Published paper
Applebaum, D. (2006) *Martingale-valued measures, Ornstein-Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space*, In Memoriam Paul-Andre Meyer, Lecture Notes in Mathematics, 1874, pp. 171-196
http://dx.doi.org/10.1007/978-3-540-35513-7_14

Martingale-valued Measures, Ornstein-Uhlenbeck Processes with Jumps and Operator Self-Decomposability in Hilbert Space

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Dedicated to the memory of Paul-André Meyer

Summary. We investigate a class of Hilbert space valued martingale-valued measures whose covariance structure is determined by a trace class positive operator valued measure. The paradigm example is the martingale part of a Lévy process. We develop both weak and strong stochastic integration with respect to such martingale-valued measures. As an application, we investigate the stochastic convolution of a C_0-semigroup with a Lévy process and the associated Ornstein-Uhlenbeck process. We give an infinite dimensional generalisation of the concept of operator self-decomposability and find conditions for random variables of this type to be embedded into a stationary Ornstein-Uhlenbeck process.

Key Words and Phrases:- martingale-valued measure, positive operator valued measure, trace class operator, nuclear, decomposable, Lévy process, C_0-semigroup, stochastic convolution, Ornstein-Uhlenbeck process, operator self-decomposability, exponentially stable semigroup.

1 Introduction

The aim of this paper is to introduce some new concepts into stochastic analysis of Hilbert space valued processes with a view to gaining deeper insights into the structure of Lévy processes and other processes which can be built from these.

We begin with an investigation of Hilbert space valued martingale-valued measures. Finite dimensional versions of these (called "martingale measures" therein) were first introduced by Walsh [38] to formulate stochastic partial differential equations (SPDEs) driven by a continuous space-time white noise. They were further developed in [23] and in [3] they were generalised to deal with SPDEs with jumps. In [2], the author found them a convenient tool for simultaneously dealing with stochastic integration with respect to the Brownian and the compensated small jumps part of a Lévy process, when the integrands depend both on time and the jump-space variable. Here we extend this

** Work carried out at The Nottingham Trent University latter construction to the infinite dimensional context. In particular, we investigate a class of martingale-valued measures whose covariance structure is determined by a trace class positive operator valued measure. This is precisely the covariance structure found in the martingale part of a Lévy process- indeed it is well known that the covariance operator of Brownian motion is trace class (see e.g. [11], proposition 2.15, p.55). Here we show that the covariance of the compensated small jumps is also determined by such operators, which in this case are a continuous superposition of finite rank operators. Our approach exploits the Lévy-Itô decomposition of a Lévy process into drift, Brownian, small jump and large jump parts which has recently been extended to type 2 Banach spaces by Albeverio and Rüdiger [1].

Having established a natural class of martingale-valued measures M, we develop both weak and strong stochastic integrals of suitable predictable processes. In the first of these the integrand $(F(t,x), t \geq 0, x \in E)$ (where E is a Lusin space) is vector valued and we generalise the approach of Kunita [24], who dealt with the case where M is a martingale, to construct the scalar valued process $\int_0^t \int_E (F(s,x), M(ds,dx))_H$, where $(\cdot,\cdot)_H$ is the inner product in the Hilbert space H. In the second of these, $(G(t,x), t \geq 0, x \in E)$ is operator-valued and we generalise the stochastic integral of Métivier [28] who dealt with the case where M is a martingale (see also [29], [11] for the case of Brownian motion), to construct the Hilbert space valued object $\int_0^t \int_E G(s,x)M(ds,dx)$.

As an application of these techniques, we first study the stochastic convolution $\int_0^t S(r)dX(r)$, of a C_0-semigroup $(S(r), r \geq 0)$ with infinitesimal generator J with a Lévy process $X = (X(t), t \geq 0)$. We then apply this to investigate the generalised Langevin equation

$$dY(t) = JY(t) + dX(t), \tag{1}$$

whose unique weak solution is the Ornstein-Uhlenbeck process. Equations of this type driven by general Lévy processes, were first considered by S.J.Wolfe ([39]) in the scalar case where J is a negative constant. Sato and Yamazoto [36], [37] generalised this to the multi-dimensional case wherein $-J$ is a matrix all of whose eigenvalues have positive real parts. The generalisation to infinite dimensions was first carried out by A.Chojnowska-Michalik [9], [8] (see also [11], [6] for the Brownian motion case). Using our stochastic integration theory we are able to give an alternative construction of the solution in which the Lévy-Itô decomposition is preserved within its structure. This is useful for later analysis as we see below.

We remark that, in the finite dimensional case, Ornstein-Uhlenbeck processes driven by non-Gaussian Lévy processes have recently been applied to the construction of self-similar processes via the Lamperti transform ([18]) and to models of stochastic volatility in the theory of option pricing [5], [31]. In the latter case, it may be that the infinite dimensional model as considered here, is more appropriate, as it can approximate the very large number of incremental market activities which lead to volatility change.

Finally, we consider an infinite-dimensional generalisation of self-decomposability. We recall that a real-valued random variable X is self-decomposable if for any $0 < c < 1$, there exists a random variable Y_c, which is independent of X such that

$$X \stackrel{d}{=} cX + Y_c. \tag{2}$$

Such random variables were first studied by Paul Lévy and they arise naturally as weak limits of normalised sums of independent (but not necessarily identically distributed) random variables (see e.g. [35], section 3.15). The definition was extended to Banach space valued random variables by Jurek and Vervaat [22] (with c still a scalar) while Jurek and Mason [21] considered the finite-dimensional case of "operator self-decomposability" where c is replaced by a semigroup $(e^{-tJ}, t \geq 0)$, with J an invertible matrix. Jurek [19] also investigated the case where J is a bounded operator in a Banach space. It is a consequence of results found in [39], [21], [22] and [37] that X is (operator) self-decomposable if and only if it can be embedded as $X(0)$ in a stationary Ornstein-Uhlenbeck process. Furthermore a necessary and sufficient condition for the required stationarity is that the Lévy measure ν of X has a certain logarithmic moment, more precisely $\int_{|x|\geq 1} \log(1 + |x|)\nu(dx) < \infty$, so we see that this is a condition on the large jumps of X (see also [16]).
Here we generalise operator self-decomposability by taking $(e^{-tJ}, t \geq 0)$ to be a contraction semigroup acting in a Hilbert space H (see [20] for the case where it is a group acting in a Banach space). We emphasise that, in contrast to the cases discussed in the previous paragraph, J is typically an unbounded operator. We are able to obtain a partial generalisation of the circle of ideas described above which relates self-decomposability, stationary Ornstein-Uhlenbeck processes and logarithmic moments of the Lévy measure. The failure to obtain a full generalisation arises from dropping the condition that J is invertible, which appears to be unnatural in this setting and also from the fact that the operators e^{-tJ} are no longer invertible. We note that the link between stationarity and logarithmic moments has also been established in [9] using different methods, and by a more indirect route than that given here (see also [15]).
The stochastic integration theory developed herein will have extensive further applications. In particular, it can be used to construct solutions to stochastic differential equations driven by Hilbert space valued processes with jumps, generalising the Brownian motion case ([11], [25]). The details will appear elsewhere (see [27] for work in a similar direction).

Notation. $\mathbb{R}^+ = [0, \infty)$. If X is a topological space, then $\mathcal{B}(X)$ denotes its Borel σ-algebra. If H is a real separable Hilbert space, $b(H)$ is the space of bounded Borel measurable real-valued functions on H and $\mathcal{L}(H)$ is the $*$-algebra of all bounded linear operators on H. The domain of a linear operator T acting in H is denoted as $\mathrm{Dom}(T)$.

Acknowledgement. I am grateful to Zbigniew Jurek for some useful comments and the referee for a number of useful observations.

