Generation of symmetric periodic orbits by a heteroclinic loop formed by two singular points and their invariant manifolds of dimension 1 and 2 in $\mathbb{R}^3$

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Abstract

In this paper we will find a continuous of periodic orbits passing near infinity for a class of polynomial vector fields in $\mathbb{R}^3$. We consider polynomial vector fields that are invariant under a symmetry with respect to a plane Σ and that possess a "generalized heteroclinic loop" formed by two singular points e^+ and e^- at infinity and their invariant manifolds Γ and Λ. Γ is an invariant manifold of dimension 1 formed by an orbit going from e^- to e^+, Γ is contained in $\mathbb{R}^3$ and is transversal to Σ. Λ is an invariant manifold of dimension 2 at infinity. In fact, Λ is the 2–dimensional sphere at infinity in the Poincaré compactification minus the singular points e^+ and e^-. The main tool for proving the existence of such periodic orbits is the construction of a Poincaré map along the generalized heteroclinic loop together with the symmetry with respect to Σ.

1 Introduction and statement of the main results

For a class of polynomial vector fields in $\mathbb{R}^3$ we will study the periodic motion around a "generalized heteroclinic loop" formed by two singular points at infinity and their invariant manifolds: an invariant manifold of dimension 1 in $\mathbb{R}^3$ and an invariant manifold of dimension 2 at infinity. Note that almost all papers studying the dynamics near homoclinic and heteroclinic loops consider that the invariant manifolds of these loops have the same dimension (usually 1), see for instance [4, 5, 7, 8, 9]. Note that the Shil'nikov homoclinic loop is really formed by a singular point and one orbit. Hence the study of the dynamic near a heteroclinic loop formed by invariant manifolds of distinct dimension is an interesting, and relatively new problem.

In order to study the behaviour of a polynomial vector field near infinity we will use the Poincaré compactification. This technique allows to extend the vector field in $\mathbb{R}^3$ to a unique analytic vector field on the *Poincaré ball* $\mathbb{D}^3 = \{\mathbf{x} \in \mathbb{R}^3 \ : \ ||\mathbf{x}|| \leqslant 1\}$, whose boundary, the sphere $\mathbb{S}^2$, plays the role of the infinity for the initial polynomial vector field. For more details on the Poincaré compactification see Appendix 1.

Let P, Q and R be polynomials in the variables x, y and z of even degree. We consider the polynomial vector fields $X = (P, Q, R)$ in $\mathbb{R}^3$ satisfying the following conditions.

(C_1) The flow of X is invariant under the symmetry $(x, y, z, t) \longrightarrow (-x, y, z, -t)$. So the phase portrait of X is symmetric with respect to the plane $x = 0$.

(C_2) The straight line $y = z = 0$ (i.e. the x–axis) is invariant by the flow of X, it does not contain any singular point and the flow on it goes in the increasing direction of the x–axis.

(C_3) The straight line $y = z = 0$ intersects the boundary of the Poincaré ball in two hyperbolic singular points, the point $e^+ = (0, 0, 0)$ of the local chart U_1 and the point $e^- = (0, 0, 0)$ of the local chart V_1, respectively. Here we are using the notation of the Poincaré compactification. Moreover the point e^+ is either an unstable focus or an unstable node on the boundary $\mathbb{S}^2$ of the Poincaré ball. Recall that a singular point is *hyperbolic* if the real part of all its eigenvalues is different from zero.

(C_4) The boundary of the Poincaré ball does not contain singular points of X different from e^+ and e^-, and it does not contain any periodic orbit.

The flow on the boundary $\mathbb{S}^2$ of the Poincaré ball always is symmetric with respect to the origin of $\mathbb{R}^3$. This symmetry reverses the orientation of the orbits because the degree of X is even. Due to this symmetry on $\mathbb{S}^2$, if the point e^+ is an unstable focus (node, respectively) on $\mathbb{S}^2$, then the point e^- is a stable focus (node, respectively) on $\mathbb{S}^2$.

We note that conditions (C_2)–(C_4) say that the Poincaré compactification of X, $p(X)$, possesses a *generalized heteroclinic loop* $\mathcal{L}$ which is formed by the two singular points at infinity e^+ and e^-, the orbit $\Gamma = \{(x, y, z) \in \mathbb{D}^3 \ : \ y = z = 0\}$ and $\Lambda = \mathbb{S}^2 \setminus \{e^+, e^-\}$. Since on $\mathbb{S}^2$ the singular point e^+ is unstable (see condition (C_3)) and consequently e^- is stable, condition (C_4) together with the Poincaré–Bendixon Theorem on $\mathbb{S}^2$ [6] assure that every orbit on Λ starts at e^+ and ends at e^-. In fact, Λ is the 2–dimensional unstable manifold of e^+ (i.e. $W^u_{e^+}$) which coincides with the 2–dimensional stable manifold of e^- (i.e. $W^s_{e^-}$). Moreover, Γ is formed by a unique orbit starting at e^- and ending at e^+; i.e. Γ is the 1–dimensional stable manifold of e^+ (i.e. $W^s_{e^+}$) which coincides with the 1–dimensional unstable manifold of e^- (i.e. $W^u_{e^-}$). From condition (C_1), the flow on the Poincaré ball $\mathbb{D}^3$ is symmetric with respect to the plane $x = 0$. With this dynamics and symmetry we can prove the existence of a continuous of symmetric periodic orbits of X near the generalized heteroclinic loop $\mathcal{L}$. Our main results are the following two theorems.

Theorem 1 *Let X be a polynomial vector field in $\mathbb{R}^3$ of even degree satisfying conditions (C_1)–(C_4). For $\varepsilon > 0$, let D_ε denote the punctured disc $\{(x, y, z) \ : \ x = 0,\ 0 < y^2 + z^2 < \varepsilon^2\}$. Then there exists ε sufficiently small such that all the solutions of X having initial conditions $(x(0), y(0), z(0)) = (0, y_0, z_0) \in D_\varepsilon$ are periodic solutions near the generalyzed heteroclinic loop $\mathcal{L}$.*

The proof of Theorem 1 is given in Section 2.

