Central configurations of nested rotated regular tetrahedra

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Abstract

In this paper we prove that there are only two different classes of central configurations with convenient masses located at the vertices of two nested regular tetrahedra: either when one of the tetrahedra is a homothecy of the other one, or when one of the tetrahedra is a homothecy followed by a rotation of Euler angles $\alpha = \gamma = 0$ and $\beta = \pi$ of the other one.

We also analyze the central configurations with convenient masses located at the vertices of three nested regular tetrahedra when one them is a homothecy of the other one, and the third one is a homothecy followed by a rotation of Euler angles $\alpha = \gamma = 0$ and $\beta = \pi$ of the other two.

In all these cases we have assumed that the masses on each tetrahedron are equal but masses on different tetrahedra could be different.

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1 Introduction

The equations of motion of the N–body problem in the 3–dimensional Euclidean space are

$$m_i\,\ddot{\mathbf{q}}_i = -\sum_{j=1,\,j\neq i}^{N} G\,m_i\,m_j\,\frac{\mathbf{q}_i-\mathbf{q}_j}{|\mathbf{q}_i-\mathbf{q}_j|^3}\;, \qquad i=1,\ldots,N\;,$$

where $\mathbf{q}_i \in \mathbb{R}^3$ is the position vector of the punctual mass m_i in an inertial coordinate system and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. We fix the center of mass $\sum_{i=1}^{N} m_i\,\mathbf{q}_i / \sum_{i=1}^{N} m_i$ of the system at the origin of $\mathbb{R}^{3N}$. The *configuration space* of the N–body problem in $\mathbb{R}^3$ is

$$\mathcal{E} = \{(\mathbf{q}_1,\ldots,\mathbf{q}_N) \in \mathbb{R}^{3N}\ :\ \sum_{i=1}^{N} m_i\,\mathbf{q}_i = 0,\ \mathbf{q}_i \neq \mathbf{q}_j,\ \text{for } i\neq j\}\;.$$

Given positive masses $m_1,\ldots,m_N$ a configuration $(\mathbf{q}_1,\ldots,\mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_i = -\lambda\,\mathbf{q}_i\;, \qquad i=1,\ldots,N\;. \tag{1}$$

That is if the acceleration $\ddot{\mathbf{q}}_i$ of each point mass m_i is proportional to its position $\mathbf{q}_i$ relative to the center of mass of the system and is directed towards the center of mass. On a central configuration system (1) can be written as

$$\sum_{j=1,\,j\neq i}^{N} m_j\,\frac{\mathbf{q}_i-\mathbf{q}_j}{|\mathbf{q}_i-\mathbf{q}_j|^{3/2}} = \lambda\,\mathbf{q}_i\;, \qquad i=1,\ldots,N\;. \tag{2}$$

Therefore a central configuration $(\mathbf{q}_1,\ldots,\mathbf{q}_N) \in \mathcal{E}$ of the N–body problem with positive masses $m_1,\ldots,m_N$ is a solution of (2) with $\lambda > 0$.

Two central configurations in $\mathbb{R}^3$ are in the same *class* if there exists a rotation and a homothecy of $\mathbb{R}^3$ which transform one into the other.

A general rotation in the 3-dimensional Euclidean space having the origin of coordinates fixed is an element of $SO(3)$ that can be parametrized by using the Euler angles (α,β,γ) via the rotation matrix $\mathcal{R}$ given by

$$\begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi)$, see for more details [4].

Consider two regular tetrahedra with vertices $\mathbf{q}_i$ and $\mathbf{q}_{i+4}$ for $i = 1, \ldots, 4$ respectively. We say that these two tetrahedra are *nested* if they have the same center and the positions of the vertices of the two tetrahedra satisfy the relation $\mathbf{q}_{i+4} = \rho \mathcal{R} \mathbf{q}_i$ for all $i = 1, \ldots, 4$, for some *scale factor* ρ and for some rotation matrix $\mathcal{R}$. Notice that if $\rho = 1$, then $R \neq Id$ where Id denotes the identity matrix.

In a similar way if we consider three regular tetrahedra with vertices $\mathbf{q}_i$, $\mathbf{q}_{i+4}$ and $\mathbf{q}_{i+8}$ for $i = 1, \ldots, 4$ respectively, then these three tetrahedra are called *nested* if they have the same center and the positions of the vertices of the three tetrahedra satisfy the relations $\mathbf{q}_{i+4} = \rho \mathcal{R} \mathbf{q}_i$ and $\mathbf{q}_{i+8} = R \mathcal{R}' \mathbf{q}_i$ for all $i = 1, \ldots, 4$, for some *scale factors* ρ and R and for some rotation matrices $\mathcal{R}$ and $\mathcal{R}'$.

In [6] and in [2] it is proved that the configuration formed by 4 equal masses m_1 located at the vertices of a regular tetrahedron and 4 additional masses m_2 located at the vertices of a second nested regular tetrahedron is central for the 8–body problem when $\mathcal{R} = Id$ and the ratio of the masses m_2/m_1 and the scale factor ρ satisfy a convenient relation. In [5] it is proved that this kind of central configurations is also central when $\mathcal{R}$ is the rotation matrix of Euler angles $\alpha = 0$, $\beta = \pi$ and $\gamma = 0$. In Section 2 we summarize these results and we prove that these are the two unique classes of central configurations formed by two nested regular tetrahedra of the 8–body problem, see Theorem 2.

In Section 3 we provide a new class of central configurations of the 12–body problem consisting of three nested regular tetrahedra with one of them rotated with respect to the other two by a rotation matrix with Euler angles $\alpha = 0$, $\beta = \pi$ and $\gamma = 0$. Of course on each tetrahedron the four masses located on its vertices are equal, see Result 3. The central configurations of the 12–body problem consisting of three nested regular tetrahedra when $\mathcal{R} = \mathcal{R}' = Id$ have been described in [3].

2 Two nested tetrahedra

In this section we study the spatial central configurations of the 8–body problem with the masses located at the vertices of two nested regular tetrahedra. In particular we give all classes of central configurations formed by two nested regular tetrahedra of the 8–body problem

We assume that the masses on each tetrahedron are equal but the masses on different tetrahedra could be different. Taking conveniently the unit of masses we can assume that the masses of the tetrahedron with scale factor 1 are equal to one, that is $m_1 = m_2 = m_3 = m_4 = 1$, and the ones of the tetrahedron with scale factor ρ are equal to m, that is $m_5 = m_6 = m_7 = m_8 = m$. We also choose the unit of length in such a way that the edges of the tetrahedron with scale factor 1 have length 2.

Recall that the set of spatial central configurations is invariant under homothecies and rotations of $SO(3)$. So without loss of generality we can assume that $\rho \geqslant 1$ and that the positions of the vertices of the tetrahedron with scale factor 1 are $\mathbf{a}_1 = (0, 0, \sqrt{3/2})^T$, $\mathbf{a}_2 = (0, 2/\sqrt{3}, -1/\sqrt{6})^T$, $\mathbf{a}_3 = (1, -1/\sqrt{3}, -1/\sqrt{6})^T$ and $\mathbf{a}_4 = (-1, -1/\sqrt{3}, -1/\sqrt{6})^T$.

In what follows a *configuration of two nested regular tetrahedra* means the configuration consisting of four equal masses $m_1 = m_2 = m_3 = m_4 = 1$ at the vertices of a regular tetrahedron having the positions $\mathbf{a}_i$ for $i = 1, \ldots, 4$ and four additional masses $m_5 = m_6 = m_7 = m_8 = m$ at the vertices of a regular tetrahedron having positions $\mathbf{q}_{i+4} = \rho \mathcal{R} \mathbf{a}_i$ for $i = 1, \ldots, 4$, for some scale factor $\rho \geqslant 1$ and for some rotation matrix $\mathcal{R}$.

We say that two configurations of two nested regular tetrahedra with rotation matrices $\mathcal{R}$ and $\mathcal{R}'$ respectively are in the same *class* if $\mathcal{R}$ and $\mathcal{R}'$ are such that the set $\{\mathbf{q}_5, \mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_8\}$ is the same for $\mathcal{R}$ and $\mathcal{R}'$; that is, if $\{\mathcal{R}\mathbf{a}_1, \mathcal{R}\mathbf{a}_2, \mathcal{R}\mathbf{a}_3, \mathcal{R}\mathbf{a}_4\} = \{\mathcal{R}'\mathbf{a}_1, \mathcal{R}'\mathbf{a}_2, \mathcal{R}'\mathbf{a}_3, \mathcal{R}'\mathbf{a}_4\}$.

As usual $< \mathbf{u}, \mathbf{v} >= \mathbf{u}^T \mathbf{v}$ denotes the scalar product of the vectors $\mathbf{u}$ and $\mathbf{v}$.

Lemma 1 *Let*

$$f(i) = \sum_{j=1}^{4} \frac{3/2 - \rho < \mathbf{a}_i, \mathcal{R}\mathbf{a}_j >}{(3\rho^2/2 - 2\rho < \mathbf{a}_i, \mathcal{R}\mathbf{a}_j > +3/2)^{3/2}},$$
$$g(i) = \sum_{j=1}^{4} \frac{3\rho/2 - < \mathcal{R}\mathbf{a}_i, \mathbf{a}_j >}{(3\rho^2/2 - 2\rho < \mathcal{R}\mathbf{a}_i, \mathbf{a}_j > +3/2)^{3/2}},$$

for $i = 1, \ldots, 4$.

A configuration of two nested regular tetrahedra is central for the 8*–body problem if and only if one of the following statements hold.*

(a) $f(1) = f(2) = f(3) = f(4)$, $g(1) = g(2) = g(3) = g(4)$, $2f(1)/3 - 1/(2\rho^3) \neq 0$,

$$\lambda = 1/2 + 2mf(i)/3 > 0 \qquad \text{and} \qquad m = \frac{2g(1)/(3\rho) - 1/2}{2f(1)/3 - 1/(2\rho^3)} > 0.$$

(b) $f(1) = f(2) = f(3) = f(4)$, $g(1) = g(2) = g(3) = g(4)$, $2f(1)/3 - 1/(2\rho^3) =$

$2g(1)/(3\rho) - 1/2 = 0$ *and* $\lambda = 1/2 + 2mf(i)/3 > 0$ *and* $m > 0$.