2 Martingale-Valued Measures With Values in a Hilbert Space

2.1 Hilbert Space Valued Martingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ be a stochastic base wherein the filtration $(\mathcal{F}_t, t \geq 0)$ satisfies the usual hypotheses of completeness and right continuity. Let H be a real separable Hilbert space with inner product $(\cdot, \cdot)_H$ and associated norm $||\cdot||$. Throughout this article, unless contra-indicated, all random variables and processes are understood to be H-valued. To any such random variable X, we associate the real-valued random variable $||X||$, where $||X||(\omega) = ||X(\omega)||$, for each $\omega \in \Omega$.
The predictable σ-algebra $\mathcal{P}$ is the smallest sub-σ-algebra of $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$ with respect to which all mappings $F : \mathbb{R}^+ \times \Omega \to H$ are measurable, wherein $(F(t), t \geq 0)$ is adapted and $t \to F(t, \omega)$ is strongly left continuous for each $\omega \in \Omega$.
If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ and X is a random variable such that $\mathbb{E}(||X||) < \infty$, the conditional expectation of X given $\mathcal{G}$ is the unique $\mathcal{G}$-measurable random variable $\mathbb{E}_{\mathcal{G}}(X)$ for which

$$\mathbb{E}(1_A X) = \mathbb{E}(1_A \mathbb{E}_{\mathcal{G}}(X)),$$

for all $A \in \mathcal{G}$ (see e.g. [11], section 1.3). Many familiar properties of conditional expectation from the case $H = \mathbb{R}$ carry over to the general case, in particular

$$\mathbb{E}_{\mathcal{G}}((X, Y)_H) = (X, \mathbb{E}_{\mathcal{G}}(Y))_H, \text{ a.s.}$$

if $\mathbb{E}(||X|| \vee ||Y||) < \infty$ and X is $\mathcal{G}$-measurable.
An adapted process $X = (X(t), t \geq 0)$ is a *martingale* if $\mathbb{E}(||X(t)||) < \infty$ and $\mathbb{E}(X(t)|\mathcal{F}_s) = X(s)$ (a.s.), for all $0 \leq s \leq t < \infty$. A martingale is said to be *square-integrable* if $\mathbb{E}(||X(t)||^2) < \infty$, for all $t \geq 0$. By proposition 3 of [24], any square-integrable martingale has a strongly càdlàg modification.
If X is a square-integrable martingale, then $(||X(t)||^2, t \geq 0)$ is a non-negative uniformly integrable submartingale, hence by the Doob-Meyer decomposition, there is a unique increasing, predictable integrable process $(\langle X \rangle(t), t \geq 0)$ such that $(||X(t)||^2 - \langle X \rangle(t), t \geq 0)$ is a real-valued martingale (see e.g. [24]). If $Y = (Y(t), t \geq 0)$ is another square-integrable martingale, we may, for each $t \geq 0$, define $\langle X, Y \rangle(t)$ in the usual way by polarisation, i.e.

$$\langle X, Y \rangle(t) = \frac{1}{4}[\langle X + Y \rangle(t) - \langle X - Y \rangle(t)].$$

Note that for all $0 \leq s \leq t < \infty$,

$$\mathbb{E}((X(t) - X(s), Y(t) - Y(s))_H|\mathcal{F}_s) = \mathbb{E}(\langle X, Y \rangle(t) - \langle X, Y \rangle(s)|\mathcal{F}(s)).$$

Two square-integrable martingales X and Y are said to be *orthogonal* if $\langle X, Y \rangle(t) = 0$, for all $t \geq 0$ (or equivalently, if $((X(t), Y(t))_H, t \geq 0)$ is a real-valued martingale).

Note. A different definition of $\langle \cdot \rangle$ for Hilbert space valued martingales is given in [11], section 3.4. We prefer to use that of [24] as it appears to be more general.

2.2 Martingale-Valued Measures

Let (S, Σ) be a Lusin space, so that S is a Hausdorff space which is the image of a Polish space under a continuous bijection and Σ is a Borel subalgebra of $\mathcal{B}(S)$ (see e.g. Chapter 8 of [10]). We assume that there is a ring $\mathcal{A} \subset \Sigma$ and an increasing sequence $(S_n, n \in \mathbb{N})$ in Σ such that

- $S = \bigcup_{n\in\mathbb{N}} S_n$
- $\Sigma_n := \Sigma|_{S_n} \subseteq \mathcal{A}$, for all $n \in \mathbb{N}$.

A *martingale-valued measure* is a set function $M : \mathbb{R}^+ \times \mathcal{A} \times \Omega \to H$ which satisfies the following (c.f [38], [23]):

1. $M(0, A) = M(t, \emptyset) = 0$ (a.s.), for all $A \in \mathcal{A}, t \geq 0$.
2. $M(t, A \cup B) = M(t, A) + M(t, B)$ (a.s.), for all $t \geq 0$ and all disjoint $A, B \in \mathcal{A}$.
3. $(M(t, A), t \geq 0)$ is a square-integrable martingale for each $A \in \mathcal{A}$ and is orthogonal to $(M(t, B), t \geq 0)$, whenever $A, B \in \mathcal{A}$ are disjoint.
4. $\sup\{\mathbb{E}(||M(t, A)||^2), A \in \Sigma_n\} < \infty$, for all $n \in \mathbb{N}, t > 0$.

Note. In Walsh's terminology [38], M is a "σ-finite L^2-valued orthogonal martingale measure".

Whenever $0 \leq s \leq t \leq \infty, M((s, t], \cdot) := M(t, \cdot) - M(s, \cdot)$. M is said to have *independent increments* if $M((s, t], A)$ is independent of $\mathcal{F}_s$ for all $A \in \mathcal{A}, 0 \leq s \leq t < \infty$.

Given a martingale valued measure M, for each $t \geq 0$, we can define a (random) real-valued set function $\langle M\rangle(t, \cdot)$ on $\mathcal{A}$ and (3) ensures that

$$\langle M\rangle(t, A \cup B) = \langle M\rangle(t, A) + \langle M\rangle(t, B) \text{ a.s.}$$

for all $t \geq 0$ and all disjoint $A, B \in \mathcal{A}$. A theorem of Walsh ([38], theorem 2.7, p.299) enables us to "regularise" $\langle M\rangle$ to obtain a (random) predictable σ-finite measure on $\mathcal{B}(\mathbb{R}^+) \otimes \Sigma$, which coincides with $\langle M\rangle$ (a.s.) on sets of the form $[0, t] \times A$, where $t > 0, A \in \mathcal{A}$. In the sequel, we will abuse notation to the extent of also denoting this measure by $\langle M\rangle$.
A *positive-operator valued measure* or (*POV measure* for short) on (S, Σ) is a family $(T_A, A \in \mathcal{A})$ of bounded positive self-adjoint operators in H for which

- $T_\emptyset = 0$,
- $T_{A\cup B} = T_A + T_B$, for all disjoint $A, B \in \mathcal{A}$

Note. This is a slightly different use of the term POV measure than that employed in the theory of measurement in quantum mechanics (see e.g. [13], section 3.1).

We say that a POV measure is *decomposable* if there exists a strongly measurable family of bounded positive self-adjoint operators in H, $\{T_x, x \in S\}$ and a σ-finite measure λ on (S, Σ) such that

$$T_A\psi = \int_A T_x \psi \lambda(dx),$$

for each $A \in \mathcal{A}, \psi \in H$, where the integral is understood in the Bochner sense.

We recall that a bounded linear operator Z on H is *trace class* if $\mathrm{tr}(|Z|) < \infty$, where $|Z| = (ZZ^*)^{\frac{1}{2}}$. Let $\mathcal{L}_1(H)$ denote the space of all trace class operators on H, then $\mathcal{L}_1(H)$ is a real Banach space with respect to the norm $||Z||_1 = \mathrm{tr}(|Z|)$ (see e.g. [34], section VI.6). A POV measure is said to be trace class if each of its constituent operators is.

Now let M be a martingale-valued measure on $\mathbb{R}^+ \times S$. We say that it is *nuclear* if there exists a pair (T, ρ) where

- $T = (T_A, A \in \mathcal{A})$ is a trace class POV measure in H,
- ρ is a σ-finite measure on $\mathbb{R}^+$,

such that for all $0 \leq s \leq t < \infty, A \in \mathcal{A}, \psi \in H$,

$$\mathbb{E}(|(M((s,t],A),\psi)_H|^2) = (\psi, T_A\psi)\rho((s,t]) \tag{3}$$

A nuclear martingale-valued measure is *decomposable* if $(T_A, A \in \mathcal{A})$ is decomposable.

