Periodic orbits of the Sitnikov problem via a Poincaré map

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Abstract. In this paper by means of a Poincaré map, we prove the existence of symmetric periodic orbits of the elliptic Sitnikov problem. Furthermore, using the presence of the Bernoulli shift as a subsystem of that Poincaré map, we prove that not all the periodic orbits of the Sitnikov problem are symmetric periodic orbits.

Keywords: Sitnikov motions, periodic orbits, Poincaré map, symmetric periodic orbits

1. Introduction

We consider the elliptic Sitnikov problem, here called simply the *Sitnikov problem*. It is an special case of the restricted three–body problem where the two primaries with equal masses are moving in an elliptic orbit of the two–body problem, and the infinitesimal mass is moving on the straight line orthogonal to the plane of motion of the primaries which passes through the center of mass. The purpose of this paper is to use the symmetries of the problem to find symmetric periodic solutions of the Sitnikov problem. We do this by means of a Poincaré map, which is essentially the Poincaré map that used Alekseev (1968) and Moser (1973) to analyze the final evolutions of the orbits of the Sitnikov problem. The presence of the Bernoulli shift as a subsystem of that Poincaré map will allow us to prove the existence of non–symmetric periodic orbits.

In Section 2 we give the equations of motion of the Sitnikov problem and we prove that the flow of the Sitnikov problem is complete (that is, all solutions are defined for all $t \in \mathbb{R}$), see Proposition 2.

In Section 3 we define the Poincaré map f_e and we analyze the properties of f_e^n for $n = 1, 2, \ldots$

In Section 4 we analyze the symmetries of the problem and we see how they can be used to obtain periodic orbits, see Propositions 5, 6 and 7. Moreover we characterize the symmetric periodic orbits and, by means of the Poincaré map, we prove the existence of symmetric periodic orbits of the Sitnikov problem for all values of the eccentricity $e \in (0, 1)$, see Propositions 12 and 15.

In Section 4 we summarize the basic results of Moser (1973) about the presence of the Bernoulli shift as a subsystem of that Poincaré map. In particular, we are interested in the fact that we can associate an orbit of the Sitnikov problem to each sequence of integers satisfying certain conditions. Then analyzing the sequences associated to the symmetric periodic orbits we prove the existence of non–symmetric periodic orbits, see Proposition 23.

2. Sitnikov problem

Let $m_1 = m_2$ be two punctual masses (called *primaries*) describing an elliptic orbit of the two body problem. We consider an infinitesimal mass m_3 that moves on the straight line ρ orthogonal to the plane of motion of the primaries that passes through their center of mass. The Sitnikov problem will consist of describing the motion of the infinitesimal mass.

We choose the units of mass, length and time so that $m_1 = m_2 = 1/2$, the gravitational constant $G = 1$, and the period of the orbit described by the primaries is 2π. If z denotes the position of the particle m_3 in a coordinate system on ρ with origin at the center of mass of the primaries (see Figure 1), then the equation of motion of the Sitnikov problem becomes

$$\ddot{z} = -\frac{z}{(z^2 + r^2(t))^{3/2}} \ , \tag{1}$$

where $r(t)$ is the distance of the primaries to their center of mass and it is given by

$$r(t) = \frac{1}{2}(1 - e\cos u(t)) \ ,$$

which is an elliptic solution of the Kepler problem

$$\ddot{r} = \frac{1 - e^2}{16\, r^3} - \frac{1}{8\, r^2} \ ,$$

with eccentricity $0 \leq e < 1$. Here $u(t)$ is the eccentric anomaly which is a function of time via the Kepler's equation

$$u - e \sin u = t - \ell \ ,$$

with ℓ the time of pericenter passage. Without loss of generality we usually take $\ell = 0$, i.e. the primaries, at $t = 0$, are at the pericenter

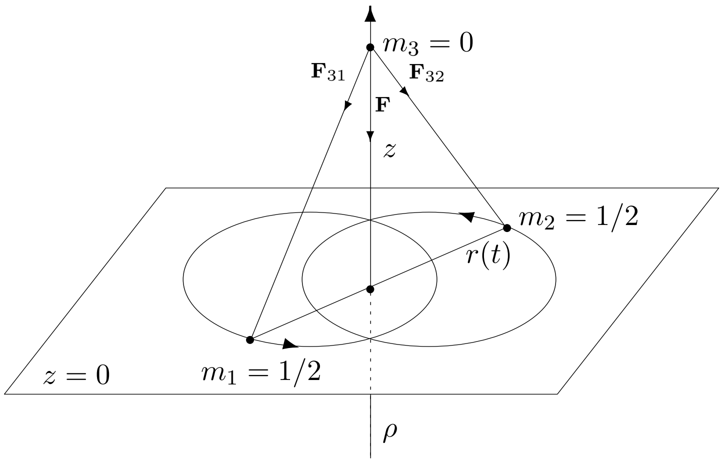

Figure 1. The Sitnikov problem.

of the ellipse. We note that here in a Sitnikov problem the primaries always describe an elliptic orbit.

Now we want to show that any solution $z(t)$ of the Sitnikov problem is defined for all $t \in \mathbb{R}$.

Let $D = \mathbb{R} \times \Omega$ where Ω is an open set in $\mathbb{R}^n$. We consider the system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \ , \tag{2}$$

where $\mathbf{f} : D \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^1$ map. Let $(t_0, \mathbf{x}_0) \in D$, we say that $\mathbf{x}(t)$ is a *solution* of (2) on the open interval $I \subset \mathbb{R}$ satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ if $\mathbf{x} : I \longrightarrow \Omega$ is a $\mathcal{C}^1$ function, $\{(t, \mathbf{x}(t)) \, : \, t \in I\} \in D$ and $\mathbf{x}(t)$ satisfies equations (2) for all $t \in I$. The interval I is called the *maximal interval of definition* of the solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$, if there are no solutions $\overline{\mathbf{x}}(t)$ satisfying $\overline{\mathbf{x}}(t_0) = \mathbf{x}_0$ defined on J, with $I \subsetneqq J$, such that $\overline{\mathbf{x}}(t)|_J = \mathbf{x}(t)$.

PROPOSITION 1. *Let $D = \mathbb{R} \times \Omega$ where Ω is an open set in $\mathbb{R}^n$, let $\mathbf{f} : D \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ map, and let $\mathbf{x}(t)$ be a solution of $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ and having maximal interval of definition $I = (\omega_-, \omega_+) \subset \mathbb{R}$. If $\omega_+ < \infty$ (respectively, $\omega_- > -\infty$), then $\mathbf{x}(t)$ tends to the boundary of Ω when $t \to \omega_+$ (respectively, $t \to \omega_-$). That is, for every compact $K \subset \Omega$ there exist $\epsilon = \epsilon(K) > 0$ such that if $t \in [\omega_+ - \epsilon, \omega_+)$ (respectively, $t \in (\omega_-, \omega_- - \epsilon]$), then $\mathbf{x}(t) \notin K$.*

Proof. See for instance (Hale, 1980) page 17. ∎

PROPOSITION 2. *The flow of the Sitnikov problem is complete; that is, any solution of the Sitnikov problem is defined for all time.*

Proof. The equation of motion of the Sitnikov problem (1) can be written like the system of first order differential equations

$$\begin{aligned} \dot{z} &= v \,, \\ \dot{v} &= -\frac{z}{(z^2 + r^2(t))^{3/2}} \,. \end{aligned} \tag{3}$$

We suppose that $(z(t), v(t))$ is a solution of (3) having maximal interval of definition (ω_-, ω_+) with $\omega_+ < \infty$, and then we will arrive to a contradiction, consequently $\omega_+ = \infty$. In a similar way we would prove that $\omega_- = -\infty$.

Since $r(t)$ is the radial component of an elliptic solution of the Kepler problem, $r(t)$ is defined for all $t \in \mathbb{R}$ and it is always different from zero. Therefore system (3) is analytic in $(t, z, v) \in D = \mathbb{R} \times \Omega = \mathbb{R} \times \mathbb{R}^2$. Thus by Proposition 1 the solution $(z(t), v(t))$ tends to the boundary of Ω when $t \to \omega_+$. That is, $\lim_{t\to\omega_+} z(t) = \pm\infty$ or $\lim_{t\to\omega_+} v(t) = \pm\infty$. From (3) and the fact that $r(t) \neq 0$ for all $t \in \mathbb{R}$, we have that $|\dot{v}|$ is bounded at any point of the phase space D. Hence it is impossible that $\lim_{t\to\omega_+} v(t) = \pm\infty$. So it is sufficient to analyze the case $\lim_{t\to\omega_+} z(t) = \pm\infty$.

We assume that $\lim_{t\to\omega_+} z(t) = \infty$ (the case $\lim_{t\to\omega_+} z(t) = -\infty$ would be similar). Then we can find $\bar{t} \in (0, \omega_+)$ such that $z(t) > 0$ for all $t \in [\,\bar{t}, \omega_+)$, so $\ddot{z}(t) < 0$ for all $t \in [\bar{t}, \omega_+)$ (see (3)). Thus the function $z(t)$ is concave and positive in the interval $t \in [\,\bar{t}, \omega_+)$, and $\lim_{t\to\omega_+} z(t) = \infty$. This is impossible with the assumption $\omega_+ < \infty$. ∎

3. Poincaré map

When the eccentricity $e = 0$ (that is, when the primaries move on a circular orbit of the Kepler problem) equation (1) becomes the circular Sitnikov problem

$$\ddot{z} = -\frac{z}{(z^2 + 1/4)^{3/2}} \,.$$

This equation defines an integrable Hamiltonian system with one degree of freedom and we know explicitly all its solutions, see for instance (Stumpff, 1965) and (Belbruno et al., 1994). These solutions are completely described in the phase space $(z, \dot{z})$ by means of the Hamiltonian.

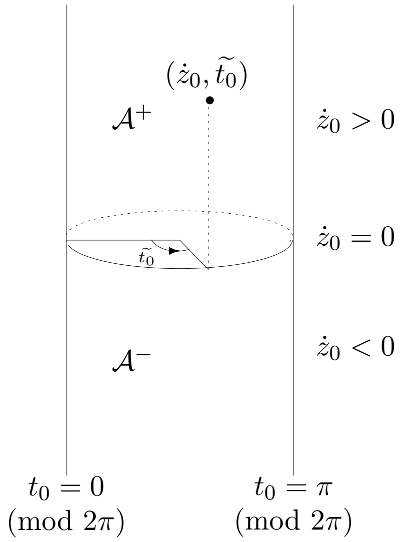

Figure 2. The cylinder $\mathcal{A}$.

When e is positive equation (1) contains explicitly the time (i.e. it is a nonautonomous system), then we must study the orbits in the phase space $(t(\text{mod } 2\pi), z, \dot{z})$. In fact, if we want to describe completely the motion of the circular Sitnikov problem, that is, the motion of the three bodies not only the motion of the infinitesimal mass, we must also consider the phase space $(t(\text{mod } 2\pi), z, \dot{z})$. In order to analyze the orbits in this phase space we will use a Poincaré map.

Using the symmetry of the Sitnikov problem (see Figure 1) it is easy to see that all solutions $z(t)$ of (1) have at least one zero. Then we can describe an arbitrary orbit giving the time t_0 at which $z(t_0) = 0$, and giving $\dot{z}(t_0) = \dot{z}_0$. Moreover the time t_0 can be given modulus 2π because the position of the primaries at a time t and at a time $t + 2\pi k$ for all integer k will be the same. Therefore the orbits of (1) can be thought like points of the cylinder

$$\mathcal{A} = \{(\dot{z}_0, t_0(\text{mod } 2\pi)) \in \mathbb{R} \times \mathbb{S}^1 \, : \, z(t_0) = 0, \; \dot{z}(t_0) = \dot{z}_0 \in \mathbb{R}\} \; ,$$

(see Figure 2). The point $(\dot{z}, t(\text{mod } 2\pi))$ of the cylinder $\mathcal{A}$ will be denoted by $(\dot{z}, \widetilde{t})$; i.e. $t(\text{mod } 2\pi) = \widetilde{t}$.

It is easy to see that equation (1) is invariant under the symmetry

$$(t, z, \dot{z}) \longmapsto (t, -z, -\dot{z}) \; , \tag{4}$$

so cylinder $\mathcal{A}$ can be thought as the union of two cylinders, one corresponding to initial conditions $\dot{z}(t_0) = \dot{z}_0 > 0$ $(\mathcal{A}^+)$, and the other one corresponding to initial conditions $\dot{z}(t_0) = \dot{z}_0 < 0$ ($\mathcal{A}^-$). These two cylinders are symmetric by symmetry (4), and they are divided by the circle of initial conditions $z(t_0) = 0$, $\dot{z}(t_0) = 0$, which corresponds to a collinear relative equilibrium solution of Euler for the 3−body problem.

We remark that the points $(t, -z, -\dot{z})$ and $(t, z, \dot{z})$ could be identified by means of symmetry (4). In particular, we could identify $(t_0, 0, -\dot{z}_0)$ with $(t_0, 0, \dot{z}_0)$. Then it would be sufficient to consider values of $\dot{z}_0 = \dot{z}_0(t_0) > 0$, as did (Moser, 1973), and we could think on $(\dot{z}_0, \widetilde{t_0})$ as polar coordinates in a plane. But here we prefer do not use this identification, because without identifying the points $(t, -z, -\dot{z})$ and $(t, z, \dot{z})$ the description of the flow becomes more realistic.

We define a Poincaré map f_e on $\mathcal{A}^+ \cup \mathcal{A}^-$ by following a solution with initial conditions $z(t_0) = 0$, $\dot{z}(t_0) = \dot{z}_0$ to its next zero of z, i.e. $z(t_1) = 0$ where t_1 is the smallest $t > t_0$ for which $z(t_1) = 0$ if it exists, otherwise we set $t_1 = \infty$, because the flow of the Sitnikov problem is complete (see Proposition 2). Then the map f_e is given by

$$f_e(\dot{z}_0, \widetilde{t_0}) = \begin{cases} f_e^+(\dot{z}_0, \widetilde{t_0}) & \text{if } \ (\dot{z}_0, \widetilde{t_0}) \in \mathcal{A}^+ \text{ and } t_1 < \infty \ , \\ f_e^-(\dot{z}_0, \widetilde{t_0}) & \text{if } \ (\dot{z}_0, \widetilde{t_0}) \in \mathcal{A}^- \text{ and } t_1 < \infty \ , \end{cases}$$

where $f_e^{\pm}(\dot{z}_0, \widetilde{t_0}) = (\dot{z}_1, \widetilde{t_1})$ with $\dot{z}_1 = \dot{z}(t_1)$. We note that f_e^+ applies points of $\mathcal{A}^+$ into $\mathcal{A}^-$, and f_e^- applies points of $\mathcal{A}^-$ into $\mathcal{A}^+$. Moreover, by symmetry (4), we see that

$$f_e^- = \rho_0^{-1} \circ f_e^+ \circ \rho_0 \ ,$$

where ρ_0 is the symmetry on $\mathcal{A}$ given by

$$\rho_0 \,:\, (\dot{z}_0, \widetilde{t_0}) \longmapsto (-\dot{z}_0, \widetilde{t_0}) \ . \tag{5}$$

We note that the map $\phi_e = f_e^2$ corresponds to a return map on the transversal section $\mathcal{A} \setminus \{\dot{z} = 0\}$.

