PERIODIC ORBITS OF A COLLINEAR RESTRICTED THREE BODY PROBLEM

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Abstract. In this paper we study symmetric periodic orbits of a collinear restricted three body problem, when the middle mass is the largest one. These symmetric periodic orbits are obtained from analytic continuation of symmetric periodic orbits of two collinear two body problems.

Keywords: Periodic orbits, collinear restricted 3 body problem, analytic continuation method.

1. Introduction

In the study of any dynamical system and, in particular, in the study of the dynamical systems associated to the $n-$body problem, it is very important to know the existence, stability and bifurcation of periodic orbits. Over the years many different methods have been used to establish the existence and the nature of periodic solutions of the $n-$body problem (for instance Poincaré's continuation method, averaging, Lagrangian manifold intersection theory, normal forms, numerical analysis, majorant series, special fixed point theorems, symbolic dynamics, variational methods, ...). In some sense, the analytic study of the periodic orbits of the $n-$body problem was started by Poincaré in (Poincaré, 1892–1899), when he studied periodic orbits for the planar circular restricted 3$-$body problem. There is a very extensive literature on the existence of periodic solutions of the n-body problem, especially in the restricted 3$-$body problems. In (Meyer, 1999) we find a good discussion on the applicability of the Poincare's continuation method to different $n-$body problems.

In this paper we study symmetric periodic orbits of a collinear restricted three body problem. We consider two primaries m_1 and m_2 moving in an elliptic collision orbit. We fix the primary m_2 at the origin of coordinates, and then we assume that the position of m_1 is on the negative x–axis, see Figure 1. The *collinear restricted three body* *problem* that we consider is to describe the motion of an infinitesimal mass $m_3 = 0$ moving on the positive x–axis under the newtonian gravitational attraction of m_1 and m_2.

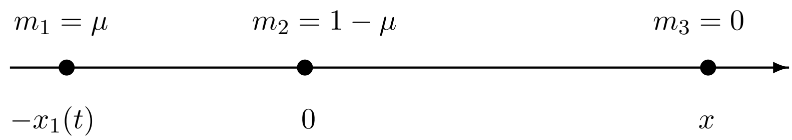

Figure 1. The collinear restricted three body problem.

We choose the units of mass, length and time in such a way that $m_1 = \mu$ and $m_2 = 1 - \mu$, with $\mu \geqslant 0$ sufficiently small; the gravitational constant is $G = 1$; the major semiaxis of the collision orbit is 1; and the time between two collisions of the primaries is 2π. Then, the equation of motion for the infinitesimal mass becomes

$$\ddot{x} = -\frac{1-\mu}{x^2} - \frac{\mu}{(x + x_1(t))^2} \; , \tag{1}$$

where $x_1(t)$ is the distance between the primaries. Of course the dot denotes the derivative with respect to the time t.

We remark that other collinear restricted three body problems can be considered. For instance, if we put the center of mass of the two primaries (moving in an elliptic collision orbit) at the origin of coordinates, and we study the motion of an infinitesimal mass located on the straight line determined by the collision orbit of the primaries and contained on a side of the primaries, then the equation of motion of this problem is not equivalent to equation (1). The periodic orbits of this different collinear restricted three body problem have been studied partially in Llibre and Simó (1980).

On the other hand, symmetric periodic orbits for the collinear three body problem, when the three masses are positive, have been studied numerically by Hénon (1977) and Broucke and Walker (1980), and analytically by Kammeyer (1983).

Using the eccentric anomaly of the collision orbit followed by the primaries, we regularize the binary collisions between the primaries. Then we consider the Hamiltonian formulation of the problem in an extended phase space by introducing the time as a new canonical coordinate. We take the eccentric anomaly as a new time variable and we use the Levi–Civita transformation to regularize the binary collisions between m_3 and m_2. After the regularization, the resulting system is analytic with respect to all its variables except at the triple collision. We show that the resulting system is invariant under two discrete symmetries.

These symmetries are used to obtain symmetric periodic orbits satisfying either one or the two symmetries (i.e. doubly–symmetric periodic orbits).

In appropriate coordinates, we reduce our collinear restricted three body problem for $\mu = 0$ (after the regularization) to an integrable system. We compute explicitly all its symmetric periodic orbits. Then, using the analytic continuation method due to Poincaré, we continue them to symmetric periodic orbits of the collinear restricted three body problem (after the regularization) for $\mu > 0$ sufficiently small. The main results of this paper are summarized in the following theorem.

THEOREM 1. *Each symmetric periodic orbit of the collinear restricted three body problem for $\mu = 0$ can be continued to a one parameter family, that depends on μ, of symmetric periodic orbits of the collinear restricted three body problem for $\mu > 0$ sufficiently small. Moreover, the continued symmetric periodic orbits satisfy the same symmetry than the initial orbit.*

To understand better the statement of Theorem 1 see Theorems 12 and 14.

This paper is structured as follows. In Section 2, we define our collinear restricted three body problem and we give its equations of motion after regularizing binary collisions. In Section 3, we analyze the symmetries of the problem. These symmetries are used in Section 4 to find symmetric periodic solutions of the collinear restricted three body problem (after regularization) when the small primary has infinitesimal mass. Finally, in Section 5, we continue those symmetric periodic solutions to symmetric periodic solutions of the collinear restricted three body problem (after regularization) for sufficiently small values of the mass of the small primary.

2. Equations of motion

System (1) has three singularities: the binary collision between the primaries (i.e. when $x_1 = 0$), the binary collision between m_2 and m_3 (i.e. when $x = 0$), and the triple collision when $x = x_1 = 0$.

We regularize the binary collision between the primaries by taking

$$x_1(t) = 1 - \cos E \ ,$$

where E is the eccentric anomaly which is a function of t via the Kepler's equation

$$t = E - \sin E \ , \qquad (2)$$

see (Roy, 1978) for more details.

Equation (1) defines a Hamiltonian system of one and half degrees of freedom with Hamiltonian

$$H(x,y,t,\mu) = \frac{y^2}{2} - \frac{1}{x} + \mu\left(\frac{1}{x} - \frac{1}{x+1-\cos E}\right) \ , \tag{3}$$

where $y = \dot{x}$ is the conjugate momentum of the variable x. We note that when μ is positive then $H(x,y,t,\mu)$ depends explicitly on the time. Moreover, $H(x,y,t,\mu)$ is periodic of period 2π with respect to the variable t.

We introduce the time (modulus 2π) as a new position variable u; i.e. $u = t$. It can be shown that its conjugate momentum, which is denoted here by v, satisfies $v = -H$. Then Hamiltonian (3) in the extended phase space $(x,y,u,v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}$ becomes

$$\overline{H}(x,y,u,v,\mu) = \frac{y^2}{2} - \frac{1}{x} + \mu\left(\frac{1}{x} - \frac{1}{x+1-\cos E}\right) + v \ ,$$

where $E = E(u)$ and it is given by (2). The flow of our collinear restricted three body problem corresponds to the flow of the vector field given by $\overline{H}$ on the energy level $\overline{H} = 0$.

The equations of motion associated to the Hamiltonian $\overline{H}$ are

$$\begin{aligned} \dot{x} &= y \ , \\ \dot{y} &= -\frac{1}{x^2} + \mu\left(\frac{1}{x^2} - \frac{1}{(x+1-\cos E)^2}\right) \ , \\ \dot{u} &= 1 \ , \\ \dot{v} &= -\mu \frac{\sin E}{(x+1-\cos E)^2} \frac{1}{1-\cos E} \ . \end{aligned} \tag{4}$$

We note that (4) continues having the three singularities: $x = 0$ that corresponds to the binary collision between m_2 and m_3; $x+1-\cos E = 0$ that corresponds to tripe collision; and $1 - \cos E = 0$ that corresponds to the binary collision between the primaries.

