at 1. N_λ takes non-negative integer values, and we have the Poisson distribution:

$$P(N_\lambda(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

The paths of N_λ are piecewise constant on each finite interval, with jumps of size 1 at the random times $\tau_n = \inf\{t \geq 0, N_\lambda(t) = n\}$.

3. The Compound Poisson Process

Let $(Y_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables with common law q and let N_λ be an independent Poisson process. The compound Poisson process is the Lévy process $Z_\lambda(t) = \sum_{j=1}^{N_\lambda(t)} Y_j$. It has characteristic exponent $\eta(u) = \int_{\mathbb{R}^d}(e^{iu\cdot y} - 1)\lambda q(dy)$. The compound Poisson process (with $d = 1$) can be used to model the takings at a till in a supermarket, where $N_\lambda(t)$ is the number of customers in the queue at time t and Y_j is the amount paid by the jth customer.

4. Interlacing Processes

We can define a Lévy process by the prescription $X(t) = C_{a,b}(t) + Z_\lambda(t)$, provided the two summands are assumed to be independent. We call this an *interlacing process* since its paths have the form of continuous motion interlaced with random jumps of size $||Y_n||$ occurring at the random times τ_n (where the Y_ns are as in Example 3 above). X has characteristic exponent

$$\eta(u) = ib \cdot u - \frac{1}{2}u \cdot au + \int_{\mathbb{R}^d}(e^{iu\cdot y} - 1)\lambda q(dy), \tag{0.2}$$

which is quite close to the general form (0.1). Indeed (0.2) was proposed as the form of the most general η by the Italian mathematician B. de Finetti in the 1920s. His error was in failing to appreciate that the finite measure λq can be replaced by a σ-finite Lévy measure ν. But if we do this, $(e^{iu\cdot y} - 1)$ may not be ν-integrable and hence we must adjust the integrand. Probabilistically, this corresponds to a lack of convergence of a countable number of "small jumps", as we will see in the next section. Although (0.2) is incorrect, the most general η can be obtained as a pointwise limit of terms of similar type, i.e., $\eta(u) = \lim_{n\to\infty} \eta_n(u)$, where each

$$\eta_n(u) = i\left(b - \int_{\frac{1}{n}<||y||<1} y\nu(dy)\right)\cdot u$$
$$-\frac{1}{2}u\cdot au + \int_{||y||\geq\frac{1}{n}}(e^{iu\cdot y}-1)\lambda q(dy),$$

and the integrals must be combined together before the passage to the limit. In the next section we will see the intuition behind this.

From the above examples, the reader may be forgiven for thinking that a Lévy process is nothing but the interplay of Gaussian and Poisson measures. In a sense this is correct; however, note that the Gaussian and Poisson measures give rise to extreme points of the convex cone of all characteristic exponents. As the following shows, there are some interesting inhabitants of the interior.

5. Stable Lévy Processes

Stable probability distributions arise as the possible weak limits of normalized sums of i.i.d. (i.e., independent, identically distributed) random variables in the central limit theorem. The normal distribution is stable and corresponds to the case in which each of the i.i.d. random variables has finite mean and variance. Stable random variables are those whose laws are stable. They are characterized by the property that if X_1 and X_2 are independent copies of a stable random variable X, then for each $c_1, c_2 > 0$, there exists $c > 0$ and $d \in \mathbb{R}^d$ such that $cX + d$ has the same law as $c_1X_1 + c_2X_2$. A Lévy process is stable if each $X(t)$ is stable in this sense. The characteristics of a stable Lévy process are either of the form $(b, a, 0)$ (so it is a Brownian motion with drift) or $(b, 0, \nu)$, where $\nu(dx) = \frac{C}{|x|^{\alpha+d}}dx$, with $0 < \alpha < 2$ and $C > 0$. α is called the index of stability. With the sole exception of the Brownian motions with drift, the random variables of a stable Lévy process all have infinite variance, and if $\alpha \leq 1$, they also have infinite mean. One example of interest (in the case $d = 1$) for which $\alpha = 1$ is the Cauchy process, which has the density $f_t(x) = \frac{t}{\pi(x^2+t^2)}$. Figure 2 presents a simulation of its paths in which jump discontinuities are represented by vertical lines.

With a little calculus, the characteristic exponent can be transformed to a more useful form. This is particularly simple when X is rotationally invariant, i.e., $P(X(t) \in OA) = P(X(t) \in A)$, for all $O \in O(d), t \geq 0$, and Borel sets A. We then obtain $\eta(u) = -\sigma^\alpha|u|^\alpha$, where $\sigma > 0$. Rotationally invariant stable processes are an important class of self-similar processes, i.e., $(X(ct), t \geq 0)$ and $(c^{\frac{1}{\alpha}}X(t), t \geq 0)$ have the same finite dimensional distributions (for each $c > 0$), and this is one reason why such processes are important in applications. Another reason, applying to general stable random variables X, is that they have "heavy tails", i.e. $P(X > y)$ behaves asymptotically like $y^{-\alpha}$ as $y \to \infty$, as opposed to the exponential decay found in the Gaussian case. Such behavior has been found in models of telecommunications traffic on the Internet.

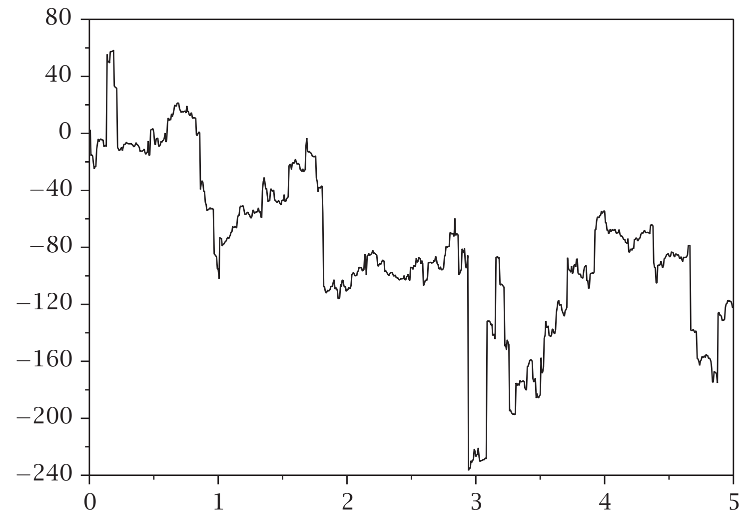

Figure 2. Simulation of the Cauchy process. The Cauchy process is stable with $\alpha = 1$. Jump discontinuities are represented by vertical lines. This process is also self-similar so the path has a fractal nature.