Proposition 2.1 *If M is a nuclear martingale-valued measure, then for all $t \geq 0, A \in \mathcal{A}$,*

$$\mathbb{E}(\langle M\rangle(t,A)) = ||T_A||_1 \rho((0,t]).$$

Proof. Let $(e_n, n \in \mathbb{N})$ be a maximal orthonormal set in H. We have

$$\begin{aligned}\mathbb{E}(\langle M\rangle(t,A)) &= \mathbb{E}(||M(t,A)||^2)\\ &= \sum_{n=1}^{\infty} \mathbb{E}(|(e_n, M(t,A))_H|^2)\\ &= \rho((0,t])\sum_{n=1}^{\infty}(e_n, T_A e_n)\\ &= \rho((0,t])\mathrm{tr}(T_A).\end{aligned}$$

□

2.3 Lévy Processes

Let X be a *Lévy process* taking values in H, so that X has stationary and independent increments, is stochastically continuous and satisfies $X(0) = 0$ (a.s.). If p_t is the law of $X(t)$ for each $t \geq 0$, then $(p_t, t \geq 0)$ is a weakly continuous convolution semigroup of probability measures on H. We have the *Lévy-Khinchine formula* (see e.g. [32]) which yields for all $t \geq 0, \psi \in H$,

$$\mathbb{E}(\exp(i(\psi, X(t))_H) = e^{ta(\psi)},$$

where

$$a(\psi) = i(\zeta, \psi)_H - \frac{1}{2}(\psi, Q\psi) + \int_{H-\{0\}} (e^{i(u,\psi)_H} - 1 - i(u, \psi)_H 1_{\{||u||<1\}})\nu(du), \quad (4)$$

where $\zeta \in H, Q$ is a positive, self-adjoint trace class operator on H and ν is a *Lévy measure* on $H - \{0\}$, i.e. $\int_{H-\{0\}}(||x||^2 \wedge 1)\nu(dx) < \infty$. We call the triple (ζ, Q, ν) the *characteristics* of the process X and the mapping a, the *characteristic exponent* of X.

Example 1 (Q- Brownian motion)

Q- Brownian motion $B_Q = (B_Q(t), t \geq 0)$ has characteristics $(0, Q, 0)$. It is a Gaussian process with continuous sample paths and covariance operator Q (see e.g. [11], section 4.1) so that $\mathbb{E}((\psi, B_Q(t))_H^2) = t(\psi, Q\psi)$, for each $\psi \in H, t \geq 0$. If $(\lambda_n, n \in \mathbb{N})$ are the eigenvalues of Q and $(e_n, n \in \mathbb{N})$ are the corresponding normalised eigenvectors, we have the useful representation of B_Q as an L^2-convergent series:

$$B_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \quad (5)$$

for each $t \geq 0$, where $(\beta_n, n \in \mathbb{N})$ are independent standard real-valued Brownian motions.

In the sequel, a Lévy process with characteristics $(\zeta, Q, 0)$ will be called a *Q-Brownian motion with drift*, while a Lévy process with characteristics $(\zeta, 0, \nu)$ will be said to be *non-Gaussian*.

Example 2 (α-Stable Lévy Processes)

A Lévy process is said to be *stable* if p_t is a stable law for each $t \geq 0$, i.e. for all $a, b > 0$, there exists $\phi \in H$ and $c > 0$ such that

$$(\tau_a p_t) * (\tau_b p_t) = \delta_\phi * (\tau_c p_t),$$

where for any measure q on H, $(\tau_a q)(E) = q(a^{-1}E)$, for all $E \in \mathcal{B}(H)$.
By a theorem of Jajte [17], a Lévy process X is stable iff it is a Q-Brownian motion with drift or it is non-Gaussian and there exists $0 < \alpha < 2$ such that $\tau_c \nu = c^\alpha \nu$, for all $c > 0$. We call this latter case an *α-stable Lévy process*. An extensive account of stable distributions in Hilbert and Banach spaces can be found in Chapters 6 and 7 of [26].

From now on we will always assume that Lévy processes have strongly càdlàg paths. We also strengthen the independent increments requirement on X by assuming that $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.
If X is a Lévy process, we write $\Delta X(t) = X(t) - X(t-)$, for all $t > 0$. We obtain a Poisson random measure N on $\mathbb{R}^+ \times (H - \{0\})$ by the prescription:

$$N(t, E) = \#\{0 \leq s \leq t; \Delta X(s) \in E\},$$

for each $t \geq 0, E \in \mathcal{B}(H - \{0\})$. The associated compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx).$$

Let $A \in \mathcal{B}(H - \{0\})$ with $0 \notin \overline{A}$. If $f : A \to H$ is measurable, we may define

$$\int_A f(x)N(t, dx) = \sum_{0 \leq s \leq t} f(\Delta X(s))1_A(\Delta X(s))$$

as a random finite sum. Let ν_A denote the restriction of the measure ν to A, so that ν_A is finite. If $f \in L^2(A, \nu_A; H)$, we define

$$\int_A f(x)\tilde{N}(t, dx) = \int_A f(x)N(t, dx) - t\int_A f(x)\nu(dx),$$

then by standard arguments (see e.g. [2], Chapter 2) we see that $(\int_A f(x)\tilde{N}(t, dx), t \geq 0)$ is a centred square-integrable martingale with

$$\mathbb{E}\left(\left|\left|\int_A f(x)\tilde{N}(t, dx)\right|\right|^2\right) = t\int_A ||f(x)||^2\nu(dx), \tag{6}$$

for each $t \geq 0$ (see also theorem 3.2.5 in [1]).
The Lévy-Itô decomposition for a càdlàg Lévy process taking values in a separable type 2 Banach space is established in [1]. We only need the Hilbert space version here:

Theorem 1. *[1] If H is a separable Hilbert space and $X = (X(t), t \geq 0)$ is a càdlàg H-valued Lévy process with characteristic exponent given by (4), then for each $t \geq 0$,*

$$X(t) = t\zeta + B_Q(t) + \int_{||x||<1} x\tilde{N}(t, dx) + \int_{||x||\geq 1} xN(t, dx), \tag{7}$$

where B_Q is a Brownian motion which is independent of N.

In (7),

$$\int_{||x||<1} x\tilde{N}(t, dx) = \lim_{n\to\infty}\int_{\frac{1}{n}<||x||<1} x\tilde{N}(t, dx),$$

where the limit is taken in the L^2-sense, and it is a square-integrable martingale.
Let $S = \{x \in H; ||x|| < 1\}$ and take Σ to be its Borel σ-algebra, then it is easy to check that M is a martingale valued measure on $\mathbb{R}^+ \times S$, where

$$M(t, A) = B_Q(t)\delta_0(A) + \int_{A-\{0\}} x\tilde{N}(t, dx), \tag{8}$$

for each $t \geq 0, A \in \mathcal{A}$. We call M a *Lévy martingale-valued measure.*

Here we take $\mathcal{A} = \mathcal{A}_0 \cup \{0\}$, where $\mathcal{A}_0 = \{A \in \Sigma; 0 \notin \overline{A}\}$ and each $S_n = \{x \in S : \frac{1}{n} < ||x|| < 1\}$.
We now aim to show that M is nuclear. To this end, we introduce the family of linear operators $(F_A, A \in \mathcal{A}_0)$ on H given by

$$F_A y = \int_A (x, y)_H x \nu(dx),$$

so that each F_A is a continuous superposition of finite-rank operators (using the Dirac notation employed in physics, we would write "$F_A = \int_A (|x\rangle\langle x|)\nu(dx)$").
It is easy to see that each $||F_A|| \leq \int_A ||x||^2 \nu(dx) < \infty$, hence F_A is bounded. Straightforward manipulations show that F_A is positive, self-adjoint. F_A is also trace class. To see this, let $(e_n, n \in \mathbb{N})$ be a maximal orthonormal set in H, then

$$\begin{aligned} \text{tr}(F_A) &= \sum_{n=1}^{\infty} (e_n, F_A e_n)_H \\ &= \sum_{n=1}^{\infty} \int_A (x, e_n)_H^2 \nu(dx) \\ &= \int_A ||x||^2 \nu(dx) < \infty. \end{aligned}$$

Theorem 2. *If M is a Lévy martingale-valued measure of the form (8), then M is nuclear with ρ being Lebesgue measure on $\mathbb{R}^+$ and*

$$T_A = Q\delta_0(A) + F_{A-\{0\}}, \tag{9}$$

for all $A \in \mathcal{A}$.