In order to avoid the singularity due to the binary collision between the primaries, we take the eccentric anomaly as the new time variable; i.e. we do the following change in the time variable

$$dt = (1 - \cos E) dE \ .$$

Then equations of motion (4) become

$$\begin{aligned}
\frac{dx}{dE} &= y(1-\cos E)\ , \\
\frac{dy}{dE} &= \left(-\frac{1}{x^2}+\mu\left(\frac{1}{x^2}-\frac{1}{(x+1-\cos E)^2}\right)\right)(1-\cos E)\ , \\
\frac{du}{dE} &= 1-\cos E\ , \\
\frac{dv}{dE} &= -\mu\frac{\sin E}{(x+1-\cos E)^2}\ .
\end{aligned} \tag{5}$$

We note that system (5) is analytic with respect to all its variables except when $x = 0$, that corresponds to binary collision between m_3 and m_2, and when $x+1-\cos E = 0$ that corresponds to triple collision. We remark that system (5) is non–autonomous with respect to the time E.

Now we regularize the binary collision between m_3 and m_2 by using the Levi–Civita transformation

$$x = \xi^2\ , \quad y = \eta\xi^{-1}\ , \quad u = u\ , \quad v = v\ , \quad dE = 2\xi^2 dw\ .$$

The equations of motion in the new variables are

$$\begin{aligned}
\frac{d\xi}{dw} &= \eta(1-\cos E)\ , \\
\frac{d\eta}{dw} &= \left(-2\xi v+\mu\left(\frac{2\xi(1-\cos E)}{(\xi^2+1-\cos E)^2}\right)\right)(1-\cos E)\ , \\
\frac{du}{dw} &= 2\xi^2(1-\cos E)\ , \\
\frac{dv}{dw} &= -\mu\frac{2\xi^2\sin E}{(\xi^2+1-\cos E)^2}\ , \\
\frac{dE}{dw} &= 2\xi^2\ .
\end{aligned} \tag{6}$$

The energy relation $\overline{H} = 0$ in the new variables becomes

$$\eta^2 - 2 + \mu\frac{2(1-\cos E)}{\xi^2+1-\cos E} + 2\xi^2 v = 0\ . \tag{7}$$

We note that system (6) is analytic with respect to all its variables except when $\xi^2+1-\cos E = 0$ which corresponds to triple collision.

The regularization of the binary collisions allows us to look for periodic orbits of m_3 containing binary collisions with m_2. Our aim is to find periodic orbits of (6) for $\mu > 0$ sufficiently small, satisfying the energy relation (7). In fact, we are looking for symmetric periodic orbits which are easier to control than general periodic orbits.

3. Symmetries

In order to analyze the symmetries of system (6), we must rewrite this system from a different point of view.

Applying to system (4) the Levi–Civita transformation

$$x = \xi^2 \ , \quad y = \eta \xi^{-1} \ , \quad u = u \ , \quad v = v \ , \quad dt = 2\xi^2 ds \ ,$$

we obtain

$$\begin{aligned} \frac{d\xi}{ds} &= \eta \ , \\ \frac{d\eta}{ds} &= -2\xi v + \mu \left(\frac{2\xi(1-\cos E)}{(\xi^2+1-\cos E)^2} \right) \ , \\ \frac{du}{ds} &= 2\xi^2 \ , \\ \frac{dv}{ds} &= -\mu \frac{2\xi^2 \sin E}{(\xi^2+1-\cos E)^2} \frac{1}{1-\cos E} \ , \end{aligned} \tag{8}$$

where $E = E(u)$ is given through the Kepler's equation (2). We note that $E(u)$ is not analytic when $u = 2k\pi$ with $k \in \mathbb{N}$, then system (8) is not analytic with respect to the variable u for all $u \in \mathbb{R}$.

By definition, $E(u)$ is the solution with respect to E of the equation $u = E - \sin E$. Then we claim that

$$E(k\pi + u) = -E(k\pi - u) + 2k\pi \ , \tag{9}$$

for all $k \in \mathbb{N}$. Now we prove the claim. Since $E(k\pi + u)$ is the solution of $k\pi + u = E - \sin E$, we must see that the equation

$$k\pi + u = -E(k\pi - u) + 2k\pi - \sin(-E(k\pi - u) + 2k\pi) \ ,$$

holds. Indeed,

$$\begin{aligned} &-E(k\pi - u) + 2k\pi - \sin(-E(k\pi - u) + 2k\pi) \\ &\quad = -E(k\pi - u) + \sin(E(k\pi - u)) + 2k\pi \\ &\quad = -(k\pi - u) + 2k\pi = k\pi + u \ . \end{aligned}$$

So, the claim is proved.

Since the variable u is taken modulus 2π and equation (9) is satisfied, it is easy to check that system (8) is invariant under symmetries

$$(\xi, \eta, k\pi + u, v, s) \longrightarrow (\xi, -\eta, k\pi - u, v, -s) \ , \tag{10}$$

and

$$(\xi, \eta, k\pi + u, v, s) \longrightarrow (-\xi, \eta, k\pi - u, v, -s) \ , \tag{11}$$

for all $k \in \mathbb{N}$.

On the other hand, after the time reparametrization $ds = (1 - \cos E)dw$, system (8) is transformed into system

$$\begin{aligned}
\frac{d\xi}{dw} &= \eta(1 - \cos E) \ , \\
\frac{d\eta}{dw} &= \left(-2\xi v + \mu\left(\frac{2\xi(1 - \cos E)}{(\xi^2 + 1 - \cos E)^2}\right)\right)(1 - \cos E) \ , \\
\frac{du}{dw} &= 2\xi^2(1 - \cos E) \ , \\
\frac{dv}{dw} &= -\mu\frac{2\xi^2 \sin E}{(\xi^2 + 1 - \cos E)^2} \ .
\end{aligned} \tag{12}$$

We note that the four equations of system (12) have the same form than the first four equations of system (6), but on the right–hand of system (6) E is a function of w by means of

$$\frac{dE}{dw} = 2\xi^2 \ ,$$

whereas in system (12) E is a function of u through

$$u = E - \sin E \ . \tag{13}$$

Differentiating equation (13) with respect to w and using (12) we obtain

$$\frac{dE}{dw} = \frac{du}{dw}\frac{1}{1 - \cos E} = 2\xi^2 \ .$$

Therefore the fifth equation of system (6) is satisfied also for the solutions of system (12), and consequently (6) and (12) define the same system of differential equations.

By means of the time transformation $ds = (1 - \cos E)dw$, we see that a change in the sign of the time s implies a change in the sign of the time w. Then system (12) is invariant under the symmetries

$$(\xi, \eta, k\pi + u, v, w) \longrightarrow (\xi, -\eta, k\pi - u, v, -w) \ , \tag{14}$$

and

$$(\xi, \eta, k\pi + u, v, w) \longrightarrow (-\xi, \eta, k\pi - u, v, -w) \ , \tag{15}$$

for all $k \in \mathbb{N}$.

We claim that

$$E(k\pi + u) = k\pi + E(u) \ , \qquad \text{and} \qquad E(k\pi - u) = k\pi - E(u) \ , \tag{16}$$

for all $k \in \mathbb{N}$. Now we prove $E(k\pi + u) = k\pi + E(u)$. The proof of $E(k\pi - u) = k\pi - E(u)$ follows immediately form the previous one. First we consider that k is even. Since $E(k\pi + u)$ is the solution of $k\pi + u = E - \sin E$, we must see that the equation

$$k\pi + u = k\pi + E(u) - \sin(k\pi + E(u)) \ ,$$

holds. Indeed, since k is even,

$$k\pi + E(u) - \sin(k\pi + E(u)) = k\pi + E(u) - \sin(E(u)) = k\pi + u \ .$$

Assume that k is odd. Then, to prove $E(k\pi+u) = k\pi+E(u)$, by (9), is equivalent to show that $-E(k\pi-u)+2k\pi = k\pi+E(u)$, i.e. $E(k\pi-u) = k\pi - E(u)$. Since $E(k\pi - u)$ is the solution of $k\pi - u = E - \sin E$, we must see that the equation

$$k\pi - u = k\pi - E(u) - \sin(k\pi - E(u)) \ ,$$

holds. Indeed, since p is odd

$$k\pi - E(u) - \sin(k\pi - E(u)) = k\pi - E(u) + \sin E(u) = k\pi - u \ .$$

Hence, the claim is proved.