6. Relativistic Processes

1905 was a busy year for Albert Einstein. As well as his work on Brownian motion, mentioned above, he also gave a quantum mechanical explanation of the photoelectric effect (for which he won his Nobel Prize) and developed the special theory of relativity. According to the latter, a particle of rest mass m moving with momentum p has kinetic energy $E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$, where c is the velocity of light. If we define $\eta(p) = -E(p)$, then η is the characteristic exponent of a Lévy process. We will explore some consequences of this below.

7. Subordinators

A *subordinator* is a one-dimensional Lévy process $(T(t), t \geq 0)$ that is nondecreasing (w.p.1). In this case, the Fourier transform that defines the characteristic function can be analytically continued to yield the Laplace transform $\mathbb{E}(e^{-uT(t)}) = e^{-t\psi(u)}$, for each $u > 0$, where

$$\psi(u) = -\eta(iu) = bu + \int_{(0,\infty)}(1 - e^{-uy})\lambda(dy).$$

Here $b \geq 0$ and λ is a Lévy measure that satisfies the additional constraints $\lambda(-\infty, 0) = 0$ and $\int_{(0,\infty)}(y \wedge 1)\lambda(dy) < \infty$.

ψ is called the *Laplace exponent* of the subordinator. The set of all of these is in one-to-one correspondence with the set of Bernstein functions for which $\lim_{x\to 0} f(x) = 0$, where we recall that an infinitely differentiable function f on $(0, \infty)$ is a Bernstein function if and only if $f \geq 0$ and $(-1)^n f^{(n)} \leq 0$, for all $n \in \mathbb{N}$.

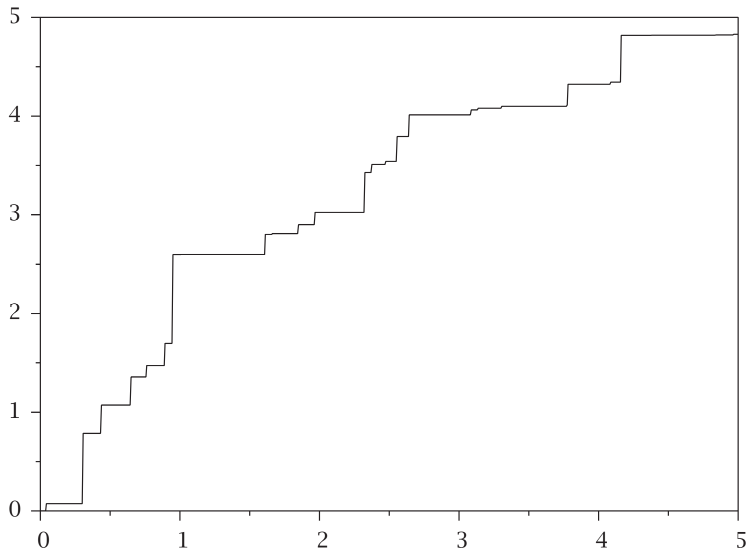

Figure 3. Simulation of the gamma subordinator. In contrast to the cases shown by the previous two figures, the sample paths of subordinators are considerably more regular. The path is a non-decreasing step function with jump discontinuities again shown as vertical lines.

Examples of subordinators include the α-stable ones ($0 < \alpha < 1$) that have Laplace exponent $\psi(u) = u^{\alpha}$. For the case $\alpha = \frac{1}{2}$, each $T(t)$ is the first hitting time of a standard Brownian motion to a level, i.e., $T(t) = \inf\{s > 0; B(s) = \frac{t}{\sqrt{2}}\}$. Furthermore, each $T(t)$ has a Lévy distribution with density $f_t(s) = \left(\frac{t}{2\sqrt{\pi}}\right) s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}}$. Another well-known example of a subordinator, where each $T(t)$ has a gamma distribution, is depicted in Figure 3.

An important application of subordinators is to the time change of Lévy processes. If X is a Lévy process with characteristic exponent η_X and T is an independent subordinator with Laplace exponent λ, then $Y(t) = X(T(t))$ is a new Lévy process with characteristic exponent $\eta_Y = -\lambda \circ -\eta_X$. This procedure was first investigated by S. Bochner in the 1950s and is sometimes called "subordination in the sense of Bochner" in his honor. In particular, if X is a Brownian motion (with a a multiple of the identity) and T is an independent α-stable subordinator, then Y is a 2α-stable rotationally invariant Lévy process.

8. The Riemann-Zeta Process

Readers who are interested in number theory may find this example of interest. If ζ is the usual Riemann zeta function, we obtain a Lévy process for each $u > 1$ by the following prescription for the characteristic exponent,

$$\eta_u(v) = \log\left(\frac{\zeta(u+iv)}{\zeta(u)}\right).$$

This was established by Khintchine in the 1930s.

The Lévy-Itô Decomposition

With the insight we obtained from Example 4, we can now return to the task of trying to understand the structure of the sample paths of Lévy processes. Given a characteristic exponent, we can always associate to it a Lévy process whose paths are right continuous with left limits (w.p.1). It follows that this process X can only have jump discontinuities, and there are, at most, a countable number of these on each closed interval. We formally write $X(t) = X_c(t) + \sum_{0 \le s \le t} \Delta X(s)$, where X_c has continuous paths (w.p.1) and $\Delta X(s) = X(s) - X(s-)$ is the "jump" at time s where $X(s-) = \lim_{u \uparrow s} X(u)$ is the left limit.