The Poincaré map f_e, identifying points $(t, -z, -\dot{z})$ and $(t, z, \dot{z})$, has been widely studied, see for instance (Alekseev, 1968), (Moser, 1973) and (Llibre and Simó, 1980). From the results of (Moser, 1973) pages 87–89 we have the following theorem for the map f_e^+.

Let X be a topological space. If $A \subset X$, we denote by $\partial_X A$ the boundary of A in X, and we denote by $\mathrm{cl}_X(A)$ the closure of A in X.

THEOREM 3. *The following statements hold for f_e^+.*

(a) The orbits of the Sitnikov problem escaping for $t \to \infty$ with $\dot{z}(\infty) = 0$ intersect the cylinder $\mathcal{A}^+$ in a simple closed curve β_e^+, which decomposes $\mathcal{A}^+$ into two components. We denote by $\mathcal{B}_e^+$ the component such that $\partial_{\mathcal{A}} \mathcal{B}_e^+ = \{\dot{z}_0 = 0\} \cup \beta_e^+$. It consists of those

points of $\mathcal{A}^+$ *through which pass orbits which return to* $z = 0$ *in forward time; that is,* $\mathcal{B}_e^+$ *is the domain of definition of* f_e^+*. The points* $(\dot{z}_0, \widetilde{t_0}) \in \mathcal{A}^+$ *outside* $\mathcal{B}_e^+$ *correspond to orbits that escape to* $z = +\infty$ *for* $t \to \infty$*. Moreover the orbit associated to* $(\dot{z}_0, \widetilde{t_0})$ *escapes parabolically (i.e.* $\dot{z}(\infty) = 0$*) when* $(\dot{z}_0, \widetilde{t_0}) \in \partial_{\mathcal{A}^+}\mathcal{B}_e^+ = \beta_e^+$*, and it escapes hyperbolically (i.e.* $\dot{z}(\infty) > 0$*) when* $(\dot{z}_0, \widetilde{t_0}) \in \mathcal{A}^+ \setminus \mathrm{cl}_{\mathcal{A}^+}(\mathcal{B}_e^+)$*.*

(b) If $e > 0$ *is small enough, then the curve* β_e^+ *satisfies* $\dot{z}_0 = 2 + 3e(-A\sin t_0 + B\cos t_0)/8 + O(e^2)$*, where*

$$A = \int_0^\infty \sin t \frac{z\dot{z}}{(z^2+1/4)^{5/2}}\, dt \ , \quad B = \int_0^\infty \cos t \frac{z\dot{z}}{(z^2+1/4)^{5/2}}\, dt \ ,$$

and $z = z(t)$ *is the solution of the differential equation*

$$\ddot{z} = -z/(z^2+1/4)^{3/2} \ ,$$

with initial conditions $z(0) = 0$*,* $\dot{z}(0) = 2$*.*

(c) Let ρ_1 *be the symmetry on* $\mathcal{A}$ *defined by* $\rho_1 : (\dot{z}_0, \widetilde{t_0}) \longmapsto (-\dot{z}_0, -\widetilde{t_0})$*. If* $\mathcal{K}_e^+ = f_e^+(\mathcal{B}_e^+) \subset \mathcal{A}^-$*, then* $\mathcal{K}_e^+ = \rho_1(\mathcal{B}_e^+)$*. Moreover* f_e^+ *is an area preserving map and* $(f_e^+)^{-1} = \rho_1^{-1} \circ f_e^+ \circ \rho_1$*. The points of* $\mathcal{K}_e^+$ *correspond to orbits that return to* $z = 0$ *in backward time, whereas the points of* $\mathcal{A}^-$ *outside* $\mathcal{K}_e^+$ *correspond to orbits escaping to* $z = +\infty$ *for* $t \to -\infty$*. Moreover the orbit associated to* $(\dot{z}_0, \widetilde{t_0})$ *escapes parabolically (i.e.* $\dot{z}(-\infty) = 0$*) when* $(\dot{z}_0, \widetilde{t_0}) \in \partial_{\mathcal{A}^-}\mathcal{K}_e^+ = \kappa_e^+ = \rho_1(\beta_e^+)$ *and it escapes hyperbolically (i.e.* $\dot{z}(-\infty) < 0$*) when* $(\dot{z}_0, \widetilde{t_0}) \in \mathcal{A}^- \setminus \mathrm{cl}_{\mathcal{A}^-}(\mathcal{K}_e^+)$*.*

(d) Let γ *be a* $\mathcal{C}^1$ *arc parameterized by* $\dot{z}_0 = \dot{z}_0(\lambda)$*,* $t_0 = t_0(\lambda)$ *with* $0 \le \lambda \le 1$*, such that* γ *meets* β_e^+ *in the endpoint* r *corresponding to* $\lambda = 0$ *nontangentially, while* $\gamma \setminus \{r\}$ *lies in* $\mathcal{B}_e^+$*. Then the image curve*

$$f_e^+(\gamma) = \{f_e^+(\dot{z}_0(\lambda), \widetilde{t_0(\lambda)}) = (\dot{z}_1(\lambda), \widetilde{t_1(\lambda)}) \,:\, 0 < \lambda \le 1\} \ ,$$

approaches the boundary κ_e^+ *spiraling; i.e.* $t_1(\lambda) \to \infty$ *as* $\lambda \to 0$ *(see Figure 3).*

Proof. See the proofs of Lemmas 1, 2, 3 and 4 of Chapter III §5(c) of (Moser, 1973). ■

REMARK . The spiraling behaviour of $f_e^+(\gamma)$ given by Theorem 3(d) does not need that the curves γ and β_e^+ intersect nontangentially. It follows from the fact that the time between two consecutive zeros of z is a continuous function of $(\dot{z}_0, t_0)$ that tends to infinity as we approach to the boundary β_e^+. □

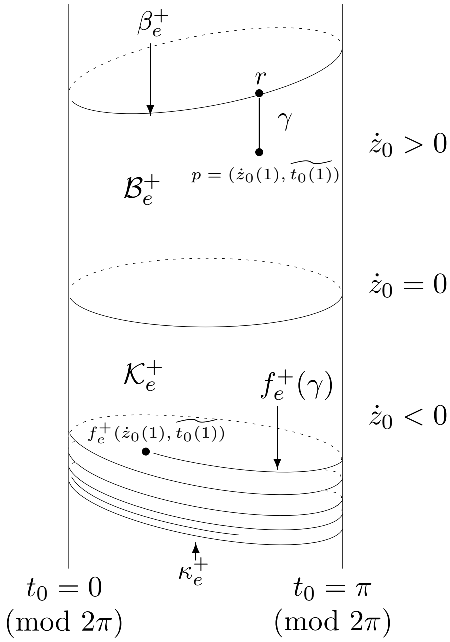

Figure 3. The image by f_e^+ of the arc γ.

By means of symmetry ρ_0, see (5), we have a similar result for f_e^- (remember that $f_e^- = \rho_0^{-1} \circ f_e^+ \circ \rho_0$).

We set $\mathcal{B}_e^- = \rho_0(\mathcal{B}_e^+) \subset \mathcal{A}^-$ (the definition domain of f_e^-), $\beta_e^- = \rho_0(\beta_e^+) = \partial_{\mathcal{A}^-}\mathcal{B}_e^-$, $\mathcal{K}_e^- = \rho_0(\mathcal{K}_e^+) \subset \mathcal{A}^+$ (the definition domain of $(f_e^-)^{-1}$) and $\kappa_e^- = \rho_0(\kappa_e^+) = \partial_{\mathcal{A}^+}\mathcal{K}_e^-$.

Using the above notation, the domain of definition of the Poincaré map f_e is

$$\mathcal{B}_e = \mathcal{B}_e^+ \cup \mathcal{B}_e^- \ ,$$

and the domain of definition of f_e^{-1} (or equivalently, the image of f_e) is

$$\mathcal{K}_e = \mathcal{K}_e^+ \cup \mathcal{K}_e^- \ .$$

THEOREM 4. *If $e > 0$, the regions $\mathcal{B}_e^+$ and $\mathcal{K}_e^-$ (respectively, $\mathcal{B}_e^-$ and $\mathcal{K}_e^+$) are different, and the boundary curves β_e^+ and κ_e^- (respectively, β_e^- and κ_e^+) intersect on $t = 0$ and $t = \pi$ at two points P and Q (respectively, P' and Q') (see Figure 4). Moreover these intersections are transversal for all $e \in (0,1)$ except perhaps for a discrete set of values of e.*

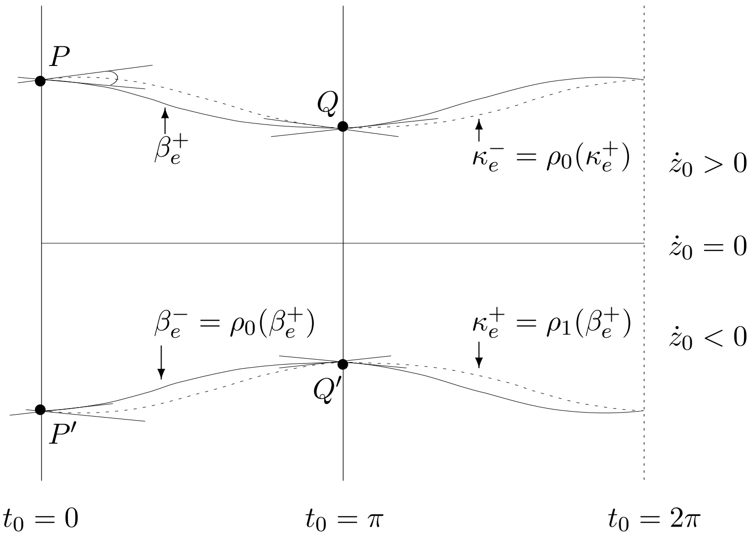

Figure 4. The curves $\beta_e^\pm$ and $\kappa_e^\pm$.

Proof. See the proof of Lemma 3 of Chapter III §5(c) of (Moser, 1973). ∎

REMARK . Numeric computations show that the constants A and B that appear in the expression of β_e^+ (see Theorem 3(b)) are given by $A = 0.67639\ldots$ and $B = 2.46685\ldots$. Using these values of A and B we see that for e small the curves β_e^+ and κ_e^- (respectively, β_e^- and κ_e^+) intersect exactly in two points, instead of the four points which appear in the figures of (Moser, 1973). This fact was noted in (Llibre and Simó, 1980). □

3.1. The domain of definition of the map f_e^k

We only analyze the domain of definition of f_e^k with $k > 1$. Then the domain of definition of f_e^k with $k < -1$ would be obtained by symmetry (see Theorem 3).

Since $\mathcal{B}_e = \mathcal{B}_e^+ \cup \mathcal{B}_e^-$ (the domain of definition of f_e) and $\mathcal{K}_e = \mathcal{K}_e^+ \cup \mathcal{K}_e^-$ (the image of f_e) do not coincide when $e > 0$, the domain of definition of f_e^k for $k > 1$ is different from $\mathcal{B}_e$.

We start analyzing the domain of definition of f_e^2 on $\mathcal{A}^+$ (i.e. the domain of definition of the map $f_e^- \circ f_e^+$), then by means of the symmetry ρ_0 we obtain the domain of definition f_e^2 on $\mathcal{A}^-$ (i.e. the domain of definition of $f_e^+ \circ f_e^-$). We consider an arbitrary segment γ on $\mathcal{B}_e^+$

that starts at a point $p \in \mathcal{B}_e^+$ and meets nontangentially $\partial\mathcal{B}_e^+ = \beta_e^+$ at a endpoint r (see Figure 3). By Theorem 3(d), the image by f_e^+ of the curve γ approaches the boundary $\partial_{\mathcal{A}^-}\mathcal{K}_e^+ = \kappa_e^+$ spiraling infinitely many times. Since $\mathcal{B}_e^- \neq \mathcal{K}_e^+$ (see Theorem 4), there are infinitely many arcs on $f_e(\gamma)$ that are not in $\mathcal{B}_e^-$ (the domain of definition of f_e^-). Thus, the domain of definition of f_e^2 restricted to γ is $\gamma \setminus \bigcup_{j=1}^{\infty} I_j$, where I_j are the closed intervals such that $f_e^+(I_j) \not\subset \mathcal{B}_e^-$. It is clear that the intervals I_j accumulate in a neighborhood of the point r when $j \to \infty$ (see Figure 5). Then, by continuity, the domain of definition $\mathcal{B}_e^{+,2}$ of f_e^2 on $\mathcal{A}^+$ is $\mathcal{B}_e^+$ minus a strip that approaches the boundary $\partial_{\mathcal{A}^+}\mathcal{B}_e^+ = \beta_e^+$ spiraling infinitely many times. Using the symmetry ρ_0, the domain of definition $\mathcal{B}_e^{-,2}$ of f_e^2 on $\mathcal{A}^-$ is $\rho_0(\mathcal{B}_e^{+,2})$. Thus the domain of definition of f_e^2 on $\mathcal{A}$ is $\mathcal{B}_e^{+,2} \cup \rho_0(\mathcal{B}_e^{+,2}) = \mathcal{B}_e^{+,2} \cup \mathcal{B}_e^{-,2}$.

Here $\mathbb{Z}$ denotes the set of integer numbers and $\mathbb{N}$ denotes the set of positive integers.