The proof follows easily from the above calculations and (6).

It is straightforward to deduce that $(T_A, A \in \mathcal{A})$ is decomposable, wherein $\lambda = \nu + \delta_0$ and

$$T(x) = \begin{cases} Q & \text{if } x = 0 \\ (x, \cdot)_H x & \text{if } x \neq 0. \end{cases}$$

3 Stochastic Integration

3.1 Weak Stochastic Integration

Let M be a martingale-valued measure. Fix $T > 0$. We denote by $\mathcal{H}_2^M(T; S)$ the space of all $\mathcal{P} \otimes \Sigma$-measurable mappings $F : [0, T] \times S \times \Omega \to H$ for which

$$\mathbb{E}\left(\int_0^T \int_S ||F(s, x)||^2 \langle M \rangle (ds, dx) \right) < \infty.$$

Then $\mathcal{H}_2^M(T;S)$ is a real Hilbert space. $\mathcal{S}(T;S)$ is the subspace of all $F \in \mathcal{H}_2^M(T;S)$ for which

$$F = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} F_{ij} 1_{(t_i,t_{i+1}]} 1_{A_j},$$

where $N_1, N_2 \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_{N_1+1} = T, A_0, \ldots, A_{N_2}$ are disjoint sets in $\mathcal{A}$ and each F_{ij} is a bounded $\mathcal{F}_{t_i}$-measurable random variable. $\mathcal{S}(T;S)$ is dense in $\mathcal{H}_2^M(T;S)$ (see e.g. [2], section 4.1). We generalise the construction of stochastic integrals with respect to martingales as developed in [24]. For each $F \in \mathcal{S}(T;S), 0 \leq t \leq T$, define

$$I_t(F) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} (F_{ij}, M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H.$$

Then

$$\begin{aligned}
&\mathbb{E}(|I_t(F)|^2) \\
&= \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} \mathbb{E}[(F_{ij}, M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H (F_{kl}, M((t \wedge t_k, t \wedge t_{k+1}], A_l))_H] \\
&= \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbb{E}[|(F_{ij}, M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H|^2] \\
&\leq \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbb{E}[||F_{ij}||^2.||M((t \wedge t_i, t \wedge t_{i+1}], A_j)||^2] \\
&= \mathbb{E}\left[\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} ||F_{ij}||^2 (\langle M \rangle((t \wedge t_i, t \wedge t_{i+1}], A_j)\right] \\
&= \mathbb{E}\left[\int_0^t \int_S ||F(s,x)||^2 \langle M \rangle (ds, dx)\right].
\end{aligned}$$

Hence I_t extends to a contraction from $\mathcal{H}_2^M(T;S)$ to $L^2(\Omega, \mathcal{F}, P)$. For each $0 \leq t \leq T, F \in \mathcal{H}_2^M(T;S)$,

$$\int_0^t \int_S (F(s,x), M(ds,dx))_H := I_t(F).$$

By standard arguments, we see that $(I_t(T); 0 \leq t \leq T)$ is a centred square-integrable real-valued martingale with

$$\mathbb{E}(|I_t(F)|^2 \leq \mathbb{E}\left[\int_0^t \int_S ||F(s,x)||^2 \langle M \rangle (ds, dx)\right],$$

for all $0 \leq s \leq T, F \in \mathcal{H}_2^M(T;S)$.

3.2 Strong Stochastic Integration

In the section we will take the martingale-valued measure M to be nuclear and decomposable. Let $(R(t,x), t \in [0,T], x \in S)$ be a family of bounded linear operators on H. We say that they are predictable if the mappings $[0,T] \times S \to H$, given by $(t,x) \to R(t,x)\psi$ are $\mathcal{P} \otimes \Sigma$-measurable, for each $\psi \in H$. Our aim is to define $\int_0^t \int_S R(s,x)M(ds,dx)$ as random vectors, for each $t \geq 0$. We follow the approach given in section 4.2 of [11] for the case of Brownian motion (see also [28], section 4.22). Let $\mathcal{H}_2(T;\lambda,\rho)$ be the real Hilbert space of all predictable R for which

$$\mathbb{E}\left(\int_0^T \int_S \operatorname{tr}(R(t,x)T_x R(t,x)^*)\lambda(dx)\rho(dt)\right) < \infty. \tag{10}$$

We denote by $S(T;\lambda,\rho)$ the dense linear space of all $R \in \mathcal{H}_2(T;\lambda,\rho)$, which take the form

$$R = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} R_{ij} 1_{(t_i,t_{i+1}]} 1_{A_j},$$

where $N_1, N_2 \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_{N_1+1} = T, A_0, \ldots, A_{N_2}$ are disjoint sets in $\mathcal{A}$ and each R_{ij} is a bounded operator valued $\mathcal{F}_{t_i}$-measurable random variable. For each $R \in S(T;\lambda,\rho), 0 \leq t \leq T$, define

$$J_t(R) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} R_{ij} M((t \wedge t_i, t \wedge t_{i+1}], A_j).$$

Let $(e_n, n \in \mathbb{N})$ be a maximal orthonormal set in H. We compute

$$\begin{aligned}
\mathbb{E}(J_t(R)) &= \sum_{n=1}^{\infty}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbb{E}((R_{ij}^* e_n, M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H) e_n \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbb{E}((R_{ij}^* e_n, e_m)_H (e_m, M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H) e_n \\
&= 0.
\end{aligned}$$

Similar arguments yield

$$\begin{aligned}
\mathbb{E}(||J_t(R)||^2) &= \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbb{E}(||R_{ij} M((t \wedge t_i, t \wedge t_{i+1}], A_j)||^2) \\
&= \sum_{n=1}^{\infty}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbb{E}(|(R_{ij} M((t \wedge t_i, t \wedge t_{i+1}], A_j), e_n)_H|^2)
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \mathbb{E}((R_{ij}^* e_n, T_{A_j} R_{ij}^* e_n)_H) \rho((t_i \wedge t, t_{i+1} \wedge t])$$
$$= \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \mathbb{E}(\mathrm{tr}(R_{ij} T_{A_j} R_{ij}^*)) \rho((t_i \wedge t, t_{i+1} \wedge t])$$
$$= \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \mathbb{E}\left(\int_{A_j} \mathrm{tr}(R_{ij} T_x R_{ij}^*) \lambda(dx)\right) \rho((t_i \wedge t, t_{i+1} \wedge t])$$

Hence each J_t extends to an isometry from $\mathcal{H}_2(T; \lambda, \rho)$ into $L^2(\Omega, \mathcal{F}, P; H)$ and we write $\int_0^t \int_S R(s,x) M(ds, dx) := J_t(R)$, for each $0 \leq t \leq T, R \in \mathcal{H}_2(T; \lambda, \rho)$. The process $(J_t, t \geq 0)$ is a square-integrable centred martingale. Henceforth we will always take a strongly càdlàg version.

Notes 1). The condition (10) can be rewritten as

$$\mathbb{E}\left(\int_0^T \int_S ||R(t,x) T_x^{\frac{1}{2}}||_2 \lambda(dx) \rho(dt)\right) < \infty,$$

where $|| \cdot ||_2$ is the Hilbert-Schmidt norm, i.e. $||C||_2 = \mathrm{tr}(CC^*)$ for $C \in L(H)$. The set of all $C \in L(H)$ for which $||C||_2 < \infty$ is a Hilbert space with respect to the inner product $(C, D)_2 = \mathrm{tr}(CD^*)$, which we denote as $\mathcal{L}_2(H)$, (see e.g. [34], section VI.6 for further details).