PROOF. It is easy to check that at the configuration of two nested tetrahedra, system (2) can be written as

$$\sum_{j=1, j\neq i}^{4} \frac{\mathbf{a}_i - \mathbf{a}_j}{|\mathbf{a}_i - \mathbf{a}_j|^3} + m \sum_{j=1}^{4} \frac{\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j}{|\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j|^3} = \lambda \mathbf{a}_i \ , \tag{3}$$

$$\sum_{j=1}^{4} \frac{\rho\mathcal{R}\mathbf{a}_i - \mathbf{a}_j}{|\rho\mathcal{R}\mathbf{a}_i - \mathbf{a}_j|^3} + m \sum_{j=1, j\neq i}^{4} \frac{\rho\mathcal{R}\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j}{|\rho\mathcal{R}\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j|^3} = \lambda \rho\mathcal{R}\mathbf{a}_i \ , \tag{4}$$

for $i = 1, \ldots, 4$. We do the scalar product of the i–th equation of (3) with the vector $\mathbf{a}_i$ and the scalar product of the i–th equation of (4) with the vector $\mathcal{R}\mathbf{a}_i$ and we get

$$\sum_{j=1, j\neq i}^{4} \frac{< \mathbf{a}_i, \mathbf{a}_i > - < \mathbf{a}_i, \mathbf{a}_j >}{|\mathbf{a}_i - \mathbf{a}_j|^3} + m \sum_{j=1}^{4} \frac{< \mathbf{a}_i, \mathbf{a}_i > -\rho < \mathbf{a}_i, \mathcal{R}\mathbf{a}_j >}{|\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j|^3} = \lambda < \mathbf{a}_i, \mathbf{a}_i > \ , \tag{5}$$

$$\sum_{j=1}^{4} \frac{\rho < \mathcal{R}\mathbf{a}_i, \mathcal{R}\mathbf{a}_i > - < \mathcal{R}\mathbf{a}_i, \mathbf{a}_j >}{|\rho\mathcal{R}\mathbf{a}_i - \mathbf{a}_j|^3} + m \sum_{j=1, j\neq i}^{4} \frac{\rho < \mathcal{R}\mathbf{a}_i, \mathcal{R}\mathbf{a}_i > -\rho < \mathcal{R}\mathbf{a}_i, \mathcal{R}\mathbf{a}_j >}{|\rho\mathcal{R}\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j|^3} = \lambda\rho < \mathcal{R}\mathbf{a}_i, \mathcal{R}\mathbf{a}_i > \ , \tag{6}$$

for $i = 1, \ldots, 4$. It is easy to check that $< \mathbf{a}_i, \mathbf{a}_i > = < \mathcal{R}\mathbf{a}_i, \mathcal{R}\mathbf{a}_i > = 3/2$, $< \mathcal{R}\mathbf{a}_i, \mathcal{R}\mathbf{a}_j > = < \mathbf{a}_i, \mathbf{a}_j >$, $|\rho\mathcal{R}\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j|^3 = \rho^3 |\mathbf{a}_i - \mathbf{a}_j|^3$ and

$$\sum_{j=1, j\neq i}^{4} \frac{< \mathbf{a}_i, \mathbf{a}_i > - < \mathbf{a}_i, \mathbf{a}_j >}{|\mathbf{a}_i - \mathbf{a}_j|^3} = \frac{3}{4} \ ,$$

for all $i = 1, \ldots, 4$. On the other hand

$$|\mathbf{a}_i - \rho\mathcal{R}\mathbf{a}_j| = \left(3\rho^2/2 - 2\rho < \mathbf{a}_i, \mathcal{R}\mathbf{a}_j > +3/2\right)^{1/2} ,$$

$$|\rho\mathcal{R}\mathbf{a}_i - \mathbf{a}_j| = \left(3\rho^2/2 - 2\rho < \mathcal{R}\mathbf{a}_i, \mathbf{a}_j > +3/2\right)^{1/2} .$$

So equations (5) and (6) become

$$\frac{3}{4} + m\, f(i) = \frac{3}{2}\lambda \ , \qquad g(i) + m\frac{3}{4\rho^2} = \frac{3\rho}{2}\lambda \ , \tag{7}$$

for $i = 1, \ldots, 4$. In short, the configuration consisting of two nested tetrahedra is central if and only if λ and m is a solution of (7) satisfying that $\lambda > 0$ and $m > 0$.

From the first equation of (7) we have that $\lambda = 1/2+2mf(i)/3$ for $i = 1, \ldots, 4$, this implies that $f(1) = f(2) = f(3) = f(4)$. From the second equation of (7) we have $\lambda = 2g(i)/(3\rho) + m/(2\rho^3)$ for $i = 1, \ldots, 4$, so $g(1) = g(2) = g(3) = g(4)$. Equating both expressions of λ for $i = 1$ we have that if $2f(1)/3 - 1/(2\rho^3) \neq 0$, then

$$m = \frac{2g(1)/(3\rho) - 1/2}{2f(1)/3 - 1/(2\rho^3)}.$$

This proves statement (a).

Moreover if $2f(1)/3-1/(2\rho^3) = 2g(1)/(3\rho)-1/2 = 0$, then the solution of (7) is $\lambda = 1/2+2mf(i)/3 > 0$ and $m > 0$, which concludes the prove of statement (b).

Theorem 2 *The following statements hold.*

(a) There are two unique classes of central configurations of two nested regular tetrahedra.

(a.1) The class of configurations of Type I with $\{\mathbf{q}_5, \mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_8\} = \{\rho\mathbf{a}_1, \rho\mathbf{a}_2, \rho\mathbf{a}_3, \rho\mathbf{a}_4\}$.

(a.2) The class of configurations of Type II with $\{\mathbf{q}_5, \mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_8\} = \{\rho\mathcal{R}\mathbf{a}_1, \rho\mathcal{R}\mathbf{a}_2, \rho\mathcal{R}\mathbf{a}_3, \rho\mathcal{R}\mathbf{a}_4\}$ *and*

$$\mathcal{R} = P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These two classes of configurations are shown in Figure 1.

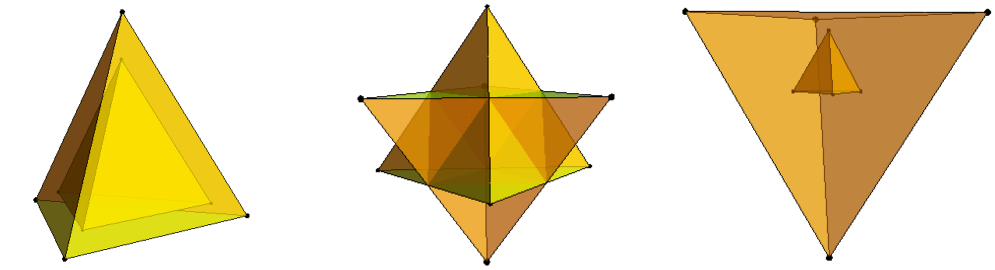
(b) (see [2,6]) The configuration of Type I is central for the spatial 8–body problem when

$$m = m_I(\rho) = \frac{n(\rho)}{d(\rho)} = \frac{\dfrac{(2/3)^{3/2}}{(\rho-1)^2} - \dfrac{\rho}{2} + \dfrac{2\sqrt{2}\,(3\rho+1)}{(3\rho^2+2\rho+3)^{3/2}}}{-\dfrac{1/2}{\rho^2} - \dfrac{(2/3)^{3/2}\rho}{(\rho-1)^2} + \dfrac{2\sqrt{2}\,\rho\,(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}}},$$

and $\rho > \alpha = 1.8899915758\ldots$, *where* α *is the unique real solution of* $n(\rho) = 0$ *for* $\rho > 1$. *Moreover, fixed a value of* $m > 0$ *there exists a unique* $\rho > \alpha$ *for which the configurations of Type I is central.*

(c) (see [5]) The configuration of Type II is central for the spatial 8–body problem when

$$m = m_{II}(\rho) = \frac{n(\rho)}{d(\rho)} = \frac{-\dfrac{\rho}{2} + \dfrac{2\sqrt{2}(3\rho-1)}{(3\rho^2-2\rho+3)^{3/2}} + \dfrac{(2/3)^{3/2}}{(\rho+1)^2}}{-\dfrac{2\sqrt{2}(\rho-3)\rho}{(3\rho^2-2\rho+3)^{3/2}} + \dfrac{(2/3)^{3/2}\rho}{(\rho+1)^2} - \dfrac{1}{2\rho^2}},$$



(a) Conf. of Type I
$\rho > 1$

(b) Conf. of Type II
$1 \leqslant \rho \leqslant 3$

(c) Conf. of Type II
$\rho > 3$

Fig. 1. Plot of the possible classes of configurations of two nested regular tetrahedra. Notice that we have two possibilities for the configurations of Type II, in (b) the faces of both tetrahedra intersect whereas in (c) they do not intersect.

and $\rho \in [1, \alpha_1) \cup (\alpha_2, \infty)$ *where* $\alpha_1 = 1.3981650369\ldots$ *is the unique real solution of* $n(\rho) = 0$ *with* $\rho \geqslant 1$ *and* $\alpha_2 = 6.5360793703\ldots$ *is the unique real solution of* $d(\rho) = 0$ *with* $\rho \geqslant 1$*. Moreover the following statements hold.*

(c.1) There is a unique central configuration of Type II with $\rho \in [1, \alpha_1)$ *for* $m \in (0, 1]$*. In this configuration the faces of both tetrahedra intersect.*

(c.2) Let $\alpha_3 = 8.7756058918\ldots$ *be the unique real solution of* $(n(\rho)/d(\rho))'$ *with* $\rho \geqslant 1$ *and* $m_0 = n(\alpha_3)/d(\alpha_3) = 2880.33\ldots$*. There are no central configurations for* $m \in (1, m_0)$*.*

(c.3) There is a unique central configuration of Type II with $\rho = \alpha_3$ *for* $m = m_0$*. In this configuration the faces of the tetrahedra do not intersect.*

(c.4) There are two central configurations of Type II, one with $\rho \in (\alpha_2, \alpha_3)$ *and the other with* $\rho \in (\alpha_3, \infty)$*, for* $m \in (m_0, \infty)$*. In this configuration the faces of the tetrahedra do not intersect.*

In [6,5] the authors also consider values of the scale factor $\rho \in (0, 1]$. The results for the values of ρ in $(0, 1]$ can be obtained from the ones given in Theorem 2 by replacing m by $1/m$ and ρ by $1/\rho$. In particular, fixed a value of $m > 0$ there exists a unique $0 < \rho < 1/\alpha = 0.5291028874\ldots$ for which the configurations of Type I is central. There is a unique central configuration of Type II with $\rho \in (1/\alpha_1, 1] = (0.7152231486\ldots, 1]$ for $m \in [1, +\infty)$. There is a unique central configuration of Type II with $\rho = 1/\alpha_3 = 0.1139522458\ldots$ for $m = 1/m_0 = \widetilde{m}_0 = 0.0003471823\ldots$. There are two central configurations of Type II, one with $\rho \in (1/\alpha_3, 1/\alpha_2) = (0.1139522458\ldots, 0.1529969280\ldots)$ and the other one with $\rho \in (0, 1/\alpha_3) = (0, 0.1139522458\ldots)$, for $m \in (0, \widetilde{m}_0)$.