Let $I_j = [a_j, b_j]$ for all $j \in \mathbb{N}$, $\gamma_1 = [p, a_1)$ and $\gamma_j = (b_{j-1}, a_j)$ for $j > 1$. We note that $f_e^+(\gamma_1)$ is an arc on $\mathcal{B}_e^-$ that meets $\partial_{\mathcal{A}^-}\mathcal{B}_e^- = \beta_e^-$ at the endpoint $f_e^+(a_1)$. Then using the properties of f_e^-, the image by f_e^- of the arc $f_e^+(\gamma_1)$ approaches the boundary $\partial_{\mathcal{A}^+}\mathcal{B}_e^+ = \beta_e^+$ spiraling infinitely many times. Thus, using the previous arguments, the domain of definition of $f_e^3 = f_e^+ \circ f_e^- \circ f_e^+$ on γ_1 is $\gamma_1 \setminus \bigcup_{j=1}^{\infty} J_{1,n}$, where $J_{1,n}$ are the closed intervals such that $f_e^2(J_{1,n}) \not\subset \mathcal{B}_e^+$ and they accumulate in a neighborhood of the point a_1 when $n \to \infty$. In a similar way, for $j > 1$, we can see that the domain of definition of f_e^3 on γ_j is

$$\gamma_j \setminus \left[\left(\bigcup_{n=1}^{\infty} J_{j,n} \right) \bigcup \left(\bigcup_{n=1}^{\infty} K_{j-1,n} \right) \right],$$

where $J_{j,n}$ are closed intervals that accumulate in a neighborhood of the point a_j when $n \to \infty$, and $K_{j-1,n}$ are closed intervals that accumulate in a neighborhood of the point b_{j-1} when $n \to \infty$. In short, the domain of definition of f_e^3 on γ is

$$\gamma \setminus \left[\bigcup_{j=1}^{\infty} \left[I_j \bigcup \left(\bigcup_{n=1}^{\infty} J_{j,n} \right) \bigcup \left(\bigcup_{n=1}^{\infty} K_{j,n} \right) \right] \right],$$

(see Figure 6). So, by continuity, the domain of definition $\mathcal{B}_e^{+,3}$ of f_e^3 on $\mathcal{A}^+$ is $\mathcal{B}_e^{+,2}$ minus two families of $\mathbb{N}$ strips, which are close to the strip that we have removed in $\mathcal{B}_e^+$ to obtain $\mathcal{B}_e^{+,2}$. Moreover the domain of

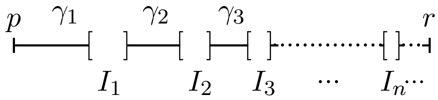

Figure 5. Domain of definition of f_e^2 restricted to γ.

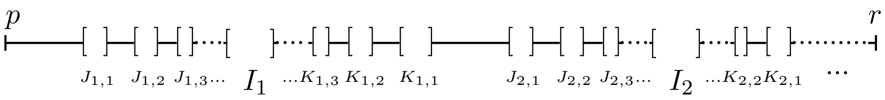

Figure 6. Domain of definition of f_e^3 restricted to γ.

definition of f_e^3 on $\mathcal{A}$ is $\mathcal{B}_e^{+,3} \cup \rho_0(\mathcal{B}_e^{+,3})$. In a similar way we would find the domain of definition of f_e^k for an arbitrary $k > 1$.

4. Symmetric periodic orbits

In this work when we say that $\varphi(t)$ is a *periodic orbit of period* τ, with $\tau > 0$, we mean that $\varphi(t+\tau) = \varphi(t)$ for all $t \in \mathbb{R}$ and there is no $\overline{\tau} \in (0, \tau)$ satisfying that condition; that is, we mean that τ is the *minimal period.*

Since we have chosen the origin of time so that at $t = 0$ the primaries are at the pericenter of the ellipse it is not difficult to see that $r(t) = r(-t)$, so the equation of motion of the Sitnikov problem (1) is invariant under the symmetry

$$(t, z, \dot{z}) \longmapsto (-t, -z, \dot{z}) \; . \tag{6}$$

We note that this symmetry corresponds to a symmetry with respect to the plane that contains the primaries, so in what follows it will be denoted by the $r-$*symmetry.*

Let $\varphi(t) = (z(t), \dot{z}(t))$ denote the solution of the Sitnikov problem with initial conditions $z(0) = z_0$, $\dot{z}(0) = \dot{z}_0$. By means of symmetry (6) be have that $\psi(t) = (-z(-t), \dot{z}(-t))$ is also a solution of (1). If $z(0) = 0$, then the two solutions $\varphi(t)$ and $\psi(t)$ coincide at time $t = 0$. Therefore, by the existence and uniqueness theorem on the solutions of an ordinary differential system, the solutions $\varphi(t)$ and $\psi(t)$ must be the same. In that case we say that $\varphi(t)$ is a $r-$*symmetric solution.*

Moreover if we can find a time $t = \tau/2 = k\pi \neq 0$, for some $k \in \mathbb{N}$, such that $z(t) = 0$, then $\varphi(\tau/2)$ and $\psi(\tau/2)$ correspond to the same point of the phase space $(t(\mathrm{mod} 2\pi), z, \dot{z})$, consequently the orbit of $\varphi(t)$ must be closed, i.e. $\varphi(t)$ is a $r-$*symmetric periodic solution* of the Sitnikov problem. If there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (0, \tau/2)$ such that $z(\overline{\tau}) = 0$, then the period of this periodic orbit is τ.

If we chose the origin of time $t = 0$ when the primaries are at the apocenter instead of the pericenter, then the $r-$symmetry (6) works also for this new time. But since in this paper we have selected the origin of time $t = 0$ when the primaries are at the pericenter, the $r-$symmetry with respect to the pericenter or the apocenter becomes

$$(t + n\pi, z, \dot{z}) \longmapsto (n\pi - t, -z, \dot{z}) \ ,$$

with $n \in \mathbb{Z}$ even or odd, respectively. By means of the above symmetry we have that $(z(n\pi + s), \dot{z}(n\pi + s)) = (-z(n\pi - s), \dot{z}(n\pi - s))$, for all s.

In short we have proved the following result.

PROPOSITION 5. *Let $\varphi(t) = (z(t), \dot{z}(t))$ be the solution of the Sitnikov problem with initial conditions $z(t_0) = z_0$ and $\dot{z}(t_0) = \dot{z}_0$, with $t_0 = n\pi$ for some $n \in \mathbb{Z}$. If $z(t)$ is zero at times $t = t_0$ and $t = t_0 + \tau/2$, with $\tau/2 = k\pi$ for some integer $k > n$, and there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (t_0, t_0 + \tau/2)$ such that $z(\overline{\tau}) = 0$, then $\varphi(t)$ is a $r-$symmetric periodic solution with period τ.*

The Sitnikov problem is also invariant under the symmetry

$$(t, z, \dot{z}) \longmapsto (-t, z, -\dot{z}) \ , \tag{7}$$

which corresponds to the time reversibility symmetry, so we will denote it by the $t-$*symmetry*. Proceeding in a similar way than in the $r-$symmetry, we can introduce the notion of $t-$*symmetric solution* and $t-$*symmetric periodic solution*. Moreover the $t-$symmetric periodic solutions are characterized by the following result.

PROPOSITION 6. *Let $\varphi(t) = (z(t), \dot{z}(t))$ be the solution of the Sitnikov problem with initial conditions $z(t_0) = z_0$ and $\dot{z}(t_0) = \dot{z}_0$ with $t_0 = n\pi$ for some $n \in \mathbb{Z}$. If $\dot{z}(t)$ is zero at times $t = t_0$ and $t = t_0 + \tau/2$, with $\tau/2 = k\pi$ for some integer $k > n$, and there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (t_0, t_0 + \tau/2)$ such that $\dot{z}(\overline{\tau}) = 0$, then $\varphi(t)$ is a $t-$symmetric periodic solution with period τ.*

We note that it could be periodic solutions of the Sitnikov problem that are simultaneously $r-$symmetric and $t-$symmetric. These periodic solutions are called *double–symmetric periodic solutions* and they are characterized by the following result.

PROPOSITION 7. *Let $\varphi(t) = (z(t), \dot{z}(t))$ be the solution of the Sitnikov problem with initial conditions $z(t_0) = z_0$ and $\dot{z}(t_0) = \dot{z}_0$ with $t_0 = n\pi$ for some $n \in \mathbb{Z}$.*

(a) If $z(t)$ is zero at time $t = t_0$ and $\dot{z}(t)$ is zero at time $t = t_0+\tau/4$, with $\tau/4 = k\pi$ for some integer $k > n$, and there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (t_0, t_0 + \tau/4)$ such that $\dot{z}(\overline{\tau}) = 0$, then $\varphi(t)$ is a double–symmetric periodic solution with period τ.

(b) If $\dot{z}(t)$ is zero at time $t = t_0$ and $z(t)$ is zero at time $t = t_0+\tau/4$, with $\tau/4 = k\pi$ for some integer $k > n$, and there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (t_0, t_0 + \tau/4)$ such that $z(\overline{\tau}) = 0$, then $\varphi(t)$ is a double–symmetric periodic solution with period τ.

Proof. Assume that the solution $\varphi(t) = (z(t), \dot{z}(t))$ with initial conditions $z(0) = 0$ and $\dot{z}(0) = \dot{z}_0$ satisfies that $\dot{z}(\tau/4) = 0$ for a given $\tau/4 = k\pi$ with $k \in \mathbb{N}$, and there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (0, \tau/4)$ such that $\dot{z}(\overline{\tau}) = 0$. We want to see that $\varphi(t)$ is a $r-$symmetric and a $t-$symmetric periodic solution with period τ.

Since at $t = 0$ the primaries are at the pericenter of the ellipse and $z(0) = 0$, we can apply the $r-$symmetry (see (6)) obtaining

$$(z(t), \dot{z}(t)) = (-z(-t), \dot{z}(-t)) \ . \tag{8}$$

We set $\varphi(\tau/4) = (z(\tau/4), \dot{z}(\tau/4)) = (\overline{z}_0, 0)$. Taking $t = \tau/4$ into expression (8) we have that

$$(z(\tau/4), \dot{z}(\tau/4)) = (-z(-\tau/4), \dot{z}(-\tau/4)) = (\overline{z}_0, 0) \ .$$

Thus $\varphi(t)$ is a solution of the Sitnikov problem with initial conditions $z(-\tau/4) = \overline{z}_0$ and $\dot{z}(-\tau/4) = 0$, such that $\dot{z}(-\tau/4 + \tau/2) = 0$ with $\tau/2 = 2k\pi$. Moreover we claim that there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (-\tau/4, \tau/4)$ such that $\dot{z}(\overline{\tau}) = 0$. Indeed, we have assumed that there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (0, \tau/4)$ such that $\dot{z}(\overline{\tau}) = 0$, consequently by means of (8) there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (-\tau/4, 0)$ such that $\dot{z}(\overline{\tau}) = 0$. Hence the claim is proved. Therefore, applying Proposition 6, we have that $\varphi(t)$ is a $t-$symmetric periodic solution of the Sitnikov problem with period τ.

On the other hand, since $\tau/4 = k\pi$ with $k \in \mathbb{N}$ (that is, at $t = \tau/4$ the primaries are at the pericenter or the apocenter of the ellipse) and $\dot{z}(\tau/4) = 0$, we can apply the $t-$symmetry (see (7)) obtaining

$$(z(\tau/4 + s), \dot{z}(\tau/4 + s)) = (z(\tau/4 - s), -\dot{z}(\tau/4 - s)) \ ,$$

for all s. Taking $s = \tau/4$ into the above expression we have that

$$(z(\tau/2), \dot{z}(\tau/2)) = (z(0), -\dot{z}(0)) = (0, -\dot{z}_0) \ .$$

So $\varphi(t) = (z(t), \dot{z}(t))$ is a solution of the Sitnikov problem with initial conditions $z(0) = 0$ and $\dot{z}(0) = \dot{z}_0$, such that $z(\tau/2) = 0$ with $\tau/2 = 2k\pi$. Moreover there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (0, \tau/2)$ such that $z(\overline{\tau}) = 0$. Indeed, if such $\overline{\tau}$ exists then by means of the $r-$symmetry we have that

$$(z(\overline{\tau} + s), \dot{z}(\overline{\tau} + s)) = (-z(\overline{\tau} - s), \dot{z}(\overline{\tau} - s)) \ ,$$

for all s. Taking $s = \tau/4 - \overline{\tau}$ we have that

$$(z(\tau/4), \dot{z}(\tau/4)) = (-z(2\overline{\tau} - \tau/4), \dot{z}(2\overline{\tau} - \tau/4)) = (\overline{z}_0, 0) \ .$$

This means that there exists t multiple of π with $t \in (-\tau/4, \tau/4)$ such that $\dot{z}(t) = 0$, which contradicts the claim. Therefore, from Proposition 5, $\varphi(t)$ is a $r-$symmetric periodic solution of the Sitnikov problem with period τ.

This proves statement (a) when $t_0 = 0$. The cases $t_0 = n\pi$ with $n \in \mathbb{Z} \setminus \{0\}$ follow using the same arguments.

Statement (b) would be proved in a similar way. ■

From the proof of the previous proposition it follows easily the next result.

COROLLARY 8. *The following statements hold.*

(a) Let $(z(t), \dot{z}(t))$ be a periodic solution of the Sitnikov problem with period $\tau = 2\pi p$, for some $p \in \mathbb{N}$, and with initial conditions $z(t_0) = 0$, $\dot{z}(t_0) = \dot{z}_0$, with $t_0 = n\pi$ for some $n \in \mathbb{Z}$. Then the periodic solution $(z(t), \dot{z}(t))$ is double–symmetric if and only if p is even and $(z(t), \dot{z}(t))$ satisfies

$$z(t_0 + \tau/2) = 0 \qquad \textit{and} \qquad \dot{z}(t_0 + \tau/2) = -\dot{z}_0 \ .$$

(b) Let $(z(t), \dot{z}(t))$ be a periodic solution of the Sitnikov problem with period $\tau = 2\pi p$, for some $p \in \mathbb{N}$, and with initial conditions $z(t_0) = z_0$, $\dot{z}(t_0) = 0$, with $t_0 = n\pi$ for some $n \in \mathbb{Z}$. Then the periodic solution $(z(t), \dot{z}(t))$ is double–symmetric if and only if p is even and $(z(t), \dot{z}(t))$ satisfies

$$z(t_0 + \tau/2) = -z_0 \qquad \textit{and} \qquad \dot{z}(t_0 + \tau/2) = 0 \ .$$

From Propositions 5 and 6, the $r-$symmetric and $t-$symmetric periodic orbits characterize for having two points of the phase space of the Sitnikov problem $(t(\mathrm{mod}\, 2\pi), z, \dot{z})$ into the sets

$$\mathcal{Z}_r = \{(t(\mathrm{mod}\, 2\pi), z, \dot{z}) \, : \, t = n\pi, \ n \in \mathbb{Z}, z = 0\} \ ,$$

and

$$\mathcal{Z}_t = \{(t(\operatorname{mod} 2\pi), z, \dot{z}) \,:\, t = n\pi,\ n \in \mathbb{Z}, \dot{z} = 0\} \ ,$$

respectively. Then the symmetric periodic orbits of the Sitnikov problem can be classified using their "positions" at the points of intersection with $\mathcal{Z}_r$ and $\mathcal{Z}_t$. At these points, the primaries could be either at the pericenter or at the apocenter of the ellipse (i.e. $t(\operatorname{mod} 2\pi) = 0$ and $t(\operatorname{mod} 2\pi) = \pi$, respectively). Moreover the infinitesimal mass could cross $\mathcal{Z}_r$ with either $\dot{z} > 0$ or $\dot{z} < 0$; and it could cross $\mathcal{Z}_t$ with either $z > 0$ or $z < 0$. This gives us eight possible positions, that will be denoted by $P_r^{\pm}$ ($A_r^{\pm}$) when we are at the pericenter (apocenter) with $r-$symmetric conditions (the signs $\pm$ correspond to $\dot{z} > 0$ and $\dot{z} < 0$, respectively), and by $P_t^{\pm}$ ($A_t^{\pm}$) when we are at the pericenter (apocenter) with $t-$symmetric conditions (the signs $\pm$ correspond to $z > 0$ and $z < 0$, respectively). We use the notation $\text{pos}_1 \stackrel{\bar{t}}{\longrightarrow} \text{pos}_2$ to say that at an instant t_0 the orbit is at position pos_1 and at $t_0 + \bar{t}$ it is at position pos_2. Then the classification of the symmetric periodic orbits of the Sitnikov problem is given by the following result.