We can describe X_c quite easily. It is a Brownian motion with drift, $X_c(t) = bt + B_a(t)$ (although this is by no means easy to prove). The second term is more problematic—in particular, the sum may not converge. It turns out to be helpful to count the jumps up to time t that are in a given Borel set A and to introduce

$$N(t, A) = \#\{0 \le s \le t; \Delta X(s) \in A\}.$$

N is a very interesting object—it is in fact a function of three variables—time t, the set A, and the sample point ω. If we fix t and ω, we get a σ-finite measure on the Borel sets of $\mathbb{R}^d$. On the other hand, if we fix the set A and ensure that it is bounded away from zero, we get a Poisson process with intensity $\lambda = \nu(A)$. For these reasons N is called a *Poisson random measure.*

In any finite time, X can have only a finite number of jumps of size greater than 1 (or indeed greater than any $\epsilon > 0$). We can write this finite sum of jumps as $\int_{||x||>1} xN(t, dx)$. Similarly, the sum of all the jumps of size greater than $\frac{1}{n}$ but less than 1 is $\int_{\frac{1}{n}<||x||<1} xN(t, dx)$; however, the limit may not converge as $n \to \infty$. Paul Lévy argued that the accumulation of a large number of very small jumps may be difficult to distinguish from bursts of deterministic motion, so one should consider $M_n(t) = \int_{\frac{1}{n}<||x||<1} x(N(t, dx) - t\nu(dx))$. $(M_n, n \in \mathbb{N})$ is a sequence of square-integrable, mean zero martingales and hence is a very pleasant object from both a probabilistic and an analytic viewpoint. In particular the sequence converges in mean square to a martingale $M(t) = \int_{0<||x||<1} x\tilde{N}(t, dx)$, where $\tilde{N}(t, dx) = N(t, dx) - t\nu(dx)$ is called a *compensated Poisson random measure.* Lévy's intuition was made precise by K. Itô, and we can now give the celebrated Lévy-Itô decomposition for the sample paths of a Lévy process:

$$X(t) = bt + B_a(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\ge 1} xN(t, dx). \qquad (0.3)$$

Readers should beware of generalizing from the Gaussian to this more general case. For example, bt is not in general the mean of $X(t)$—indeed, as we saw in Example 5, this may not exist. The "martingale part" of $X(t)$, i.e., the process $M(t) + B_a(t)$, has moments to all orders, so if $X(t)$ itself fails to have an nth moment this is entirely due to the influence of "large jumps".

Applications to Finance

A sociologist investigating the behavior of the probability community during the early 1990s would surely report an interesting phenomenon. Many of the best minds of this (or any other) generation began concentrating their research in the area of mathematical finance. The main reason for this can be summed up in two words—option pricing.

Essentially, an option is a contract that confers upon the holder the right, but not the obligation, to purchase (or sell) a unit of a certain stock for a fixed price k on (or perhaps before) a fixed expiry date T, after which the option becomes worthless. For the option to make sense, k should be considerably less than the current price of the stock. If the stock price rises above k, the holder of the option may make a considerable profit; on the other hand, if the stock price falls dramatically, losses will be considerably less through buying options than by purchasing the stock itself.

The key question is—does the market determine a unique price for a given option, and if so, can this price be explicitly computed? Much of the current interest in the subject derives from Nobel-prize winning work of F. Black, M. Scholes and R. Merton in the 1970s who gave a positive answer to this question. Underlying their analysis was a model of stock prices that improved upon that of Bachelier by using *geometric Brownian motion*; i.e., the price $S(t)$ of a given stock at time t is

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right\}.$$

The constant $\mu \in \mathbb{R}$ is the (logarithmic) expected rate of return, while $\sigma > 0$, called the *volatility,* is a measure of the excitability of the market. We will have more to say about volatility below. Black and Scholes obtained an exact formula for the unique price of a *European option* (i.e., one that can only be exercised at time T) using the normal distribution. The derivation of this formula involves the use of tools such as martingales and Girsanov transforms, and it is this link with stochastic analysis that so excited the probabilistic community.

Although very elegant, the Black-Scholes-Merton model has limitations and possible defects that have led many probabilists to query it. Indeed, empirical studies of stock prices have found evidence of heavy tails, which is incompatible with a Gaussian model, and this suggests that it might be fruitful to replace Brownian motion with a more general Lévy process. Indeed, H. Geman, D. Madan and M. Yor have argued that this is quite natural from the point of view of the Lévy-Itô decomposition (0.3), where the small jumps term $\int_{|x|<1} x\tilde{N}(t,dx)$ describes the day-to-day jitter that causes minor fluctuations in stock prices, while the big jumps term $\int_{|x|\geq 1} xN(t,dx)$ describes large stock price movements caused by major market upsets arising from, e.g., earthquakes or terrorist atrocities.

If we set aside Brownian motion, there are a plethora of Lévy processes to choose from, and our choice must enable us to derive a pricing formula that market analysts can compute with. One interesting group of candidates is the (symmetric) *hyperbolic Lévy processes*, whose financial applications have been extensively developed by E. Eberlein and his group in Freiburg, Germany. These are processes with no Brownian motion part in (0.3), and the characteristic function is given by

$$\phi_t(u) = \left(\frac{\zeta}{K_1(\zeta)}\frac{K_1(\sqrt{\zeta^2+\delta^2u^2})}{\sqrt{\zeta^2+\delta^2u^2}}\right)^t,$$

where K_1 is a Bessel function of the third kind, and ζ and δ are non-negative parameters.

Hyperbolic Lévy processes were discovered by O. Barndorff-Nielsen the 1970s and used as models for the distribution of particle size in wind-blown sand deposits. N. H. Bingham and R. Keisel make an interesting analogy between the dynamics of sand production and stock prices in that just as large rocks are broken down to smaller and smaller particles "this 'energy cascade effect' might be paralleled in the 'information cascade effect', whereby price-sensitive information originates in, say, a global newsflash and trickles down through national and local level to smaller and smaller units of the economic and social environment."

A problem with non-Gaussian option pricing is that the market is "incomplete", i.e., there may be more than one possible pricing formula. This is clearly undesirable, and a number of selection principles, such as entropy minimization, have been employed to overcome this problem. For hyperbolic processes, a pricing formula has been developed that has minimum entropy and that is claimed to be an improvement on the Black-Scholes formula.