PROOF. [Proof of Theorem 2] We know that the spatial central configurations are invariant under rotations of $SO(3)$. Consider the rotation

$$A = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ 1/(2\sqrt{3}) & -1/6 & 2\sqrt{2}/3 \\ \sqrt{2/3} & -\sqrt{2}/3 & -1/3 \end{pmatrix}.$$

It is easy to check that $A\mathbf{a}_1 = \mathbf{a}_2$, $A\mathbf{a}_2 = \mathbf{a}_3$, $A\mathbf{a}_3 = \mathbf{a}_1$ and $A\mathbf{a}_4 = \mathbf{a}_4$. Since the rotation A leaves invariant the tetrahedron with scale factor 1, it also must leave invariant the one with scale factor ρ. This means that the sets $\{\mathbf{q}_5, \mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_8\}$ and $\{A\mathbf{q}_5, A\mathbf{q}_6, A\mathbf{q}_7, A\mathbf{q}_8\}$ must be the same. Then either $A\mathbf{q}_5 = \mathbf{q}_5$, or $A\mathbf{q}_5 = \mathbf{q}_6$, or $A\mathbf{q}_5 = \mathbf{q}_7$, or $A\mathbf{q}_5 = \mathbf{q}_8$. On the other hand, by Lemma 1 the rotation matrix $\mathcal{R}$ provides a central configuration of two nested tetrahedra if $f(1) = f(2) = f(3) = f(4)$ and $g(1) = g(2) = g(3) = g(4)$.

We start analyzing the solutions of system $\mathbf{f}_5 = A\mathbf{q}_5 - \mathbf{q}_5 = \mathbf{0}$ where

$$\mathbf{f}_5 = \begin{pmatrix} -\dfrac{\sin\beta\left(\sqrt{3}\sin\gamma - 3\cos\gamma\right)}{2\sqrt{2}} \\ \dfrac{2\cos\beta}{\sqrt{3}} + \dfrac{1}{12}\sin\beta\left(3\sqrt{2}\sin\gamma - 7\sqrt{6}\cos\gamma\right) \\ \sin\beta\left(\sin\gamma - \dfrac{\cos\gamma}{\sqrt{3}}\right) - 2\sqrt{\dfrac{2}{3}}\cos\beta \end{pmatrix}.$$

From the first equation of $\mathbf{f}_5 = \mathbf{0}$ we have that either $\beta = 0$, or $\beta = \pi$, or $\gamma = \pi/3$, or $\gamma = 4\pi/3$. It is easy to check that $\beta = 0$ and $\beta = \pi$ do not provide solutions of the full system $\mathbf{f}_5 = \mathbf{0}$. If $\gamma = \pi/3$, then there is a unique solution of $\mathbf{f}_5 = \mathbf{0}$ with $\beta \in [0, \pi]$, the solution $(\alpha, \beta, \gamma) = (\alpha, 2\arctan(1/\sqrt{2}), \pi/3) = \sigma_1(\alpha)$. When $\gamma = 4\pi/3$ system $\mathbf{f}_5 = \mathbf{0}$ has a unique solution with $\beta \in [0, \pi]$ which is given by $(\alpha, \beta, \gamma) = (\alpha, 2\arctan\sqrt{2}, 4/3) = \sigma_2(\alpha)$. These two solutions of $\mathbf{f}_5 = \mathbf{0}$ leave invariant the tetrahedron with scale factor ρ.

It only remains to find the values of α for which $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ satisfy that $f(1) = f(2) = f(3) = f(4)$ and $g(1) = g(2) = g(3) = g(4)$. The functions $f(i)$ and $g(i)$ evaluated at $\sigma_1(\alpha)$ are given by

$$\begin{aligned}
f(1) = f(2) = f(3) &= -\frac{\sqrt{2}(\rho - 3)}{(3\rho^2 - 2\rho + 3)^{3/2}} + F(\alpha) + F(\alpha + 2\pi/3) + F(\alpha + 4\pi/3), \\
f(4) &= \frac{\sqrt{2/3}}{(\rho+1)^2} - \frac{3\sqrt{2}(\rho - 3)}{(3\rho^2 - 2\rho + 3)^{3/2}}, \\
g(1) &= \frac{3\sqrt{2}(3\rho - 1)}{(3\rho^2 - 2\rho + 3)^{3/2}} + \frac{\sqrt{2/3}}{(\rho+1)^2}, \\
g(2) = g(3) = g(4) &= \frac{\sqrt{2}(3\rho - 1)}{(3\rho^2 - 2\rho + 3)^{3/2}} + G(\alpha) + G(\alpha + 2\pi/3) + G(\alpha + 4\pi/3),
\end{aligned}$$

where

$$F(\alpha) = \frac{\sqrt{6}(8\rho\cos\alpha + \rho + 9)}{(9\rho^2 + 16\rho\cos\alpha + 2\rho + 9)^{3/2}},$$
$$G(\alpha) = \frac{\sqrt{6}(9\rho + 8\cos\alpha + 1)}{(9\rho^2 + 16\rho\cos\alpha + 2\rho + 9)^{3/2}}.$$

We must find the values of α for which $\sigma_1(\alpha)$ is a solution of system $e_1(\alpha) = f(1) - f(4) = 0$ and $e_2(\alpha) = g(1) - g(2) = 0$. We consider the additional equation

$$e(\alpha) = \frac{e_1(\alpha) + e_2(\alpha)}{\rho - 1} = H_0 - H(\alpha) - H(\alpha + 2\pi/3) - H(\alpha + 4/\pi/3) = 0,$$

where

$$H_0 = \frac{8\sqrt{2}}{(3\rho^2 - 2\rho + 3)^{3/2}},$$
$$H(\alpha) = \frac{8\sqrt{6}(1 - \cos\alpha)}{(9\rho^2 + 16\rho\cos\alpha + 2\rho + 9)^{3/2}}.$$

Notice that $H_0 > 0$ for all $\rho \geqslant 1$ and $H(\alpha) \geqslant 0$ for all $\rho \geqslant 1$ and $\alpha \in [0, 2\pi)$. Next we prove that $e(\alpha) \leqslant 0$ for all $\rho \geqslant 1$ and $\alpha \in [0, 2\pi)$. First we find the minimum values of the function $h(\alpha) = H(\alpha) + H(\alpha + 2\pi/3) + H(\alpha + 4/\pi/3)$.

It is easy to check that $h(\alpha)$ is a periodic function of period $2\pi/3$. Moreover $h(2\pi/3 - \alpha) = h(\alpha)$ so it is sufficient to study the behaviour of $h(\alpha)$ for $\alpha \in [0, \pi/3]$. The possible minimums of $h(\alpha)$ are given by the solutions of $h'(\alpha) = 0$ where

$$h'(\alpha) = H'(\alpha) + H'(\alpha + 2\pi/3) + H'(\alpha + 4\pi/3),$$

and

$$H'(\alpha) = \frac{8\sqrt{6}\,(9\rho^2 - 8\rho\cos\alpha + 26\rho + 9)\sin\alpha}{(9\rho^2 + 16\rho\cos\alpha + 2\rho + 9)^{5/2}}.$$

In order to solve equation $h'(\alpha) = 0$, we eliminate the radicals and the denominators of the equation by using the procedure described in the Appendix and we set $c = \cos\alpha$. The result is the polynomial equation

$$21233664(c - 1)(c + 1)(2c - 1)^2(2c + 1)^2\rho^2\, E(c, \rho) = 0, \qquad (8)$$

in the variable c with coefficients depending on ρ where $E(c, \rho)$ is a polynomial of degree 21 in c. Notice that we are only interested in solutions of $h'(\alpha) = 0$ with $\alpha \in [0, \pi/3]$, so we only consider solutions of (8) with $c \in [1/2, 1]$.

We solve symbolically with the help of the algebraic manipulator Mathematica the equation $E(c,\rho)=0$. We see that only three of the 21 solutions $c(\rho)$ are real for all $\rho \geqslant 1$, one satisfies $c(\rho) \geqslant 1.8638527724\ldots$ for all $\rho \geqslant 1$, the other one satisfies $c(\rho) \geqslant 1.9150397104\ldots$, and finally the third one $c(\rho)=\tilde{c}(\rho)$ satisfies $\tilde{c}(\rho) \in [0.5088804885\ldots,1]$ for $\rho \in [1,\rho^*]$ and $\tilde{c}(\rho) > 1$ for $\rho > \rho^*$. Here $\rho^* = 6.1180170733\ldots$ is the unique real solution of $E(1,\rho)=0$ with $\rho \geqslant 1$. Two of the solutions of $E(c,\rho)=0$ are real for $\rho \in [1,\rho^*]$ and in $\mathbb{C}\setminus\mathbb{R}$ for $\rho > \rho^*$, moreover for these two real solutions $c(\rho) \notin [1/2,1]$. The rest of the solutions of $E(c,\rho)=0$ are in $\mathbb{C}\setminus\mathbb{R}$ for all $\rho \geqslant 1$. In short, there is a unique solution $c(\rho)=\tilde{c}(\rho)$ with $\rho \in [1,\rho^*]$ satisfying that $c(\rho) \in [1/2,1]$. This solution provides a solution of the initial equation $h'(\alpha)=0$ only when $\rho=\rho^*$ and $c=1$. Consequently the solutions of (8) with $c(\rho) \in [1/2,1]$ that provide solutions of $h'(\alpha)=0$ are $c=1/2$ and $c=1$ for all $\rho \geqslant 1$. Therefore $h'(\alpha)=0$ has two unique solutions in the interval $[0,\pi/3]$ that are independent of ρ, $\alpha=0$ which corresponds to a minimum and $\alpha=\pi/3$ which corresponds to a maximum. Thus the minimum values of $h(\alpha)$ in the interval $[0,2\pi)$ are $\alpha=0$, $\alpha=2\pi/3$ and $\alpha=4\pi/3$. Furthermore $h(0)=h(2\pi/3)=h(4\pi/3)=H_0$. Consequently $e(\alpha) \leqslant 0$ for all $\rho \geqslant 1$ and $e(\alpha)=0$ if and only if either $\alpha=0$, $\alpha=2\pi/3$ or $\alpha=4\pi/3$. It is easy to check that all these solutions satisfy system $e_1(\alpha)=0$ and $e_2(\alpha)=0$. On the other hand, in a similar way it can be proved that the solutions of system $e_1(\alpha)=0$ and $e_2(\alpha)=0$ for $\rho=1$ are also $\alpha=0$, $\alpha=2\pi/3$ or $\alpha=4\pi/3$.