Since (12) and (6) define the same system of differential equations, using symmetries (14) and (15) together with (16) the following result follows easily.

PROPOSITION 2. *System (6) is invariant under symmetries*

$$S_\eta : (\xi, \eta, k\pi + u, v, k\pi + E, w) \longrightarrow (\xi, -\eta, k\pi - u, v, k\pi - E, -w) \ ,$$

and

$$S_\xi : (\xi, \eta, k\pi + u, v, k\pi + E, w) \longrightarrow (-\xi, \eta, k\pi - u, v, k\pi - E, -w) \ ,$$

with $k \in \mathbb{N}$.

We note that the above symmetries can be written as

$$S_\eta : (\xi, \eta, u, v, E, w) \longrightarrow (\xi, -\eta, 2k\pi - u, v, 2k\pi - E, -w) \ ,$$

and

$$S_\xi : (\xi, \eta, u, v, E, w) \longrightarrow (-\xi, \eta, 2k\pi - u, v, 2k\pi - E, -w) \ ,$$

We remark that in fact S_η and S_ξ are a family of symmetries depending on $k \in \mathbb{N}$.

Symmetries S_η and S_ξ can be exploited in a similar way as for instance in (Howison and Meyer, 2000; Corbera and Llibre, 2000, 2001a, 2001b) to obtain symmetric periodic orbits of system (6). These symmetric periodic orbits are characterized by the following result.

LEMMA 3. *Let* $\varphi(w) = (\xi(w), \eta(w), u(w), v(w), E(w))$ *be a solution of (6).*

(a) If $\eta(0) = 0$, $u(0) = k\pi$, $E(0) = k\pi$ *and there exists* $W > 0$ *such that* $\eta(W/2) = 0$, $u(W/2) = (k+p)\pi$ *and* $E(W/2) = (k+p)\pi$ *for some* $k, p \in \mathbb{N}$, *but there is no* $\overline{W} \in (0, W/2)$ *such that* $\eta(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ *and* $E(\overline{W}) = (k+p)\pi$, *then* $\varphi(w)$ *is a* $S_\eta-$*symmetric periodic solution of (6) of period* W.

(b) If $\xi(0) = 0$, $u(0) = k\pi$, $E(0) = k\pi$ *and there exists* $W > 0$ *such that* $\xi(W/2) = 0$, $u(W/2) = (k+p)\pi$ *and* $E(W/2) = (k+p)\pi$ *for some* $k, p \in \mathbb{N}$, *but there is no* $\overline{W} \in (0, W/2)$ *such that* $\xi(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ *and* $E(\overline{W}) = (k+p)\pi$, *then* $\varphi(w)$ *is a* $S_\xi-$*symmetric periodic solution of (6) of period* W.

It could be periodic solutions of (6) that are simultaneously $S_\eta-$ and $S_\xi-$symmetric. These periodic solutions will be called *doubly–symmetric periodic solutions* and they are characterized by the following result.

LEMMA 4. *Let* $\varphi(w) = (\xi(w), \eta(w), u(w), v(w), E(w))$ *be a solution of (6).*

(a) If $\eta(0) = 0$, $u(0) = k\pi$, $E(0) = k\pi$ *and there exists* $W > 0$ *such that* $\xi(W/4) = 0$, $u(W/4) = (k+p)\pi$ *and* $E(W/4) = (k+p)\pi$ *for some* $k, p \in \mathbb{N}$, *but there is no* $\overline{W} \in (0, W/4)$ *such that* $\xi(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ *and* $E(\overline{W}) = (k+p)\pi$, *then* $\varphi(w)$ *is a doubly–symmetric periodic solution of (6) of period* W.

(b) If $\xi(0) = 0$, $u(0) = k\pi$, $E(0) = k\pi$ *and there exists* $W > 0$ *such that* $\eta(W/4) = 0$, $u(W/4) = (k+p)\pi$ *and* $E(W/4) = (k+p)\pi$ *for some* $k, p \in \mathbb{N}$, *but there is no* $\overline{W} \in (0, W/4)$ *such that* $\xi(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ *and* $E(\overline{W}) = (k+p)\pi$, *then* $\varphi(w)$ *is a doubly–symmetric periodic solution of (6) of period* W.

We note that the condition on E is not really necessary in order to have symmetric periodic solutions of (6), because, by means of (13), if $u = (k+p)\pi$ for some $k, p \in \mathbb{N}$, then $E = (k+p)\pi$. On the other hand, we are only interested in periodic solutions of (6) satisfying the energy relation (7). It is easy to check that relation (7) is constant along the solutions of (6). So it is sufficient to prove that the energy relation is satisfied for a fixed value of w. For our convenience we choose $w = 0$. Then the necessary and sufficient conditions given in Lemmas 3 and 4 in order to have symmetric periodic solutions, taken into account the energy relation (7), can be written as follows.

PROPOSITION 5. *Let $\varphi(w) = (\xi(w), \eta(w), u(w), v(w), E(w))$ be a solution of (6) and let*

$$f(\xi, \eta, u, v, E, \mu) = \eta^2 - 2 + \mu\frac{2(1-\cos E)}{\xi^2 + 1 - \cos E} + 2\xi^2 v \ .$$

(a) If $\eta(0) = 0$, $u(0) = k\pi$ and there exists $W > 0$ such that $\eta(W/2) = 0$, $u(W/2) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$ for some $k, p \in \mathbb{N}$, but there is no $\overline{W} \in (0, W/2)$ such that $\eta(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$, then $\varphi(w)$ is a $S_\eta-$symmetric periodic solution of (6) of period W.

(b) If $\xi(0) = 0$, $u(0) = k\pi$ and there exists $W > 0$ such that $\xi(W/2) = 0$, $u(W/2) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$ for some $k, p \in \mathbb{N}$, but there is no $\overline{W} \in (0, W/2)$ such that $\xi(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$, then $\varphi(w)$ is a $S_\xi-$symmetric periodic solution of (6) of period W.

(c) If $\eta(0) = 0$, $u(0) = k\pi$ and there exists $W > 0$ such that $\xi(W/4) = 0$, $u(W/4) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$ for some $k, p \in \mathbb{N}$, but there is no $\overline{W} \in (0, W/4)$ such that $\xi(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$, then $\varphi(w)$ is a doubly–symmetric periodic solution of (6) of period W.

(d) If $\xi(0) = 0$, $u(0) = k\pi$ and there exists $W > 0$ such that $\eta(W/4) = 0$, $u(W/4) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$ for some $k, p \in \mathbb{N}$, but there is no $\overline{W} \in (0, W/4)$ such that $\xi(\overline{W}) = 0$, $u(\overline{W}) = (k+p)\pi$ and $f(\xi(0), \eta(0), u(0), v(0), E(0), \mu) = 0$, then $\varphi(w)$ is a doubly–symmetric periodic solution of (6) of period W.

We remark that, in system (6), the triple collision is not regularized. Therefore in our study we avoid the orbits of the collinear restricted three body problem which start or end in triple collision. In the variables that we are working, triple collision takes place when $\xi = 0$ and simultaneously $E = 0$ (mod. 2π) (or equivalently, $u = 0$ (mod. 2π)).