THEOREM 9. *The following statements hold.*

(a) Each $r-$symmetric periodic orbit of the Sitnikov problem with period τ is of one of the following types:

(i) $P_r^+ \stackrel{\tau/2}{\longrightarrow} A_r^+ \stackrel{\tau/2}{\longrightarrow} P_r^+$,

(ii) $P_r^+ \stackrel{\tau/2}{\longrightarrow} A_r^- \stackrel{\tau/2}{\longrightarrow} P_r^+$,

(iii) $P_r^- \stackrel{\tau/2}{\longrightarrow} A_r^+ \stackrel{\tau/2}{\longrightarrow} P_r^-$,

(iv) $P_r^- \stackrel{\tau/2}{\longrightarrow} A_r^- \stackrel{\tau/2}{\longrightarrow} P_r^-$,

(v) $P_r^+ \stackrel{\tau/2}{\longrightarrow} P_r^- \stackrel{\tau/2}{\longrightarrow} P_r^+$,

(vi) $A_r^+ \stackrel{\tau/2}{\longrightarrow} A_r^- \stackrel{\tau/2}{\longrightarrow} A_r^+$.

(vii) $P_r^+ \stackrel{\tau/2}{\longrightarrow} P_r^+ \stackrel{\tau/2}{\longrightarrow} P_r^+$.

(viii) $P_r^- \stackrel{\tau/2}{\longrightarrow} P_r^- \stackrel{\tau/2}{\longrightarrow} P_r^-$.

(ix) $A_r^+ \stackrel{\tau/2}{\longrightarrow} A_r^+ \stackrel{\tau/2}{\longrightarrow} A_r^+$.

(x) $A_r^- \stackrel{\tau/2}{\longrightarrow} A_r^- \stackrel{\tau/2}{\longrightarrow} A_r^-$.

Moreover from these ten types of periodic orbits only the types (v) and (vi) can correspond to double–symmetric periodic orbits.

(b) Each $t-$symmetric periodic orbit of the Sitnikov problem with period τ is of one of the following types:

(i) $P_t^+ \xrightarrow{\tau/2} A_t^+ \xrightarrow{\tau/2} P_t^+$,

(ii) $P_t^+ \xrightarrow{\tau/2} A_t^- \xrightarrow{\tau/2} P_t^+$,

(iii) $P_t^- \xrightarrow{\tau/2} A_t^+ \xrightarrow{\tau/2} P_t^-$,

(iv) $P_t^- \xrightarrow{\tau/2} A_t^- \xrightarrow{\tau/2} P_t^-$,

(v) $P_t^+ \xrightarrow{\tau/2} P_t^- \xrightarrow{\tau/2} P_t^+$,

(vi) $A_t^+ \xrightarrow{\tau/2} A_t^- \xrightarrow{\tau/2} A_t^+$.

(vii) $P_t^+ \xrightarrow{\tau/2} P_t^+ \xrightarrow{\tau/2} P_t^+$.

(viii) $P_t^- \xrightarrow{\tau/2} P_t^- \xrightarrow{\tau/2} P_t^-$.

(ix) $A_t^+ \xrightarrow{\tau/2} A_t^+ \xrightarrow{\tau/2} A_t^+$.

(x) $A_t^- \xrightarrow{\tau/2} A_t^- \xrightarrow{\tau/2} A_t^-$.

Moreover from these ten types of periodic orbits only the types (v) and (vi) can correspond to double–symmetric periodic orbits.

(c) Each double–symmetric periodic orbit of the Sitnikov problem with period τ is of one of the following types:

(i) $P_r^+ \xrightarrow{\tau/4} A_t^+ \xrightarrow{\tau/4} P_r^- \xrightarrow{\tau/4} A_t^- \xrightarrow{\tau/4} P_r^+$,

(ii) $P_r^+ \xrightarrow{\tau/4} A_t^- \xrightarrow{\tau/4} P_r^- \xrightarrow{\tau/4} A_t^+ \xrightarrow{\tau/4} P_r^+$,

(iii) $A_r^+ \xrightarrow{\tau/4} P_t^+ \xrightarrow{\tau/4} A_r^- \xrightarrow{\tau/4} P_t^- \xrightarrow{\tau/4} A_r^+$,

(iv) $A_r^+ \xrightarrow{\tau/4} P_t^- \xrightarrow{\tau/4} A_r^- \xrightarrow{\tau/4} P_t^+ \xrightarrow{\tau/4} A_r^+$,

(v) $P_r^+ \xrightarrow{\tau/4} P_t^+ \xrightarrow{\tau/4} P_r^- \xrightarrow{\tau/4} P_t^- \xrightarrow{\tau/4} P_r^+$,

(vi) $P_r^+ \xrightarrow{\tau/4} P_t^- \xrightarrow{\tau/4} P_r^- \xrightarrow{\tau/4} P_t^+ \xrightarrow{\tau/4} P_r^+$,

(vii) $A_r^+ \xrightarrow{\tau/4} A_t^+ \xrightarrow{\tau/4} A_r^- \xrightarrow{\tau/4} A_t^- \xrightarrow{\tau/4} A_r^+$,

(viii) $A_r^+ \xrightarrow{\tau/4} A_t^- \xrightarrow{\tau/4} A_r^- \xrightarrow{\tau/4} A_t^+ \xrightarrow{\tau/4} A_r^+$.

Proof. We only give the proof of statement (a). The other two statements would be proved in a similar way.

The $r-$symmetric periodic orbits of the Sitnikov problem with period τ cross $\mathcal{Z}_r$ at two points: one for time t_0 and the other one for time $t_0 + \tau/2$. Each intersection point must be of type P_r^+, P_r^-, A_r^+ or A_r^-. Therefore we have sixteen possible configurations for the the pair of points of intersection of the orbit with $\mathcal{Z}_r$.

Apart from configurations (vii), (viii), (ix) and (x), we have another twelve configurations from which only configurations (i), (ii), (iii), (iv), (v) and (vi) are different. Indeed, configuration $P_r^+ \xrightarrow{\tau/2} A_r^+ \xrightarrow{\tau/2} P_r^+$ corresponds to an orbit such that at time t_0 is at position P_r^+ and at time $t_0 + \tau/2$ is at position A_r^+. But it also corresponds to an orbit such that at time $\bar{t}_0 = t_0 + \tau/2$ is at position A_r^+ and at time $\bar{t}_0 + \tau/2$ is at position P_r^+. So configurations $P_r^+ \xrightarrow{\tau/2} A_r^+ \xrightarrow{\tau/2} P_r^+$ and $A_r^+ \xrightarrow{\tau/2} P_r^+ \xrightarrow{\tau/2} A_r^+$ are the same. The other configuration would be treated in a similar way.

We note that configurations (i), (ii), (iii) and (iv) correspond to symmetric periodic orbits $\varphi(t) = (z(t), \dot{z}(t))$ such that the primaries are located at the pericenter at time t_0 and they are at the apocenter at time $t_0 + \tau/2$. Thus the period of these periodic orbits is $\tau = 2\pi k$ for some $k \in \mathbb{N}$ odd. Consequently they cannot be double–symmetric periodic orbits because $\tau/4$ is not multiple of π (see Proposition 7(a)). Therefore the only configurations that can correspond to double–symmetric periodic orbits are configurations (v) and (vi). ■

REMARK 10. *It seems that configurations (vii), (viii), (ix) and (x) of statements (a) and (b) of Theorem 9 are only possible near the escaping orbits.*

We show the remark working with the orbits having initial conditions on $\mathcal{A}^+$. A similar analysis could be done for the orbits having initial conditions on $\mathcal{A}^-$.

Given an even $q \in \mathbb{N}$, let $\overline{\gamma} = \{(\dot{z}_0, 0) \in \mathcal{A}^+\}$ and let $\gamma_e = \{(\dot{z}_0, 0) \in \mathcal{B}_e^{+,q} \ : \ 0 < \dot{z}_0 < a\} \subset \overline{\gamma}$, where a is the smallest point such that $f_e^{q-1}(a, 0) \in \partial_{\mathcal{A}^-}\mathcal{B}_e^- = \beta_e^-$. It is clear that $f_e^{q-1}(\gamma_e)$ is a $\mathcal{C}^1$–arc that meets β_e^- in the endpoint $f_e^{q-1}(a, 0)$. Thus, by the results of Section 3, we have that the image by f_e^q of the curve γ_e approaches the boundary $\partial_{\mathcal{A}^+}\mathcal{K}_e^- = \kappa_e^-$ spiraling infinitely many times. Consequently, $f_e^q(\gamma_e)$ and $\overline{\gamma}$ have infinitely many intersection points $(\dot{z}_k, 0)$. We follow the curve γ_e from the point $(0, 0)$ to the point $(a, 0)$ and we order the points of intersection so that $(\dot{z}_1, 0)$ is the first point of intersection, as one traverses γ_e from $(0, 0)$ to $(a, 0)$, $(\dot{z}_2, 0)$ is the second, etc.

Consider configuration $P_r^+ \xrightarrow{\tau/2} P_r^+ \xrightarrow{\tau/2} P_r^+$ of statement (a) (the other ones would be treated in a similar way). It corresponds to a $\tau-$periodic solution $(z(t), \dot{z}(t))$ of the Sitnikov problem with initial conditions $z(0) = 0$, $\dot{z}(0) = a_0$ for some $a_0 > 0$ and such that $z(\tau/2) = 0$ and $\dot{z}(\tau/2) = a_1$ with $a_1 > 0$ and $\tau/2 = 2\pi k$ for some $k \in \mathbb{N}$.

The solution $(z(t), \dot{z}(t))$ is defined by the initial conditions $(a_0, 0) \in \mathcal{A}^+$ (see Section 3 for details), moreover the point $(a_0, 0)$ satisfies that $f_e^q(a_0, 0) = (a_1, 0)$ for some even $q \in \mathbb{N}$, where q is the number of zeros of z between the times $t = 0$ and $t = \tau/2$. We note that q cannot be odd because if q were odd, then $f_e^q(a_0, 0) \in \mathcal{A}^-$, and consequently a_1 would be negative. On the other hand, since $(z(t), \dot{z}(t))$ is a $r-$symmetric periodic solution, we have also that $f_e^q(a_1, 0) = (a_0, 0)$.

Assume that the points $(a_0, 0)$ and $(a_1, 0)$ are in γ_e and that the intersection points $(\dot{z}_k, 0)$ satisfy the following condition

$$0 < \dot{z}_1 < \dot{z}_2 < \dot{z}_3 < \dots \tag{9}$$

Since $f_e^q(a_0, 0) = (a_1, 0)$, it is clear that $(a_1, 0) \in f_e^q(\gamma_e) \cap \overline{\gamma}$, so $(a_1, 0) = (\dot{z}_k, 0)$ for some $k \in \mathbb{N}$. On the other hand, $f_e^q(a_1, 0) = (a_0, 0)$, so $(a_0, 0) = (\dot{z}_j, 0)$ for some $j \in \mathbb{N}$. Hence, taking into account the order of intersection we see that this is only possible when $\dot{z}_k = \dot{z}_j$; that is when $a_0 = a_1$.

We note that near the escaping orbits, we cannot assure if the intersection points of $f_e^q(\gamma_e)$ with $\overline{\gamma}$ belong to γ_e or, on the contrary, they belong to another component of the domain of definition of f_e^q restricted to $\overline{\gamma}$. In short, if condition (9) is satisfied and $(a_0, 0)$, $(a_1, 0) \in \gamma_e$, then configuration $P_r^+ \xrightarrow{\tau/2} P_r^+ \xrightarrow{\tau/2} P_r^+$ corresponds to a periodic orbit of period $\tau/2$. So in this case configuration $P_r^+ \xrightarrow{\tau/2} P_r^+ \xrightarrow{\tau/2} P_r^+$ is not possible with period τ.

By the uniqueness theorem on the solutions of an ordinary differential equation, the map $f_e^q|_{\gamma_e}$ is injective. If

$$\frac{\partial t_1}{\partial \dot{z}_0} > 0 \ , \tag{10}$$

for all $(\dot{z}_0, 0) \in \gamma_e$, with t_1 defined by $f_e^q(\dot{z}_0, 0) = (\dot{z}_1, t_1 \, (\mathrm{mod}\, 2\pi))$, then condition (9) is satisfied.

When $e = 0$, the map f_e^q restricted to $\mathcal{A}^+$ is given by

$$f_0^q(\dot{z}_0, 0) = (\dot{z}_0, \tau(\dot{z}_0) q/2 \, (\mathrm{mod}\, 2\pi)) \ ,$$

where $\tau(\dot{z}_0)$ is the period of the periodic orbit having initial conditions $z(0) = 0$, $\dot{z}(0) = \dot{z}_0$ with $\dot{z}_0 \in (0, 2)$. The definition domain of the map f_0^q on $\mathcal{A}^+$ does not depend on q and it given by the set $\mathcal{B}_0^+ = \{(\dot{z}_0, \widetilde{t}_0) \in \mathcal{A}^+ \ : \ 0 < \dot{z}_0 < 2\}$; moreover, since q is even, $f_0^q(\mathcal{B}_0^+) = \mathcal{B}_0^+$. We note that if q where odd, then $f_0^q(\mathcal{B}_0^+) = \mathcal{B}_0^- = \rho_0(\mathcal{B}_0^+)$. On the other hand,

$$\frac{\partial \tau(\dot{z}_0)}{\partial \dot{z}_0} > 0 \ ,$$

for all $\dot{z}_0 \in (0,2)$, see for more details (Llibre and Simó, 1980). We set $\gamma_0 = \{(\dot{z}_0, 0) \in \mathcal{B}_0^+\}$. Since $f_0^q(\mathcal{B}_0^+) = \mathcal{B}_0^+$, it is easy to see that $f_0^q(\gamma_0)$ intersects γ_0 at infinitely many points satisfying condition (9). Therefore, configuration $P_r^+ \xrightarrow{\tau/2} P_r^+ \xrightarrow{\tau/2} P_r^+$ is not possible when $e = 0$.