In short, there are three rotations for $\sigma_1(\alpha)$ that satisfy conditions of Lemma 1, they have Euler rotation angles (α,β,γ) equal to

$$\begin{aligned}
&(0, 2\arctan(1/\sqrt{2}), \pi/3),\\
&(2\pi/3, 2\arctan(1/\sqrt{2}), \pi/3),\\
&(4\pi/3, 2\arctan(1/\sqrt{2}), \pi/3).
\end{aligned}$$

All these rotations give configurations of Type II.

Proceeding in a similar way with the solution $\sigma_2(\alpha)$ we obtain three additional rotations that satisfy conditions of Lemma 1, they have Euler rotation angles (α,β,γ) equal to

$$\begin{aligned}
&(\pi/3, 2\arctan\sqrt{2}, 4\pi/3),\\
&(\pi, 2\arctan\sqrt{2}, 4\pi/3),\\
&(5\pi/3, 2\arctan\sqrt{2}, 4\pi/3).
\end{aligned}$$

All these rotations give configurations of Type I.

We have just proved that all the rotations satisfying $\mathbf{f}_5=\mathbf{0}$ and the conditions of Lemma 1 provide configurations of Type either I or II.

Type I	Type II
$(\alpha, 0, 2\pi-\alpha)$ with $\alpha\in[0,2\pi)$	(α,π,α) with $\alpha\in[0,2\pi)$
$(\pi, 2\arctan\sqrt{2}, 2\pi/3)$	$(2\pi/3, 2\arctan(1/\sqrt{2}), \pi)$
$(\pi/3, 2\arctan\sqrt{2}, 0)$	$(0, 2\arctan(1/\sqrt{2}), 5\pi/3)$

Table 1
The Euler rotation angles (α,β,γ) of all the rotations that satisfy system $A\mathbf{q}_5-\mathbf{q}_6=\mathbf{0}$ and the conditions of Lemma 1 classified by the type of configurations that they provide.

Type I	Type II
$(\alpha, 0, 4\pi/3-\alpha \mod 2\pi)$, $\alpha\in[0,2\pi)$	$(\alpha,\pi,\alpha+2\pi/3 \mod 2\pi)$, $\alpha\in[0,2\pi)$
$(\pi/3, 2\arctan\sqrt{2}, 2\pi/3)$	$(4\pi/3, 2\arctan(1/\sqrt{2}), 5\pi/3)$
$(5\pi/3, 2\arctan\sqrt{2}, 0)$	$(0, 2\arctan(1/\sqrt{2}), \pi)$

Table 2
The Euler rotation angles (α,β,γ) of all the rotations that satisfy system $A\mathbf{q}_5-\mathbf{q}_7=\mathbf{0}$ and the conditions of Lemma 1 classified by the type of configurations that they provide.

Type I	Type II
$(\alpha, 0, 2\pi/3-\alpha \mod 2\pi)$, $\alpha\in[0,2\pi)$	$(\alpha,\pi,\alpha-2\pi/3 \mod 2\pi)$, $\alpha\in[0,2\pi)$
$(\pi, 2\arctan\sqrt{2}, 0)$	$(2\pi/3, 2\arctan(1/\sqrt{2}), 5\pi/3)$
$(5\pi/3, 2\arctan\sqrt{2}, 2\pi/3)$	$(4\pi/3, 2\arctan(1/\sqrt{2}), \pi)$

Table 3
The Euler rotation angles (α,β,γ) of all the rotations that satisfy system $A\mathbf{q}_5-\mathbf{q}_8=\mathbf{0}$ and the conditions of Lemma 1 classified by the type of configurations that they provide.

By using similar arguments, we find all the rotations that satisfy system $A\mathbf{q}_5-\mathbf{q}_i=\mathbf{0}$, for $i=6,7,8$, and the conditions of Lemma 1, moreover we see that all these rotations provide configurations of Type either I or II. The Euler rotation angles (α,β,γ) of those rotations classified by the type of configurations that they provide are given in Tables 1, 2, and 3.

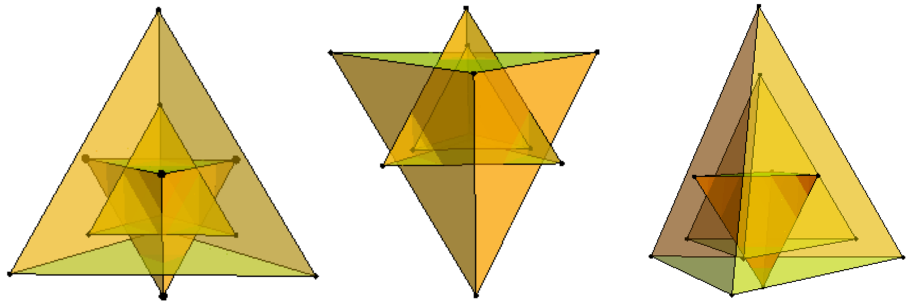
In short, all rotations that satisfy conditions of Lemma 1 provide configurations of Type either I or II. This completes the proof of statement (a).

Statement (b) is proved in [2] by using the choice of the units of mass and length that we use here and it is proved in [6] with a different choice of the units. Here by using Lemma 1 we give a new and shorter proof of this result. Indeed, statement (b) is an immediate consequence of Lemma 1 together with statement (a). In statement (a) we have proved that if $\mathbf{q}_5=\rho\mathbf{a}_1$, $\mathbf{q}_6=\rho\mathbf{a}_2$, $\mathbf{q}_7=\rho\mathbf{a}_3$, and $\mathbf{q}_8=\rho\mathbf{a}_4$ then the hypotheses of Lemma 1 hold. So applying Lemma 1 we obtain the expressions of λ and m, these expressions provide a central configuration if $\lambda > 0$ and $m > 0$. We see that $2f(1)/3 - 1/(2\rho^3) \neq 0$ for all $\rho > 1$ and $2g(1)/(3\rho) - 1/2 = 0$ when $\rho = \alpha$, in particular, $\lambda > 0$ and $m > 0$ when $\rho > \alpha$. Moreover m is an increasing function of ρ, so for each value of $m > 0$ there exists a unique value of $\rho > \alpha$ for which the configuration is central. This completes the proof of the statement (b).

Statement (c) is proved in [5] by using different choices of the units of mass and length that the ones used here. Statement (c) is also an immediate consequence of Lemma 1 together with statement (a). As above in statement (a) we have proved that if $\mathbf{q}_5 = \rho P \mathbf{a}_1$, $\mathbf{q}_6 = \rho P \mathbf{a}_2$, $\mathbf{q}_7 = \rho P \mathbf{a}_3$, and $\mathbf{q}_8 = \rho P \mathbf{a}_4$ then the hypotheses of Lemma 1 hold. So applying this Lemma 1 we obtain the expressions of λ and m which provide a central configuration if $\lambda > 0$ and $m > 0$. We see that $2f(1)/3 - 1/(2\rho^3) = 0$ at $\rho = \alpha_2$ and $2g(1)/(3\rho) - 1/2 = 0$ at $\rho = \alpha_1$, in particular, $\lambda > 0$ and $m > 0$ when $\rho \in [1, \alpha_1) \cup (\alpha_2, \infty)$. Moreover $m = m(\rho)$ is decreasing at the interval $[1, \alpha_2)$ with $m(1) = 1$ and $\lim_{\rho \to \alpha_2^-} m(\rho) = -\infty$, it is decreasing at the interval (α_2, α_3), increasing at the interval (α_3, ∞), the point $\rho = \alpha_3$ is a minimum with $m(\rho) = m_0$ and $\lim_{\rho \to \alpha_2^+} m(\rho) = \infty$ and $\lim_{\rho \to +\infty} m(\rho) = \infty$. These properties of $m(\rho)$ together with the fact the faces of the nested tetrahedra intersect when $1 \leqslant \rho \leqslant 3$ prove statements (c.1), (c.2), (c.3), and (c.4).

3 Three nested tetrahedra

In this section we study the spatial central configurations of the 12–body problem when the masses are located at the vertices of three nested regular tetrahedra with scale factors 1, ρ and R and some of them rotated with respect to the others by a rotation of Euler angles $\alpha = 0$, $\beta = \pi$ and $\gamma = 0$ (i.e. by a rotation with rotation matrix P). Taking conveniently the unit of masses we can assume that all the masses of the tetrahedron with scale factor 1 are equal to one. We also choose the unit of length in such a way that the edges of the tetrahedron with scale factor 1 have length 2. Without loss of generality we can assume that the configuration has only one rotated tetrahedra which could be either the inner, the medium or the outer one. This is due to the fact that central configurations are invariant under rotations and a configuration with two rotated tetrahedra can be transformed into a configuration with only one rotated tetrahedra by doing a rotation of Euler angles $\alpha = 0$, $\beta = \pi$ and $\gamma = 0$. Central configurations are also invariant under homothecies, so we can assume that the rotated tetrahedra is the one with scale factor ρ, and that $1 \leqslant \rho < R$ when the rotated tetrahedra is the medium one, $1 < R \leqslant \rho$ when the rotated tetrahedra is the outer one, and finally $0 < \rho < 1 < R$ when the rotated tetrahedra is the inner one (see Figure 2). We define



(a) $1 \leqslant \rho < R$ (b) $1 < R \leqslant \rho$ (c) $0 < \rho < 1 < R$

Fig. 2. Three possible central configurations with three nested regular tetrahedra with scale factors 1, ρ and R and the one with scale factor ρ rotated with respect to the other two by a rotation of Euler angles $\alpha = 0$, $\beta = \pi$ and $\gamma = 0$. In (a) the medium tetrahedra is the rotated one, in (b) is the outermost and in (c) is the innermost. In any of these three possibilities the tetrahedra can intersect or not.

$$\begin{aligned}\mathcal{C}_m &= \{(\rho, R) \in \mathbb{R}^2 \,:\, 1 \leqslant \rho < R\},\\ \mathcal{C}_o &= \{(\rho, R) \in \mathbb{R}^2 \,:\, 1 < R \leqslant \rho\},\\ \mathcal{C}_i &= \{(\rho, R) \in \mathbb{R}^2 \,:\, 0 < \rho < 1 < R\}.\end{aligned}$$

Result 3 *Consider four equal masses $m_1 = m_2 = m_3 = m_4 = 1$ at the vertices of a regular tetrahedron with edge length 2 having positions $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$. Consider four additional equal masses $m_5 = m_6 = m_7 = m_8 = m$ at the vertices of a second nested regular tetrahedron having positions $\mathbf{q}_{i+4} = \rho P \mathbf{a}_i$ for all $i = 1, \ldots, 4$ with $\rho > 0$. Finally we consider four additional equal masses $m_9 = m_{10} = m_{11} = m_{12} = M$ at the vertices of a third nested regular tetrahedron having positions $\mathbf{q}_{i+8} = R\mathbf{a}_i$ for all $i = 1, \ldots, 4$ with $R > 1$ (see Figure 2). Then the following statements hold.*

(a) Such configuration is central for the spatial 12–body problem when $m = m(\rho, R)$ and $M = M(\rho, R)$ are given by the expression (11) and $(R, \rho) \in \mathcal{D} = \{(R, \rho) \in \mathbb{R}^2 \,:\, \det(A) \neq 0,\, m(R, \rho) > 0,\, M(R, \rho) > 0,\, \rho > 0,\, R > 1\}$ (see (11) for the definition of A and see Figure 3 for the plot of $\mathcal{D}$).