2) $\mathcal{L}_2(H)$ is a two-sided $L(H)$-ideal with $||C_1 D C_2||_2 \leq ||C_1||.||C_2||.||D||_2$, for all $C_1, C_2 \in L(H), D \in \mathcal{L}_2(H)$. From this we easily deduce that

$$\int_0^T \int_S \mathbb{E}(||R(t,x)||^2) \mathrm{tr}(T_x) \lambda(dx) \rho(dt) < \infty \tag{11}$$

is a sufficient condition for (10).

3) The construction of this section is easily extended to the conceptually simpler case of deterministic operator-valued families $(R(t,x), t \in [0,T], x \in S)$ satisfying $\int_0^T \int_S \mathrm{tr}(R(t,x) T_x R(t,x)^*) \lambda(dx) \rho(dt) < \infty$.

If $C \in L(H)$ and $R = (R(t,x), t \in [0,T], x \in S)$, we define $CR = (CR(t,x), t \in [0,T], x \in S)$. We will need the following result in section 4.3 below.

Theorem 3. *If $C \in L(H)$ and $R \in \mathcal{H}_2(T; \lambda, \rho)$ then $CR \in \mathcal{H}_2(T; \lambda, \rho)$ and*

$$C \int_0^t \int_S R(s,x) M(ds, dx) = \int_0^t \int_S CR(s,x) M(ds, dx),$$

for all $t \geq 0$.

Proof. $CR \in \mathcal{H}_2(T;\lambda,\rho)$ follows easily from Note 2 above. The identity is immediate if $R \in S(T;\lambda,\rho)$. More generally, let $(R_n, n \in \mathbb{N})$ be a sequence in $S(T;\lambda,\rho)$ converging to $R \in \mathcal{H}_2(T;\lambda,\rho)$, then for all $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left(\left|\int_0^t \int_S CR(s,x)M(ds,dx) - \int_0^t \int_S CR_n(s,x)M(ds,dx)\right|^2\right) \\
&= \mathbb{E}\left(\left|\int_0^t \int_S C[R(s,x) - R_n(s,x)]M(ds,dx)\right|^2\right) \\
&= \mathbb{E}\left(\int_0^t \int_S \mathrm{tr}(C[R(s,x) - R_n(s,x)]F_x[R(s,x)^* - R_n(s,x)^*C^*])\lambda(dx)\rho(dt)\right) \\
&\leq ||C||^2 \mathbb{E}\left(\int_0^t \int_S \mathrm{tr}([R(s,x) - R_n(s,x)]F_x[R(s,x)^* - R_n(s,x)^*])\lambda(dx)\rho(dt)\right) \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

and the result follows. □

3.3 Weak-Strong Compatibility

In this subsection we will assume that the operator-valued family $(R(t,x), t \in [0,T], x \in S)$ is such that the mappings $[0,T] \times S \to H$, given by $(t,x) \to R(t,x)^*\psi$ are $\mathcal{P} \otimes \Sigma$-measurable, for each $\psi \in H$.

Theorem 4. *If M is a decomposable nuclear martingale-valued measure with independent increments and the operator-valued family $(R(t,x), t \in [0,T], x \in S)$ satisfies (11) then for all $0 \leq t \leq T, \psi \in H$*

$$\left(\psi, \int_0^t \int_S R(s,x)M(ds,dx)\right)_H = \int_0^t \int_S (R(s,x)^*\psi, M(ds,dx))_H. \tag{12}$$

Proof. First note that since (11) holds, the strong integral appearing on the left hand side of (12) exists. The weak integral on the right hand side also exists, since by the independent increments property of M, proposition 2.1 and (11),

$$\begin{aligned}
\mathbb{E}\left(\int_0^T \int_S ||R(s,x)^*\psi||^2 \langle M\rangle(ds,dx)\right) &= \int_0^T \int_S \mathbb{E}(||R(s,x)^*\psi||^2)\mathbb{E}(\langle M\rangle(ds,dx)) \\
&\leq \int_0^T \int_S \mathbb{E}(||R(s,x)||^2)\mathrm{tr}(T_x)\lambda(dx)\rho(ds)||\psi||^2 \\
&< \infty.
\end{aligned}$$

To establish the result, first let $R \in S(T;\lambda,\rho)$, then

$$\left(\psi, \int_0^t \int_S R(s,x)M(ds,dx)\right)_H = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}(\psi, R_{ij}M((t_i,t_{i+1}],A_j))_H$$
$$= \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}(R_{ij}^*\psi, M((t_i,t_{i+1}],A_j))_H$$
$$= \int_0^t \int_S (R(s,x)^*\psi, M(ds,dx))_H.$$

The general result follows by a straightforward limiting argument. □

3.4 A Stochastic Fubini Theorem

The result to be established is in some respects quite simple, however it is adequate for our later needs. Let N be a Poisson random measure defined on $\mathbb{R}^+ \times (H - \{0\})$ as in section 1.3 and let ν be its intensity measure, which we will assume to be a Lévy measure. Let $E \in \mathcal{B}(H - \{0\})$. If $F : \mathbb{R}^+ \times H \to \mathbb{R}$ is $\mathcal{P} \otimes E$-measurable and $\int_0^t \int_E \mathbb{E}(|F(s,x)|^2)\nu(dx)ds < \infty$, we can construct the stochastic integral $\int_0^t \int_E F(s,x)\tilde{N}(ds,dx)$. It is a centred square-integrable martingale with

$$\mathbb{E}\left(\left|\int_0^t \int_E F(s,x)\tilde{N}(ds,dx)\right|^2\right) = \int_0^t \int_E \mathbb{E}(|F(s,x)|^2)\nu(dx)ds,$$

see e.g. Chapter 4 of [2].

Now let $(W, \mathcal{W}, \mu)$ be a finite measure space and let $\mathcal{H}_2(T,E,W)$ be the real Hilbert space of all $\mathcal{P} \otimes \mathcal{B}(E) \otimes \mathcal{W}$-measurable functions G from $[0,T] \times E \times W \to \mathbb{R}$ for which $\int_W \int_0^t \int_E \mathbb{E}(|G(s,x,w)|^2)\nu(dx)ds\mu(dw) < \infty$. The space $S(T,E,W)$ is dense in $\mathcal{H}_2(T,E,W)$, where $G \in S(T,E,W)$ if

$$G = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_3} G_{ijk}1_{(t_i,t_{i+1}]}1_{A_j}1_{B_k},$$

where $N_1, N_2, N_3 \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_{N_1+1} = T, A_0, \ldots, A_{N_2}$ are disjoint sets in $\mathcal{A}, B_0, \ldots, B_{N_3}$ is a partition of W, wherein each $B_k \in \mathcal{W}$ and each G_{ijk} is a bounded $\mathcal{F}_{t_i}$-measurable random variable.

Theorem 5. *If $G \in \mathcal{H}_2(T,E,W)$, then for each $0 \leq t \leq T$,*

$$\int_W \left(\int_0^t \int_E G(s,x,w)\tilde{N}(ds,dx)\right)\mu(dy) = \int_0^t \int_E \left(\int_W G(s,x,w)\mu(dy)\right)\tilde{N}(ds,dx) \quad a.e. \tag{13}$$

Proof. First note that both integrals in (13) are easily seen to exist in $L^2(\Omega, \mathcal{F}, P)$. If $G \in S(T,E,W)$, then the result holds with both sides of (13) equal to

$$\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_3} G_{ijk}\tilde{N}((t_i, t_{i+1}], A_j)\mu(B_k).$$

Now suppose that $(G_n, n \in \mathbb{N})$ is a sequence of mappings in $S(T, E, W)$ converging to $G \in \mathcal{H}_2(T, E, W)$, then

$$\begin{aligned}
&\mathbb{E}\left(\left|\int_0^t \int_E \left(\int_W [G(s,x,w) - G_n(s,x,w)]\mu(dw)\right)\tilde{N}(ds,dx)\right|^2\right) \\
&= \int_0^t \int_E \mathbb{E}\left(\left|\int_W [G(s,x,w) - G_n(s,x,w)]\mu(dw)\right|^2\right)\nu(dx)ds \\
&\leq \mu(W)\int_0^t \int_E \int_W \mathbb{E}(|G(s,x,w) - G_n(s,x,w)|^2)\mu(dw)\nu(dx)ds \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