The class $\mathcal{C}$ of polynomial vector fields satisfying conditions (C_1)–(C_4) is not empty. In this paper we characterize the polynomial vector fields of the class $\mathcal{C}$ having degree 2. In the statement of the next theorem the sets of conditions form (i) to (xi) are given in Subsection 3.3.

Theorem 2 *Let X be the polynomial vector field*

$$(a_0+a_1\,y+a_2\,z+a_3\,x^2+a_4\,y^2+a_5\,y\,z+a_6\,z^2, b_1\,x\,y+b_2\,x\,z, c_1\,x\,y+c_2\,x\,z).$$

(a) *Suppose that e^+ and e^- are foci and that $a_6 < 0$. Then X satisfies conditions (C_1)–(C_4) if and only if either (i) or (ii) holds.*

(b) *Assume that e^+ and e^- are nodes. Then X satisfies conditions (C_1)–(C_4) if and only if either (iii), or (iv), ... , or (xi) holds.*

(c) *There are vector fields X satisfying the sets of conditions either (i) or (ii) of statement (a) with $a_6 < 0$, and satisfying the sets of conditions either (iii), or (iv), ... , or (xi) of statement (b).*

In Section 3 we prove Theorem 2.

We note that for polynomial vector fields that are invariant under symmetry (C_1) its periodic orbits in a neighborhood of a heteroclinic loop formed by the same orbit Γ, the points e^+ and e^- but being saddles instead of foci when we restrict them at $\mathbb{S}^2$, and an orbit at infinity connecting e^+ and e^-, have been studied in [2]. Note that this heteroclinic loop has only invariant manifolds of dimension 1.

We can consider a polynomial vector field satisfying conditions (C_2)–(C_4) and being invariant under a symmetry with respect to the straight line $x = z = 0$ instead of being invariant by a symmetry with respect to the plane $x = 0$ (condition (C_1)). In this case, in order to assure the existence of symmetric periodic orbits using similar techniques than the ones used in this paper, we would need that e^+ be a focus on $\mathbb{S}^2$. If we could find a polynomial vector field satisfying those conditions, we could prove the existence of a discrete set of infinitely many symmetric periodic orbits near the generalized heteroclinic loop $\mathcal{L}$. Unfortunately there are no polynomial vector fields of even degree satisfying those conditions (see Appendix 2). Nevertheless in [3] you can find an example of a non–polynomial vector field exhibiting this dynamics.

2 Proof of Theorem 1

Recall that $\Gamma = \{(x, y, z) \in \mathbb{D}^3 \,:\, y = z = 0\}$ and $\Lambda = \mathbb{S}^2 \setminus \{e^+, e^-\}$ where $\mathbb{S}^2$ denotes the boundary of the Poincaré ball $\mathbb{D}^3$.

From condition (C_1), the vector field X is invariant under symmetry $(x, y, z, t) \longrightarrow (-x, y, z, -t)$, this means that if $\phi(t) = (x(t), y(t), z(t))$

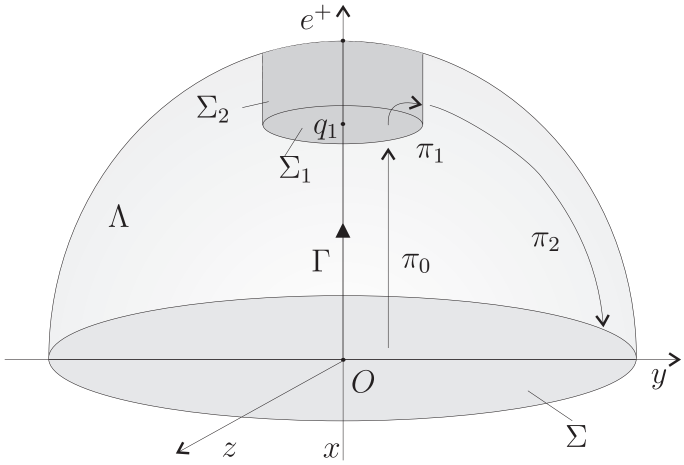

Figure 1: The map π.

is an orbit of X, then $\psi(t) = (-x(-t), y(-t), z(-t))$ is also an orbit. This symmetry can be used in order to obtain symmetric periodic orbits in the following way. Using the symmetry and the uniqueness theorem on the solutions of the differential system associated to X it is easy to see that if $x(0) = 0$, then the orbits $\phi(t)$ and $\psi(t)$ must be the same. Moreover, if there exists a time $\tau > 0$ such that $x(\tau) = 0$ and $x(\tau) \neq 0$ for all $0 < t < \tau$, then the orbit is periodic with period 2τ. In other words, if an orbit intersects the plane of symmetry $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \ : \ x = 0\}$ in two different points, then it is a symmetric periodic orbit.

We start giving some definitions and notations. Assume that $\varepsilon_1, \varepsilon_2 > 0$ are sufficiently small. We consider a small topological cylinder in a neighbourhood of the equilibrium point e^+, with base on Λ and boundaries Σ_1 and Σ_2, see Figure 1. The expression of Σ_1 and Σ_2 on the local chart U_1 are given by $\Sigma_1 = \{(z_1, z_2, z_3) \in \mathrm{Int}(\mathbb{D}^3) \ : z_3 = \varepsilon_1, z_1^2 + z_2^2 \leqslant \varepsilon_2\}$ and $\Sigma_2 = \{(z_1, z_2, z_3) \in \mathrm{Int}(\mathbb{D}^3) \ : z_3 \leqslant \varepsilon_1, z_1^2 + z_2^2 = \varepsilon_2\}$, respectively.