Another problem with the Black-Scholes-Merton formula is the constancy of the volatility. Empirical studies suggest that this should vary to give a curve called the "volatility smile". This has prompted some authors to propose "stochastic volatility models" wherein σ is replaced in the standard Black-Scholes model by a random process that solves a stochastic differential equation. There are a number of different approaches to this; e.g., O. Barndorff-Nielsen and N. Shephard have recently proposed that $(\sigma(t)^2, t \geq 0)$ should be an Ornstein-Uhlenbeck process driven by a subordinator $(T(t), t \geq 0)$, i.e.,

$$\sigma(t)^2 = e^{-\lambda t}\sigma(0)^2 + \int_0^t e^{-\lambda(t-s)}dT(\lambda s),$$

where $\lambda > 0$. As T has finite variation (w.p.1), the integral is well defined in the random Lebesgue-Stieltjes sense.

Readers who want to learn more about "Lévy finance" should consult [12], [4] , chapter 5 of [1], and references therein.

Markov Processes, Semigroups, and Pseudodifferential Operators

Lévy processes are, in particular, Markov processes, i.e., their past and future are independent, given the present. This is formulated precisely using the conditional expectation: $\mathbb{E}(f(X(t+u))|\mathcal{F}_t) = \mathbb{E}(f(X(t+u))|X(t))$, for all $t, u \geq 0$ and all $f \in B_b(\mathbb{R}^d)$—the Banach space, under the supremum norm, of all bounded Borel measurable functions on $\mathbb{R}^d$. Here "the past" $\mathcal{F}_t$ is the smallest sub-σ-algebra of $\mathcal{F}$ with respect to which all $X(s) (0 \leq s \leq t)$ are measurable. We define a two-parameter family of linear contractions $(T_{s,t}; 0 \leq s \leq t < \infty)$ on $B_b(\mathbb{R}^d)$ by the prescription $(T_{s,t}f)(x) = \mathbb{E}(f(X(t))|X(s) = x) = \int_{\mathbb{R}^d} f(x+y)p_t(dy)$. Then the Markov property implies that these form an *evolution*, i.e., $T_{r,s}T_{s,t} = T_{r,t}$, for all $r \leq s \leq t$. Note that these operators all commute with the natural action of the translation group of $\mathbb{R}^d$ on $B_b(\mathbb{R}^d)$.

Lévy processes form a nice subclass of Markov processes. First, they are time-homogeneous, i.e., $T_{s,t} = T_{0,t-s}$ for all $s \leq t$. If we now write $T_t = T_{0,t}$, the evolution property becomes the semigroup law $T_sT_t = T_{s+t}$. Second, Lévy processes are Feller processes, i.e., each T_t preserves the Banach space $C_0(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ that vanish at infinity and $\lim_{t\downarrow 0}||T_tf - f|| = 0$, for all $f \in C_0(\mathbb{R}^d)$. Hence $(T_t, t \geq 0)$ is a strongly continuous, one-parameter contraction semigroup on $C_0(\mathbb{R}^d)$, and by the general theory of such semigroups, we can assert the existence of the generator $Af = \lim_{t\downarrow 0}\dfrac{T(t)f - f}{t}$, for all $f \in D_A$. The domain D_A is a linear space that is dense in $C_0(\mathbb{R}^d)$ and A is a closed linear operator. We can explicitly compute the semigroup and its generator as pseudodifferential operators. For convenience, we work in Schwartz space $S(\mathbb{R}^d)$—the space of all smooth functions on $\mathbb{R}^d$ that are such that they and all their derivatives decay to zero at infinity faster than any negative power of $|x|$. $S(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ and is a natural domain for the Fourier transform $\hat{f}(u) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-i(u,x)}f(x)dx$. Fourier inversion then yields $f(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\hat{f}(u)e^{i(u,x)}du$. Applying theorem 0.1, we compute

$$(T_tf)(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i(u,x)}e^{t\eta(u)}\hat{f}(u)du,$$

so that T_t is a pseudodifferential operator with symbol $e^{t\eta}$. Formal differentiation can be justified, and we find that

$$(Af)(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i(u,x)}\eta(u)\hat{f}(u)du,$$

so A is also a pseudodifferential operator, with symbol η. Using the Lévy-Khintchine formula (0.1) and elementary properties of the Fourier transform, we obtain the following explicit form for the action of the generator on $S(\mathbb{R}^d)$

(0.4)

$$(Af)(x) = \sum_{i=1}^d b_i\partial_i f(x) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial_i\partial_j f(x) + \int_{\mathbb{R}^d-\{0\}}\left[f(x+y) - f(x) - \sum_{i=1}^d y_i\partial_i f(x)1_{||y||<1}(y)\right]\nu(dy).$$

Using more sophisticated methods the domain in (0.4) can be extended to a larger space of twice differentiable functions in $C_0(\mathbb{R}^d)$. Here are some specific examples of interesting generators:

1. Brownian motion (with $a = I$) is generated by (one-half times) the Laplacian, i.e., $A = \frac{1}{2}\sum_{i=1}^d \partial_i^2 = \frac{1}{2}\Delta$.
2. Rotationally invariant α-stable processes (with $\sigma = 1$) are generated by fractional powers of the Laplacian: $A = -(-\Delta)^{\frac{\alpha}{2}}$.
3. For the relativistic process, we have $A = -(\sqrt{m^2c^4 - c^2\Delta} - mc^2)$.

In the last example, $-A$ is called a *relativistic Schrödinger operator* in quantum theory. Note that A is obtained from its symbol through the correspondence $p \leftrightarrow -i\nabla$, which is precisely the usual rule for quantization, although this is more naturally carried out in a Hilbert space setting (see below).

If A_Z is the generator of the Lévy process $Z(t) = X(T(t))$ obtained from a Lévy process X with characteristic exponent η_X, associated semigroup $(T_t^X, t \geq 0)$, and generator A_X using an independent subordinator T with Laplace exponent ψ, then the identity $\eta_Z = -\psi \circ -\eta_X$, quantizes nicely to yield $A_Z = -\psi(-A_X)$. In particular, we can use the α-stable subordinators to define fractional powers of $-A_X$ using the following beautiful formula

$$-(-A_X)^\alpha f = \frac{\alpha}{\Gamma(1-\alpha)}\int_{(0,\infty)}(T_s^Xf - f)\frac{ds}{s^{1+\alpha}}.$$

A deep generalization due to R. S. Phillips allows the replacement of A_X and T_t^X with the generator of a general contraction semigroup on a Banach space.