4. Symmetric periodic solutions for $\mu = 0$

For $\mu = 0$ system (6) becomes

$$\begin{aligned} \frac{d\xi}{dw} &= \eta(1-\cos E) \ , & \frac{du}{dw} &= 2\xi^2(1-\cos E) \ , \\ \frac{d\eta}{dw} &= -2\xi v(1-\cos E) \ , & \frac{dv}{dw} &= 0 \ , \\ \frac{dE}{dw} &= 2\xi^2 \ . \end{aligned} \tag{17}$$

In order to solve system (17), we take the first four equations of system (17) and we do the following change in the time variable,

$$(1-\cos E)dw = ds \ , \tag{18}$$

obtaining in this way

$$\begin{array}{ll} \dfrac{d\xi}{ds} = \eta \ , & \dfrac{du}{ds} = 2\xi^2 \ , \\ \dfrac{d\eta}{ds} = -2\xi v \ , & \dfrac{dv}{ds} = 0 \ . \end{array} \tag{19}$$

System (19) can be integrated easily, and their solutions are given in the following proposition.

PROPOSITION 6. *The solution* $(\xi(s), \eta(s), u(s), v(s))$ *of (19) with initial conditions*

$$\xi(0) = \xi_0^*, \quad \eta(0) = \eta_0^*, \quad u(0) = u_0^*, \quad v(0) = v_0^*, \tag{20}$$

is given by

$$\begin{aligned} \xi(s) &= \xi_0^* \cos(\sqrt{2v_0^*}s) + \frac{\eta_0^*}{\sqrt{2v_0^*}} \sin(\sqrt{2v_0^*}s) \ , \\ \eta(s) &= -\xi_0^* \sqrt{2v_0^*} \sin(\sqrt{2v_0^*}s) + \eta_0^* \cos(\sqrt{2v_0^*}s) \ , \\ u(s) &= \frac{\eta_0^{*2} + 2v_0^* \xi_0^{*2}}{2v_0^*} s + \frac{\eta_0^* \xi_0^*}{v_0^*} \sin^2(\sqrt{2v_0^*}s) - \\ & \quad \frac{\sqrt{2}(\eta_0^{*2} - 2v_0^* \xi_0^{*2})}{4v_0^{*3/2}} \sin(\sqrt{2v_0^*}s) \cos(\sqrt{2v_0^*}s) + u_0^* \ , \\ v(s) &= v_0^* \ . \end{aligned}$$

The solutions of (17) can be obtained from solutions of (19) as it is shown in the next result.

PROPOSITION 7. *Let* $(\xi(s), \eta(s), u(s), v(s))$ *be a solution of (19) with initial conditions (20). Assume that there is no* $\overline{s} \in \mathbb{R}$ *such that* $\xi(\overline{s}) = 0$ *and* $u(\overline{s}) = 0$ *(mod.* 2π*). Then* $(\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ *is a solution of (17) with initial conditions (20). Here* $s(w)$ *is the inverse function of*

$$w(s) = \int_0^s \frac{d\sigma}{1 - \cos E(\sigma)} \ , \tag{21}$$

and $E(s)$ is given implicitly by equation

$$E - \sin E = u(s) \ . \tag{22}$$

Proof. Using the change in the time variable (18) and the last equation of (17) we have that

$$(1 - \cos E)\, dE = 2\xi^2\, ds \ .$$

Integrating the last equation, we obtain E as a function of s through

$$E - \sin E = \int 2\xi^2(s)\, ds + C = u(s) + C \ ,$$

where C is an integration constant. For our convenience we take $C = 0$ obtaining in this way (22). Integrating equation (18), with $E = E(s)$ given through (22), we get w as a function of s through (21). We note that in the last integration we have chosen the integration constant in such a way that $w = 0$ when $s = 0$.

The integrand of (21) becomes singular when $E = 0$ (mod. 2π). Thus, if there is $\overline{s} \in [0, s]$ such that $E(\overline{s}) = 0$ (mod. 2π), then (21) is an improper integral. Now we analyze the convergence of (21), in particular we claim that if there is no $\overline{s} \in \mathbb{R}$ such that $\xi(\overline{s}) = 0$ and $E(\overline{s}) = 0$ (mod. 2π), then (21) is convergent for all $s \in \mathbb{R}$.

Now, we shall prove the claim. From the Kepler's equation (2), we have that

$$E = \int_0^t \frac{1}{1 - \cos E(\overline{t})}\, d\overline{t} \ . \tag{23}$$

The integrand of (23) becomes singular when $t = 0$ (mod. 2π). Nevertheless the integral (23) is convergent for all $t \in \mathbb{R}$ because, by (2), $E = \infty$ if and only if $t = \infty$. On the other hand, from (22), $E(\overline{s}) = 0$ (mod. 2π) if and only if $u(\overline{s}) = 0$ (mod. 2π).

Assume that $\overline{s} \in [0, s]$ is such that $u(\overline{s}) = 0$ (mod. 2π), or equivalently, $E(\overline{s}) = 0$ (mod. 2π) (see (22)). Expanding $u(s)$ in power series of $(s - \overline{s})$ we get

$$u(s) = u(\overline{s}) + 2\xi^2(\overline{s})(s - \overline{s}) + 0((s - \overline{s})^2) \ .$$

Since $\xi(\overline{s}) \neq 0$ when $u(\overline{s}) = 0$ (mod. 2π), we have that $u(s) \sim u(\overline{s}) + 2\xi^2(\overline{s})(s - \overline{s})$ when s is close to $\overline{s}$. Then, for s near $\overline{s}$, we have that

$$\frac{1}{1 - \cos E(u(s))} \sim \frac{1}{1 - \cos E(2\xi^2(\overline{s})(s - \overline{s}))} \ .$$

Therefore the integral (21) converges in a neighborhood of $s = \overline{s}$ if and only if the integral (23) converges in a neighborhood of $t = 0$. This proves the claim.

In short, we have just seen that if there is no $\overline{s} \in \mathbb{R}$ such that $\xi(\overline{s}) = 0$ and $u(\overline{s}) = 0$ (mod. 2π), then $w(s)$ given by (21) is defined for all $s \in \mathbb{R}$. Moreover, since the integrand of (21) is positive, $w(s)$ is always injective, therefore we can find the inverse function $s(w)$.

Finally, it is easy to see that $(\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ with $s(w)$ and $E(s(w))$ defined as above is a solution of (17) with initial conditions (20), which proves the result. ∎

We are interested in periodic solutions of (17) satisfying the relation (7), which for $\mu = 0$ becomes

$$\eta^2 - 2 + 2\xi^2 v = 0 \ . \tag{24}$$

These periodic solutions are characterized in the following proposition.

PROPOSITION 8. *Let $(\xi(s), \eta(s), u(s), v(s))$ be a solution of (19) with initial conditions (20).*

(a) If $v_0^* = \left(\dfrac{q}{\sqrt{2}p}\right)^{2/3}$ for some $p, q \in \mathbb{N}$ coprime and ${\eta_0^*}^2 - 2 + 2{\xi_0^*}^2 v_0^* = 0$, then $(\xi(s), \eta(s), u(s), v(s))$ is a periodic solution of (19) with period $S^* = \dfrac{q2\pi}{\sqrt{2v_0^*}}$ that satisfies the energy relation (24).

(b) If there is no $\overline{s} \in \mathbb{R}$ such that $\xi(\overline{s}) = 0$ and $u(\overline{s}) = 0$ (mod. 2π), then the solution $(\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ of (17) defined as in Proposition 7 is periodic with period W^*, where $W^* = w(S^*)$ is given by (21).