We define a map $\pi : \Sigma \longrightarrow \Sigma$ in the following way. We denote by $\varphi(t, q)$ the flow generated by X, satisfying $\varphi(0, q) = q$. We consider the diffeomorphism $\pi_0 : \Sigma \to \Sigma_1$ defined by $\pi_0(q) = p$, where p is the point at which the orbit $\varphi(t, q)$ intersects the cross section Σ_1 for the first time. By the continuity of the flow φ with respect to initial conditions, if q is sufficiently close to the origin, then the orbit $\varphi(t, q)$

is close to the orbit Γ for all t in a finite interval of time. Since the orbit Γ expends a finite time for going from the origin to $q_1 = \Gamma \cap \Sigma_1$, we can guarantee that for q sufficiently close to the origin the orbit $\varphi(t, q)$ intersects Σ_1. Consequently π_0 is well defined in a sufficiently small neighbourhood of the origin O in Σ.

We consider a second diffeomorphism $\pi_1 : \Sigma_1 \longrightarrow \Sigma_2$ defined by $\pi_1(q) = p$, where p is the point at which the orbit $\varphi(t, q)$ intersects Σ_2 for the first time. If ε_2 is sufficiently small, then the orbit $\varphi(t, q)$ intersects Σ_2 for all $q \in \Sigma_1 \setminus \{q_1\}$, because e^+ is a hyperbolic equilibrium point with $W^u_{e^+} = \Lambda$ and $W^s_{e^+} = \Gamma$.

We define a third map $\pi_2 : \Sigma_2 \longrightarrow \Sigma$, defined by $\pi_2(q) = p$, where p is the point at which the orbit $\varphi(t, q)$ intersects Σ for the first time. Since e^+ is the α–limit of all the orbits of X on Λ, if ε_1 and ε_2 are sufficiently small then the point p is well defined.

Finally, we consider the map $\pi : \Sigma \longrightarrow \Sigma$ defined by $\pi = \pi_2 \circ \pi_1 \circ \pi_0$. Clearly, if $\varepsilon > 0$ is sufficiently small, then the map π is well defined on all D_ε. Since $\pi(D_\varepsilon) \subset \Sigma$, all the orbits of X passing through points of D_ε intersect the plane of symmetry Σ at two different points one near the origin and the other one near Λ. Therefore all the points of D_ε correspond to initial conditions of symmetric periodic orbits of X, which proves Theorem 1.

3 Proof of Theorem 2

We consider an arbitrary polynomial vector field $X = (P, Q, R)$ of degree $n = 2$ in $\mathbb{R}^3$ with

$$\begin{aligned}
P &= \sum_{0 \leqslant i+j+k \leqslant n} a_{ijk}\, x^i y^j z^k \ , \\
Q &= \sum_{0 \leqslant i+j+k \leqslant n} b_{ijk}\, x^i y^j z^k \ , \\
R &= \sum_{0 \leqslant i+j+k \leqslant n} c_{ijk}\, x^i y^j z^k \ .
\end{aligned} \tag{1}$$

Now we analyze the conditions on the coefficients that we obtain after imposing conditions (C_1)–(C_4) to the vector field X.

3.1 Conditions (C_1)–(C_2)

Assuming that the straight line $y = z = 0$ is invariant under the flow of X we have that

$$b_{000} = b_{100} = b_{200} = c_{000} = c_{100} = c_{200} = 0\ .$$

Imposing that the system associated to X is invariant under the symmetry (C_1) we get that

$$a_{100} = a_{110} = a_{101} = b_{010} = b_{020} = b_{001} = b_{011} = b_{002} = c_{010} = \\ c_{020} = c_{001} = c_{011} = c_{002} = 0\ .$$

After imposing these conditions, the system associated to the quadratic vector field X becomes

$$\begin{aligned} \dot{x} &= a_0 + a_1\, y + a_2\, z + a_3\, x^2 + a_4\, y^2 + a_5\, y\, z + a_6\, z^2, \\ \dot{y} &= b_1\, x\, y + b_2\, x\, z, \\ \dot{z} &= c_1\, x\, y + c_2\, x\, z. \end{aligned} \tag{2}$$

We want that the straight line $y = z = 0$ does not contain any singular point and the flow on it goes in the increasing direction of the x–axis. The flow on this straight line is given by $\dot{x} = a_0 + a_3\, x^2$, then the coefficients a_0 and a_3 must satisfy that $a_0 > 0$ and $a_3 \geqslant 0$.

3.2 Conditions (C_3)–(C_4)

Now we analyze system (2) at infinity. The expression of system $p(X)$ (the Poincaré compactification of X) in the local chart U_1 is

$$\begin{aligned} \dot{z}_1 &= (b_1 - a_3)\, z_1 + b_2\, z_2 - a_4\, z_1{}^3 - a_5\, z_1{}^2\, z_2 - a_6\, z_1\, z_2{}^2 - \\ &\quad a_1\, z_1{}^2\, z_3 - a_2\, z_1\, z_2\, z_3 - a_0\, z_1\, z_3{}^2\ , \\ \dot{z}_2 &= c_1\, z_1 + (c_2 - a_3)\, z_2 - a_4\, z_1{}^2\, z_2 - a_5\, z_1\, z_2{}^2 - a_6\, z_2{}^3 - \\ &\quad a_1\, z_1\, z_2\, z_3 - a_2\, z_2{}^2\, z_3 - a_0\, z_2\, z_3{}^2\ , \\ \dot{z}_3 &= -a_3\, z_3 - a_4\, z_1{}^2\, z_3 - a_5\, z_1\, z_2\, z_3 - a_6\, z_2{}^2\, z_3 - a_1\, z_1\, z_3{}^2 - \\ &\quad a_2\, z_2\, z_3{}^2 - a_0\, z_3{}^3\ . \end{aligned} \tag{3}$$

Clearly the origin e^+ of the chart U_1 is a singular point. The linear part of (3) at the origin has the matrix

$$M = \begin{pmatrix} b_1 - a_3 & b_2 & 0 \\ c_1 & c_2 - a_3 & 0 \\ 0 & 0 & -a_3 \end{pmatrix}\ .$$