A similar argument shows that

$$\lim_{n\to\infty}\mathbb{E}\left(\left|\int_W \left(\int_0^t \int_E [G_n(s,x,w) - G(s,x,w)]\tilde{N}(ds,dx)\right)\mu(dy)\right|^2\right) = 0,$$

and the result follows. □

4 Ornstein-Uhlenbeck Processes

4.1 Stochastic Convolution

Let X be a strongly càdlàg Lévy process and let $(S(t), t \geq 0)$ be a C_0-semigroup (i.e. a strongly continuous one-parameter semigroup of linear operators) acting in H. Basic facts about such semigroups can be found in e.g. Chapter 1 of [14]. We note in particular that there exists $M > 1, \beta \in \mathbb{R}$ such that

$$||S(t)|| \leq Me^{\beta t}, \tag{14}$$

for all $t \geq 0$. J will denote the infinitesimal generator of $(S(t), t \geq 0)$. It is a closed, densely defined linear operator in H and hence its adjoint J^* is also densely defined. Let $C \in L(H)$. Our aim in this subsection is to define the *stochastic convolution*

$$X_{J,C}(t) := \int_0^t S(t-s)CdX(s), \tag{15}$$

for all $t \geq 0$. We do this by employing the Lévy-Itô decomposition (7) to write each

$$X_{J,C}(t) = \int_0^t S(t-s)C\zeta ds + \int_0^t S(t-s)CdB_Q(s) + \int_0^t \int_{||x||<1} S(t-s)Cx\tilde{N}(ds,dx)$$
$$+ \int_0^t \int_{||x|\geq 1} S(t-s)CxN(ds,dx). \qquad (16)$$

We need to establish condition under which the process $(X_{J,C}(t), t \geq 0)$ exists. To do this we consider each term in (16) in turn. We define $\int_0^t S(t-s)C\zeta ds$ as a standard Bochner integral. Indeed using (14) we obtain

$$\left|\left|\int_0^t S(t-s)C\zeta ds\right|\right| \leq \left(\int_0^t ||S(t-s)||ds\right) ||C\zeta||$$
$$\leq \begin{cases} M\beta^{-1}(e^{\beta t}-1)||C\zeta|| & \text{if } \beta \neq 0 \\ Mt||C\zeta|| & \text{if } \beta = 0. \end{cases}$$

The terms $\int_0^t S(t-s)CdB_Q(s)$ and $\int_0^t \int_{||x||<1} S(t-s)Cx\tilde{N}(ds,dx)$ are dealt with using the (deterministic version) of strong stochastic integration as described in section 3.2. In fact the first of these terms was discussed in [11], section 5.1.2. (see also [7]). Using the estimate (11) we find that $\int_0^t S(t-s)CdB_Q(s)$ exists as a strong integral provided $\left(\int_0^t ||S(t-s)||^2 ds\right) \text{tr}(Q) < \infty$, which by (14) is always satisfied.

Note. [11] impose the weaker condition (10) as they want to explore the degenerate case where $Q = I$. This falls outside the context of the current work as B_Q is not then a Lévy process when H is infinite dimensional.

Again using (11) for the compensated Poisson integral, we must estimate

$$\int_0^t ||S(t-s)||^2 ds \int_{||x||<1} \text{tr}(T_x)\nu(dx) = \int_0^t ||S(t-s)||^2 ds \int_{||x||<1} ||x||^2 \nu(dx)$$
$$< \infty$$

Hence we see that $\int_0^t \int_{||x||<1} S(t-s)Cx\tilde{N}(ds,dx)$ also always exists.
Finally, we may define the final Poisson integral as a finite (random) sum:

$$\int_0^t \int_{||x|\geq 1} S(t-s)CxN(ds,dx) = \sum_{0\leq s\leq t} S(t-s)C\Delta X(s)1_{\{||x||\geq 1\}}(\Delta X(s)).$$

In conclusion, we have established the following:

Theorem 6. *If X is a càdlàg Lévy process, $C \in B(H)$ and $(S(t), t \geq 0)$ is a C_0-semigroup with generator J, the stochastic convolution $X_{J,C}(t) = \int_0^t S(t-s)CdX(s)$ exists in H for all $t \geq 0$.*

We note that the process $X_{J,C} = (X_{J,C}(t), t \geq 0)$ inherits strongly càdlàg paths from X.

Note. An alternative approach to defining the stochastic convolution is to employ (weak) integration by parts to write, for each $\psi \in \mathrm{Dom}(J^*)$,

$$(\psi, X_{J,C}(t))_H = (\psi, [S(0)CX(t) - S(t)CX(0)])_H + \int_0^t (C^* J^* \psi, S(t-s)X(s-))_H ds.$$

Another approach, using convergence in probability rather than L^2-convergence can be found in [9].

4.2 Existence and Uniqueness for Ornstein-Uhlenbeck Processes

The development of this section closely parallels that of [11], section 5.2. We consider the *generalised Langevin equation* in Hilbert space, i.e

$$dY(t) = JY(t)dt + CdX(t), \tag{17}$$

with the initial condition $Y(0) = Y_0$ (a.s.), where Y_0 is a given $\mathcal{F}_0$-measurable random variable. We consider (17) as a *weak sense stochastic differential equation.* By this we mean that $Y = (Y(t), t \geq 0)$ is a solution to (17) if for all $t \geq 0, \psi \in \mathrm{Dom}(J^*)$,

$$(\psi, Y(t) - Y_0)_H = (C^*\psi, X(t))_H + \int_0^t (J^*\psi, Y(s))_H ds. \tag{18}$$

Our candidate solution to (17) is given by the usual stochastic version of the variation of constants formula

$$Z(t) = S(t)Y_0 + \int_0^t S(t-s)CdX(s), \tag{19}$$

for each $t \geq 0$. It follows from Theorem 6 that $Z(t)$ exists for all $t \geq 0$.

Note. Da Prato and Zabczyk [11] consider (18) in the case where X is a Brownian motion. In their formalism, the operators J and the process X are associated to different Hilbert spaces H_1 and H_2, respectively and C maps H_2 to H_1. Our approach herein is easily extended to this level of generality (in fact one can just take $H = H_1 \oplus H_2$).

Theorem 7. *(19) is the unique weak solution to (18).*

Proof. We extend the argument used to prove theorem 5.4 in [11]. See [9] for an alternative approach.

Existence. First note that if $X_{J,C}$ solves (18) with the initial condition $Y_0 = 0$ (a.s.), then it is clear that Z, as given by (19) solves (18) with the arbitrary initial condition. Hence we may restrict ourselves to the former problem.

For each $t \geq 0, \psi \in \mathrm{Dom}(J^*)$, using (12) we obtain

$$\begin{aligned}(\psi, X_{J,C}(t))_H - (C^*\psi, X(t))_H) &= \left(\psi, \int_0^t [S(t-s) - I]CdX(s)\right)_H \\ &= \int_0^t (C^*[S(t-s)^* - I]\psi, dX(s))_H \\ &= \int_0^t \left(\int_0^{t-s} C^*S(r)^*J^*dr\psi, dX(s)\right)_H \\ &= \int_0^t \left(\int_s^t 1_{[0,r)}(s)C^*S(r-s)^*J^*dr\psi, dX(s)\right)_H\end{aligned}$$

We now need to change the order of integration. Using (16), we employ (13) for the compensated Poisson integral, the stochastic Fubini theorem of [11] (theorem 4.18) for the Brownian integral and the usual Fubini theorem for the Lebesgue integral to deduce that

$$\begin{aligned}(\psi, X_{J,C}(t))_H - (C^*\psi, X(t))_H) &= \int_0^t \left(J^*\psi, \int_0^t 1_{[0,r)}(s)S(r-s)CdX(s)\right)_H \\ &= \int_0^t (J^*\psi, X_{J,C}(r))_H dr,\end{aligned}$$

as was required.

Uniqueness. This is established in exactly the same way as in [11] (pp. 122-3). □

It follows immediately from (19) that Y has strongly càdlàg paths.