The semigroup associated with each Lévy process also operates in each $L^p(\mathbb{R}^d) (1 \leq p < \infty)$ and is again strongly continuous and contractive.

Since $S(\mathbb{R}^d)$ is dense in each $L^p(\mathbb{R}^d)$, the pseudodifferential operator representations discussed above still hold here. From now on, we take $p = 2$. The generator corresponding to the symbol η has maximal domain $\mathcal{H}_\eta(\mathbb{R}^d)$—the *nonisotropic Sobolev space* of all $f \in L^2(\mathbb{R}^d)$ for which $\int_{\mathbb{R}^d} |\eta(u)|^2 |\hat{f}(u)|^2 du < \infty$.

Standard semigroup theory tells us that a necessary and sufficient condition for each T_t to be self-adjoint is that $-A$ is positive, self-adjoint. A necessary and sufficient condition for this is that the associated Lévy process is symmetric, i.e., $P(X(t) \in A) = P(X(t) \in -A)$, and this holds if and only if

$$\eta(u) = -\frac{1}{2} u \cdot au + \int_{\mathbb{R}^d - \{0\}} (\cos(u \cdot y) - 1)\nu(dy).$$

This yields a probabilistic proof of self-adjointness (on $\mathcal{H}_\eta(\mathbb{R}^d)$) of each of the three operators discussed above.

Let A be the self-adjoint generator of a symmetric Lévy process and for each $f, g \in C_c^\infty(\mathbb{R}^d)$, define $\mathcal{E}(f, g) = -< f, Ag >$, then $\mathcal{E}$ extends to a *symmetric Dirichlet form* in $L^2(\mathbb{R}^d)$, i.e., a closed symmetric form in H with domain D, such that $f \in D \Rightarrow (f \vee 0) \wedge 1 \in D$ and $\mathcal{E}((f \vee 0) \wedge 1) \le \mathcal{E}(f)$ for all $f \in D$, where we have written $\mathcal{E}(f) = \mathcal{E}(f, f)$. A straightforward calculation yields

$$\begin{aligned}\mathcal{E}(f,g) \quad = \quad & \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \int_{\mathbb{R}^d} (\partial_i f)(x)(\partial_j g)(x) dx \\ + \quad & \frac{1}{2} \int_{(\mathbb{R}^d \times \mathbb{R}^d) - D} (f(x) - f(x+y)) \cdot \\ & (g(x) - g(x+y))\nu(dy)dx,\end{aligned}$$

where D is the diagonal, $D = \{(x,x), x \in \mathbb{R}^d\}$. This is the prototype for the *Beurling-Deny formula* for symmetric Dirichlet forms.

Now we return to the space $C_0(\mathbb{R}^d)$. The ideas we explored there have a far-reaching generalization, originally due to W. von Waldenfels and P. Courrège in the early 1960s and recently systematically explored by N. Jacob and his school in Erlangen and Swansea [7]. The main starting point of this is that if X is a general Feller process defined on $\mathbb{R}^d$ that has the property that the smooth functions of compact support are contained in the domain of its generator A, then we can always represent A as a pseudodifferential operator

$$(Af)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(u,x)} \eta(x,u) \hat{f}(u) du.$$

Note that the symbol η now has an additional x-dependence; however, each $\eta(x, \cdot)$ is still a characteristic exponent, so that we get an appealing intuitive understanding of X as a "field of Lévy processes" indexed by space. Aficionados of pseudodifferential operators should be aware that the map $x \to \eta(x, u)$ does not, in general, have nice smoothness properties.

Recurrence, Transience, and Bound States

From an intuitive point of view a stochastic process is *recurrent* at a point x if it visits any arbitrarily small neighborhood of that point an infinite number of times (w.p.1), and it is *transient* if each such neighborhood is only visited finitely many times (w.p.1). More precisely, a Lévy process is recurrent (at the origin) if $\liminf_{t\to\infty} |X(t)| = 0$ (w.p.1) and transient (at the origin) if $\lim_{t\to\infty} |X(t)| = \infty$ (w.p.1). The recurrence/transience dichotomy holds in that every Lévy process is either recurrent or transient. In the 1960s, S. C. Port and C. J. Stone proved that a Lévy process is recurrent if and only if $\int_{||u||<a} \Re\left(\frac{1}{-\eta(u)}\right) du = \infty$ for any $a > 0$. It follows that Brownian motion is recurrent for $d = 1, 2$ and that for $d = 1$ every α-stable process is recurrent if $1 \le \alpha < 2$ and transient if $0 < \alpha < 1$. For $d \ge 3$, every Lévy process is transient.

In the 1990s, R. Carmona, W. C. Masters, and B. Simon studied the spectral properties of Hamiltonian operators acting in $L^2(\mathbb{R}^d)$ of the form $H = H_0 + V$, where H_0 is (minus) the generator of a symmetric Lévy process X and V is a suitable potential. In particular, they were able to show that H has at least one bound state (i.e., a negative eigenvalue) if and only if X is recurrent. In particular, in the physically interesting case in which H_0 is a relativistic Schrödinger operator, bound states are obtained only in dimension 1 and 2.

Lévy Processes in Groups

So far we have dealt exclusively with Lévy processes taking values in a Euclidean space. Now we will replace $\mathbb{R}^d$ with a topological group G. First some general remarks. The interaction between probability theory and groups has been an active area of research since the 1960s—indeed, this is the natural setting for studying the interaction of "chance" with "symmetry". One area of research that is currently attracting enormous interest is random matrix theory [5], partly because of intriguing links between the asymptotics of uniformly distributed matrices in the unitary group $U(n)$ and the zeros of the Riemann zeta function. A survey on random walks and invariant diffusions in groups can be found in [10], with particular emphasis on the relationship between the asymptotic behavior of the process and the volume growth of the group.

A Lévy process on a topological group G is defined exactly as in the Euclidean case, but within the axioms (L1) and (L3), the increment $X(t) - X(s)$ is replaced by $X(s)^{-1}X(t)$ (with the group operation written multiplicatively), whereas in (L2), the role of 0 is played by the neutral element that we denote by e. If p_t is the law of $X(t)$, then $(p_t, t \ge 0)$ is a weakly continuous convolution semigroup of probability measures on G, so that in particular $p_{s+t}(A) = \int_G p_t(\tau^{-1}A)p_s(d\tau)$.