Proof. Using the expression of the solutions of (19) given in Proposition 6, we see that $\xi(s)$ and $\eta(s)$ are periodic functions of period $S = \dfrac{2\pi}{\sqrt{2v_0^*}}$. Then S^* must be a multiple of S; that is, $S^* = q\dfrac{2\pi}{\sqrt{2v_0^*}}$ for some $q \in \mathbb{N}$. On the other hand, since the variable u is taken modulus 2π, in order to have a periodic solution of (19) with period S^*, we need that $u(s) = u(s + S^*) + p2\pi$ for some $p \in \mathbb{N}$; that is we need that

$$\frac{q2\pi}{\sqrt{2v_0^*}} \frac{{\eta_0^*}^2 + 2v_0^*{\xi_0^*}^2}{2v_0^*} = p2\pi \ , \tag{25}$$

for some $p, q \in \mathbb{N}$. Since we only are interested in periodic solutions satisfying the energy relation (24), from (25), we have that v_0^* must verify

$$v_0^* = \left(\frac{q}{\sqrt{2}p} \right)^{2/3} ,$$

for some $p, q \in \mathbb{N}$. We note that if p and q are coprime, then S^* is the minimal period. This proves statement (a).

Statement (b) is an immediate consequence of Proposition 7. ■

We remark that the number q in Proposition 8 represents the number of binary collisions between m_3 and m_2 during a period, whereas p represents the number of binary collisions between the primaries.

In the next proposition we give initial conditions for the symmetric periodic solutions of (17) satisfying the energy relation (24).

PROPOSITION 9. *Let* $(\xi(s), \eta(s), u(s), v(s))$ *be a periodic solution of (19) satisfying (24) with initial conditions*

$$\xi(0) = \xi_0^*, \quad \eta(0) = \eta_0^*, \quad u(0) = u_0^*, \quad v(0) = v_0^*,$$

and period $S^* = \dfrac{q2\pi}{\sqrt{2v_0^*}}$, *with* $v_0^* = \left(\dfrac{q}{\sqrt{2}p} \right)^{2/3}$ *for some* $p, q \in \mathbb{N}$ *coprime.*

(a) If $\xi_0^* = \pm\dfrac{1}{\sqrt{v_0^*}}$, $\eta_0^* = 0$, $u_0^* = 0$ *and* $p \neq 4n$ *for all* $n \in \mathbb{N}$ *or* $\xi_0^* = \pm\dfrac{1}{\sqrt{v_0^*}}$, $\eta_0^* = 0$, $u_0^* = \pi$ *and* $p \neq 4n-2$ *for all* $n \in \mathbb{N}$, *then* $(\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ *defined as in Proposition 7 is a* $S_\eta-$*symmetric periodic solution of (17) with period* $W^* = w(S^*)$. *Moreover, if p is even, then it is a doubly–symmetric periodic solution, whereas if p is odd, then it is a* $S_\eta-$*symmetric periodic solution but not a doubly–symmetric periodic solution.*

(b) If $\xi_0^* = 0$, $\eta_0^* = \pm\sqrt{2}$, $u_0^* = \pi$ *and p is even, then* $(\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ *defined as in Proposition 7 is a* $S_\xi-$*symmetric periodic solution of (17) with period* $W^* = w(S^*)$. *These periodic solutions are always doubly–symmetric periodic solutions.*

Proof. We only prove statement (a), statement (b) would be proved in a similar way.

If $(\xi(s), \eta(s), u(s), v(s))$ satisfies the hypotheses of Proposition 8(b), then $\varphi(w) = (\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ is a periodic solution of (17) with period $W^* = w(S^*)$. Since we are interested in solutions of (17) satisfying the energy relation (24), the initial conditions of $\varphi(w)$ must verify $\eta_0^{*2} - 2 + 2\xi_0^{*2} v_0^* = 0$. Hence, if $\eta(0) = \eta_0^* = 0$ (or equivalently, $\xi_0 = \pm 1/\sqrt{v_0^*}$) and $u(0) = u_0^* = k\pi$ for some $k \in \mathbb{Z}$, then $\varphi(w)$ is a S_η−symmetric periodic solution (see Proposition 5(a)).

We claim that

$$s(W^*/2) = S^*/2 \text{ for all } p, \text{ and } s(W^*/4) = S^*/4 \text{ for all even } p. \quad (26)$$

Now we prove the claim. From (21), $w(S^*)$ is given by

$$w(S^*) = \int_0^{S^*} \frac{ds}{1 - \cos E(s)} \ , \quad (27)$$

where $E(s)$ is given through (22). Since $u(s)$ is a periodic function of period S^*, also is $E(s)$. Moreover, using symmetry (10), equation (9) and the fact that the cosinus function is even, we see that $\cos(E(s)) = \cos(E(-s))$. Hence (27) becomes

$$w(S^*) = \int_{-S^*/2}^{S^*/2} \frac{ds}{1 - \cos E(s)} = 2 \int_0^{S^*/2} \frac{ds}{1 - \cos E(s)} \ .$$

Therefore $w(S^*/2) = w(S^*)/2$, or equivalently, $s(W^*/2) = S^*/2$.

On the other hand, if p is even, then $u(s)$ is also a periodic function of period $S^*/2$. Thus, proceeding in a similar way we see $w(S^*/4) = w(S^*)/4$, or equivalently, $s(W^*/4) = S^*/4$. This completes the prove of the claim.

It is easy to see from the expression of $\xi(s)$ and $u(s)$ given in Proposition 6, that if p is even, then $\xi(S^*/4) = 0$ and $u(S^*/4) = (k + p/2)\pi$. Thus, by means of (26), $\xi(W^*/4) = 0$ and $u(W^*/4) = (k + p/2)\pi$. Therefore $(\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ is a doubly–symmetric periodic solution (see Proposition 5(c)).

It only remains to see for which values of p and q the solution $(\xi(s), \eta(s), u(s), v(s))$ satisfies the hypotheses of Proposition 8(b); that is, there is no $\overline{s} \in \mathbb{R}$ such that $\xi(\overline{s}) = 0$ and $u(\overline{s}) = 0$ (mod. 2π). From the expression of $\xi(s)$, we see that $\xi(\overline{s}) = 0$ if and only if $\overline{s} = \dfrac{\pi/2 + m\pi}{\sqrt{2v_0^*}}$ for some $m \in \mathbb{Z}$. Since u is taken modulus 2π, we are only interested in values of $u_0^* = k\pi$ for $k = 0, 1$. If $k = 0$, then using the expression of $u(s)$, we see that $u(\overline{s}) = 0$ (mod. 2π) if and only if

$$\frac{p(2m+1)}{2q} = 2l \ ,$$

for some $l \in \mathbb{Z}$; that is, if and only if $p = 4n$ for some $n \in \mathbb{N}$. If $k = 1$, then $u(\overline{s}) = 0$ (mod. 2π) if and only if

$$\frac{p(2m+1) + 2q}{2q} = 2l \ ,$$

for some $l \in \mathbb{Z}$. Thus $u(\overline{s}) = 0$ (mod. 2π) if and only if $p = 4n - 2$ for some $n \in \mathbb{N}$. ■

In Proposition 9, we give different initial conditions for the symmetric periodic solutions of the collinear restricted three–body problem with $\mu = 0$. Now we will analyze which of these initial conditions really give different periodic orbits.

The $S_\eta-$ and $S_\xi-$symmetric periodic orbits of the collinear restricted three body problem are characterized for having two points of the phase space into the sets

$$\mathcal{Z}_\eta = \{(\xi, \eta, u(\mathrm{mod.}\, 2\pi), v, E(\mathrm{mod.}\, 2\pi)) \,:\, \eta = 0, u = E = k\pi, k = 0, 1\},$$

and

$$\mathcal{Z}_\xi = \{(\xi, \eta, u(\mathrm{mod.}\, 2\pi), v, E(\mathrm{mod.}\, 2\pi)) \,:\, \xi = 0, u = E = \pi\} \ ,$$

respectively (see Proposition 5). Then these symmetric periodic orbits can be classified, as in (Corbera and Llibre, 2000), using their "positions" at the points of intersection with $\mathcal{Z}_\eta$ and $\mathcal{Z}_\xi$. The points of $\mathcal{Z}_\eta$ (respectively, $\mathcal{Z}_\xi$) are denoted $P_\eta^\pm$ and $A_\eta^\pm$ (respectively, $A_\xi^\pm$) where the symbol P corresponds to $k = 0$, the symbol A corresponds to $k = 1$ and the sign $+$ corresponds to $\xi > 0$ and the sign $-$ corresponds $\xi < 0$ (respectively, the sign $+$ corresponds to $\eta > 0$ and the sign $-$ corresponds $\eta < 0$). We note that positions $P_\xi^\pm$ are not possible because they correspond to triple collision orbits.