Example Let $H = L^2(U)$ where U is a regular domain in $\mathbb{R}^d$. If Δ denotes the usual (Dirichlet) Laplacian acting in H then for each $0 < \alpha < 2$, we can define the fractional power $-(-\Delta^{\frac{\alpha}{2}})$ by e.g. spectral theory, or as a pseudo-differential operator. Indeed when $U = \mathbb{R}^d$, $-(-\Delta^{\frac{\alpha}{2}})$ is a positive self-adjoint operator on the domain $\mathcal{H}_\alpha(\mathbb{R}^d) = \left\{f \in L^2(\mathbb{R}^d); \int_{\mathbb{R}^d} |v|^{2\alpha}|\hat{f}(v)|^2 dv < \infty\right\}$, where $\hat{f}$ denotes the Fourier transform of f, and $-(-\Delta^{\frac{\alpha}{2}})$ generates a self-adjoint contraction semigroup on H (see e.g. [2], Chapter 3). By the results of this section we know there is a unique weak solution to the equation

$$dY(t) = -(-\Delta^{\frac{\alpha}{2}})Y(t)dt + CdX(t),$$

for any $C \in B(H)$ and any Lévy process X. In particular, one can take X to be α-stable (c.f. [30]).

4.3 Flow and Markov Properties

For each $0 \leq s \leq t < \infty$, define a two-parameter family of mappings $\Phi_{s,t} : H \times \Omega \to H$ by

$$\Phi_{s,t}(y) = S(t-s)y + \int_s^t S(t-r)CdX(r).$$

The following establishes that $\{\Phi_{s,t}; 0 \leq s \leq t < \infty\}$ is a *stochastic flow.*

Proposition 4.1 *For all* $0 \leq r \leq s \leq t < \infty$*,*

$$\Phi_{s,t} \circ \Phi_{r,s} = \Phi_{r,t}$$

Using the semigroup property and Theorem 3, for each $y \in H$, we obtain

$$\begin{aligned}
\Phi_{s,t}(\Phi_{r,s}(y)) &= S(t-s)\Phi_{r,s}(y) + \int_s^t S(t-u)CdX(u) \\
&= S(t-s)S(s-r)y + S(t-s)\int_r^s S(s-u)CdX(u) + \int_s^t S(t-u)CdX(u) \\
&= S(t-r)y + \int_r^s S(t-u)CdX(u) + \int_s^t S(t-u)CdX(u) \\
&= \Phi_{r,t}(y). \qquad \square
\end{aligned}$$

By the construction of stochastic integrals, we deduce that each $\Phi_{s,t}(y)$ is $\mathcal{G}_{s,t}$-measurable where $\mathcal{G}_{s,t} = \sigma\{X(u) - X(v); s \leq u < v \leq t\}$, and hence by the independent increment property of X, it follows that $\Phi_{s,t}(y)$ is independent of $\mathcal{F}_s$.
From this fact and Proposition 4.1, we can apply standard arguments (see e.g [2], section 6.4 or [33], section 5.6) to establish the *strong Markov property* for the solution to (18), i.e. if τ is a stopping time with $P(\tau < \infty) = 1$ then for each $f \in b(H), t \geq 0$

$$\mathbb{E}(f(Y(\tau + t))|\mathcal{F}_\tau) = \mathbb{E}(f(Y(\tau + t))|Y(\tau)),$$

where $\mathcal{F}_\tau$ is the usual stopped σ-algebra.
By the stationary increments of X it follows that Y is a time-homogeneous Markov process and hence we obtain a contraction semigroup of linear operators $(\mathcal{T}_t, t \geq 0)$ on $b(H)$ via the prescription

$$(\mathcal{T}_t f)(y) = \mathbb{E}(f(Y(t))|Y(0) = y),$$

for each $t \geq 0, f \in b(H), y \in H$. We easily verify that $\mathcal{T}_t : C_b(H) \subseteq C_b(H)$ for each $t \geq 0$, by a routine application of dominated convergence. In fact $(\mathcal{T}_t, t \geq 0)$ is a *generalised Mehler semigroup* in the sense of [6], [15].

5 Operator Self-decomposability

Generalising ideas developed in [21], we say that a random variable Z is *operator self-decomposable* with respect to a C_0-semigroup $(S(t), t \geq 0)$ if for all $t \geq 0$, there exists a random variable Z_t which is independent of Z, such that

$$Z \stackrel{d}{=} S(t)Z + Z_t. \tag{20}$$

We aim to show that random variables of the form $Z = \int_0^\infty S(r)dX(r)$, where X is a Lévy process are operator self-decomposable, when the limit makes sense. For each $t \geq 0$, we define $\int_0^t S(r)dX(r)$ by employing the Lévy-Itô decomposition, as in (16). We assume throughout this section that the semigroup $(S(t), t \geq 0)$ is *exponentially stable*, i.e. (14) holds with $\beta < 0$, e.g. all self-adjoint semigroups whose generator has a spectrum which is bounded away from zero are exponentially stable. In [12], it is shown that a C_0-semigroup $(S(t), t \geq 0)$ is exponentially stable if and only if $\int_0^\infty ||S(t)x||^2 dt < \infty$, for all $x \in H$.
Under this assumption, given any sequence $(t_n, n \in \mathbb{N})$ in $[0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$, we can assert the existence of the following limits:

$$\int_0^\infty S(r)\zeta dr = \lim_{n\to\infty} \int_0^{t_n} S(r)\zeta dr$$

$$\int_0^\infty S(r)dB_Q(r) = \lim_{n\to\infty} \int_0^{t_n} S(r)dB_Q(r)$$

$$\int_0^\infty \int_{||x||<1} S(r)x\tilde{N}(dr, dx) = \lim_{n\to\infty} \int_0^{t_n} \int_{||x||<1} S(r)x\tilde{N}(dr, dx),$$

where the first limit is taken in H and the other two in $L^2(\Omega, \mathcal{F}, P; H)$. We need to work harder to consider the limiting behaviour as $t \to \infty$ of $\Pi_{S,N}(t) := \int_0^t \int_{||x||\geq 1} S(r)xN(dr, dx)$.

Lemma 1. *Let $A \in L(H)$ with $||A|| \leq 1$ and $(\xi_n, n \in \mathbb{N})$ be a sequence of iid random variables. If $\mathbb{E}(\log(1 + ||\xi_1||)) < \infty$, then $\sum_{n=1}^\infty A^n \xi_n$ converges a.s.*

The proof is exactly as in [21], lemma 3.6.5 (p.121). Note that these authors are able to prove 'if and only if' by assuming that A is invertible. That assumption would be unnatural in our context.
This next result is related to Proposition 1.8.13 in [21], p.36, although the proof is quite different.

Lemma 2. *Let $f : H \to \mathbb{R}^+$ be measurable and subadditive. If $\int_0^t \int_{||x||\geq 1} f(S(r)x)\nu(dx)dr < \infty$ then $\mathbb{E}(f(\Pi_{S,N}(t))) < \infty$, for each $t \geq 0$.*

Proof. By subadditivity of f, for each $t \geq 0$ we have

$$\begin{aligned} f(\Pi_{S,N}(t)) &= f\left(\sum_{0\leq r\leq t} S(r)\Delta X(r)1_{\{||x||\geq 1\}}(\Delta X(r))\right) \\ &\leq \sum_{0\leq r\leq t} f(S(r)\Delta X(r))1_{\{||x||\geq 1\}}(\Delta X(r)) \\ &= \int_0^t \int_{||x||\geq 1} f(S(r)x)N(dr, dx). \end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}(f(\Pi_{S,N}(t))) &\leq \mathbb{E}\left(\int_0^t \int_{||x||\geq 1} f(S(r)x)N(dr,dx)\right) \\ &= \int_0^t \int_{||x||\geq 1} f(S(r)x)\nu(dx)dr < \infty. \qquad \square\end{aligned}$$

Theorem 8. *(c.f. [39], [22], [21]) Let $(S(t), t \geq 0)$ be an exponentially stable contraction semigroup in H. If $\int_{||x||\geq 1} \log(1+||x||)\nu(dx) < \infty$ then $\lim_{t\to\infty} \int_0^t \int_{||x||<1} S(r)xN(dr,dx)$ exists in distribution.*