Let $\beta = b_1 + c_2 - 2\,a_3$ and let $\Delta = 4\,b_2\,c_1 + (b_1 - c_2)^2$. Then the eigenvalues of M are

$$\mu_1 = -a_3\ , \qquad \mu_{2,3} = \frac{1}{2}\left(\beta \pm \sqrt{\Delta}\right)\ .$$

We assume that $a_0 > 0$ and $a_3 \geqslant 0$ (these are the conditions on the coefficients obtained in Section 3.1). First we impose condition (C_3); that is, we impose that e^+ is a hyperbolic singular point which restricted to the boundary $\mathbb{S}^2$ of the Poincaré ball is either an unstable focus or node. Hence $a_3 > 0$. Since $\mathbb{S}^2$ is invariant under the flow of $p(X)$, if $\Delta < 0$ and $\beta > 0$, then e^+ is a hyperbolic singular point which is an unstable focus on $\mathbb{S}^2$; whereas if $\Delta \geqslant 0$ and $\mu_{2,3} > 0$, then e^+ is a hyperbolic singular point which is an unstable node on $\mathbb{S}^2$.

Here we characterize the sign of Δ. Let $\alpha_1 = -(b_1 - c_2)^2/(4\,c_1)$. It is easy to see that if $c_1 > 0$, then $\Delta < 0$ if $b_2 < \alpha_1$; $\Delta = 0$ if $b_2 = \alpha_1$; and $\Delta > 0$ if $b_2 > \alpha_1$. If $c_1 < 0$, then $\Delta < 0$ if $b_2 > \alpha_1$; $\Delta = 0$ if $b_2 = \alpha_1$ and $\Delta > 0$ if $b_2 < \alpha_1$. Finally, if $c_1 = 0$, then $\Delta = 0$ when $b_1 = c_2$; and $\Delta > 0$ when $b_1 \neq c_2$.

Now we study the sign of the real part of the eigenvalues $\mu_{2,3}$. We need that $\mathrm{Re}(\mu_{2,3}) > 0$, so $\beta > 0$. Let $\alpha_2 = (a_3 - b_1)(a_3 - c_2)/c_1$. Assuming that $\beta > 0$, $\mu_3 = 0$ if and only if $b_2 = \alpha_2$ and μ_2 is always positive. We can see that $\alpha_2 > \alpha_1$ for all $\beta \neq 0$ and $c_1 > 0$; $\alpha_2 < \alpha_1$ for all $\beta \neq 0$ and $c_1 < 0$; and $\alpha_1 = \alpha_2$ only when $\beta = 0$. Moreover, if $c_1 = 0$, then $\mu_{2,3} > 0$ if and only if $b_1 > a_3$ and $c_2 > a_3$. In short, analyzing the sign of Δ and of $\mathrm{Re}(\mu_{2,3}) > 0$, we have that e^+ is a hyperbolic singular point which is either an unstable focus or node at infinity if and only if the coefficients must satisfy one of the following conditions

(p.1) $a_0, a_3 > 0$, $c_1 > 0$, $\Delta < 0$, $\beta > 0$ and $b_2 < \alpha_1$;

(p.2) $a_0, a_3 > 0$, $c_1 < 0$, $\Delta < 0$, $\beta > 0$ and $b_2 > \alpha_1$;

(p.3) $a_0, a_3 > 0$, $c_1 > 0$, $\Delta \geqslant 0$, $\beta > 0$ and $b_2 \in [\alpha_1, \alpha_2)$;

(p.4) $a_0, a_3 > 0$, $c_1 < 0$, $\Delta \geqslant 0$, $\beta > 0$ and $b_2 \in (\alpha_2, \alpha_1]$;

(p.5) $a_0, a_3 > 0$, $c_1 = 0$, $b_1 > a_3$ and $c_2 > a_3$.

From condition (C_4) the vector field $p(X)$ cannot have singular points on Λ. Therefore we impose that $p(X)$ has no singular points neither on the local chart U_2, nor on the straight line $z_2 = z_3 = 0$ in the local chart U_3.

The expression of $p(X)$ in the local chart U_2 is given by

$$\begin{aligned}
\dot{z}_1 &= a_4 + a_5\, z_2 + a_1\, z_3 + (a_3 - b_1)\, {z_1}^2 - b_2\, {z_1}^2\, z_2 + a_6\, {z_2}^2 + \\
&\quad a_2\, z_2\, z_3 + a_0\, {z_3}^2 \ , \\
\dot{z}_2 &= c_1\, z_1 + (c_2 - b_1)\, z_1\, z_2 - b_2\, z_1\, {z_2}^2 \ , \\
\dot{z}_3 &= -b_1\, z_1\, z_3 - b_2\, z_1\, z_2\, z_3 \ ;
\end{aligned}$$

and the expression of $p(X)$ in the local chart U_3 is given by

$$\begin{aligned}
\dot{z}_1 &= a_6 + a_5\, z_2 + a_2\, z_3 + (a_3 - c_2)\, {z_1}^2 - c_1\, {z_1}^2\, z_2 + a_4\, {z_2}^2 + \\
&\quad a_1\, z_2\, z_3 + a_0\, {z_3}^2 \ , \\
\dot{z}_2 &= b_2\, z_1 + (b_1 - c_2)\, z_1\, z_2 - c_1\, z_1\, {z_2}^2 \ , \\
\dot{z}_3 &= -c_2\, z_1\, z_3 - c_1\, z_1\, z_2\, z_3 \ .
\end{aligned}$$

The singular points of $p(X)$ at infinity in the local chart U_2 are given by the solutions of system

$$\begin{aligned}
a_4 + a_5\, z_2 + (a_3 - b_1)\, {z_1}^2 - b_2\, {z_1}^2\, z_2 + a_6\, {z_2}^2 &= 0 \ , \\
z_1 \big(c_1 + (c_2 - b_1)\, z_2 - b_2\, {z_2}^2 \big) &= 0 \ .
\end{aligned} \tag{4}$$

From the last equation of (4) we have that either $z_1 = 0$, or

$$c_1 + (c_2 - b_1)\, z_2 - b_2\, {z_2}^2 = 0 \ . \tag{5}$$

If $z_1 = 0$, then the first equation of (4) becomes

$$a_4 + a_5\, z_2 + a_6\, z_2^2 = 0, \tag{6}$$

so if we want that system (4) has no real solutions, then we need that one of the following two conditions be satisfied

(q.1) $\Delta_1 = a_5^2 - 4\, a_4\, a_6 < 0$;

(q.2) $a_5 = a_6 = 0$ and $a_4 \neq 0$.