(b) The set $\mathcal{D}$ is formed by the disjoint union of the sets $\mathcal{D}_i$ for $i = 1, \ldots, 6$ defined in Figure 3. Then the regions $\mathcal{D}_i$ provide central configurations of the 12–body problem with the masses located at the vertices of three nested regular tetrahedra satisfying the following.

(b.1) The rotated tetrahedra is the inner one and it intersects only the medium when $(\rho, R) \in \mathcal{D}_2 \cap \mathcal{C}_i$, it intersects the medium and the outer when $(\rho, R) \in \mathcal{D}_4 \cap \mathcal{C}_i$, and the three tetrahedra do not intersect when $(\rho, R) \in \mathcal{D}_1$.

(b.2) The rotated tetrahedra is the medium one and it intersects only the inner when $(\rho, R) \in \mathcal{D}_2 \cap \mathcal{C}_m$ and when $(\rho, R) \in \mathcal{D}_3 \cap \{\rho \leqslant 3\}$, it intersects the inner and the outer when $(\rho, R) \in \mathcal{D}_5 \cap \mathcal{C}_m \cap \{\rho \leqslant 3\}$ and when $(\rho, R) \in \mathcal{D}_4 \cap \mathcal{C}_m$, it intersects only the outer one when $(\rho, R) \in \mathcal{D}_5 \cap \mathcal{C}_m \cap \{\rho > 3\}$, and the three tetrahedra do not intersect when $(\rho, R) \in \mathcal{D}_3 \cap \{\rho > 3\}$.

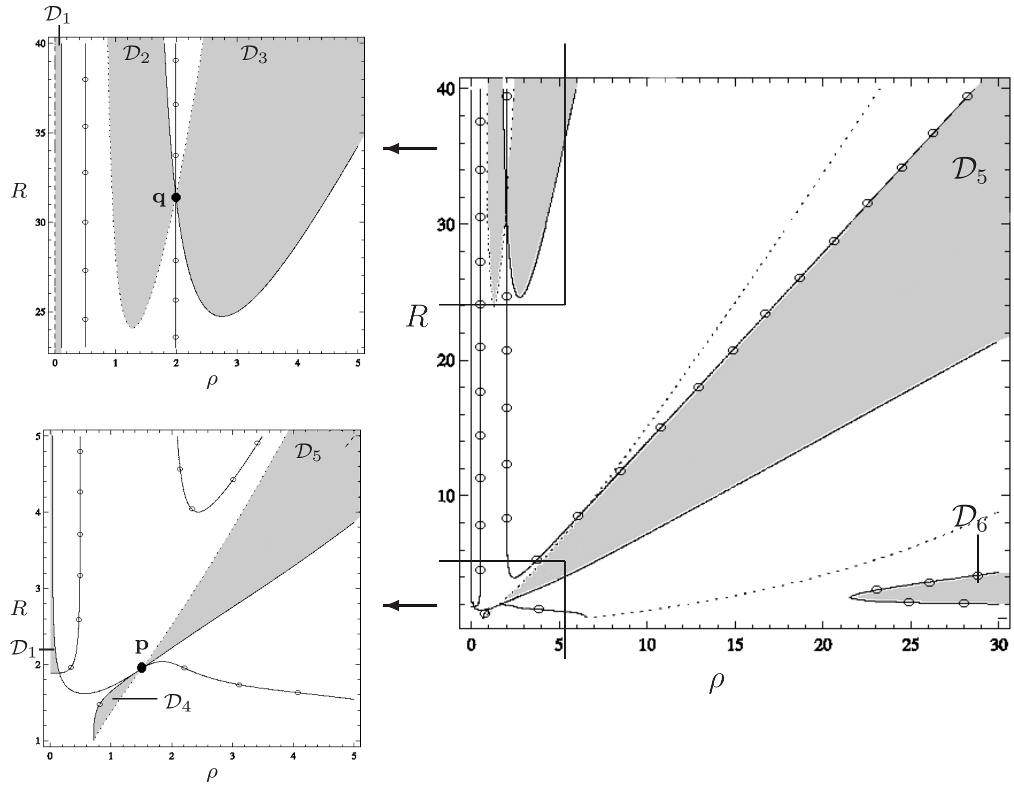


Fig. 3. The set $\mathcal{D}$. The dotted curves correspond to points where $\det(A) = 0$, the continuous curves correspond to points where $m(\rho, R) = 0$, and finally, the curves with small circles correspond to points where $M(\rho, R) = 0$.

(b.3) *The rotated tetrahedra is the outer one and it intersects only the medium when* $(\rho, R) \in \mathcal{D}_5 \cap \mathcal{C}_o \cap \{\rho > 3\}$, *it intersects the inner and the medium when* $(\rho, R) \in \mathcal{D}_5 \cap \mathcal{C}_o \cap \{\rho \leqslant 3\}$, *and the three tetrahedra do not intersect when* $(\rho, R) \in \mathcal{D}_6$.

(c) *Let*

$$m(M, \rho, R) = \frac{\rho^2(\rho + 2f(\rho, 1))}{2\rho^3 f(1, \rho) + 1} - M\frac{2\rho^2(\rho g_1(R) - f(\rho, R))}{2\rho^3 f(1, \rho) + 1},$$

and $M_0(\rho, R) = (\rho^2(\rho+2f(\rho, 1)))/(2\rho^2(\rho g_1(R)-f(\rho, R))$ *(see (9) for the definitions of* f *and* g_1*), and let* $\mathbf{p} = (p_1, p_2) = (1.5094116757\dots, 1.9479968088\dots)$ *and* $\mathbf{q} = (q_1, q_2) = (1.9926247501\dots, 31.4606148079\dots)$ *be the solutions of system* $\det(A) = \det(A_2) = \det(A_3) = 0$ *(see (11) for the definitions of* A_2 *and* A_3*). Then the following statements hold.*

(c.1) *For each* $M > M_0(\mathbf{p}) = 0.1252891302\dots$ *and* $m = m(M, \mathbf{p}) = -0.4001317738\dots + 3.1936671053\dots M$ *we have a central configuration of the spatial 12–body problem when* $(\rho, R) = \mathbf{p}$. *In this configuration the rotated tetrahedra is the medium and it intersects with the inner and the outer.*

(c.2) *For each* $M > M_0(\mathbf{q}) = 25204.620455\dots$ *and* $m = m(M, \mathbf{q}) = -3.5811747685\dots + 0.0001420840585\dots M$ *we have a central configuration of the spatial 12–body problem when* $(\rho, R) = \mathbf{q}$. *In this configuration the rotated tetrahedra is the medium and it intersects only the inner.*

The values of the masses $m(\rho, R)$ and $M(\rho, R)$ that provide central configurations on sets $\mathcal{D}_i$ are analyzed in Subsection 3.1.

We call the previous result as Result 3 instead of Theorem 3 because part of its proof is done numerically with the help of Mathematica.

PROOF. [Proof of Result 3] The positions and the values of the masses have been taken so that the center of mass of the configuration is located at the origin of coordinates.

We substitute the positions and the values of the masses into (2). After some computations we obtain that system (2) over the nested three tetrahedra configuration with $R > 1$ is equivalent to system $Ax = b$, more precisely

$$\begin{pmatrix} 1 & f(1,\rho) & g_1(R) \\ \rho & -1/(2\rho^2) & f(\rho, R) \\ R & f(R,\rho) & -1/(2R^2) \end{pmatrix} \begin{pmatrix} \lambda \\ m \\ M \end{pmatrix} = \begin{pmatrix} 1/2 \\ -f(\rho, 1) \\ g_2(R) \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} f(x,y) &= \frac{2\sqrt{2}(y-3x)}{(3x^2-2yx+3y^2)^{3/2}} - \frac{(2/3)^{3/2}}{(x+y)^2}, \\ g_1(x) &= -\frac{2\sqrt{2}(x+3)}{(3x^2+2x+3)^{3/2}} + \frac{(2/3)^{3/2}}{(x-1)^2}, \\ g_2(x) &= \frac{2\sqrt{2}(3x+1)}{(3x^2+2x+3)^{3/2}} + \frac{(2/3)^{3/2}}{(x-1)^2}. \end{aligned}$$

Since $3x^2 - 2yx + 3y^2 > 0$ in $D = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$, the function $f(x, y)$ is defined for all $(x, y) \in D$. The functions $g_1(x)$ and $g_2(x)$ are defined for all $x > 1$, moreover

$$\lim_{x \to 1^+} g_1(x) = \lim_{x \to 1^+} g_2(x) = +\infty. \tag{10}$$

On the other hand $f(x, y) < 0$ for all $(x, y) \in D$. Indeed the set D can be written as $D = \{(x, y) \in \mathbb{R}^2 : y = a\,x, x > 0, a > 0\}$. It is easy to check that $f(x, a\,x) = f_1(a)/x^2$ where

$$f_1(a) = \frac{2\sqrt{2}(a-3)}{(3a^2-2a+3)^{3/2}} - \frac{(2/3)^{3/2}}{(a+1)^2}.$$

As in the Appendix we find the zeroes of $f_1(a)$ by reducing the problem to find the zeroes of a polynomial of one variable and we see that $f_1(a)=0$ has no real solutions for $a>0$. In particular $f_1(a)<0$ for $a>0$.