Consider now the case $e > 0$ small. By the theorem of analytical dependence of the solutions of an ordinary system of differential equations on the initial conditions and parameters we can find $\overline{a} < a$, that depends on e, such that on $\overline{\gamma}_e = \{(\dot{z}_0, 0) \in \mathcal{B}_e^{+,q} \ : \ 0 < \dot{z}_0 < \overline{a}\}$ the image by f_e^q and the image by f_0^q are close. Consequently, the points of intersection of $f_e^q(\overline{\gamma}_e)$ with $\overline{\gamma}$ satisfy condition (9).

Finally, we have analyzed numerically, near $z = 0$ and $\dot{z}_0 = 0$, the points of intersection of $f_e^2(\gamma_e)$ and $f_e^4(\gamma_e)$ with $\overline{\gamma}$, for different values of e, and all the intersection points that we have found satisfy condition (9). Probably this fact also happens for the other even values of q. Thus it seems that configuration $P_r^+ \xrightarrow{\tau/2} P_r^+ \xrightarrow{\tau/2} P_r^+$ is only possible near the escaping orbits. $\square$

REMARK 11. *It seems that, near $z = 0$ and $\dot{z}_0 = 0$, configurations (v) and (vi) of statements (a) and (b) of Theorem 9 correspond to double–symmetric periodic orbits.*

Consider configuration $P_r^+ \xrightarrow{\tau/2} P_r^- \xrightarrow{\tau/2} P_r^+$ of statement (a), the other ones would be treated in a similar way. This configuration corresponds to a $\tau-$periodic solution $(z(t), \dot{z}(t))$ with initial conditions $z(0) = 0$, $\dot{z}(0) = a_0$ for some $a_0 > 0$ and such that $z(\tau/2) = 0$ and $\dot{z}(\tau/2) = -a_1$ with $a_1 > 0$ and $\tau/2 = 2\pi k$ for some $k \in \mathbb{N}$. Using the Poincaré map f_e in a similar way than in the previous remark, it seems that, unless $(a_0, 0)$ be near the escaping orbits, this is only possible when $a_0 = a_1$.

Assume that $a_0 = a_1$, then $(z(t), \dot{z}(t))$ is a periodic solution of the Sitnikov problem with initial conditions $z(0) = 0$, $\dot{z}(0) = a_0$ for some $a_0 > 0$ and period $\tau = 4\pi k$ such that $z(\tau/2) = 0$ and $\dot{z}(\tau/2) = -a_0$. Thus, from Corollary 8, we have that $(z(t), \dot{z}(t))$ is a double–symmetric periodic solution. $\square$

Now we want to prove the existence of symmetric periodic solutions for the Sitnikov problem.

PROPOSITION 12. *Fixed $e \in (0,1)$ and $q \in \mathbb{N}$, we can find $\overline{p}(e,q)$ in such a way that for all $p \in \mathbb{N}$ coprime with q and $p > \overline{p}(e,q)$, there exists at least a $r-$symmetric periodic orbit of the Sitnikov problem with period $\tau = 2\pi p$ such that the primaries make p revolutions during $2q$ consecutive crossings of m_3 with $z = 0$. Moreover if p is odd, then this periodic orbit is $r-$symmetric but not double–symmetric.*

Proof. In order to prove the existence of $r-$symmetric periodic solutions for the Sitnikov problem we will use the Poincaré map $f_e : \mathcal{B}_e \subset \mathcal{A} \longrightarrow \mathcal{A}$, defined in Section 3, and its iterates.

Consider the solution $\varphi(t) = (z(t), \dot{z}(t))$ of the Sitnikov problem having initial conditions $z(t_0) = 0$ and $\dot{z}(t_0) = \dot{z}_0$ with $t_0 = 0$ (the case $t_0 = \pi$ would be studied in a similar way). We note that this solution corresponds to the point $(\dot{z}_0, 0) \in \mathcal{A}$.

From Proposition 5 $\varphi(t)$ is a $r-$symmetric periodic solution with period τ if $z(\tau/2) = 0$, $\tau/2 = k\pi$ for some $k \in \mathbb{N}$ and there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (0, \tau/2)$ such that $z(\overline{\tau}) = 0$. Then the $r-$symmetric periodic solutions of the Sitnikov problem correspond to points $(\dot{z}_0, 0) \in \mathcal{A}$ such that either $f_e^q(\dot{z}_0, 0) = (\dot{z}_q, 0)$, or $f_e^q(\dot{z}_0, 0) = (\dot{z}_q, \pi)$, for some $q \in \mathbb{N}$.

Let $\sigma(e, q, \dot{z}_0)$ denote the time necessary for going from $(\dot{z}_0, 0)$ to the $q-$th next zero of $z(t)$, if it exists; and let

$$\overline{p}(e,q) = \left[\frac{\lim\limits_{\dot{z}_0 \to 0} \sigma(e, q, \dot{z}_0)}{\pi} \right] ,$$

where $[\,\cdot\,]$ denotes the *integer part function.*

Consider the curve $\gamma = \{(\dot{z}_0, 0) \in \mathcal{B}_e^+\}$ (see Section 3 for details). It is clear that γ is a $\mathcal{C}^1-$arc that meets nontangentially the boundary β_e^+ in the endpoint P. Thus by Theorem 3(d), the image curve $f_e^+(\gamma)$ approaches the boundary κ_e^+ spiraling (see Figure 7). Therefore $f_e^+(\gamma)$ intersects infinitely many times $\{t = 0\}$ and $\{t = \pi\}$. More precisely, for each $k \in \mathbb{N}$ we can find at least a point $\dot{z}_{k1}$ such that either $f_e^+(\dot{z}_{k1}, 0) = (v_{k1}, 0)$ or $f_e^+(\dot{z}_{k1}, 0) = (v_{k1}, \pi)$. Moreover $(\dot{z}_{k1}, 0)$ correspond to a $r-$symmetric periodic orbit of the Sitnikov problem with period $\tau = 2\pi p$ such that the primaries make $p = \overline{p}(e,1) + k$ revolutions during 2 consecutive crossings of m_3 with $z = 0$. If p is odd, then these $r-$symmetric periodic orbits are not double–symmetric, because $\tau/4$ is not multiple of π (see Proposition 7).

Consider now the image by the map f_e^q, with $q \in \mathbb{N}$ and $q > 1$, of the curve γ. We note that f_e^q is not defined for all γ (see Section 3 for the domain of definition of the map f_e^q). Set $\gamma_1 = \{(\dot{z}_0, 0) \in \mathcal{B}_e^+ : 0 < \dot{z}_0 < a\} \subset \gamma$, where a is the smallest point such that $f_e^{q-1}(a, 0) \in \partial_{\mathcal{A}^+}\mathcal{B}_e^+ =$

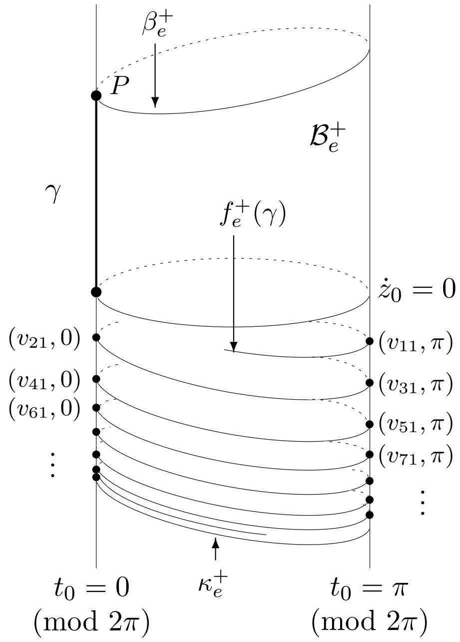

Figure 7. Image by f_e of the curve γ.

β_e^+, if q is odd; and $f_e^{q-1}(a,0) \in \partial_{\mathcal{A}^-}\mathcal{B}_e^- = \beta_e^-$, if q is even. It is clear that $f_e^{q-1}(\gamma)$ is a $\mathcal{C}^1$–arc that meets β_e^+ (respectively, β_e^-) in the endpoint $f_e^{q-1}(a,0)$. Thus, by the results of Section 3, we have that the image by f_e^q of the curve γ approaches the boundary $\partial_{\mathcal{A}^-}\mathcal{K}_e^+ = \kappa_e^+$ (respectively, the boundary $\partial_{\mathcal{A}^+}\mathcal{K}_e^- = \kappa_e^-$) spiraling. Consequently $f_e^q(\gamma)$ intersects infinitely many times $\{t = 0\}$ and $\{t = \pi\}$. Then, proceeding as in the case $q = 1$, we see that for all $p \in \mathbb{N}$ with $p > \overline{p}(e,q)$ we can find at least a point $(\dot{z}_{pq}, 0)$ that corresponds to a $r-$symmetric periodic orbit of the Sitnikov problem such that the primaries make p complete revolutions during $2q$ consecutive crossings of m_3 with $z = 0$. It is clear that if p, q coprime, then the period of this $r-$symmetric periodic solution is $\tau = 2\pi p$, and otherwise it is $\tau = 2\pi p/\gcd(p,q)$, where $\gcd(p,q)$ denotes the *greatest common divisor* of the integers p and q. Moreover if p is odd and p, q are coprime, then the $r-$symmetric periodic solutions obtained in this way are not double–symmetric because $\tau/4$ is not multiple of π. Finally, we can apply similar arguments on the other components of the domain of definition of f_e^q on γ, obtaining in this way different $r-$symmetric periodic orbits for large periods. ■

REMARK 13. According to Remark 11 it seems that the $r-$symmetric periodic orbits with period $\tau = 2\pi p$, for some even $p \in \mathbb{N}$, given Proposition 12 that are near $z = 0$ and $\dot{z} = 0$, are double–symmetric periodic solutions. In particular, for e sufficiently small there are double–symmetric periodic solutions. □

REMARK 14. *In the proof of Proposition 12 we have analyzed the image by f_e^q, with $q \in \mathbb{N}$, of the curve $\gamma = \{(\dot{z}_0, 0) \in \mathcal{B}_e^+\} \subset \mathcal{A}^+$. Applying the same arguments we can also analyze the image by f_e^q of the curve $\overline{\gamma} = \{(\dot{z}_0, 0) \in \mathcal{B}_e^-\} \subset \mathcal{A}^-$. In general, the periodic orbits that we obtain from these two ways will not be the same.*

Analyzing the image by f_e^q of the curve $\gamma = \{(\dot{z}_0, 0) \in \mathcal{B}_e^+\} \subset \mathcal{A}^+$ we have proved that, given $0 < e < 1$ and $q \in \mathbb{N}$, and for all $p \in \mathbb{N}$ coprime with q and $p > \overline{p}(e, q)$ we can find a point $(\dot{z}_{pq}, 0) \in \mathcal{A}^+$ corresponding to a $r-$symmetric periodic orbit with period $\tau = 2\pi p$ such that the primaries make p complete revolutions during $2q$ consecutive crossings of m_3 with $z = 0$. On the other hand, analyzing the image by f_e^q of the curve $\overline{\gamma} = \{(\dot{z}_0, 0) \in \mathcal{B}_e^-\} \subset \mathcal{A}^-$ we can prove that, given $0 < e < 1$ and $q \in \mathbb{N}$, and for all $p \in \mathbb{N}$ coprime with q and $p > \overline{p}(e, q)$ we can find a point $(\dot{\overline{z}}_{pq}, 0) \in \mathcal{A}^-$ corresponding to a $r-$symmetric periodic orbit of the Sitnikov problem with period $\tau = 2\pi p$ satisfying the above properties.

If p is odd and q is even, then the point $(\dot{z}_{pq}, 0)$ corresponds to a $r-$symmetric periodic orbit of type $P_r^+ \xrightarrow{\tau/2} A_r^+ \xrightarrow{\tau/2} P_r^+$, whereas the point $(\dot{\overline{z}}_{pq}, 0)$ corresponds to a $r-$symmetric periodic orbit of type $P_r^- \xrightarrow{\tau/2} A_r^- \xrightarrow{\tau/2} P_r^-$. Therefore the points $(\dot{z}_{pq}, 0)$ and $(\dot{\overline{z}}_{pq}, 0)$ correspond to different periodic orbits (see Theorem 9). The same fact occurs when p odd and q odd. In this last case $(\dot{z}_{pq}, 0)$ corresponds to a $r-$symmetric periodic orbit of type $P_r^+ \xrightarrow{\tau/2} A_r^- \xrightarrow{\tau/2} P_r^+$, whereas the point $(\dot{\overline{z}}_{pq}, 0)$ corresponds to a $r-$symmetric periodic orbit of type $P_r^- \xrightarrow{\tau/2} A_r^+ \xrightarrow{\tau/2} P_r^-$.

If p is even, then the points $(\dot{z}_{pq}, 0)$ and $(\dot{\overline{z}}_{pq}, 0)$ could correspond to the same periodic orbit. Indeed, since $(\dot{z}_{pq}, 0)$ corresponds to a $r-$symmetric periodic orbit and p is even (q is odd), we have that $f_e^q(\dot{z}_{pq}, 0) = (v_{pq}, 0) \in \overline{\gamma}$ and $f_e^q(v_{pq}, 0) = (\dot{z}_{pq}, 0) \in \gamma$. Thus if $(\dot{z}_{pq}, 0)$ is the unique point of $\mathcal{A}^+$ corresponding to a $r-$symmetric periodic orbit with period $\tau = 2\pi p$ such that the primaries make p complete revolutions during $2q$ consecutive crossings of m_3 with $z = 0$, then $\dot{\overline{z}}_{pq} = v_{pq}$ and consequently the points $(\dot{z}_{pq}, 0)$ and $(\dot{\overline{z}}_{pq}, 0)$ correspond to the same orbit. This fact occurs when f_e^q satisfies condition (10). □

PROPOSITION 15. *Fixed $e \in (0,1)$ and $q \in \mathbb{N}$, we can find $\overline{p}(e,q)$ in such a way that for all $p \in \mathbb{N}$ coprime with q and $p > \overline{p}(e,q)$, there exists at least a $t-$symmetric periodic orbit of the Sitnikov problem with period $\tau = 2\pi p$ such that the primaries make p revolutions during $2q$ consecutive crossings of m_3 with $\dot{z} = 0$. Moreover if p is odd, then this periodic orbit is $t-$symmetric but not double–symmetric.*

Proof. Consider the solution $\varphi(t) = (z(t), \dot{z}(t))$ of the Sitnikov problem having initial conditions $z(t_0) = z_0$ and $\dot{z}(t_0) = 0$ with $t_0 = 0$ (the case $t_0 = \pi$ would be treated in a similar way). From Proposition 6 $\varphi(t)$ is a $t-$symmetric periodic solution with period τ if $\dot{z}(\tau/2) = 0$, $\tau/2 = k\pi$ for some $k \in \mathbb{N}$ and there is no $\overline{\tau}$ multiple of π with $\overline{\tau} \in (0, \tau/2)$ such that $\dot{z}(\overline{\tau}) = 0$.