Notice that if $a_6 = a_5 = a_4 = 0$, then $z_1 = 0$ restricted to $\mathbb{S}^2$ is a straight line of singular points.

The solutions of equation (5) are $z_2 = \zeta^{\pm} = \left(c_2 - b_1 \pm \sqrt{\Delta} \right) / (2\, b_2)$ when $b_2 \neq 0$; $z_2 = \zeta = c_1/(b_1 - c_2)$ when $b_2 = 0$ and $b_1 \neq c_2$; and finally (5) has no real solutions when

(r.1) $b_2 = 0$ and $b_1 = c_2$.

First we consider the case $b_2 \neq 0$. Notice that if

(r.2) $\Delta < 0$, $b_2 \neq 0$,

then (5) has no real solutions, therefore (4) has no real solutions. Now we analyze the case $\Delta \geqslant 0$. Let $\xi_1 = a_6\, {b_1}^2 - a_5\, b_1\, b_2 + 2\, a_4\, {b_2}^2 + 2\, a_6\, b_2\, c_1 - 2\, a_6\, b_1\, c_2 + a_5\, b_2\, c_2 + a_6\, {c_2}^2$, $\xi_2 = a_6\, b_1 - a_5\, b_2 - a_6\, c_2$, $\eta_1 = \xi_1 - \xi_2\sqrt{\Delta}$ and $\eta_2 = \xi_1 + \xi_2\sqrt{\Delta}$. The first equation of (4) is equivalent to equation $2\,\mu_2\, b_2^2\, z_1^2 - \eta_1 = 0$ when $z_2 = \zeta^+$, and to $2\,\mu_3\, b_2^2\, z_1^2 - \eta_2 = 0$ when $z_2 = \zeta^-$. Since $\mu_{2,3} > 0$, the first equation of (4) has no real solutions when

(r.3) $\Delta \geqslant 0$, $b_2 \neq 0$, $\eta_1 < 0$ and $\eta_2 < 0$.

Now we analyze the case $b_2 = 0$ and $b_1 \neq c_2$. Notice that when $b_2 = 0$ the eigenvalues of M are $\mu_2 = b_1 - a_3$ and $\mu_3 = c_2 - a_3$ when $b_1 > c_2$, and $\mu_2 = c_2 - a_3$ and $\mu_3 = b_1 - a_3$ when $b_1 < c_2$. Let $\eta_3 = a_4\, {b_1}^2 + a_5\, b_1\, c_1 + a_6\, {c_1}^2 - 2\, a_4\, b_1\, c_2 - a_5\, c_1\, c_2 + a_4\, {c_2}^2$. If $z_2 = \zeta$, then the first equation of (4) is equivalent to equation $\mu_2\,(b_1 - c_2)^2\, z_1^2 - \eta_3 = 0$ when $b_1 > c_2$, and to $\mu_3\,(b_1 - c_2)^2\, z_1^2 - \eta_3 = 0$ when $b_1 < c_2$. Since $\mu_{2,3} > 0$, then the first equation of system (4) has no real solutions when

(r.4) $b_2 = 0$, $b_1 \neq c_2$ and $\eta_3 < 0$.

The singular points of $p(X)$ on the straight line $z_2 = z_3 = 0$ in the local chart U_3 are given by the solutions of system

$$\begin{aligned} a_6 + (a_3 - c_2)\, z_1^2 &= 0\ , \\ b_2\, z_1 &= 0\ . \end{aligned} \tag{7}$$

Since $\mathrm{Re}(\mu_{2,3}) > 0$ we have that if $b_2 = 0$, then $a_3 - c_2 < 0$. Therefore, analyzing the cases for which (7) has no real solutions we obtain that the coefficients must satisfy one of the conditions

(s.1) $b_2 \neq 0$ and $a_6 \neq 0$;

(s.2) $b_2 = 0$, $a_6 < 0$.

Notice that conditions (s.$*$) are incompatible with condition (q.2). So from now on condition (q.1) always holds.

We have just seen which conditions must verify the coefficients of X in order that $p(X)$ has no singular points on Λ. Now we impose that $p(X)$ has no periodic orbits on Λ.

Assume that $\Delta \geqslant 0$. Then it is easy to see that the vector field

$p(X)$ in the local chart U_1 possesses one or two (it depends on the values of the coefficients) invariant straight lines passing through the origin, they are given by

$$
\begin{aligned}
&2b_2 z_2 - (c_2 - b_1 \pm \sqrt{\Delta}) z_1 = 0 \text{ when } \Delta > 0 \text{ and } b_2 \neq 0\ ,\\
&2b_2 z_2 - (c_2 - b_1) z_1 = 0 \text{ when } \Delta = 0 \text{ and } b_2 \neq 0\ ,\\
&(b_1 - c_2) z_2 - c_1 z_1 = 0 \text{ and } z_1 = 0 \text{ when } \Delta > 0 \text{ and } b_2 = 0\ ,\\
&z_1 = 0 \text{ when } \Delta = 0 \text{ and } b_2 = 0\ .
\end{aligned}
$$

Each invariant straight line gives an invariant maximal circle under the flow of $p(X)$ on $\mathbb{S}^2$ that passes through the singular points e^+ and e^-. Since any periodic orbit on $\mathbb{S}^2$ must surround the singular point e^+ (or equivalently e^-), then the existence of such invariant maximal circles guarantees that $p(X)$ has no periodic orbits on Λ.