If $\det(A)\neq 0$, then the solution of system (9) is

$$\lambda=\lambda(\rho,R)=\frac{\det(A_1)}{\det(A)}=\frac{1}{\det(A)}\begin{vmatrix} 1/2 & f(1,\rho) & g_1(R) \\ -f(\rho,1) & -1/(2\rho^2) & f(\rho,R) \\ g_2(R) & f(R,\rho) & -1/(2R^2) \end{vmatrix},$$

$$m=m(\rho,R)=\frac{\det(A_2)}{\det(A)}=\frac{1}{\det(A)}\begin{vmatrix} 1 & 1/2 & g_1(R) \\ \rho & -f(\rho,1) & f(\rho,R) \\ R & g_2(R) & -1/(2R^2) \end{vmatrix}, \qquad (11)$$

$$M=M(\rho,R)=\frac{\det(A_3)}{\det(A)}=\frac{1}{\det(A)}\begin{vmatrix} 1 & f(1,\rho) & 1/2 \\ \rho & -1/(2\rho^2) & -f(\rho,1) \\ R & f(R,\rho) & g_2(R) \end{vmatrix}.$$

The solution $\lambda(\rho,R)$, $m(\rho,R)$ and $M(\rho,R)$ gives a central configuration of the 12–body problem if and only if R and ρ are such that $\lambda(\rho,R)>0$, $m(\rho,R)>0$ and $M(\rho,R)>0$.

It is easy to check that

$$\det(A)=\frac{\rho}{2R^2}f(1,\rho)+\frac{1}{4R^2\rho^2}+Rf(\rho,R)f(1,\rho)-f(R,\rho)f(\rho,R)+ \frac{R}{2\rho^2}\,g_1(R)+\rho f(R,\rho)\,g_1(R)\;.$$

We analyze the sign of $\det(A)$ on the boundaries of the regions $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$.

Clearly

$$\lim_{\rho\to 0^+}\det(A)=\operatorname{sign}\left(\frac{1}{4R^2}+\frac{R}{2}\,g_1(R)\right)\cdot\infty\;,$$

and from (10)

$$\lim_{R\to 1^+}\det(A)=\operatorname{sign}\left(\frac{1}{2\rho^2}+\rho f(1,\rho)\right)\cdot\infty\;.$$

We solve equation $f_2(R)=1/(4R^2)+Rg_1(R)/2=0$ as in the Appendix and we see that it has no real solutions for $R>1$, moreover $f_2(R)>0$ for all $R>1$. Solving equation $f_3(\rho)=1/(2\rho^2)+\rho f(1,\rho)=0$ as in the Appendix we find two unique real roots $\rho = b_1 = 0.7152231486\ldots$ and $\rho = b_2 = 6.5360789462\ldots$. Moreover $f_3(\rho) > 0$ for $\rho \in (0, b1) \cup (b_2, \infty)$ and $f_3(\rho) < 0$ for $\rho \in (b_1, b_2)$. In short, $\lim_{\rho\to 0^+} \det(A) = +\infty$ for all $R > 1$, $\lim_{R\to 1^+} \det(A) = +\infty$ when $\rho \in (0, b1) \cup (b_2, \infty)$ and $\lim_{R\to 1^+} \det(A) = -\infty$ when $\rho \in (b_1, b_2)$. The regions of these two limits are indicated in Figure 4.

After some computations we see that when $\rho = 1$ and $R > 1$

$$
\begin{aligned}
\det(A) = f_4(R) = &-\frac{-9+9\sqrt{2}+\sqrt{6}}{36R^2} + \frac{3\sqrt{6}R\left(R^2+2R+1\right)-8}{27(R-1)^2(R+1)^2} \\
&+\frac{2\left(3\sqrt{3}R(R+1)^2+R(R+1)^2-4\right)}{27(R+1)^4} + \frac{8\left(3R^2-10R+3\right)}{\left(3R^2-2R+3\right)^3} \\
&-\frac{2\left(9+\sqrt{3}\right)(R-3)R}{9\left(3R^2-2R+3\right)^{3/2}} - \frac{8\left(5R^2-2R+1\right)}{3\sqrt{3}(R-1)^2(R+1)\left(3R^2-2R+3\right)^{3/2}} \\
&-\frac{(R+3)\left(9\sqrt{2}R^3+18\sqrt{2}R^2+9\sqrt{2}R-8\sqrt{3}\right)}{9(R+1)^2\left(3R^2+2R+3\right)^{3/2}} \\
&+\frac{8\left(3R^2+8R-3\right)}{\left(3R^2-2R+3\right)^{3/2}\left(3R^2+2R+3\right)^{3/2}}.
\end{aligned}
$$

We solve equation $f_4(R) = 0$ as in the Appendix and we find two real solutions with $R > 1$, they are $R = c_1 = 1.3711124500\ldots$ and $R = c_2 = 28.2412927769\ldots$. Moreover $f_4(R) > 0$ for $R \in (c_1, c_2)$ and $f_4(R) < 0$ for $R \in (1, c_1) \cup (c_2, \infty)$.

For $R = \rho$ and $\rho > 1$ we have that

$$
\begin{aligned}
\det(A) = f_5(\rho) = &\frac{\left(-18+9\sqrt{2}-2\sqrt{3}\right)(\rho-3)}{9\rho\left(3\rho^2-2\rho+3\right)^{3/2}} + \frac{-2-6\sqrt{3}+3\sqrt{6}}{27(\rho-1)^2\rho} + \\
&\frac{\left(18-9\sqrt{2}+2\sqrt{3}\right)(\rho+3)}{9\rho\left(3\rho^2+2\rho+3\right)^{3/2}} + \frac{2+6\sqrt{3}-3\sqrt{6}}{27\rho(\rho+1)^2} - \frac{29+12\sqrt{3}}{108\rho^4}.
\end{aligned}
$$

We solve equation $f_5(\rho) = 0$ as in the Appendix and we see that it has no real solutions with $\rho > 1$. Moreover $f_5(R) < 0$ for all $\rho > 1$.

Notice that $\det(A)$ is continuous for all $\rho > 0$, $R > 1$. This means that if $\det(A)$ has different signs on two different curves γ_1 and γ_2, then there exist an odd number of curves of zeroes of $\det(A)$ (taking into account their multiplicities) between γ_1 and γ_2, whereas if $\det(A)$ has the same sign on the curves γ_1 and γ_2, then there exist an even number (that could be 0) of curves of zeroes of $\det(A)$ (taking into account their multiplicities) between γ_1 and γ_2. Moreover these curves are continuous.

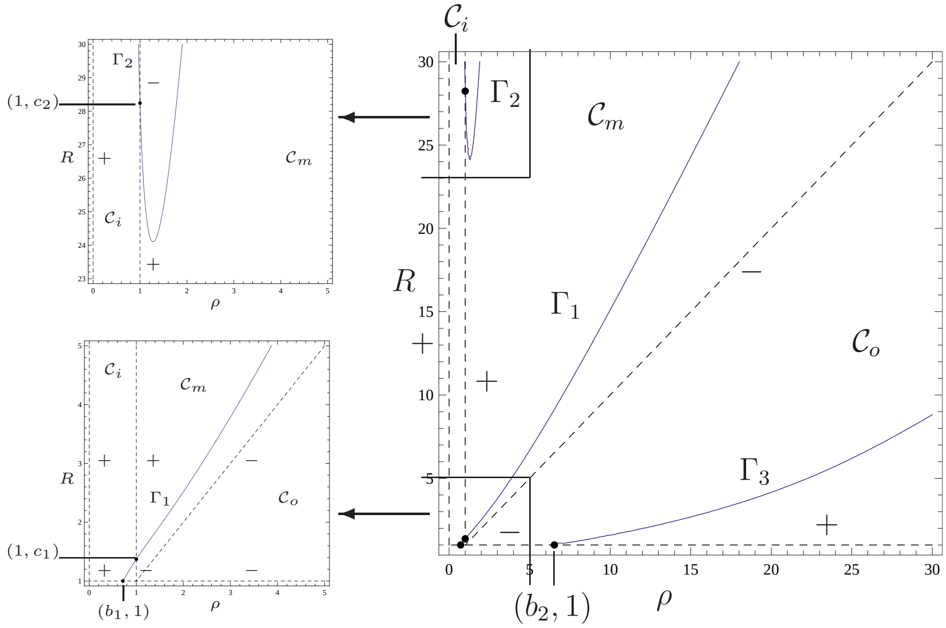


Fig. 4. The level curves $\det(A) = 0$. The dashed lines correspond to the boundaries of the regions $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$.

By analyzing the sign of $\det(A)$ on the boundary of $\mathcal{C}_i$ we have seen that $\det(A) > 0$ on $\{(\rho, R) \in \mathbb{R}^2 \ : \ \rho = 0, R > 1\} \cup \{(\rho, R) \in \mathbb{R}^2 \ : \ \rho \in (0, b_1), R = 1\} \cup \{(\rho, R) \in \mathbb{R}^2 \ : \ \rho = 1, R \in (c_1, c_2)\}$, $\det(A) < 0$ on $\{(\rho, R) \in \mathbb{R}^2 \ : \ \rho = 1, R \in (1, c_1)\} \cup \{(\rho, R) \in \mathbb{R}^2 \ : \ \rho = 1, R \in (c_2, +\infty)\}$ and that $\det(A) = 0$ at the points $(1, c_1)$ and $(1, c_2)$. Therefore on $\mathcal{C}_i$ there exists a curve Γ_1 starting at the point $(b_1, 1)$ and passing through the point $(1, c_1)$ and another one Γ_2 coming from $R = +\infty$ and passing through the point $(1, c_2)$ on which $\det(A) = 0$ (see Figure 4).

On the boundary of $\mathcal{C}_m$ we have that $\det(A) < 0$ everywhere with the exception of the set $\{(\rho, R) \in \mathbb{R}^2 \ : \ \rho = 1, R \in (c_1, c_2)\}$ and $\det(A) = 0$ at the points $(1, c_1)$ and $(1, c_2)$. Therefore we can guarantee the existence of a curve in $\mathcal{C}_m$ with $R \in (c_1, c_2)$ on which $\det(A) = 0$. This curve starts at the point $(1, c_1)$, so it is the continuation on the region $\mathcal{C}_m$ of the curve Γ_1. By plotting the level curves $\det(A) = 0$ with the help of Mathematica (see Figure 4), we see that the curve Γ_2 can be continued to the region $\mathcal{C}_m$ and that the curves Γ_1 and Γ_2 do not coincide.

Finally on the boundary of $\mathcal{C}_o$ we have that $\det(A) < 0$ at everywhere with the exception of the set $\{(\rho, R) \in \mathbb{R}^2 \ : \ \rho \in (b_2, +\infty), R = 1\}$ and $\det(A) = 0$ at the point $(b_2, 1)$. Therefore there exists a curve Γ_3 on $\mathcal{C}_o$ starting at the point $(b_2, 1)$ and going to $\rho = +\infty$ (see again Figure 4).