We note that all periodic orbits of the Sitnikov problem have at least one zero of $\dot{z}$. Then we can describe an arbitrary periodic orbit giving the time t_0 at which $\dot{z}(t_0) = 0$, and giving $z(t_0) = z_0$. Moreover the time t_0 can be given modulus 2π. Therefore the periodic orbits of the Sitnikov problem can be thought like points of the cylinder

$$\mathcal{D} = \{(z_0, t_0(\text{mod } 2\pi)) \in \mathbb{R} \times \mathbb{S}^1 \, : \, z(t_0) = z_0 \in \mathbb{R}, \ \dot{z}(t_0) = 0\} \, .$$

The point $(z, t(\text{mod } 2\pi))$ of the cylinder $\mathcal{D}$ is denoted by $(z, \widetilde{t})$; i.e. $t(\text{mod } 2\pi) = \widetilde{t}$. On the other hand, the cylinder $\mathcal{D}$ can be thought as union of two cylinders, one corresponding to initial conditions $z(t_0) = z_0 > 0$ ($\mathcal{D}^+$), and the other one corresponding to initial conditions $z(t_0) = z_0 < 0$ ($\mathcal{D}^-$). These two cylinders are symmetric by symmetry $(t, z, \dot{z}) \longmapsto (t, -z, -\dot{z})$ (see Section 3), and they are divided by the circle of initial conditions $z(t_0) = 0$, $\dot{z}(t_0) = 0$.

We define a Poincaré map g_e on $\mathcal{D}^+ \cup \mathcal{D}^-$ by following a solution with initial conditions $z(t_0) = z_0$, $\dot{z}(t_0) = 0$ to its next zero of $\dot{z}$, i.e. $\dot{z}(t_1) = 0$ where t_1 is the smallest $t > t_0$ for which $\dot{z}(t_1) = 0$ if it exists, otherwise we set $t_1 = \infty$. Then the map g_e is given by

$$g_e(z_0, \widetilde{t_0}) = \begin{cases} g_e^+(z_0, \widetilde{t_0}) & \text{if } \ (z_0, \widetilde{t_0}) \in \mathcal{D}^+ \text{ and } t_1 < \infty \, , \\ g_e^-(z_0, \widetilde{t_0}) & \text{if } \ (z_0, \widetilde{t_0}) \in \mathcal{D}^- \text{ and } t_1 < \infty \, , \end{cases}$$

where $g_e^{\pm}(z_0, \widetilde{t_0}) = (z_1, \widetilde{t_1})$ with $z_1 = z(t_1)$. Notice that g_e^+ applies points of $\mathcal{D}^+$ into $\mathcal{D}^-$ and g_e^- applies points of $\mathcal{D}^-$ into $\mathcal{D}^+$. Moreover,

$$g_e^- = \overline{\rho}_0^{\,-1} \circ g_e^+ \circ \overline{\rho}_0 \, ,$$

where $\overline{\rho}_0$ is the symmetry on $\mathcal{D}$ given by

$$\overline{\rho}_0 \, : \, (z_0, \widetilde{t_0}) \longmapsto (-z_0, \widetilde{t_0}) \, .$$

We note that the $t-$symmetric periodic solutions of the Sitnikov problem correspond to points $(z_0, 0)$ of $\mathcal{D}$ such that either $g_e^q(z_0, 0) = (z_q, 0)$, or $g_e^q(z_0, 0) = (z_q, \pi)$, for some $q \in \mathbb{N}$.

Let $\sigma(e, q, z_0)$ denote the time necessary for going from $(z_0, 0)$ to the $q-$th next zero of $\dot{z}(t)$, if it exists; and let

$$\overline{p}(e, q) = \left[\frac{\lim\limits_{z_0 \to 0} \sigma(e, q, z_0)}{\pi} \right] .$$

Let $\mathcal{D}_e = \mathcal{D}_e^+ \cup \mathcal{D}_e^-$ denote the domain of definition of g_e, where $\mathcal{D}_e^+ \subset \mathcal{D}^+$ and $\mathcal{D}_e^- \subset \mathcal{D}^-$. Consider the curves $\gamma = \{(z_0, 0) \in \mathcal{D}^+\}$ and $\gamma_1 = \{(z_0, 0) \in \mathcal{D}_e^+ \,:\, 0 < z_0 < a\} \subset \gamma$, where a is the smallest point such that $(a, 0) \in \partial_{\mathcal{D}^+}\mathcal{D}_e^+$. It is clear that when $(z_0, 0) \in \gamma_1$ tends to the point $(a, 0)$, the image curve $g_e^+(\gamma_1)$ tends to $z = -\infty$ spiraling, because the boundary $\partial_{\mathcal{D}^+} D_e^+$ corresponds to orbits that escape parabolically (i.e. $\dot{z}(\infty) = 0$) to $z = -\infty$. Thus $g_e^+(\gamma_1)$ intersects infinitely many times $\{t = 0\}$ and $\{t = \pi\}$. Then, proceeding as in Proposition 12, we see that for all $p \in \mathbb{N}$ with $p > \overline{p}(e, 1)$ we can find at least a point $(z_{p1}, 0)$ that corresponds to a $t-$symmetric periodic orbit of the Sitnikov problem such that the primaries make p complete revolutions during 2 consecutive crossings of m_3 with $\dot{z} = 0$.

Consider now the image by the map g_e^q, with $q \in \mathbb{N}$ and $q > 1$, of the curve γ. Let now $\gamma_1 = \{(z_0, 0) \in \mathcal{D}_e^+ \,:\, 0 < z_0 < a\} \subset \gamma$, where a is the smallest point such that $g_e^{q-1}(a, 0) \in \partial_{\mathcal{D}^+}\mathcal{D}_e^+$, if q is odd; and $g_e^{q-1}(a, 0) \in \partial_{\mathcal{D}^-}\mathcal{D}_e^-$, if q is even. It is clear that the image curve $g_e^q(\gamma_1)$ tends to $z = \pm\infty$ spiraling ($+\infty$ for even q and $-\infty$ for odd q) as $z_0 \to a$. Thus, proceeding as in Proposition 12, we see that for all $p \in \mathbb{N}$ with $p > \overline{p}(e, q)$ we can find at least a point $(z_{pq}, 0)$ that corresponds to a $t-$symmetric periodic orbit of the Sitnikov problem such that the primaries make p complete revolutions during $2q$ consecutive crossings of m_3 with $\dot{z} = 0$. If p and q are coprime, then the period of this $t-$symmetric periodic solution is $\tau = 2\pi p$, and otherwise it is $\tau = 2\pi p/\gcd(p, q)$. Moreover if p is odd and p, q are coprime, then the $t-$symmetric periodic solutions obtained in this way are not double–symmetric, because $\tau/4$ is not multiple of π (see Proposition 7). Finally we can apply similar arguments on the other components of the domain of definition of g_e^q on γ, obtaining in this way different $t-$symmetric periodic orbits of the Sitnikov problem for large periods. ■

REMARK 16. According to Remark 11 it seems that the $t-$symmetric periodic orbits with period $\tau = 2\pi p$, for some even $p \in \mathbb{N}$, given by Proposition 15 that are near $z = 0$ and $\dot{z} = 0$ are double–symmetric periodic solutions. In particular, for $e > 0$ sufficiently small, there are double–symmetric periodic orbits. $\square$

REMARK 17. In the proof of Proposition 15 we have analyzed the image by g_e^q, with $q \in \mathbb{N}$, of the curve $\gamma = \{(z_0, 0) \in \mathcal{D}^+\}$. Using similar arguments than in Remark 14, we see that we also could analyze the image by g_e^q, with $q \in \mathbb{N}$, of the curve $\overline{\gamma} = \{(z_0, 0) \in \mathcal{D}^-\}$, obtaining in general different $t-$symmetric periodic orbits of the Sitnikov problem (see Remark 14). $\square$

In (Corbera and Llibre, 2000) the existence of double–symmetric, $r-$symmetric and $t-$symmetric periodic orbits of the Sitnikov problem is proved, for small values of e, from the analytic continuation method applied to periodic orbits of the circular Sitnikov problem; that is, the Sitnikov problem when $e = 0$.

5. Non–symmetric periodic orbits

Up to here we have proved the existence of $r-$symmetric, $t-$symmetric periodic orbits of the Sitnikov problem. We have also proved the existence of some double–symmetric periodic orbits, see Remarks 13 and 16. Here we will prove the existence of periodic orbits of the Sitnikov problem that do not satisfy those symmetries. Those periodic orbits will be called *non–symmetric* periodic orbits.

In order to prove the existence of non–symmetric periodic orbits, we will use the results of (Alekseev, 1968) and (Moser, 1973) about the presence of the Bernoulli shift as a subsystem of f_e. Remember that (Alekseev, 1968) and (Moser, 1973) identify points $(t_0, 0, -\dot{z}_0)$ with $(t_0, 0, \dot{z}_0)$ and here we do not use this identification. So we start summarizing those results without identifying points.

5.1. The Bernoulli shift as a subsystem of the Poincaré map f_e

Let $W^S(+\infty)$ and $W^U(+\infty)$ (respectively, $W^S(-\infty)$ and $W^U(-\infty)$) denote the manifolds formed by orbits which escape to $z = +\infty$ (respectively, to $z = -\infty$) parabolically for $t \to \infty$ and $t \to -\infty$, respectively. These are $2-$dimensional manifolds in the $3-$dimensional phase space $(t(\text{mod } 2\pi), z, \dot{z})$, in fact they are cylinders, see (Moser, 1973).

Using McGehee's coordinates to blow-up the singularity $z = +\infty$, Moser proves that the manifolds $W^S(+\infty)$ and $W^U(+\infty)$ are the stable and unstable manifolds of a periodic orbit Υ^+ at the infinity $z = +\infty$.

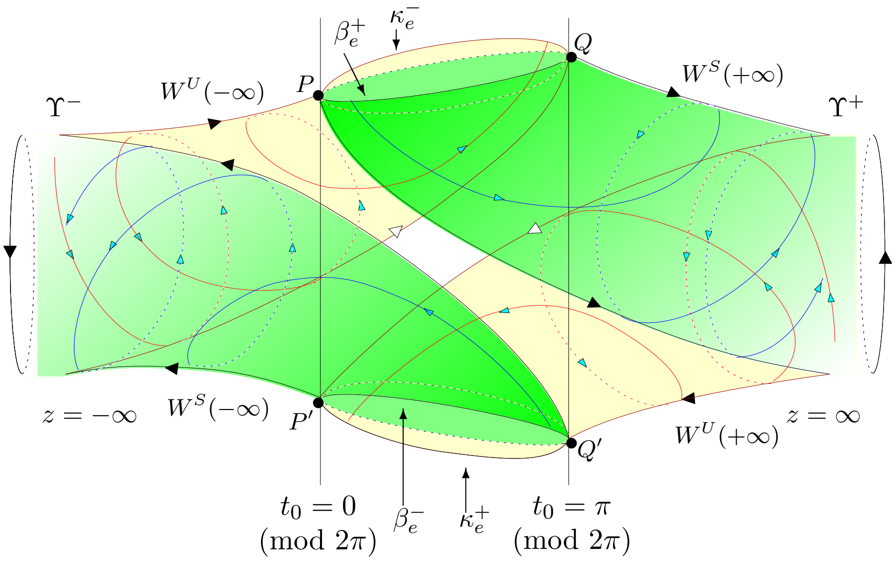

Figure 8. The periodic orbits Υ^+ and Υ^- at $z = +\infty$ and $z = -\infty$, respectively.

He also proves that $W^S(+\infty)$ and $W^U(+\infty)$ meet $z = 0$ in the curves β_e^+ and κ_e^+, respectively (see Chapter VI of (Moser, 1973) for more details). Then by means of the symmetry (4) the manifolds $W^S(-\infty)$ and $W^U(-\infty)$ are the stable and unstable manifolds of a periodic orbit Υ^- at the infinity $z = -\infty$. Moreover $W^S(-\infty)$ and $W^U(-\infty)$ meet $z = 0$ in the curves β_e^- and κ_e^-, respectively (see Figure 8).

We note that the point P (respectively, Q) corresponds to an orbit γ_P (respectively, γ_Q) that comes parabolically from $z = -\infty$, cross once $z = 0$ and escapes parabolically to $z = \infty$. In the same way, the point P' (respectively, Q') corresponds to an orbit $\gamma_{P'}$ (respectively, $\gamma_{Q'}$) that comes parabolically from $z = \infty$, cross once $z = 0$ and escapes parabolically to $z = -\infty$. Thus we have four different heteroclinic loops each one formed by two parabolic orbits of the Sitnikov problem, one that comes from $-\infty$ and escapes to $+\infty$ and the other one that comes from $+\infty$ and escapes to $-\infty$. These heteroclinic loops are, the one formed by the parabolic orbits γ_P and $\gamma_{P'}$, the one formed by γ_Q and $\gamma_{Q'}$, the one formed by γ_P and $\gamma_{Q'}$ and finally the one formed by γ_Q and $\gamma_{P'}$. They are denoted by ξ_1, ξ_2, ξ_3 and ξ_4, respectively. In Moser's identification these four heteroclinic loops become two homoclinic loops.

Using techniques of symbolic dynamics, we study the qualitative dynamics in a neighborhood of the heteroclinic loop ξ_1. All types of orbits that we find here could be find in a neighborhood of the other heteroclinic loops, but here we do not give the details for these other loops.

We start introducing the shift map. Let A be the set $\mathbb{N} \cup \{0, \infty\}$. We introduce a topological space S whose elements s are sequences of the form

(i) $s = (\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots)$ with $s_n \neq \infty$ for all $n \in \mathbb{Z}$;

(ii) $s = (\infty, s_{k+1}, s_{k+2}, \ldots)$ with $k \leq 0$ and $s_n \neq \infty$ for all $n \in \mathbb{Z}$, $n > k$;

(iii) $s = (\ldots, s_{l-2}, s_{l-1}, \infty)$ with $l \geq 1$ and $s_n \neq \infty$ for all $n \in \mathbb{Z}$, $n < l$;

(iv) $s = (\infty, s_{k+1}, \ldots, s_{l-1}, \infty)$ with $k \leq 0$, $l \geq 1$ and $s_n \neq \infty$ for all $n \in \mathbb{Z}$, $k < n < l$.

We consider in S the topology given by the base of neighborhoods $\{U_j(s) \,:\, s \in S, j \in \mathbb{N}\}$ where

$U_j(s) = \{s' \in S \,:\, s'_n = s_n,\ |n| \leq j\}$ when s is of type (i),
$U_j(s) = \{s' \in S \,:\, s'_n = s_n,\ k < n \leq j,\ s'_k \geq j\}$ when s is of type (ii),
$U_j(s) = \{s' \in S \,:\, s'_n = s_n,\ -j \leq n < l,\ s'_l \geq j\}$ when s is of type (iii),
$U_j(s) = \{s' \in S \,:\, s'_n = s_n,\ k < n < l,\ s'_k, s'_l \geq j\}$ when s is of type (iv).