There are three cases of interest—locally compact abelian groups (LCA groups), Lie groups, and general locally compact groups. The LCA case was extensively studied during the 1960s. The fact that the dual group $\hat{G}$ of characters is itself an LCA group allows a natural generalization of the Fourier transform from $\mathbb{R}^d$ to G, and a Lévy-Khintchine formula that characterizes Lévy processes can hence be developed similarly to the Euclidean case. We will not dwell further on this topic here; interested readers are directed to section 5.6 in [6].

The case in which G is a Lie group has been extensively studied. For non-abelian G, there is no direct analogue of the Fourier transform available, and one of the joys of the subject is the challenge of surmounting this obstacle using tools from semigroup theory, stochastic analysis, group representations, and noncommutative harmonic analysis. The first important step in this direction was taken by G. A. Hunt in 1956. He effectively characterized Lévy processes in Lie groups by generalizing the formula (0.4) for the generator in $\mathbb{R}^d$. To be precise, let $X = (X(t), t \geq 0)$ be a Lévy process on a d-dimensional Lie group G and let p_t be the law of each $X(t)$. We obtain a one-parameter, strongly continuous, contraction semigroup $(T_t, t \geq 0)$ with generator A on $C_0(G)$ by the prescription

$$(T_t f)(\tau) = \mathbb{E}(f(\tau X(t))) = \int_G f(\tau\sigma)p_t(d\sigma).$$

Note that T_t commutes with left translations. Now let $\{Y_1, \ldots, Y_d\}$ be a fixed basis for the Lie algebra $\mathbf{g}$ of left-invariant vector fields on G. Define a linear manifold $C_2(G)$ that is dense in $C_0(G)$ by the prescription $C_2(G) = \{f \in C_0(G); Y_i f \in C_0(G)$ and $Y_i Y_j f \in C_0(G)$ for all $1 \leq i, j \leq n\}$. Hunt showed that there exist functions $x_i \in C_2(G), 1 \leq i \leq n$ so that $(x_1, \ldots, x_n)$ is a system of canonical coordinates for G at e. A Lévy measure ν is a Borel measure on $G - \{e\}$ for which $\int_{G-\{e\}} \left[\left(\sum_{i=1}^d x_i(\sigma)^2\right) \wedge 1\right] \nu(d\sigma) < \infty$. Hunt was then able to obtain the following key result:

Theorem 0.2 [Hunt's Theorem]. *If X is a Lévy process in G with infinitesimal generator $\mathcal{A}$, then*

1. $C_2(G) \subseteq \mathrm{Dom}(\mathcal{A})$.
2. For each $\tau \in G, f \in C_2(G)$,

(0.5)

$$(\mathcal{A}f)(\tau) = \sum_{i=1}^d b_i Y_i f(\tau) + \frac{1}{2}\sum_{i,j=1}^d a_{ij} Y_i Y_j f(\tau) + \int_{G-\{e\}} \left[f(\tau\sigma) - f(\tau) - \sum_{i=1}^d x_i(\sigma) Y_i f(\tau)\right] \nu(d\sigma),$$

where $b = (b^1, \ldots, b^n) \in \mathbb{R}^n$, $a = (a_{ij})$, is a non-negative definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

Conversely, any linear operator with a representation as in (0.5) is the restriction to $C_2(G)$ of the infinitesimal generator of some Lévy process.

The characteristics (b, a, ν) of a Lévy process determine its law, just as in the Euclidean case.

In the 1990s, H. Kunita and the author were able to generalize the Lévy-Itô decomposition to the extent that for each $f \in C_2(G)$, the real-valued process $f(X) = (f(X(t), t \geq 0)$ can be described (using stochastic integrals in the sense of K. Itô) in terms of a Brownian motion on $\mathbb{R}^d$ and a Poisson random measure on $\mathbb{R}^+ \times (G - \{e\})$. We now give some examples of Lévy processes on a Lie group G:

1. Brownian motion in G.

This is a Lévy process that has characteristics $(0, I, 0)$. It has continuous sample paths (w.p.1), and its generator is (up to the usual factor of one-half) a left-invariant Laplacian on G, $\Delta_G = \sum_{i=1}^d Y_j^2$. The basis dependence is a nuisance here. It can be dispensed with by equipping G with a left-invariant Riemannian metric m, with respect to which $\{Y_1, \ldots, Y_d\}$ is orthonormal. Δ_G is then the Laplace-Beltrami operator associated to (G, m) and the corresponding Brownian motion is a geometrically intrinsic object—indeed, it has played a central role in recent years within the development of analysis in path and loop spaces.

2. The Compound Poisson Process

Let $(\gamma_n, n \in \mathbf{N})$ be a sequence of i.i.d. random variables taking values in G with common law μ and let $(N(t), t \geq 0)$ be an independent Poisson process with intensity $\lambda > 0$. We define the compound Poisson process in G by $Y(t) = \gamma_1\gamma_2 \ldots \gamma_{N(t)}$. In this case the generator is bounded and is given by $(\mathcal{A}f)(\tau) = \int_G (f(\tau\sigma) - f(\tau))\nu(d\sigma)$, for each $f \in C_0(G)$ where the Lévy measure $\nu(\cdot) = \lambda\mu(\cdot)$ is finite.

3. Stable Processes

The theory of stable processes in Lie groups was developed by H. Kunita in the 1990s. His approach was to generalize the self-similarity property, and for this he needed a notion of scaling. This is provided by a *dilation*, i.e., a family of automorphisms $\delta = (\delta(r), r > 0)$ for which $\delta(r)\delta(s) = \delta(rs)$ for all $r, s > 0$, which also possess suitable continuity properties. A Lévy process X in G is *stable* with respect to the dilation δ if $\delta(r)X(s)$ has the same law as $X(rs)$ for each $r, s > 0$. Dilations (and hence stable Lévy processes) can exist only on simply connected nilpotent groups. Stable processes in such groups have some surprising properties, e.g., Kunita has shown that there is no dilation with respect to which Brownian motion in the Heisenberg group is stable. It is however possible to construct a stable process on this group whose first two components are Brownian motion whereas the third is a Cauchy process.