By plotting the level curves $\det(A) = 0$ for larger values of ρ and R we see that these are the unique three curves with $\det(A) = 0$.

Boundary	Sign of $\det(A_2)$
$\rho > 0,\ R = 1$	$\lim_{R\to 1^+} \det(A_2) = +\infty$
$\rho = 0,\ R > 1$	$\det(A_2) > 0$
$\rho = 1,\ R > 1$	$\det(A_2) > 0$ when $R \in (1, c_3)$ $\det(A_2) < 0$ when $R \in (c_3, \infty)$
$R = \rho,\ \rho > 1$	$\det(A_2) > 0$ when $\rho \in (1, d_1)$ $\det(A_2) < 0$ when $\rho \in (d_1, \infty)$

Table 4
The sign of $\det(A_2)$ on the boundaries of $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$. Here $c_3 = 1.7097032687\ldots$ is the unique solution of equation $\det(A_2)|_{\rho=1} = 0$ with $R > 1$, $d_1 = 2.4741715435\ldots$ is the unique solution of equation $\det(A_2)|_{R=\rho} = 0$ with $\rho > 1$.

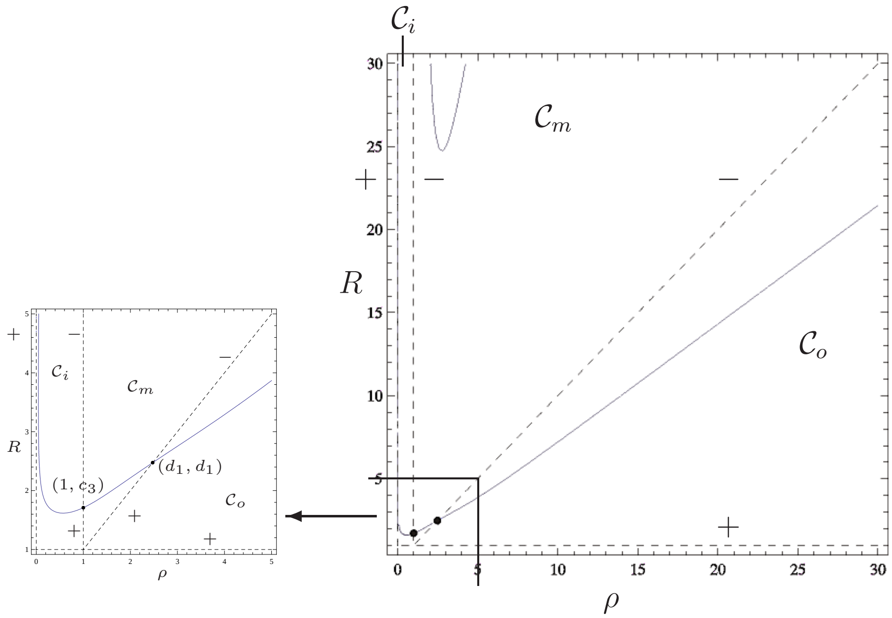


Fig. 5. The level curves $\det(A_2) = 0$. The dashed lines correspond to the boundaries of the regions $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$.

Now we analyze the sign of $\det(A_2)$ on the boundaries of $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$. From (11) we have that

$$\det(A_2) = g_1(R)g_2(R)\rho + \frac{\rho}{4R^2} + \frac{f(\rho,1)}{2R^2} + \frac{1}{2}Rf(\rho,R) + Rf(\rho,1)g_1(R) \\ - f(\rho,R)g_2(R)\ .$$

Since the procedure is exactly the same than for the sign of $\det(A)$ we omit the computations and we only summarize the results in Table 4.

The plot of the level curves $\det(A_2) = 0$ is given in Figure 5. As in the previous case the plot of these level curves for larger values of ρ and R does not provide new curves with $\det(A_2) = 0$.

Boundary	Sign of $\det(A_3)$
$\rho > 0,\ R = 1$	$\lim_{R\to 1^+} \det(A_3) = -\infty$ when $\rho \in (1, b_1) \cup (b_2, \infty)$ $\lim_{R\to 1^+} \det(A_3) = +\infty$ when $\rho \in (b_1, b_2)$
$\rho = 0,\ R > 1$	$\lim_{\rho\to 0^+} \det(A_3) = -\infty$ when $R \in (1, e_1)$ $\lim_{\rho\to 0^+} \det(A_3) = +\infty$ when $R \in (e_1, \infty)$
$\rho = 1,\ R > 1$	$\det(A_3) > 0$ when $R \in (1, c_4)$ $\det(A_3) < 0$ when $R \in (c_4, \infty)$
$R = \rho,\ \rho > 1$	$\det(A_3) > 0$ when $\rho \in (1, d_2)$ $\det(A_3) < 0$ when $\rho \in (d_2, \infty)$

Table 5
The sign of $\det(A_3)$ on the boundaries of $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$. Here $e_1 = 1.8899915758\ldots$ is the unique real solution of equation $R/4 - g_2(R)/2 = 0$ with $R > 1$, the values b_1 and b_2 defined above are the unique real solutions of equation $-\rho f(1,\rho) - 1/(2\rho^2) = 0$ with $\rho \geqslant 1$, $c_4 = 1.6436467629\ldots$ is the unique real solution of $\det(A_3)|_{\rho=1} = 0$ with $R > 1$, and $d_2 = 2.0151307882\ldots$ is the unique real solution of equation $\det(A_3)|_{R=\rho} = 0$ with $\rho > 1$.

Finally we analyze the sign of $\det(A_3)$ on the boundaries of $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$. From (11) we have that

$$\det(A_3) = -f(1,\rho)f(\rho,1)R + \frac{R}{4\rho^2} + \frac{1}{2}\rho f(R,\rho) + f(R,\rho)f(\rho,1) - \frac{g_2(R)}{2\rho^2} \\ -\rho f(1,\rho)g_2(R)\ .$$

As above, and since the procedure is exactly the same than for the sign of $\det(A)$ we omit the computations and we only summarize the results in Table 5.

The plot of the level curves $\det(A_3) = 0$ is given in Figure 6. As in the previous cases, the plot of these level curves for larger values of ρ and R does not provide new curves with $\det(A_3) = 0$.

Since $f(R,\rho) < 0$ and $g_2(R) > 0$ in the region $\rho > 0$, $R > 1$, from the third equation of (9) we have that if $m > 0$ and $M > 0$, then $\lambda > 0$. So it is not necessary to find the set of level curves $\det(A_1) = 0$.

Analyzing the sign of m and M on the regions $\mathcal{C}_m$, $\mathcal{C}_o$, and $\mathcal{C}_i$ we prove that the region $\mathcal{D} = \{(R,\rho) \in \mathbb{R}^2\ :\ \det(A) \neq 0,\ m(R,\rho) > 0,\ M(R,\rho) > 0,\ \rho > 0,\ R > 1\}$ where the solution of (9) given by (11) gives a central configuration of the spatial 12–body problem is the one plotted in Figure 2. This proves statement (a).

It is easy to check that the faces of the rotated tetrahedron, which has scale factor ρ, intersect with the ones of the tetrahedron with scale factor 1 when $\rho \in [1/3, 3]$, and they intersect with the ones of the tetrahedron with scale factor R when $R \in [\rho/3, 3\rho]$. Moreover the rotated tetrahedron can be either

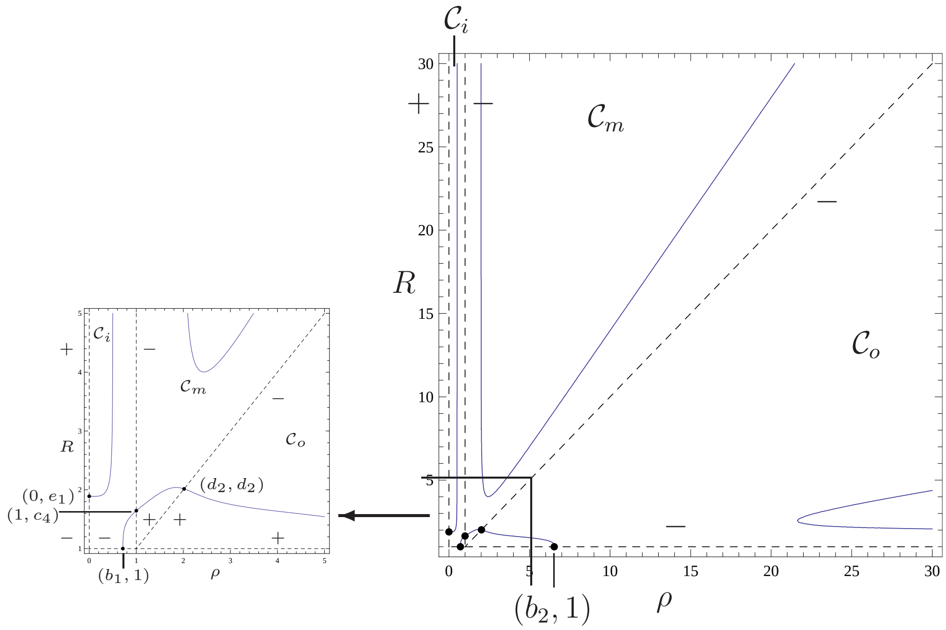


Fig. 6. The level curves $\det(A_3) = 0$. The dashed lines correspond to the boundaries of the regions $\mathcal{C}_m$, $\mathcal{C}_o$ and $\mathcal{C}_i$.