Now we study the case $\Delta < 0$. Let $q(X)$ be the vector field $p(X)$ restricted to $\mathbb{S}^2$. If the divergence $\operatorname{div}(q(X))$ on the local chart U_1 is either no–negative or no–positive for all $z_1, z_2 \in \mathbb{R}$, then we can assure that $q(X)$ has no periodic orbits strictly contained on the local chart U_1.

The divergence $\operatorname{div}(q(X))$ on the local chart U_1 is

$$
\operatorname{div}(q(X)) = \beta - 4\, a_4\, z_1^2 - 4\, a_5\, z_1\, z_2 - 4\, a_6\, z_2^2\ . \tag{8}
$$

Solving equation $\operatorname{div}(q(X)) = 0$ with respect to the variable z_2 we obtain $z_2 = (a_5\, z_1 \pm \sqrt{\Delta_2})/(2\, a_6)$, where $\Delta_2 = \beta\, a_6 + \Delta_1\, z_1^2$. Condition (q.1) says that $\Delta_1 < 0$ and consequently $a_6 \neq 0$. Moreover $\beta > 0$. Therefore if $a_6 < 0$, then $\Delta_2 < 0$ for all z_1. In short, if

(t.1) $\Delta < 0$, (q.1) and $a_6 < 0$,

then expression (8) is different from zero for all z_1, z_2. Notice that condition (t.1) implies that $a_4 < 0$.

There could be periodic orbits of $q(X)$ that are not strictly contained on the local chart U_1. In order to prove that such periodic orbits do not exist, we will see that the flow of $q(X)$ on the maximal circle that separates de local charts U_1 and V_1 is transversal and it goes from U_1 to V_1. It is sufficient to prove that the flow on the z_2 axis of the local chart U_2 is transversal, i.e. $\dot{z}_1$ is negative for all $(0, z_2, 0)$; and that the flow on the origin of the local chart U_3 satisfies that $\dot{z}_1$ is negative.

On the z_2 axis of the local chart U_2 we have that

$$
\dot{z}_1|_{(z_1=0,\, z_3=0)} = a_6\, z_2^2 + a_5\, z_2 + a_4\ .
$$

Note that $\dot{z}_1|_{(z_1=0,\, z_3=0)} = 0$ if and only if $z_2 = (-a_5 \pm \sqrt{\Delta_1})/(2\, a_6)$. Therefore from condition (t.1), $\dot{z}_1|_{z_1=0,\, z_3=0} < 0$ for all z_2.

On the origin of the local chart U_3 we have that

$$\dot{z}_1|_{(z_1=0,\, z_2=0,\, z_3=0)} = a_6 \ .$$

From condition (t.1) again, $\dot{z}_1|_{(z_1=0,\, z_2=0,\, z_3=0)} < 0$.

In short if $\Delta < 0$ and condition (t.1) is satisfied, then there are no periodic orbits on Λ strictly contained in the local chart U_1 (consequently there are no periodic orbits strictly contained in the local chart V_1) and there are no periodic orbits passing through the maximal circle that separates de local charts U_1 and V_1, because the flow on it is transversal and goes from U_1 to V_1. Therefore there are no periodic orbits of $p(X)$ on Λ.

In short, when $\Delta \geqslant 0$, conditions (C_1)–(C_4) are satisfied if and only if the coefficients of X satisfy one of the conditions (p.$*$), the condition (q.1), one of the conditions (r.$*$) and one of the conditions (s.$*$). When $\Delta > 0$, if the coefficients of X satisfy one of the conditions (p.$*$), one of the conditions (r.$*$), one of the conditions (s.$*$), and the condition (t.1), then conditions (C_1)–(C_4) are satisfied.

3.3 Summary

Analyzing all the possible combinations of the conditions on the coefficients of X obtained in the previous section we have that when the singular point e^+ is an unstable focus on $\mathbb{S}^2$ and $a_6 < 0$, conditions (C_1)–(C_4) are satisfied if and only if one of the following two conditions is satisfied. Notice that if $b_2 = 0$ then $\Delta > 0$ when $b_1 \neq c_2$, and $\Delta = 0$ when $b_1 = c_2$.

(i) $a_0, a_3 > 0$, $\Delta < 0$, $c_1 > 0$, $\beta > 0$, $b_2 < \alpha_1$, $b_2 \neq 0$, $\Delta_1 < 0$;

(ii) $a_0, a_3 > 0$, $\Delta < 0$, $c_1 < 0$, $\beta > 0$, $b_2 > \alpha_1$, $b_2 \neq 0$, $\Delta_1 < 0$.

The necessary and sufficient conditions in order that conditions (C_1)–(C_4) be satisfied when the singular point e^+ is an unstable node on $\mathbb{S}^2$ are the following.

(iii) $a_0, a_3 > 0$, $\Delta \geqslant 0$, $c_1 > 0$, $\beta > 0$, $b_2 \in [\alpha_1, \alpha_2)$, $\Delta_1 < 0$, $b_2 \neq 0$, $\eta_1 < 0$, $\eta_2 < 0$, $a_6 \neq 0$;

(iv) $a_0, a_3 > 0$, $\Delta \geqslant 0$, $c_1 > 0$, $\beta > 0$, $b_2 = 0 \in [\alpha_1, \alpha_2)$, $\Delta_1 < 0$, $b_1 \neq c_2$, $\eta_3 < 0$, $a_6 < 0$;

(v) $a_0, a_3 > 0$, $\Delta \geqslant 0$, $c_1 > 0$, $\beta > 0$, $b_2 = 0 \in [\alpha_1, \alpha_2)$, $\Delta_1 < 0$, $b_1 = c_2$, $a_6 < 0$;

(vi) $a_0, a_3 > 0$, $\Delta \geqslant 0$, $c_1 < 0$, $\beta > 0$, $b_2 \in (\alpha_2, \alpha_1]$, $\Delta_1 < 0$, $b_2 \neq 0$, $\eta_1 < 0$, $\eta_2 < 0$, $a_6 \neq 0$;