Proof. We follow the approach of [21], theorem 3.6.6 (p.123). By stationary increments of X and the semigroup property, for each $n \in \mathbb{N}$,

$$\begin{aligned}\int_0^n \int_{||x|\geq 1} S(r)xN(dr,dx) &= \sum_{k=0}^{n-1} \int_k^{k+1} \int_{||x||\geq 1} S(r)xN(dr,dx) \\ &= \sum_{k=0}^{n-1} \int_0^1 \int_{||x||\geq 1} S(r+k)xN(dr+k,dx) \\ &= \sum_{k=0}^{n-1} S(1)^k \int_0^1 \int_{||x||\geq 1} S(r)xN(dr+k,dx) \\ &\stackrel{d}{=} \sum_{k=0}^{n-1} S(1)^k \int_0^1 \int_{||x||\geq 1} S(r)xN(dr,dx) \\ &\stackrel{d}{=} \sum_{k=0}^{n-1} S(1)^k M_k,\end{aligned}$$

where each $M_k := \int_k^{k+1} \int_{||x||\geq 1} S(r-k)xN(dr,dx)$. The M_k's are independent by the independent increment property of N. Moreover by the stationary increment property of N, each

$$M_k = \int_0^1 \int_{||x||\geq 1} S(r)xN(dr+k,dx) \stackrel{d}{=} \int_0^1 \int_{||x||\geq 1} S(r)xN(dr,dx).$$

We deduce the convergence in distribution as $n \to \infty$ of $\int_0^n S(r)xN(dr,dx)$ by lemmas 1 and 2 together with the estimate

$$\begin{aligned}\int_0^t \int_{||x||\geq 1} \log(1+||S(r)x||)\nu(dx)dr &\leq \int_0^t \int_{||x||\geq 1} \log(1+||x||)\nu(dx)dr \\ &= t\int_{||x||\geq 1} \log(1+||x||)\nu(dx).\end{aligned}$$

Now let $(s_n, n \in \mathbb{N})$ be an arbitrary sequence in $[0,1]$. By stationary increments of N, for each $n \in \mathbb{N}$,

$$\int_n^{n+s_n} \int_{||x||\geq 1} S(r)xN(dr,dx) \stackrel{d}{=} S(n) \int_0^{s_n} \int_{||x||\geq 1} S(r)xN(dr,dx).$$

Since $t \to \int_0^t S(r)xN(dr,dx)$ is a.s. càdlàg, we deduce that

$$\left|\left| S(n) \int_0^{s_n} \int_{||x||\geq 1} S(r)xN(dr,dx) \right|\right| \leq ||S(n)|| \sup_{t\in[0,1]} \left|\left| \int_0^t S(r) \int_{||x||\geq 1} xN(dr,dx) \right|\right| \\ \to 0 \text{ as } n \to \infty \text{ a.s.}$$

Hence, given any sequence $(t_n, n \in \mathbb{N})$ diverging to ∞, we can deduce the convergence in distribution as $t_n \to \infty$ of

$$\int_0^{t_n} S(r) \int_{||x||\geq 1} xN(dr,dx) = \int_0^{[t_n]} \int_{||x||\geq 1} S(r)xN(dr,dx) + \int_{[t_n]}^{t_n} \int_{||x|\geq 1} S(r)xN(dr,dx). \square$$

Note. In [9] it is shown that the following conditions are necessary and sufficient for the existence (in distribution) of $\lim_{t\to\infty} \int_0^t S(r)dK(r)$ where K is the jump part of X, i.e. $K(t) = X(t) - t\zeta - B_Q(t)$, for each $t \geq 0$:

$$\int_0^\infty \int_{H-\{0\}} (||S(r)x||^2 \wedge 1)\nu(dx)dr < \infty$$

$$\lim_{t\to\infty} \int_0^t \int_{H-\{0\}} S(r)x[1_{B_1}(S(r)(x)) - 1_{B_1}(x)]\nu(dx)ds \text{ exists .} \tag{21}$$

These may be difficult to verify in practice.

The main result of this section is the following:

Theorem 9. *If $(S(t), t \geq 0)$ is an exponentially stable contraction semigroup in H and X is a Lévy process with Lévy measure ν for which $\int_{||x||\geq 1} \log(1 + ||x||)\nu(dx) < \infty$, then $\lim_{t\to\infty} \int_0^t S(r)dX(r)$ exists in distribution and is operator self-decomposable with respect to $(S(t), t \geq 0)$.*

Proof. It follows from the Lévy-Itô decomposition that $\int_0^t S(r)dX(r) - \Pi_{S,N}(t)$ and $\Pi_{S,N}(t)$ are independent. Since each of these terms converges in distribution as $t \to \infty$, it follows that their sum also does. For the self-decomposability, we define $Z = \int_0^\infty S(r)dX(r)$, then

$$Z = \int_0^t S(r)dX(r) + \int_t^\infty S(r)dX(r),$$

and these terms are independent, by the independent increment property of X. Now

$$\int_t^\infty S(r)dX(r) = \int_0^\infty S(r+t)dX(r+t) \stackrel{d}{=} S(t)\int_0^\infty S(r)dX(r),$$

by the stationary increment property of X. Hence we have (20) with $Z_t = \int_0^t S(r)dX(r)$.□

Finally, there is an interesting link between self-decomposability and Ornstein-Uhlenbeck processes (c.f. [39], [4] for the finite-dimensional case).

Suppose that X is a Lévy process with characteristics (ζ, Q, ν), and define the process $\tilde{X} = (-X(t), t \geq 0)$, then $\tilde{X}$ is a Lévy process with characteristics $(-\zeta, Q, \tilde{\nu})$, where $\tilde{\nu}(A) = \nu(-A)$, for all $A \in \mathcal{B}(H - \{0\})$. In the following, we define $(X(t), t < 0)$ to be an independent copy of $\tilde{X}$.

We recall the Ornstein-Uhlenbeck process (19)

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)dX(s),$$

for each $t \geq 0$, where we have taken $C = I$.

Theorem 10. *If the Ornstein-Uhlenbeck process $(Y(t), t \geq 0)$ is stationary, then Y_0 is self-decomposable. Conversely, if $(S(t), t \geq 0)$ is an exponentially stable contraction semigroup in H and $\int_{||x|| \geq 1} \log(1 + ||x||)\nu(dx) < \infty$, then there exists a self-decomposable Y_0 such that $Y = (Y(t), t \geq 0)$ is stationary.*

Proof. Suppose that $Y = (Y(t), t \geq 0)$ is stationary, then for each $t \geq 0$,

$$Y_0 \stackrel{d}{=} Y(t) = S(t)Y_0 + \int_0^t S(t-r)dX(r),$$

so Y_0 is self-decomposable. Conversely, define $Y_0 := \int_{-\infty}^0 S(-r)dX(r)$, then Y_0 is self-decomposable by theorem 9. By theorem 3 and the semigroup property, for each $t \geq 0$, we have

$$\begin{aligned} Y(t) &= \int_{-\infty}^0 S(t-r)dX(r) + \int_0^t S(t-r)dX(r) \\ &= \int_{-\infty}^t S(t-r)dX(r). \end{aligned}$$

Clearly $Y(t+h) \stackrel{d}{=} Y(t)$, for each $h > 0$. More generally, by stationary increments of Z we can easily deduce (as in [2], theorem 4.3.16) that

$$\mathbb{E}\left(\exp\left\{i\sum_{j=1}^n (u_j, Y(t_j+h))_H\right\}\right) = \mathbb{E}\left(\exp\left\{i\sum_{j=1}^n (u_j, Y(t_j))_H\right\}\right),$$

for each $n \in \mathbb{N}, u_1, \ldots, u_n \in H, t_1, \ldots, t_n \in \mathbb{R}^+$ □.

Note. In [9], it is shown that the conditions (21) are necessary and sufficient for Y to have a stationary solution and the condition $\int_{||x|| \geq 1} \log(1 + ||x||)\nu(dx) < \infty$ is demonstrated to be sufficient for these to hold. In [8] an example is constructed which demonstrates that this condition is not necessary when H is infinite dimensional.

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