4. Subordinated Processes

Let $Y = (Y(t), t \geq 0)$ be a Lévy process on G and $T = (T(t), t \geq 0)$ be a subordinator that is independent of Y. Just as in the Euclidean case, we can construct a new Lévy process $Z = (Z(t), t \geq 0)$ by the prescription $Z(t) = Y(T(t))$, for each $t \geq 0$.

Lévy processes in Lie groups is a subject that is currently undergoing intense development—see the author's survey article in [2] and the recent book by M. Liao [8]. The latter contains a lot of interesting material on the asymptotics of Lévy processes on noncompact semisimple Lie groups, as $t \to \infty$.

Liao has also found some classes of Lévy processes on compact Lie groups that have L^2-densities. The density then has a "noncommutative Fourier series" expansion via the Peter-Weyl theorem. In the special case of Brownian motion on $SU(2)$, Liao obtains the following beautiful formula for its density ρ_t at time t:

$$\rho_t(\theta) = \sum_{n=1}^{\infty} n \exp\left(-\frac{(n^2-1)t}{64\pi^2}\right) \frac{\sin(2\pi n\theta)}{\sin(2\pi\theta)},$$

where $\theta \in (0, 1]$ parameterizes the maximal torus $\{\text{diag}\left(e^{2\pi i\theta}, e^{-2\pi i\theta}\right), \theta \in [0, 1)\}$.

Another important theme, originally due to R. Gangolli in the 1960s, is to study *spherically symmetric Lévy processes* on semisimple Lie groups G (i.e., those whose laws are bi-invariant under the action of a fixed compact subgroup K). Using Harish-Chandra's theory of spherical functions, one can carry out "Fourier analysis" and obtain a Lévy-Khintchine-type formula. One of the reasons why this is interesting is that G/K is a Riemannian (globally) symmetric space and all such spaces can be obtained in this way. The Lévy process in G projects to a Lévy process in G/K, and this is the prototype for constructions of Lévy processes in more general Riemannian manifolds.

Before leaving the subject of Lévy processes in groups, we briefly mention the general locally compact case. Work on Hilbert's fifth problem during the 1950s established that every such group has an open subgroup of the identity that is a projective limit of Lie groups. This enables the use of Lie group methods within the more general case, and there has been intensive work on this subject since the 1970s by the German school of H. Heyer, W. Hazod, E. Siebert, and their students ([6]). Quite recently, the path properties of Brownian motion in general locally compact groups have been investigated by A. Bendikov and L. Saloff-Coste at Cornell. It will be interesting to see if the new techniques they've developed can be applied to more general classes of Lévy processes.

Lévy Processes in Quantum Groups

Through the work of physicists such as N. Bohr, M. Born, and W. Heisenberg and its mathematical formulation by J. von Neumann, we came to a dual understanding of quantum mechanics. On the one hand, physical observables such as position, momentum, energy, and spin should be described as (not necessarily bounded) self-adjoint linear operators acting in a complex Hilbert space. On the other hand, these observables are also random quantities whose statistical properties are determined by a unit vector in Hilbert space (for pure states) or a more general density matrix (for mixed states). However, the celebrated Heisenberg uncertainty principle tells us that certain pairs of these operators, such as those representing position and momentum, fail to commute. Consequently they cannot both be described together as measurable functions on the same probability space using Kolmogorov's prescription, and hence they cannot have a joint probability distribution.

To describe the probabilistic features of quantum theoretic phenomena systematically, we need to take an algebraic viewpoint. We define a *quantum probability space* to be a pair (B, ω) where B is a complex $*$-algebra (with identity I) and ω is a state on B, i.e., a positive, linear map for which $\omega(I) = 1$. If B is a C*-algebra, we can recover a Hilbert space viewpoint by taking the Gelfand-Naimark-Segal representation.

Quantum stochastic processes were introduced by L. Accardi, A. Frigerio, and J. T. Lewis in the 1980s. Every "classical" stochastic process $(X(t), t \geq 0)$ with state space E gives rise to a family of $*$-homomorphisms $(j_t, t \geq 0)$ from the $*$-algebra $B_b(E)$ of bounded measurable functions on E into the $*$-algebra $L^\infty(\Omega, \mathcal{F}, P)$ by the prescription $j_t(f) = f \circ X(t)$. Given a quantum probability space (B, ω) and a $*$-algebra A, a *quantum stochastic process* is a family $(j_t, t \geq 0)$ of $*$-homomorphisms from A into B. Many concrete examples of these have been constructed using the quantum stochastic calculus of R. L. Hudson and K. R. Parthasarathy as solutions of operator-valued stochastic differential equations driven by "quantum noise", i.e., the creation, conservation, and annihilation processes acting in a suitable Fock space.

In order to clarify the last remark, we make a brief diversion. *Fock space* $\Gamma(h)$ over a complex Hilbert space h is $\Gamma(h) := \bigoplus_{n=0}^{\infty} h^{(n)}$, where $h^{(0)} = \mathbb{C}, h^{(1)} = h$, and $h^{(n)}$ is the tensor product of n copies of h. It is often desirable to restrict to boson (symmetric) or fermion (antisymmetric) Fock space, which are the closed subspaces obtained by restricting to symmetric or antisymmetric tensors, respectively. For each $f \in h$ the creation operator $a^\dagger(f)$ maps each $h^{(n)}$ to $h^{(n+1)}$ while the annihilation operator $a(f)$ maps each $h^{(n)}$ to $h^{(n-1)}$. For each self-adjoint T acting in h, the conservation operator $d\Gamma(T)$ maps $h^{(n)}$ to itself. All three types of operator are densely defined linear operators in $\Gamma(h)$ (see, e.g., [9] for precise definitions). As a by-product of work on factorizable representations of current groups in the 1960s and 1970s it was found that any Lévy process $X = (X(t), t \geq 0)$ on $\mathbb{R}^d$ can be realized as a family of self-adjoint operators acting in a symmetric Fock space, where the Lévy-Itô decomposition (0.3) appears as a certain combination of creation, conservation, and annihilation operators. In the 1980s, Hudson and Parthasarathy realized that they could build interesting classes of quantum stochastic processes by developing a stochastic calculus in which each of the creation, conservation, and annihilation parts is treated as a separate operator-valued process rather than in a special "classical" self-adjoint combination.