(vii) $a_0, a_3 > 0$, $\Delta \geqslant 0$, $c_1 < 0$, $\beta > 0$, $b_2 = 0 \in (\alpha_2, \alpha_1]$, $\Delta_1 < 0$, $b_1 \neq c_2$, $\eta_3 < 0$, $a_6 < 0$;

(viii) $a_0, a_3 > 0$, $\Delta \geqslant 0$, $c_1 < 0$, $\beta > 0$, $b_2 = 0 \in (\alpha_2, \alpha_1]$, $\Delta_1 < 0$, $b_1 = c_2$, $a_6 < 0$;

(ix) $a_0, a_3 > 0$, $c_1 = 0$, $b_1 > a_3$, $c_2 > a_3$, $\Delta_1 < 0$, $b_2 \neq 0$, $\eta_1 < 0$, $\eta_2 < 0$;

(x) $a_0, a_3 > 0$, $c_1 = 0$, $b_1 > a_3$, $c_2 > a_3$, $\Delta_1 < 0$, $b_2 = 0$, $b_1 \neq c_2$, $\eta_3 < 0$, $a_6 < 0$;

(xi) $a_0, a_3 > 0$, $c_1 = 0$, $b_1 = c_2 > a_3$, $\Delta_1 < 0$, $b_2 = 0$, $a_6 < 0$.

Next we give examples of coefficients satisfying each one of these 11 condition sets.

(i) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 4$, $b_2 = -4$, $c_1 = 2$, $c_2 = -1$;

(ii) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 4$, $b_2 = 4$, $c_1 = -2$, $c_2 = -1$;

(iii) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 6$, $b_2 = -6$, $c_1 = 2$, $c_2 = -1$;

(iv) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 3$, $b_2 = 0$, $c_1 = 2$, $c_2 = 2$;

(v) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 2$, $b_2 = 0$, $c_1 = 2$, $c_2 = 2$;

(vi) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 6$, $b_2 = 6$, $c_1 = -2$, $c_2 = -1$;

(vii) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 3$, $b_2 = 0$, $c_1 = -2$, $c_2 = 2$;

(viii) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 2$, $b_2 = 0$, $c_1 = -2$, $c_2 = 2$;

(ix) $a_0 = 2$, $a_1 = -1$, $a_2 = 2$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1$, $a_6 = -2$, $b_1 = 3$, $b_2 = 1$, $c_1 = 0$, $c_2 = 2$;

(x) $a_0 = 2,\ a_1 = -1,\ a_2 = 2,\ a_3 = 1,\ a_4 = -1,\ a_5 = 1,\ a_6 = -2,\ b_1 = 3,\ b_2 = 0,\ c_1 = 0,\ c_2 = 2$;

(xi) $a_0 = 2,\ a_1 = -1,\ a_2 = 2,\ a_3 = 1,\ a_4 = -1,\ a_5 = 1,\ a_6 = -2,\ b_1 = 2,\ b_2 = 0,\ c_1 = 0,\ c_2 = 2$.

Appendix 1

A polynomial vector field X in $\mathbb{R}^n$ can be extended to a unique analytic vector field on the sphere $\mathbb{S}^n$. The technique for making such an extension is called the Poincaré compactification. The Poincaré compactification allows us to study the vector field in a neighbourhood of infinity which is represented by the equator $\mathbb{S}^{n-1}$ of the sphere $\mathbb{S}^n$. Poincaré introduced this technique for polynomial vector fields in $\mathbb{R}^2$, its extension to $\mathbb{R}^n$ can be found in [1]. Here we only consider the Poincaré compactification for polynomial vector fields in $\mathbb{R}^3$.

Let $X = (P^1, P^2, P^3)$ be a polynomial vector field in $\mathbb{R}^3$, let $x = (x_1, x_2, x_3)$ and let $m = \max\{\deg(P^1), \deg(P^2), \deg(P^3)\}$ be the degree of X.

We consider the unit sphere in $\mathbb{R}^4$, $\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : ||y|| = 1\}$, which is called the *Poincaré sphere*; and we consider the hyperplane $\Pi = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_4 = 1\}$ which is the tangent to $\mathbb{S}^3$ at the northern pole $(0,0,0,1)$. We note that Π is diffeomorphic to $\mathbb{R}^3$, then we identify $\mathbb{R}^3$ with Π. Let $H_+ = \{y \in \mathbb{S}^3 \ : \ y_4 > 0\}$ and $H_- = \{y \in \mathbb{S}^3 \ : \ y_4 < 0\}$ be the northern and southern hemispheres of $\mathbb{S}^3$, respectively.

We consider the central projections $f_+ : \Pi = \mathbb{R}^3 \longrightarrow H_+$ and $f_- : \Pi = \mathbb{R}^3 \longrightarrow H_-$, defined by $f_+(x) = (x_1, x_2, x_3, 1)/\Delta(x)$ and $f_-(x) = -(x_1, x_2, x_3, 1)/\Delta(x)$ respectively, where $\Delta(x) = (1 + \sum_{i=1}^{3} x_i^2)^{1/2}$. Through these central projections, $\mathbb{R}^3$ can be identified with the northern and southern hemispheres of $\mathbb{S}^3$ respectively. So the vector field X induces a vector field $\widetilde{X}$ in $H_+ \cup H_-$ defined by $\widetilde{X}(y) = (Df_+)_x X(x)$ when $y = f_+(x)$, and by $\widetilde{X}(y) = (Df_-)_x X(x)$ when $y = f_-(x)$.

We note that $\widetilde{X}(y)$ gives two copies of X one on the northern hemisphere H_+ and the other one on the southern hemisphere H_-. Moreover $\widetilde{X}(y)$ is defined in $H_+ \cup H_-$, but in general it is not defined on the equator $\mathbb{S}^2 = \{y \in \mathbb{S}^3 \ : \ y_4 = 0\}$ of $\mathbb{S}^3$. We can extend analytically the vector field $\widetilde{X}(y)$ to the whole sphere $\mathbb{S}^3$ in the following way

$$p(X)(y) = y_4^{m-1} \widetilde{X}(y).$$

The vector field $p(X)$ is called the *Poincaré compatification of* X.