On this topological space we consider the homeomorphism σ defined by $(\sigma(s))_k = s_{k-1}$ (i.e. σ shifts the sequence s by one position to the right) which is known as the *Bernoulli shift of* S. The domain of definition of σ is $D(\sigma) = \{s \in S \,:\, s_0 \neq \infty\}$.

Set $U = U^+ \cup U^-$ where U^+ and U^- are two copies of the square $[0,1] \times [0,1]$, and set $\widetilde{S} = S^+ \cup S^-$ where S^+ and S^- are two copies of S. We define a map $\widetilde{\sigma} : \widetilde{S} \longrightarrow \widetilde{S}$ in such a way that $\widetilde{\sigma}^+ = \widetilde{\sigma}|_{S^+} : S^+ \longrightarrow S^-$ and $\widetilde{\sigma}^- = \widetilde{\sigma}|_{S^-} : S^- \longrightarrow S^+$ coincide with the Bernoulli shift σ. Then the domain of definition of $\widetilde{\sigma}$ is $D(\widetilde{\sigma}) = \{s \in \widetilde{S} \,:\, s_0 \neq \infty\}$. Let F be a continuous map of U into itself such that $F^+ = F|_{U^+} : U^+ \longrightarrow U^-$ and $F^- = F|_{U^-} : U^- \longrightarrow U^+$. We say that the map F *possesses two copies of the Bernoulli shift* σ *as a subsystem* if there exist a homeomorphism $h : \widetilde{S} \longrightarrow h(\widetilde{S}) \subset U$, with $h^+ = h|_{S^+} : S^+ \longrightarrow h(S^+) \subset U^+$ and $h^- = h|_{S^-} : S^- \longrightarrow h(S^-) \subset U^-$, such that

$$h \circ \widetilde{\sigma} = F \circ h|_{D(\widetilde{\sigma})} \ . \tag{11}$$

In Theorems 8.4 and 8.6 of (Alvarez and Llibre, 1999) we can find the hypotheses on F that are needed to guarantee the presence of two copies of the Bernoulli shift σ as a subsystem of F.

If the map F possesses two copies of the Bernoulli shift σ as a subsystem, then there is a one–to–one correspondence between the sequences of $\widetilde{S}$ and the points of $I = h(\widetilde{S}) \subset U$. Moreover the action of F on I is topologically equivalent to the action of $\widetilde{\sigma}$ on $\widetilde{S}$. Then the dynamical behaviour of the orbits of points of I under the action of F and the behaviour of the corresponding sequences of $\widetilde{S}$ under the action of $\widetilde{\sigma}$ are the same. Here we only are interested in the presence of the Bernoulli shift σ as a subsystem of F in order to prove the existence of infinitely many periodic orbits, and specially of infinitely many non–symmetric periodic orbits.

We see that the periodic sequences of $S = S^+ = S^-$ correspond to periodic points of the map F. Indeed, let $s = (s_n)$ be a $\bar{l}-$ periodic sequence of S; that is, $s_{j+\bar{l}} = s_j$ for all $j \in \mathbb{Z}$ and there is no $1 \le \bar{l}_0 < \bar{l}$ such that $s_{j+\bar{l}_0} = s_j$ for all $j \in \mathbb{Z}$. Assume that $s \in S^+$ (the case $s \in S^-$ would be treated in a similar way). It is easy to see that if $\bar{l}$ is even, then $\widetilde{\sigma}^{\bar{l}}(s) = s \in S^+$ and there is no $1 \le \bar{l}_0 < \bar{l}$ such that $\widetilde{\sigma}^{\bar{l}_0}(s) = s \in S^+$; that is, the sequence s is a periodic point of the map $\widetilde{\sigma}$ with period $l = \bar{l}$. If $\bar{l}$ is odd, then $\widetilde{\sigma}^{\bar{l}}(s) = s \in S^-$, nevertheless $\widetilde{\sigma}^{2l}(s) = s \in S^+$ and there is no $1 \le \bar{l}_0 < 2\bar{l}$ such that $\widetilde{\sigma}^{\bar{l}_0}(s) = s \in S^+$. Thus in this case the sequence s is a periodic point of the map $\widetilde{\sigma}$ with period $l = 2\bar{l}$. Let $p = h(s) \in U$, since $\widetilde{\sigma}^l(s) = s$, it follows from (11) that $p = F^l(p)$; that is, p is a periodic point of the map F with period l. Since the sequences (s_n) are arbitrary and the periodic sequences are dense in S, see (Denker et al., 1976); we can conclude that the periodic points of F are dense on I.

We return now to the Sitnikov problem. By Theorem 3.6 of (Moser, 1973), it is easy to see that the Poincaré map f_e possesses two copies of the Bernoulli shift σ as a subsystem in a neighborhood of the heteroclinic loop ξ_1 (i.e. the heteroclinic loop formed by the parabolic orbits γ_P and $\gamma_{P'}$), for more details see (Alvarez and Llibre, 1999).

The first one in proving that the Bernoulli shift σ is a subsystem of the Poincaré map f_e of the Sitnikov problem was Alekseev (1968). Later on Moser (1973) gave a more geometric and simpler proof of the results of Alekseev. We remark that Alekseev in his analysis also identifies points $(t_0, 0, -\dot{z}_0)$ with points $(t_0, 0, \dot{z}_0)$.

The presence of two copies of the Bernoulli shift σ as a subsystem of the Poincaré map f_e in a neighborhood of the heteroclinic loop formed by the parabolic orbits γ_P and $\gamma_{P'}$ leads to the following theorem.

THEOREM 18. *The following statements hold.*

(a) We can associate to every solution $z(t)$ of the Sitnikov problem a sequence $(a_n) \in \widetilde{S}$ of type (i), (ii), (iii) or (iv) in such a way that

a_n measure the number of complete revolutions of the primaries (m_1 and m_2) between the $n-$th and the $(n+1)-$th zero of $z(t)$.

(i) If the solution $z(t)$ does infinitely many oscillations in forward and in backward times, then we associate to $z(t)$ a sequence of type (i).

(ii) If the solution $z(t)$ comes from infinity and performs infinitely many oscillations in forward time, then we associate to $z(t)$ a sequence of type (ii).

(iii) If the solution $z(t)$ performs infinitely many oscillations in backward time and escapes to infinity in forward time, then we associate to $z(t)$ a sequence of type (iii).

(iv) If the solution $z(t)$ comes from infinity and after doing a finite number of oscillations it escapes to infinity, then we associate to $z(t)$ a sequence of type (iv).

(b) Given a sufficiently small $e > 0$ there exists an integer $\overline{s} = \overline{s}(e)$ such that for any sequence $(s_n) \in S$ of type (i), (ii), (iii) or (iv) with $s_n \geq \overline{s}$ there are two unique orbits (one for $(s_n) \in S^+ \subset \widetilde{S}$ and another one for $(s_n) \in S^- \subset \widetilde{S}$) of the Sitnikov problem sufficiently near the heteroclinic loop formed by the parabolic orbits γ_P and $\gamma_{P'}$, such that its associated sequence is (s_n). Eventually these two periodic orbits can be the same.

Proof. See Chapter III §5 of (Moser, 1973). ∎

REMARK 19. In Theorem 18, the smallness assumption on e is not really necessary. It holds true for all $e \in (0,1)$ except perhaps for a discrete set of values of e, see (Moser, 1973). We note that the construction of the Bernoulli shift σ as a subsystem of f_e done in (Moser, 1973) needs that the boundary curves $\partial_{\mathcal{A}^+}(\mathcal{B}_e^+) = \beta_e^+$ and $\kappa_e^- = \partial_{\mathcal{A}^+} f_e^-(\mathcal{B}_e^-)$ intersect transversally at the point P. The exclusion of a finite set of values of e is needed in order to guarantee the transversality, using the analyticity of the angle between the tangents at P of the curves β_e^+ and κ_e^- (see Theorem 4).

A recent work of Rayskin (1998) shows that, under certain conditions, we can find a transversal homoclinic point as close as we want of a given non–transversal homoclinic point. Consequently, under suitable conditions, in a neighborhood of non–transversal homoclinic points we also can find a Bernoulli shift as a subsystem.

In short Theorem 18 is valid for all $e \in (0,1)$ except perhaps for a discrete set of values of e. In the line of the results of (Rayskin, 1998), it seems that this assumption is only a technical question that could be removed by using the techniques developed in that paper. But here we do not apply directly these techniques because our hypotheses are quite different. □

We note that the sequence (s_n) in Theorem 18 can be chosen completely independently (only with the restriction $s_n \geq m$). Taking into account the different types of sequences (s_n), Theorem 18 allows to classify the final evolutions (i.e. the behaviour of the orbits when $t \to \infty$ or $t \to -\infty$) of the Sitnikov problem, see for instance (Alekseev, 1968); but this topic is not treated in this work. Here we will use Theorem 18 only to find infinitely many non–symmetric periodic orbits, which come from choosing convenient periodic sequences (s_n). We note that the periodic orbits obtained in this way are always hyperbolic periodic orbits, see (Moser, 1973).

5.2. Sequences associated to symmetric periodic orbits

From Theorem 18(a), we can associate to every solution $(z(t), \dot{z}(t))$ of the Sitnikov problem a sequence $(a_n) \in \widetilde{S}$ in such a way that a_n measure the number of complete revolutions of the primaries (m_1 and m_2) between the $n-$th and the $(n+1)-$th zeros of $z(t)$ if such zeros exist. Now we want to characterize the sequences associated to symmetric periodic orbits. In order to do this we need the following two lemmas.

LEMMA 20. *Let $\varphi(t) = (z(t), \dot{z}(t))$ be a $r-$symmetric solution of the Sitnikov problem with initial conditions $z(0) = 0$ and $\dot{z}(0) = \dot{z}_0$ for some $\dot{z}_0 \in \mathbb{R}$. Then the following statements hold.*

(a) If t_k and t_{-k} denote the $k-$th next zero of $z(t)$ in forward time and in backward time respectively (in the case that they exist), then $t_{-k} = -t_k$.

(b) If $\bar{t}_k$ and $\bar{t}_{-k}$ denote the next $k-$th zero of $\dot{z}(t)$ in forward time and in backward time respectively (in the case that they exist), then $\bar{t}_{-k} = -\bar{t}_k$.

Proof. Since at $t = 0$ the primaries are at the pericenter of the ellipse and $z(0) = 0$ we can use the $r-$symmetry obtaining

$$(z(-t), \dot{z}(-t)) = (-z(t), \dot{z}(t)) \ .$$

Therefore $z(-t_k) = -z(t_k) = 0$ and $\dot{z}(-\bar{t}_k) = \dot{z}(\bar{t}_k) = 0$. Moreover, since t_k (respectively, $\bar{t}_k$) is the $k-$th zero of $z(t)$ (respectively, $\dot{z}(t)$) in forward time, $-t_k$ (respectively, $-\bar{t}_k$) is the $k-$th zero of $z(t)$ (respectively, $\dot{z}(t)$) in backward time. Therefore $t_{-k} = -t_k$ (respectively, $\bar{t}_{-k} = -\bar{t}_k$). ■

LEMMA 21. *Let $\varphi(t) = (z(t), \dot{z}(t))$ be a $t-$symmetric solution of the Sitnikov problem with initial conditions $z(0) = z_0$ and $\dot{z}(0) = 0$ for some $z_0 \in \mathbb{R}$. Then the following statements hold.*

(a) If t_k and t_{-k} denote the $k-$th next zero of $\dot{z}(t)$ in forward time and in backward time respectively (in the case that they exist), then $t_{-k} = -t_k$.

(b) If $\bar{t}_k$ and $\bar{t}_{-k}$ denote the next $k-$th zero of $z(t)$ in forward time and in backward time respectively (in the case that they exist), then $\bar{t}_{-k} = -\bar{t}_k$.

Proof. The proof is the same than the proof of Lemma 20 but using the $t-$symmetry instead of the $r-$symmetry. ∎

THEOREM 22. *Let $\varphi(t) = (z(t), \dot{z}(t))$ be a periodic solution of the Sitnikov problem with period τ and let $(a_n) = (\ldots, a_{-1}, a_0, a_1, a_2, \ldots)$ be the sequence associated to this periodic solution. The solution $\varphi(t)$ corresponds to a periodic point $(\dot{z}_0, \widetilde{t_0}) \in \mathcal{A}$ of the Poincaré map f_e (see Section 3), where t_0 is such that $z(t_0) = 0$, $\widetilde{t_0} = t_0 (\mathrm{mod}\ 2\pi)$ and $\dot{z}_0 = \dot{z}(t_0)$. Assume that $l \in \mathbb{N}$ is the period of the periodic point $(\dot{z}_0, \widetilde{t_0})$ under f_e. Then the following statements hold.*

(a) l is even and (a_n) is a periodic sequence of period l or a divisor of l.

(b) If $\varphi(t)$ is a $r-$symmetric periodic solution, then the associated sequence (a_n) verifies that

$$a_j = a_{l-j-1} \quad \textit{for} \quad j = 0, 1, \ldots, (l-2)/2 \ . \tag{12}$$

(c) If $\varphi(t)$ is a $t-$symmetric periodic solution, then the associated sequence (a_n) verifies that

$$a_j = a_{l-j} \ \ \textit{for} \ \ j = 1, 2, \ldots, (l-2)/2 \ . \tag{13}$$

(d) If $\varphi(t)$ is a double–symmetric periodic solution, then the associated sequence (a_n) is the constant sequence; that is,

$$a_j = a_0 \ \ \textit{for} \ \ j \in \mathbb{N} \ . \tag{14}$$

(e) If $f_e^{l/2}(\dot{z}_0, \widetilde{t_0}) = (-\dot{z}_0, \widetilde{t_0})$, then (a_n) is a periodic sequence of period $\bar{l} = l/2$ or a divisor of $\bar{l}$.

Proof. Let t_0 be the first positive zero of $z(t)$ and set $\dot{z}_0 = \dot{z}(t_0)$. Then the solution $\varphi(t)$ corresponds to the point $(\dot{z}_0, \widetilde{t_0}) \in \mathcal{A}$ with $\widetilde{t_0} = t_0 (\text{mod } 2\pi)$ (see Section 3 for details). Let t_k be the next $k-$th zero of $z(t)$ in forward time if $k > 0$, and the next $k-$th zero in backward time if $k < 0$, and set $\dot{z}_k = \dot{z}(t_k)$; that is, $(\dot{z}_k, \widetilde{t_k}) = f_e^k(\dot{z}_0, \widetilde{t_0})$ where $\widetilde{t_k} = t_k (\text{mod } 2\pi)$. Since $\varphi(t)$ is a periodic solution, $(\dot{z}_0, \widetilde{t_0})$ is a periodic point of the map f_e with period l for some even $l \in \mathbb{N}$ (i.e. $f_e^l(\dot{z}_0, \widetilde{t_0}) = (\dot{z}_0, \widetilde{t_0})$ and $f_e^k(\dot{z}_0, \widetilde{t_0}) \neq (\dot{z}_0, \widetilde{t_0})$ for $k = 1, 2, \ldots, l-1$). We note that the period l cannot be odd because, if l were odd, then f_e^l would apply points of $\mathcal{A}^+$ into $\mathcal{A}^-$ and viceversa, so in this case f_e^l could not have fixed points.