We can now make an attempt at defining a "quantum Lévy process". At the very least this should be a quantum stochastic process $(j_t, t \geq 0)$ where each j_t is embedded as $k_{0,t}$ into an associated two-parameter family of $*$-homomorphisms $(k_{s,t}, 0 \leq s \leq t < \infty)$ which are the "increments" of the process. We generalize the key axiom (L1). The stationary increments requirement becomes $\omega(k_{s,t}(a)) = \omega(k_{0,t-s}(a))$, for each $a \in A$. For independent increments, we have a choice from a number of competing algebraic notions of independence, each of which will yield a distinct notion of Lévy process. The simplest, called *tensor* (or *bosonic) independence*, requires that

$$\omega(k_{s_1,t_1}(a_1)k_{s_2,t_2}(a_2)\cdots k_{s_n,t_n}(a_n)) = \prod_{i=1}^{n} \omega(k_{s_i,t_i}(a_i)),$$

for all $n \in \mathbb{N}, a_1, \ldots, a_n \in A, 0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \cdots \leq s_n \leq t_n < \infty$, whenever each pair $k_{s_i,t_i}(a_i)$ and $k_{s_j,t_j}(a_j)$ commute. Other notions of independence that could be used include the fermionic (or $\mathbb{Z}_2$ graded version) or the free independence of D. Voiculescu. Axioms (L2) and (L3) translate rather easily into this framework; however, the concept we have thus obtained is too general, as it is not clear how $k_{s,t}$ has captured the notion of "increment".

To overcome this problem, we need to generalize the group concept algebraically, and this is precisely the purpose of quantum groups. More precisely, we need A to be a $*$-*bialgebra*, i.e., a $*$-algebra in which the multiplication and identity have been dualized to give a compatible co-algebra structure. We thus require that there are two $*$-homomorphisms, a comultiplication $\Delta : A \to A \otimes A$ and a co-unit $\varepsilon : A \to \mathbb{C}$ which satisfy the co-associativity and co-unit axioms:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta\ ,$$
$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta,$$

where id is the identity mapping.

If A is a $*$-bialgebra, we obtain a *quantum Lévy process* on A when we augment the generalizations of (L1) to (L3) with an additional axiom

(L0) $k_{r,s} * k_{s,t} = k_{r,t}$, for all $0 \leq r \leq s \leq t < \infty$,

where the *convolution* is given by

$$k_{r,s} * k_{s,t} = m_B \circ (k_{r,s} \otimes k_{s,t}) \circ \Delta;$$

here m_B denotes the multiplication in B.

To understand the meaning of (L0) in the simplest possible context, let X be a Lévy process in a finite group G, and take A to be the $*$-bialgebra of all complex valued functions on G with the usual pointwise algebra operations and comultiplication $(\Delta f)(\sigma_1, \sigma_2) = f(\sigma_1\sigma_2)$ and co-unit $\varepsilon(f) = f(e)$. Take $B = L^\infty(\Omega, \mathcal{F}, P)$ and each $k_{s,t}f = f \circ X(s)^{-1}X(t)$. Then (L0) precisely expresses the "increment property", $X(r)^{-1}X(s)X(s)^{-1}X(t) = X(r)^{-1}X(t)$.

Quantum Lévy processes first arose in work by W. von Waldenfels on a model of the emission and absorption of light by atoms interacting with "noise". The quantum stochastic process obtained appeared to be a noncommutative analogue of a Lévy process on the unitary group $U(d)$, and this was made precise in terms of quantum Lévy processes when $U(d)$ was replaced by a noncommutative $*$-bialgebra that generalizes the coefficient algebra of $U(d)$. The theory of quantum Lévy processes has been extensively developed by M. Schürmann and U. Franz in Greifswald, Germany (see [13] or Chapter 7 of [9]). In particular, all quantum Lévy processes are equivalent to solutions of quantum stochastic differential equations driven by creation, conservation, and annihilation processes acting in a suitable Fock space.

We briefly describe one interesting application of quantum Lévy processes to classical probability. Let $B = (B(t), t \geq 0)$ be a one-dimensional Brownian motion and $g(t) = \sup\{0 \leq s \leq t; B(s) = 0\}$. *Azéma's martingale* $M(t) = \sqrt{\frac{\pi}{2}}\,\mathrm{sign}(B(t))\sqrt{t - g(t)}$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{M(s); 0 \leq s \leq t\}$. This process has many intriguing features, e.g., M. Emery proved that it shares with Brownian motion and the compensated Poisson process the rare property of being "chaotically complete" (i.e., the linear span of all multiple Wiener integrals is dense in the natural L^2 space), but it is not a Lévy process on $\mathbb{R}$ in the usual sense. However, Schürmann has shown that it is a quantum Lévy process on a certain $*$-bialgebra generated by two indeterminates.

Conclusion

One way of assessing the health of an area of mathematics is to explore the extent to which it permeates other aspects of the subject. Another way is to examine its use in applications. Regarding both of these criteria, Lévy processes appears to be flourishing. Indeed, limitations of space in this article have prevented me from discussing a host of other topics, including new theoretical advances in the fluctuation theory of real-valued Lévy processes due to J. Bertoin and R. A. Doney and applications to turbulence, time series, and the codification of branching processes. Readers are invited to join the author in speculating that the interplay of Gaussian continuous motion with Poisson jumps, or alternatively its quantum theoretic manifestation within the dance of creation, conservation, and annihilation operators, is a universal feature of a class of random motions (both classical and quantum) that is sufficiently wide to keep mathematicians busy for many years to come.

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About the Cover

Kleinian Pearls

This month's cover was created by David Wright, who explains it in a brief article in this issue (pages 1332–1333). He entered it in the 2003 NSF Visualization Challenge, in which it was a semifinalist. Limit sets of Kleinian groups and how to draw them are major themes of the well-illustrated book *Indra's Pearls* (Cambridge University Press, 2002), written by Wright with coauthors David Mumford and Caroline Series.

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