We use the notation $\mathrm{pos}_1 \xrightarrow{\overline{w}} \mathrm{pos}_2$ to say that at an instant w_0 the orbit is at position pos_1 and at $w_0 + \overline{w}$ it is at position pos_2. Then the classification of the positions for the symmetric periodic orbits of the collinear restricted three body problem with $\mu = 0$ is given by the following result.

THEOREM 10. *The following statements hold.*

(a) Each $S_\eta$symmetric periodic orbit of the collinear restricted three body problem for $\mu = 0$ with period W is of one of the following types:

(i) $P_\eta^+ \xrightarrow{W/2} A_\eta^+ \xrightarrow{W/2} P_\eta^+$ when p is odd and q is even,

(ii) $P_\eta^+ \xrightarrow{W/2} A_\eta^- \xrightarrow{W/2} P_\eta^+$ *when* p *is odd and* q *is odd,*

(iii) $P_\eta^- \xrightarrow{W/2} A_\eta^+ \xrightarrow{W/2} P_\eta^-$ *when* p *is odd and* q *is odd,*

(iv) $P_\eta^- \xrightarrow{W/2} A_\eta^- \xrightarrow{W/2} P_\eta^-$ *when* p *is odd and* q *is even,*

(v) $P_\eta^+ \xrightarrow{W/2} P_\eta^- \xrightarrow{W/2} P_\eta^+$ *when* p *is even,*

(vi) $A_\eta^+ \xrightarrow{W/2} A_\eta^- \xrightarrow{W/2} A_\eta^+$ *when* p *is even.*

Moreover the types (v) and (vi) correspond to doubly–symmetric periodic orbits.

(b) *Each doubly–symmetric periodic orbit of the collinear restricted three body problem for* $\mu = 0$ *with period* W *is of one of the following types:*

(i) $P_\eta^+ \xrightarrow{W/4} A_\xi^+ \xrightarrow{W/4} P_\eta^- \xrightarrow{W/4} A_\xi^- \xrightarrow{W/4} P_\eta^+$ *when* $p = 4n - 2$ *and* $q = 4l + 3$*,*

(ii) $P_\eta^+ \xrightarrow{W/4} A_\xi^- \xrightarrow{W/4} P_\eta^- \xrightarrow{W/4} A_\xi^+ \xrightarrow{W/4} P_\eta^+$ *when* $p = 4n - 2$ *and* $q = 4l + 1$*,*

(iii) $A_\eta^+ \xrightarrow{W/4} A_\xi^+ \xrightarrow{W/4} A_\eta^- \xrightarrow{W/4} A_\xi^- \xrightarrow{W/4} A_\eta^+$ *when* $p = 4n$ *and* $q = 4l + 3$*,*

(iv) $A_\eta^+ \xrightarrow{W/4} A_\xi^- \xrightarrow{W/4} A_\eta^- \xrightarrow{W/4} A_\xi^+ \xrightarrow{W/4} A_\eta^+$ *when* $p = 4n$ *and* $q = 4l + 1$*.*

Here $n \in \mathbb{N}$ *and* $l \in \mathbb{N} \cup \{0\}$*.*

Proof. Let $\varphi(s) = (\xi(s), \eta(s), u(s), v(s))$ denote a symmetric periodic solution of (19) with period $S^* = \dfrac{q2\pi}{\sqrt{2v_0^*}}$, where $v_0^* = \left(\dfrac{q}{\sqrt{2}p}\right)^{2/3}$ for some $p, q \in \mathbb{N}$ coprime. In the following tables we give the points into the sets $\mathcal{Z}_\eta$ and $\mathcal{Z}_\xi$ corresponding to the solution $\varphi(s)$ for each one of the initial conditions given in Proposition 9. By means of (26), these points will be the same for the solution $\varphi(w) = (\xi(s(w)), \eta(s(w)), u(s(w)), v(s(w)), E(s(w)))$ of (17) given by Proposition 7, except when there is $\overline{s} \in \mathbb{R}$ such that $\xi(\overline{s}) = 0$ and $u(\overline{s}) = 0$ (mod. 2π). In this last case the solution $\varphi(w)$ is not defined because it corresponds to a triple collision orbit.

We distinguish six cases: when p and q are odd, when p is odd and q is even, when $p = 4n-2$ and $q = 4l+1$ for some $n \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{0\}$, when $p = 4n - 2$ and $q = 4l + 3$, when $p = 4n$ and $q = 4l + 1$, and finally when $p = 4n$ and $q = 4l + 3$. We note that the case p even and q even is not possible because p and q are coprime.

In order to simplify the notation, we set $\xi^* = \dfrac{1}{\sqrt{v_0^*}}$ and $\eta^* = \sqrt{2}$. In the last column of each table we give the configuration type of the orbit according with Theorem 10.

$\varphi(0)$	$\varphi(S^*/2)$	
$(\xi^*, 0, 0, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	(a, ii)
$(-\xi^*, 0, 0, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	(a, iii)
$(\xi^*, 0, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	(a, iii)
$(-\xi^*, 0, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	(a, ii)

Case p odd, q odd.

$\varphi(0)$	$\varphi(S^*/2)$	
$(\xi^*, 0, 0, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	(a, i)
$(-\xi^*, 0, 0, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	(a, iv)
$(\xi^*, 0, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	(a, i)
$(-\xi^*, 0, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	(a, iv)

Case p odd, q even.

$\varphi(0)$	$\varphi(S^*/4)$	$\varphi(S^*/2)$	$\varphi(3S^*/4)$	
$(\xi^*, 0, 0, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	(b, ii)
$(-\xi^*, 0, 0, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	(b, ii)
$(0, \eta^*, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	(b, ii)
$(0, -\eta^*, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	(b, ii)

Case $p = 4n - 2$, $q = 4l + 1$.

$\varphi(0)$	$\varphi(S^*/4)$	$\varphi(S^*/2)$	$\varphi(3S^*/4)$	
$(\xi^*, 0, \pi, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	(b, iv)
$(-\xi^*, 0, \pi, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	(b, iv)
$(0, \eta^*, \pi, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	(b, iv)
$(0, -\eta^*, \pi, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	(b, iv)

Case $p = 4n$, $q = 4l + 1$.

$\varphi(0)$	$\varphi(S^*/4)$	$\varphi(S^*/2)$	$\varphi(3S^*/4)$	
$(\xi^*, 0, 0, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	(b, i)
$(-\xi^*, 0, 0, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	(b, i)
$(0, \eta^*, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	(b, i)
$(0, -\eta^*, \pi, v_0^*)$	$(\xi^*, 0, 0, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(-\xi^*, 0, 0, v_0^*)$	(b, i)

Case $p = 4n - 2$, $q = 4l + 3$.

$\varphi(0)$	$\varphi(S^*/4)$	$\varphi(S^*/2)$	$\varphi(3S^*/4)$	
$(\xi^*, 0, \pi, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	(b, iii)
$(-\xi^*, 0, \pi, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	(b, iii)
$(0, \eta^*, \pi, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	$(0, -\eta^*, \pi, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	(b, iii)
$(0, -\eta^*, \pi, v_0^*)$	$(\xi^*, 0, \pi, v_0^*)$	$(0, \eta^*, \pi, v_0^*)$	$(-\xi^*, 0, \pi, v_0^*)$	(b, iii)

Case $p = 4n$, $q = 4l + 3$.

We note that if two different initial conditions for fixed values of p and q give the same configuration type, then they are different initial conditions of the same periodic orbit.