The closed northern hemisphere is a closed ball of $\mathbb{R}^3$, called the *Poincaré ball* $\mathbb{D}^3$, its interior is diffeomorphic to $\mathbb{R}^3$ and its boundary $\mathbb{S}^2$ correspond to the infinity of $\mathbb{R}^3$. We note that the boundary of the Poincaré ball is invariant by the flow of $p(X)$. So $p(X)$ allows us to study the behaviour of X in a neighbourhood of infinity.

To compute the analytical expression for $p(X)$ we shall consider $\mathbb{S}^3$ as a differentiable manifold and we choose the eight coordinate neighbourhoods $U_i = \{y \in S^3 : y_i > 0\}$ and $V_i = \{y \in S^3 : y_i < 0\}$, for $i = 1, \ldots, 4$, with the corresponding coordinate maps $F_i : U_i \longrightarrow \mathbb{R}^3$ and $G_i : V_i \longrightarrow \mathbb{R}^3$ defined by

$$F_i(y) = G_i(y) = \frac{1}{y_i}\,\overline{y}_i = (z_1, z_2, z_3) \ ,$$

where $\overline{y}_i$ is the point (y_1, y_2, y_3, y_4) without the component y_i.

We do the computations of $p(X)$ on the local chart U_1. The coordinate map on U_1 is given by $F_1(y) = (y_2/y_1, y_3/y_1, y_4/y_1) = (z_1, z_2, z_3)$. We note that the map F_1 is the inverse of the central projection from the origin to the tangent space of $\mathbb{S}^3$ at the point $(1, 0, 0, 0)$. The expression of $p(X)$ in this local chart U_1 is given by $(DF_1)_y(p(X)(y))$, which after doing the computations becomes

$$\frac{z_3^m}{(\Delta z)^{m-1}} \left(-z_1 P^1 + P^2, -z_2 P^1 + P^3, -z_3 P^1\right) \ ,$$

where $P^i = P^i(1/z_3, z_1/z_3, z_2/z_3)$ and $\Delta z = (1 + \sum_{i=1}^3 z_i^2)^{1/2}$.

In a similar way we can deduce the expressions for $p(X)$ in the local charts U_2 and U_3. These are

$$\frac{z_3^m}{(\Delta z)^{m-1}} \left(-z_1 P^2 + P^1, -z_2 P^2 + P^3, -z_3 P^2\right) \ ,$$

where $P^i = P^i(z_1/z_3, 1/z_3, z_2/z_3)$, and

$$\frac{z_3^m}{(\Delta z)^{m-1}} \left(-z_1 P^3 + P^1, -z_2 P^3 + P^2, -z_3 P^3\right) \ ,$$

where $P^i = P^i(z_1/z_3, z_2/z_3, 1/z_3)$, respectively.

The expression for $p(X)$ in the local chart U_4 is $(\Delta z)^{1-m}(P^1, P^2, P^3)$ where $P^i = P^i(z_1, z_2, z_3)$. Finally, the expression for $p(X)$ in the local charts V_i is the same as in U_i multiplied by $(-1)^{m-1}$.

We note that with a convenient change of the time we shall omit the factor $1/(\Delta z)^{m-1}$ in the expressions of $p(X)$.

Appendix 2

We consider the polynomial vector field $X = (P, Q, R)$ of even degree n given by (1). The system associated to the vector field X in the local chart U_1 is given by

$$\begin{aligned}
\dot{z}_1 &= -\sum_{0\leqslant i+j+k\leqslant n} a_{ijk}\, z_1^{j+1}\, z_2^k\, z_3^{n-i-j-k} + \sum_{0\leqslant i+j+k\leqslant n} b_{ijk}\, z_1^j\, z_2^k\, z_3^{n-i-j-k}\,, \\
\dot{z}_2 &= -\sum_{0\leqslant i+j+k\leqslant n} a_{ijk}\, z_1^{j}\, z_2^{k+1}\, z_3^{n-i-j-k} + \sum_{0\leqslant i+j+k\leqslant n} c_{ijk}\, z_1^j\, z_2^k\, z_3^{n-i-j-k}\,, \\
\dot{z}_3 &= -\sum_{0\leqslant i+j+k\leqslant n} a_{ijk}\, z_1^{j}\, z_2^{k}\, z_3^{n-i-j-k+1}\,.
\end{aligned}$$

The lineal part of this system at the origin has the matrix

$$M = \begin{pmatrix} b_{n-1,1,0} - a_{n00} & b_{n-1,0,1} & b_{n-1,0,0} \\ c_{n-1,1,0} & c_{n-1,0,1} - a_{n00} & c_{n-1,0,0} \\ 0 & 0 & -a_{n00} \end{pmatrix}.$$

Assuming that the straight line $y = z = 0$ is invariant by the flow of X we have that $b_{i00} = c_{i00} = 0$ for all $i \in \{0, 1, \ldots, n\}$. Imposing that the system associated to X is invariant under the symmetry $(x, y, z, t) \longrightarrow (-x, y, -z, t)$, we get that $a_{ijk} = c_{ijk} = 0$ for all $i, k \in \{0, 1, \ldots, n\}$ such that $i + k$ is odd, and $b_{ijk} = 0$ for all $i, k \in \{0, 1, \ldots, n\}$ such that $i + k$ is even.

In this case $b_{n-1,0,1} = b_{n-1,0,0} = c_{n-1,1,0} = c_{n-1,0,0} = 0$, so the eigenvalues of M are given by

$$\mu_1 = -a_{n00}, \qquad \mu_2 = b_{n-1,1,0} - a_{n00}, \qquad \mu_3 = c_{n-1,0,1} - a_{n00}.$$

Notice that all the eigenvalues of M are real. Therefore in this case e^+ cannot be a focus on $\mathbb{S}^2$.

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