Let $\tau_{i,j} = |t_i - t_j|$ denote the time that the orbit needs for going from the point $(\dot{z}_i, \widetilde{t_i})$ to the point $(\dot{z}_j, \widetilde{t_j})$. Since $\tau_{i,j} = \tau_{j,i}$, in what follows when $\tau_{i,j}$ or $\tau_{j,i}$ appears we will always write $\tau_{i,j}$ with $i < j$.

Using the fact that $(\dot{z}_0, \widetilde{t_0})$ is a $l-$periodic point of the map f_e, we have that $\tau_{k,(k+1)} = \tau_{(l+k),(l+k+1)}$ for all $k \in \mathbb{Z}$. On the other hand, the primaries have completed a_k revolutions between $(\dot{z}_k, \widetilde{t_k})$ and $(\dot{z}_{k+1}, \widetilde{t_{k+1}})$, and they have completed a_{l+k} revolutions between $(\dot{z}_{l+k}, \widetilde{t_{l+k}})$ and $(\dot{z}_{l+k+1}, \widetilde{t_{l+k+1}})$. This means that $\tau_{k,(k+1)} = 2\pi(a_k + \theta_k)$ for some $\theta_k \in [0,1)$, and $\tau_{(l+k),(l+k+1)} = 2\pi(a_{l+k} + \theta_{l+k})$ for some $\theta_{l+k} \in [0,1)$. Therefore $a_{l+k} = a_k$ and $\theta_k = \theta_{l+k}$ for all $k \in \mathbb{N}$. Thus (a_n) is a periodic sequence of period l or divisor of l, which proves statement (a).

Assume now that $\varphi(t)$ is a $r-$symmetric periodic solution of the Sitnikov problem. From Proposition 5, it is easy to see that we can find $0 \leq \overline{k} < l/2$ such that either $f_e^{\overline{k}}(\dot{z}_0, \widetilde{t_0}) = (\dot{z}_k, 0)$ or $f_e^{\overline{k}}(\dot{z}_0, \widetilde{t_0}) = (\dot{z}_k, \pi)$. Changing the origin of time if necessary we can assume that $\overline{k} = 0$. Then applying Lemma 20 (with a change of the origin of time if necessary), the time $\tau_{0,k}$ that the orbit needs for going from $(\dot{z}_0, \widetilde{t_0})$ to $(\dot{z}_k, \widetilde{t_k})$, for all $k \in \mathbb{N}$, is the same than the time $\tau_{-k,0}$ needed for going from $(\dot{z}_0, \widetilde{t_0})$ to $(\dot{z}_{-k}, \widetilde{t_{-k}})$ in backward motion. In particular, $\tau_{0,1} = \tau_{-1,0}$. Using the fact that $(\dot{z}_0, \widetilde{t_0})$ is a $l-$periodic point of f_e, we have that $\tau_{-1,0} = \tau_{(l-1),l}$, so $\tau_{0,1} = \tau_{(l-1),l}$ (see Figure 9). On the other hand, the primaries have completed a_0 revolutions between $(\dot{z}_0, \widetilde{t_0})$ and $(\dot{z}_1, \widetilde{t_1})$, and they have completed a_{l-1} revolutions between $(\dot{z}_{l-1}, \widetilde{t_{l-1}})$ and $(\dot{z}_l, \widetilde{t_l})$. This means that $\tau_{0,1} = 2\pi(a_0 + \theta)$ for some $\theta \in [0,1)$, and $\tau_{(l-1),l} = 2\pi(a_{l-1} + \overline{\theta})$ for some $\overline{\theta} \in [0,1)$. Therefore $a_0 = a_{l-1}$ and $\theta = \overline{\theta}$.

In the same way, $\tau_{0,2} = \tau_{-2,0} = \tau_{(l-2),l}$. Since $\tau_{0,1} = \tau_{(l-1),l}$, we see that $\tau_{1,2} = \tau_{(l-2),(l-1)}$. On the other hand, $\tau_{1,2} = 2\pi(a_1 + \theta)$ for some

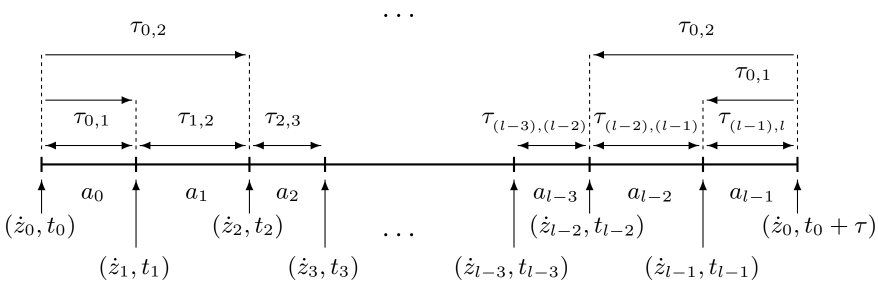

Figure 9. The times $\tau_{i,j}$.

$\theta \in [0,1)$ and $\tau_{(l-2),(l-1)} = 2\pi(a_{l-2} + \overline{\theta})$ for some $\overline{\theta} \in [0,1)$. Therefore $a_1 = a_{l-2}$ and $\theta = \overline{\theta}$.

Analyzing $\tau_{i,j}$ for $i, j \in [0, l]$ and $i < j$, we see that the sequences associated to $r-$symmetric periodic orbits of the Sitnikov problem verify that

$$a_j = a_{l-j-1} \text{ for } j = 0, 1, \ldots, (l-2)/2 \ ,$$

which proves statement (b).

Assume now that $\varphi(t)$ is a $t-$symmetric periodic solution of the Sitnikov problem.

We note that between two consecutive zeros of $z(t)$ there exist always a zero of $\dot{z}(t)$. Let $\bar{t}_k$ be the zero of $\dot{z}(t)$ between $(\dot{z}_k, \widetilde{t_k})$ and $(\dot{z}_{k+1}, \widetilde{t_{k+1}})$ and let $z_k = z(\bar{t}_k)$. This zero corresponds to the point $(\widetilde{\bar{t}_k}, z_k, 0)$, with $\widetilde{\bar{t}_k} = \bar{t}_k (\text{mod } 2\pi)$, of the phase space $(t(\text{mod } 2\pi), z, \dot{z})$. To simplify, this point is denoted by $(z_k, \widetilde{\bar{t}_k})$. From Proposition 6, it is easy to see that we can find $0 \leq \overline{k} < l/2$ such that either $(z_{\overline{k}}, \widetilde{\bar{t}_{\overline{k}}}) = (z_{\overline{k}}, 0)$ or $(z_{\overline{k}}, \widetilde{\bar{t}_{\overline{k}}}) = (z_{\overline{k}}, \pi)$. Changing the origin of time if necessary we can assume that $\overline{k} = 0$.

Let $\overline{\tau}_{0,j} = |\bar{t}_j - \bar{t}_0|$ denote the time that the orbit needs for going from the point $(z_0, \widetilde{\bar{t}_0})$ to the point $(\dot{z}_j, \widetilde{t_j})$. Using Lemma 21 (with a change of the origin of time if necessary) the time $\overline{\tau}_{0,1}$ needed for going from $(z_0, \widetilde{\bar{t}_0})$ to $(\dot{z}_1, \widetilde{t_1})$ is the same than the time $\overline{\tau}_{0,0}$ needed for going from $(z_0, \widetilde{\bar{t}_0})$ to $(\dot{z}_0, \widetilde{t_0})$ in backward motion. In the same way, $\overline{\tau}_{0,2} = \overline{\tau}_{0,-1}$ and in general $\overline{\tau}_{0,k} = \overline{\tau}_{0,-(k-1)}$ for all $k \in \mathbb{N}$. Condition $\overline{\tau}_{0,2} = \overline{\tau}_{0,-1}$ implies that $\tau_{1,2} = \tau_{-1,0}$ because $\overline{\tau}_{0,1} = \overline{\tau}_{0,0}$ (see Figure 10). Using the fact that $(\dot{z}_0, \widetilde{t_0})$ is a $l-$periodic point of f_e, we have that $\tau_{-1,0} = \tau_{(l-1),l}$. Therefore $a_1 = a_{l-1}$. In the same way, condition $\overline{\tau}_{0,3} = \overline{\tau}_{0,-2}$ implies that $\tau_{2,3} = \tau_{-2,-1} = \tau_{(l-2),(l-1)}$, because $\overline{\tau}_{0,2} = \overline{\tau}_{0,-1}$. Therefore $a_2 = a_{l-2}$.

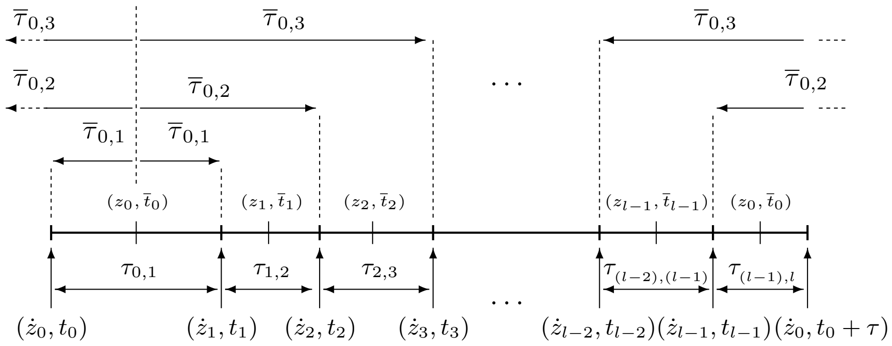

Figure 10. The times $\overline{\tau}_{0,j}$.

Analyzing the remaining cases, we see that the sequences associated to $t-$symmetric periodic orbits of the Sitnikov problem verify that

$$a_j = a_{l-j} \text{ for } j = 1, 2, \ldots, (l-2)/2 \ ,$$

which proves statement (c).

Analyzing the sequences associated to $r-$symmetric and $t-$symmetric periodic solutions (i.e. (12) and (13)), we see that the sequences associated to double–symmetric periodic orbits of the Sitnikov problem are the constant sequences (i.e. $a_0 = a_n$ for all $n \in \mathbb{N}$). So statement (d) is proved.

Finally assume that $f_e^{l/2}(\dot{z}_0, \widetilde{t_0}) = (-\dot{z}_0, \widetilde{t_0})$, then by means of symmetry $\rho_0 \, : \, (\dot{z}, \widetilde{t}) \longmapsto (-\dot{z}, \widetilde{t})$ we have that $f_e^{l/2+k}(\dot{z}_0, \widetilde{t_0}) = \rho_0(f_e^k(\dot{z}_0, \widetilde{t_0}))$ for all $k \in \mathbb{Z}$. Thus in this case $a_{l/2+k} = a_k$ for all $k \in \mathbb{N}$ and consequently (a_n) is a periodic sequence with period $\bar{l} = l/2$ or a divisor of it, which proves statement (e).

■

5.3. Existence of non–symmetric periodic orbits

Once we have characterized the sequences associated to periodic orbits, we can prove the existence of non–symmetric periodic orbits.

PROPOSITION 23. *For all $e \in (0,1)$ except perhaps for a discrete set of values of e, there exist infinitely many non–symmetric periodic orbits of the Sitnikov problem with sufficiently large periods that are close to the heteroclinic loop ξ_1.*

Proof. The existence of non–symmetric periodic orbits of the Sitnikov problem follows from Theorem 18, Remark 19 and Theorem 22.

Indeed, by Theorem 18, we can associate to any periodic sequence $(s_n) \in \widetilde{S}$, with $s_n \geq \overline{s}$ and $\overline{s} = \overline{s}(e)$ a sufficiently large integer, a unique periodic orbit of the Sitnikov problem, near the heteroclinic loop ξ_1, such that the primaries have completed s_n revolutions between the $n-$th and the $(n+1)-$th zero of $z(t)$. On the other hand, there exist infinitely many sequences that cannot be associated to symmetric periodic orbits, all sequences which are not of the forms (12), (13) or (14) (see Theorem 22). Moreover Theorem 18 holds for $e \in (0,1)$ except perhaps a discrete set of values of e (see Remark 19). Therefore we have proved the existence of infinitely many non–symmetric periodic orbits of the Sitnikov problem for $e \in (0,1)$, except perhaps for a discrete set of values of e. ■

REMARK . Here we have only proved the existence of infinitely many non–symmetric periodic solutions near the heteroclinic loop ξ_1. But similar arguments would prove the existence of infinitely many non–symmetric periodic solutions near the other three heteroclinic loops (ξ_2, ξ_3 and ξ_4). □

Here we have used the existence of the Bernoulli shift as a subsystem of f_e in order to prove the existence of infinitely many non–symmetric periodic orbits, but the Bernoulli shift also gives infinitely many symmetric periodic orbits. For instance, consider the image by f_e of the curve $\gamma = \{(\dot{z}_0, 0) \in \mathcal{B}_e^+ \cap R\}$, where R is the subset of $\mathcal{B}_e^+$, containing the point P, where we have constructed the Bernoulli shift as a subsystem. Following the arguments of (Moser, 1973, page 98), we see that $f_e(\gamma)$ intersects $\rho_0(\gamma)$ at infinitely many points $(\dot{z}_k, 0)$ which correspond to periodic orbits associated to constant sequences $(s_n) = (m) \in S^+$ for m large enough. Moreover, using the arguments of Proposition 12, we see that these periodic orbits correspond to $r-$symmetric periodic orbits that are not double–symmetric. On the other hand, by Theorem 18, we can associate to any periodic sequence $(s_n) \in S^+$, with $s_n \geq \overline{s}$ and $\overline{s} = \overline{s}(e)$ a sufficiently large integer, a unique periodic orbit of the Sitnikov problem, sufficiently near the heteroclinic loop ξ_1. Thus the sequences $(s_n) = (m) \in S^+$ for m large enough are associated to $r-$symmetric periodic orbits.

We note that the sequences $(s_n) = (m)$ for m large enough satisfy condition (14), but they are not associated to double–symmetric periodic orbits. In general, using the Bernoulli shift, we can find infinitely many periodic orbits associated to sequences of the form (12) or (13) but a priori we cannot guarantee that those periodic orbits are symmetric periodic orbits.

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