■

5. Continuation of symmetric periodic solutions

In this section using the classical analytic continuation method of Poincaré (see for details (Siegel and Moser, 1971), or (Meyer, 1999)) we shall continue the symmetric periodic orbits of the collinear restricted three body problem (6) for $\mu = 0$ to symmetric periodic orbits of (6) for $\mu > 0$ sufficiently small.

First we will analyze the continuation to doubly–symmetric periodic solutions. After we will analyze the continuation to $S_\eta-$ and $S_\xi-$symmetric periodic solutions.

5.1. Continuation to doubly–symmetric periodic solutions

Let $\varphi(w;\xi_0,0,k\pi,v_0,\mu) = (\xi(w;\xi_0,v_0,\mu),\eta(w;\xi_0,v_0,\mu),u(w;\xi_0,v_0,\mu), v(w;\xi_0,v_0,\mu),E(w;\xi_0,v_0,\mu))$ denote the solution of (6) with initial conditions

$$\xi(0) = \xi_0 \ , \quad \eta(0) = 0 \ , \quad u(0) = k\pi \ , \quad v(0) = v_0 \ , \quad E(0) = k\pi \ .$$

From Proposition 5(c), $\varphi(w;\xi_0,0,k\pi,v_0,\mu)$ is a doubly–symmetric periodic solution of the collinear restricted three body problem with period W if

$$\begin{array}{rcl} \xi(W/4;\xi_0,v_0,\mu) & = & 0 \ , \\ u(W/4;\xi_0,v_0,\mu) & = & (k+l)\pi \ , \\ f(\xi_0,0,k\pi,v_0,k\pi,\mu) & = & 0 \ , \end{array} \tag{28}$$

for some $l \in \mathbb{N}$. Since u is taken modulus 2π, we are only interested in $k = 0, 1$.

By Proposition 9(a), we have that if p is even and $p \neq 4n$ for all $n \in \mathbb{N}$ when $k = 0$ and if p is even and $p \neq 4n-2$ for all $n \in \mathbb{N}$ when $k = 1$, then $W = W^* = w(S^*)$, $\xi_0 = \xi_0^* = \dfrac{1}{\sqrt{v_0^*}}$, $v_0 = v_0^* = \left(\dfrac{q}{\sqrt{2}p}\right)^{2/3}$, $\mu = 0$, and $W = W^*$, $\xi_0 = \xi_0^* = -\dfrac{1}{\sqrt{v_0^*}}$, $v_0 = v_0^* = \left(\dfrac{q}{\sqrt{2}p}\right)^{2/3}$, $\mu = 0$, are two solutions of (28). Moreover these solutions correspond to the known doubly–symmetric periodic solutions $\varphi(w;\xi_0^*,0,k\pi,v_0^*,0)$ of (6), for $\mu = 0$. Our aim is to find solutions of (28) near the known solutions for $\mu > 0$ sufficiently small.

We note that, for fixed values of p and q, some of the initial conditions $\xi_0 = \pm 1/\sqrt{v_0^*}$, $\eta_0 = 0$, $u_0 = k\pi$, $v_0 = v_0^* = \left(\dfrac{q}{\sqrt{2}p}\right)^{2/3}$ given by Proposition 9(a) provide the same periodic orbit of (6) (according with Theorem 10). Since the collinear restricted three body problem (6) is autonomous, if we continue different initial conditions defining the same periodic orbit, then we will obtain the same continued periodic orbits.

Nevertheless here we will analyze the continuation of the four above initial conditions, and then we will see which of them give different periodic orbits.

Applying the Implicit Function Theorem to system (28) in a neighborhood of the known solution we have that if

$$\left| \begin{array}{ccc} \dfrac{\partial \xi}{\partial w} & \dfrac{\partial \xi}{\partial \xi_0} & \dfrac{\partial \xi}{\partial v_0} \\ \dfrac{\partial u}{\partial w} & \dfrac{\partial u}{\partial \xi_0} & \dfrac{\partial u}{\partial v_0} \\ \dfrac{\partial f}{\partial w} & \dfrac{\partial f}{\partial \xi_0} & \dfrac{\partial f}{\partial v_0} \end{array} \right|_{\substack{w=W^*/4 \\ \xi_0=\xi_0^* \\ v_0=v_0^* \\ \mu=0}} \neq 0 \tag{29}$$

then we can find unique analytic functions $\xi_0 = \xi_0(\mu), v_0 = v_0(\mu), W = W(\mu)$ defined for $\mu \in [0, \mu_0)$ with μ_0 sufficiently small, such that $\xi_0(0) = \xi_0^*, v_0(0) = v_0^*, W(0) = W^*$ and the solution of (6) with initial conditions $\xi(0) = \xi_0$, $\eta(0) = 0$, $u(0) = k\pi$, $v(0) = v_0$ is a doubly–symmetric periodic solution with period W. That is, if determinant (29) is different from zero, then the periodic solution $\varphi(w; \xi_0^*, 0, k\pi, v_0^*, 0)$ can be continued to a family of doubly–symmetric periodic solutions of (6) for $\mu > 0$ sufficiently small.

The derivatives $\partial\xi/\partial w$ and $\partial u/\partial w$ evaluated at $w = W^*/4, \xi_0 = \xi_0^*, v_0 = v_0^*, \mu = 0$ can be obtained directly from system (6) for $\mu = 0$, evaluating it at the solution $\varphi(w; \xi_0^*, 0, k\pi, v_0^*, 0)$ at time $W^*/4$. On the other hand, since f does not depend on w, $\partial f/\partial w$ is equal to zero.

The derivatives of $\varphi(w; \xi_0, 0, k\pi, v_0, \mu)$, with respect to the initial conditions ξ_0 and v_0, evaluated at $w = W^*/4, \xi_0 = \xi_0^*, v_0 = v_0^*, \mu = 0$ are given by the derivatives of $\varphi(w; \xi_0^*, 0, k\pi, v_0^*, 0)$, with respect to the initial conditions ξ_0 and v_0, evaluated at $w = W^*/4, \xi_0 = \xi_0^*, v_0 = v_0^*$. Let $\gamma(s) = (\xi(s), \eta(s), u(s), v(s))$ be the solution of (19) with initial conditions

$$\xi(0) = \xi_0^*, \quad \eta(0) = 0, \quad u(0) = k\pi, \quad v(0) = v_0^* \ .$$

Since we know explicitly the expression of $\gamma(s)$ as a function of the initial conditions (see Proposition 6) we can obtain, by simple derivation, the derivatives of $\gamma(s)$ with respect to the initial conditions ξ_0 and v_0. We can see that the derivatives of $\varphi(w; \xi_0^*, 0, k\pi, v_0^*, 0)$, with respect to the initial conditions ξ_0 and v_0, evaluated at $w = W^*/4, \xi_0 = \xi_0^*, v_0 = v_0^*$ are given by the derivatives of $\gamma(s)$, with respect to the initial conditions ξ_0 and v_0, evaluated at $s = s(W^*/4) = S^*/4, \xi_0 = \xi_0^*, v_0 = v_0^*$. In this way we can compute the four derivatives $\partial\xi/\partial\xi_0$, $\partial\xi/\partial v_0$, $\partial u/\partial\xi_0$ and $\partial u/\partial v_0$ evaluated at $w = W^*/4, \xi_0 = \xi_0^*, v_0 = v_0^*$.

After some computations we see that determinant (29) is given by

$$((-1)^k \cos\left(\frac{\pi p}{2}\right) - 1) \left[\frac{2^{5/3}((-1)^q+1)}{(q/p)^{4/3}} \cos\left(\frac{\pi q}{2}\right) + \right.$$
$$\left. \frac{3}{q} 2^{2/3} \pi p^2 \left(\frac{q}{p}\right)^{2/3} \sin\left(\frac{\pi q}{2}\right)\right] .$$

We note that p is even (i.e. $p = 2n$ for some $n \in \mathbb{N}$) and consequently q must be odd. Thus the last expression can be written as

$$((-1)^{k+n} - 1) \left[\frac{12}{q} \pi n^2 \left(\frac{q}{n}\right)^{2/3} \sin\left(\frac{\pi q}{2}\right)\right] .$$

Thus determinant (29) is different from zero if and only if n and k have different parity. In short we have proved the following result.

PROPOSITION 11. *Let $p = 2n$ for some $n \in \mathbb{N}$. If n and k have different parity, then the symmetric periodic solution $\varphi(w; \pm 1/\sqrt{v_0^*}, 0, k\pi, v_0^*, 0)$ can be continued to a family of doubly–symmetric periodic solutions $\varphi(w; \xi_0(\mu), 0, k\pi, v_0(\mu), \mu)$ of the collinear restricted three body problem with period $W(\mu)$, for $\mu \in [0, \mu_0)$ with μ_0 sufficiently small.*

Analyzing which of the initial conditions correspond to different doubly–symmetric periodic orbits (see Theorem 10), we arrive to the following theorem.

THEOREM 12. *Each doubly–symmetric periodic orbit of the collinear restricted three body problem (6) for $\mu = 0$ can be continued to a one parameter family, depending on μ, of doubly–symmetric periodic orbits of the collinear restricted three body problem (6) for $\mu > 0$ sufficiently small.*

We note that we could continue the doubly–symmetric periodic solutions of (6), for $\mu = 0$, that have initial conditions $\varphi(w; 0, \sqrt{2}, \pi, v_0^*, 0)$ and $\varphi(w; 0, -\sqrt{2}, \pi, v_0^*, 0)$, but the continuation of those periodic solutions does not give us new periodic orbits of (6).

5.2. Continuation of $S_\eta-$ and $S_\xi-$symmetric periodic solutions

We start continuing the S_η−symmetric periodic solutions. Let $\varphi(w; \xi_0, 0, k\pi, v_0, \mu)$ be as in Subsection 5.1. From Proposition 5(a), $\varphi(w; \xi_0, 0,$

$k\pi, v_0, \mu)$ is a $S_\eta-$symmetric periodic solution of the collinear restricted three body problem with period W if

$$\begin{aligned} \eta(W/2;\xi_0,v_0,\mu) &= 0 \ , \\ u(W/2;\xi_0,v_0,\mu) &= (k+p)\pi \ , \\ f(\xi_0,0,k\pi,v_0,k\pi,\mu) &= 0 \ , \end{aligned} \tag{30}$$

for some $p \in \mathbb{N}$.

We note that if $p \neq 4n$ for all $n \in \mathbb{N}$ when $k=0$ and if $p \neq 4n-2$ for all $n \in \mathbb{N}$ when $k=1$, then $W = W^* = w(S^*)$, $\xi_0 = \xi_0^* = \dfrac{1}{\sqrt{v_0^*}}$, $v_0 = v_0^* = \left(\dfrac{q}{\sqrt{2}p}\right)^{2/3}$, $\mu = 0$, and $W = W^*$, $\xi_0 = \xi_0^* = -\dfrac{1}{\sqrt{v_0^*}}$, $v_0 = v_0^* = \left(\dfrac{q}{\sqrt{2}p}\right)^{2/3}$, $\mu = 0$, are two solutions of (30). These solutions correspond to the $S_\eta-$symmetric periodic solutions $\varphi(w;\xi_0^*,0,k\pi,v_0^*,0)$ of (6), for $\mu = 0$, (see Proposition 9(a)). As in the previous subsection we will analyze the continuation of all of these initial conditions although some of them give the same $S_\eta-$symmetric periodic orbits.

Applying the Implicit Function Theorem to system (30) in a neighborhood of the known solution we have that if

$$\left| \begin{array}{ccc} \dfrac{\partial \eta}{\partial w} & \dfrac{\partial \eta}{\partial \xi_0} & \dfrac{\partial \eta}{\partial v_0} \\ \dfrac{\partial u}{\partial w} & \dfrac{\partial u}{\partial \xi_0} & \dfrac{\partial u}{\partial v_0} \\ \dfrac{\partial f}{\partial w} & \dfrac{\partial f}{\partial \xi_0} & \dfrac{\partial f}{\partial v_0} \end{array} \right|_{\substack{w=W^*/2 \\ \xi_0=\xi_0^* \\ v_0=v_0^* \\ \mu=0}} \neq 0 \ , \tag{31}$$

then we can find unique analytic functions $\xi_0 = \xi_0(\mu), v_0 = v_0(\mu), W = W(\mu)$ defined for $\mu \in [0,\mu_0)$ with μ_0 sufficiently small, such that $\xi_0(0) = \xi_0^*, v_0(0) = v_0^*, W(0) = W^*$ and the solution of (6) with initial conditions $\xi(0) = \xi_0$, $\eta(0) = 0$, $u(0) = k\pi$, $v(0) = v_0$ is a $S_\eta-$symmetric periodic solution with period W.

Evaluating the terms that appear in (31) we see that determinant (31) is given by

$$12\pi p\,(-1)^q \left[(-1)^{k+p} - 1\right] \ .$$

Thus determinant (31) is different from zero if and only if p and k have different parity . In short we have proved the following result.

PROPOSITION 13. *If p and k have different parity, then the $S_\eta-$symmetric periodic solution $\varphi(w;\pm 1/\sqrt{v_0^*},0,k\pi,v_0^*,0)$ can be continued to* *a family of $S_\eta-$symmetric periodic solutions $\varphi(w;\xi_0(\mu),0,k\pi,v_0(\mu),\mu)$ of the collinear restricted three body problem with period $W(\mu)$, for $\mu\in[0,\mu_0)$ with μ_0 sufficiently small.*

If p is odd, then $\varphi(w;\pm 1/\sqrt{v_0^*},0,k\pi,v_0^*,0)$ is a $S_\eta-$symmetric periodic solution but it is not a doubly–symmetric periodic solution because $\xi(W^*/4)$ and $u(W^*/4)$ (mod. 2π) are not simultaneously zero (see Proposition 5). Hence, if $\mu>0$ is sufficiently small, then $\xi(W(\mu))$ and $u(W(\mu))$ (mod. 2π) are not simultaneously zero. Therefore the continued periodic solutions are not doubly–symmetric periodic solutions.

On the other hand, if p is even, then $\varphi(w;\pm 1/\sqrt{v_0^*},0,k\pi,v_0^*,0)$ is a doubly–symmetric periodic solution. So this solution has already been continued, in the previous subsection, to a family of doubly–symmetric periodic solutions. Since the doubly–symmetric periodic solutions are also $S_\eta-$symmetric periodic solutions, we have that a solution of (28) is also a solution of (30). Therefore, if p is even and the solution $\varphi(w;\pm 1/\sqrt{v_0^*},0,k\pi,v_0^*,0)$ can be continued, then the continued periodic solutions are doubly–symmetric periodic solutions.

Analyzing as in the previous subsection which of the initial conditions correspond to different $S_\eta-$symmetric periodic orbits (see Theorem 10), we have the following.

THEOREM 14. *Each $S_\eta-$symmetric but not doubly–symmetric periodic orbit of the collinear restricted three body problem (6) for $\mu=0$ can be continued to a one parameter family, that depends on μ, of $S_\eta-$symmetric but not doubly–symmetric periodic orbits of the collinear restricted three body problem (6) for $\mu>0$ sufficiently small.*

We note that all $S_\xi-$symmetric periodic solutions of the collinear restricted three body problem for $\mu=0$ are doubly–symmetric periodic solutions (see Proposition 9(b)). So they have already been continued in Subsection 5.1 to doubly–symmetric periodic solution. We could continued them to $S_\xi-$symmetric periodic solutions, but if we do that, then we will not obtain new symmetric periodic orbits.

In short, Theorem 12 together with Theorem 14 prove Theorem 1 stated in the introduction.

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