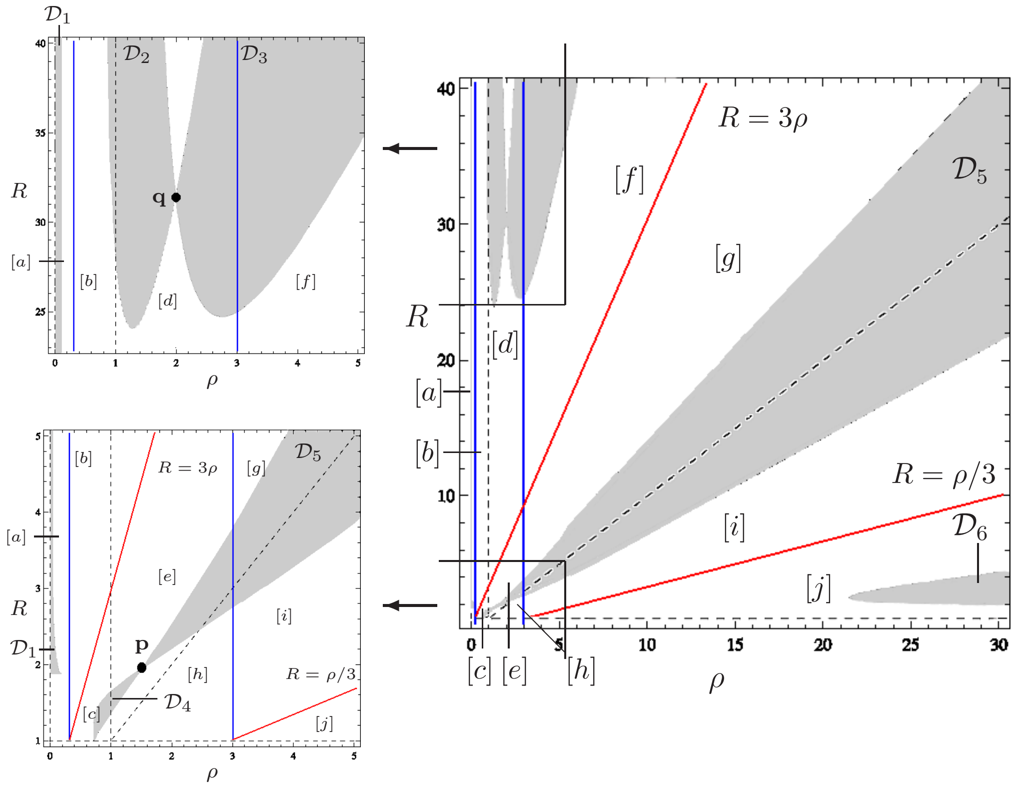


Fig. 7. Geometric properties of the nested tetraheda in $\mathcal{D}$. We use the notation $[\{a, b, c\}, a \cap b, a\overline{\cap}c]$ where $\{a, b, c\}$ denotes that a, b, and c are the scale factors of the inner, medium and outer tetrahedra, respectively, $a \cap b$ denotes that the faces of the tetrahedron with scale factor a intersect with the ones of the scale factor b, and $a\overline{\cap}c$ denotes that the faces of the tetrahedron with scale factor a does not intersect with the ones of the scale factor c. Using this notation we obtain ten different regions $[a]$: $[\{\rho, 1, R\}, \rho\overline{\cap}1, \rho\overline{\cap}R]$, $[b]$: $[\{\rho, 1, R\}, \rho\cap 1, \rho\overline{\cap}R]$, $[c]$: $[\{\rho, 1, R\}, \rho\cap 1, \rho\cap R]$, $[d]$: $[\{1, \rho, R\}, \rho \cap 1, \rho\overline{\cap}R]$, $[e]$: $[\{1, \rho, R\}, \rho \cap 1, \rho \cap R]$, $[f]$: $[\{1, \rho, R\}, \rho\overline{\cap}1, \rho\overline{\cap}R]$, $[g]$: $[\{1, \rho, R\}, \rho\overline{\cap}1, \rho \cap R]$, $[h]$: $[\{1, R, \rho\}, \rho \cap 1, \rho \cap R]$, $[i]$: $[\{1, R, \rho\}, \rho\overline{\cap}1, \rho \cap R]$, $[j]$: $[\{1, R, \rho\}, \rho\overline{\cap}1, \rho\overline{\cap}R]$. These ten regions are limited by the dashed and continuous lines in the figures.

the inner, the medium and the outer one. Statement (b) can be proved by analyzing these geometric properties of the three nested tetrahedra on the regions $\mathcal{D}_i$ for $i = 1, \ldots, 6$ (see Figure 7 for more details).

Finally we analyze the solution of (9) at the points $\mathbf{p}$ and $\mathbf{q}$. The numerical values of $\mathbf{p}$ and $\mathbf{q}$ can be found by solving system $\det(A) = \det(A_2) = \det(A_3) = 0$.

Let

$$\overline{A}(\rho) = \begin{pmatrix} 1 & f(1, \rho) \\ \rho & -1/(2\rho^2) \end{pmatrix} .$$

We note that $\det(\overline{A}(\rho)) = -f_3(\rho)$, so $\det(\overline{A}(\rho)) \neq 0$ for all $\rho \neq b_1$ and $\rho \neq b_2$. In particular, $\det(\overline{A}(p_1)) \neq 0$ and $\det(\overline{A}(q_1)) \neq 0$.

Since $\det(A) = \det(A_3) = 0$ and $\det(\overline{A}(\rho)) \neq 0$, system (9) is equivalent to system

$$\begin{pmatrix} 1 & f(1,\rho) \\ \rho & -1/(2\rho^2) \end{pmatrix} \begin{pmatrix} \lambda \\ m \end{pmatrix} = \begin{pmatrix} 1/2 - Mg_1(R) \\ -f(\rho,1) - Mf(\rho,R) \end{pmatrix} .$$

and it has infinitely many solutions which are given by

$$\begin{aligned} \lambda = \lambda(M) &= \frac{1}{\det(\overline{A})} \begin{vmatrix} 1/2 - Mg_1(R) & f(1,\rho) \\ -f(\rho,1) - Mf(\rho,R) & -1/(2\rho^2) \end{vmatrix} \\ &= -\frac{4\rho^2 f(1,\rho) f(\rho,1) - 1}{2\left(2\rho^3 f(1,\rho) + 1\right)} - M\frac{2\rho^2 f(1,\rho) f(\rho,R) + g_1(R)}{2\rho^3 f(1,\rho) + 1} , \\ m = m(M) &= \frac{1}{\det(\overline{A})} \begin{vmatrix} 1 & 1/2 - Mg_1(R) \\ \rho & -f(\rho,1) - Mf(\rho,R) \end{vmatrix} \\ &= \frac{\rho^2(\rho + 2f(\rho,1))}{2\rho^3 f(1,\rho) + 1} - M\frac{2\rho^2(\rho g_1(R) - f(\rho,R))}{2\rho^3 f(1,\rho) + 1} . \end{aligned} \tag{12}$$

This solution provides a central configuration of the 12–body problem when $\lambda(M) > 0$, $m(M) > 0$ and $M > 0$. Moreover $m(M) = 0$ when $M = (\rho^2(\rho + 2f(\rho,1)))/(2\rho^2(\rho g_1(R) - f(\rho,R)))$.

Evaluating (12) at the point $\mathbf{p}$ we get $\lambda = 0.3706146625\cdots + 0.6060692429\ldots M$, and $m = -0.4001317738\cdots + 3.1936671053\ldots M$. Moreover this solution provides a central configuration of the 12–body problem when $M > 0.1252891302\ldots$. Finally evaluating (12) at the point $\mathbf{q}$ we get the solution $\lambda = -0.0001803428746\cdots + 0.00001606707675\ldots M$, and $m = -3.5811747685\cdots + 0.0001420840585\ldots M$, which provides a central configuration of the 12–body problem when $M > 25204.620455\ldots$. This completes the proof of statement (c).

3.1 The functions $m(\rho, R)$ and $M(\rho, R)$

In this section we analyze some of the properties of the functions $m(\rho, R)$ and $M(\rho, R)$ on the regions $\mathcal{D}_i$ for $i = 1, \ldots, 6$.

After some tedious computations we can prove that

$$\lim_{R\to\infty} M(\rho, R) = \begin{cases} +\infty \text{ if } \rho \in K_1, \\ -\infty \text{ if } \rho \in K_2, \end{cases}$$

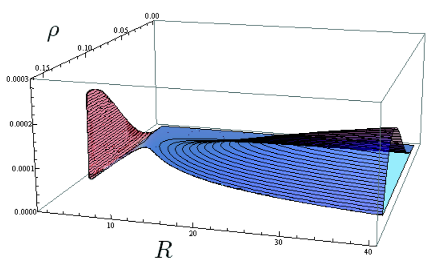
where

$$K_1 = (0, 0.501930\ldots) \cup (0.715223\ldots, 1.992306\ldots) \cup (6.536078, \infty),$$
$$K_2 = (0.501930\ldots, 0.715223\ldots) \cup (1.992306\ldots, 6.536078\ldots),$$

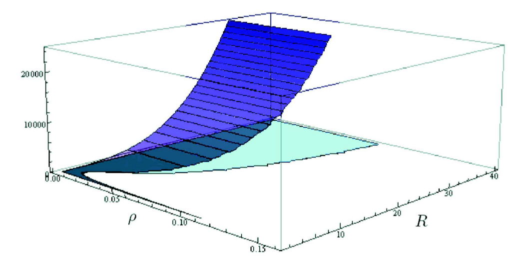
$$\lim_{\rho\to\infty} M(\rho,R) = m_I(R), \qquad \lim_{\rho\to 0} M(\rho,R) = m_I(R),$$
$$\lim_{R\to\infty} m(\rho,R) = m_{II}(\rho), \qquad \lim_{\rho\to\infty} m(\rho,R) = +\infty, \qquad \lim_{\rho\to 0} m(\rho,R) = 0,$$

where $m_I(\rho)$ and $m_{II}(\rho)$ are the solutions given in Theorem 2.

With the help of Mathematica, we plot the functions $m(\rho,R)$ and $M(\rho,R)$ defined in (11) on the regions $\mathcal{D}_i$ for $i = 1,\ldots,6$. We also find numerically the minimum and the maximum values of m and M in each region. The results that we have obtained are the following.

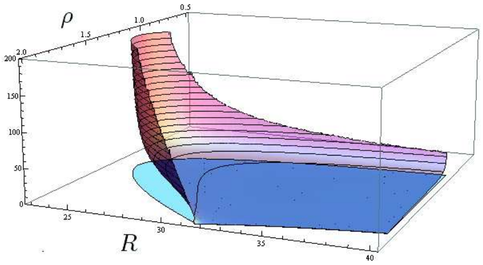


(a) The plot of $m(\rho,R)$ on the region $\mathcal{D}_1$. Here $m \in (0, 0.000347182\ldots)$.

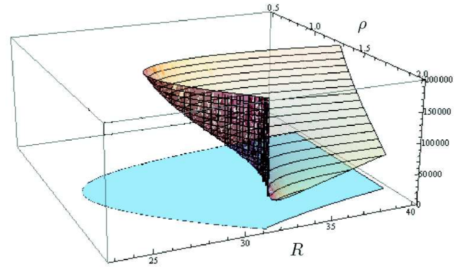


(b) The plot of $M(\rho,R)$ on the region $\mathcal{D}_1$. Here $M \in (0,\infty)$.

On the region $\mathcal{D}_1$ the function $m(\rho,R)$ is equal to zero on the boundaries $\rho = 0$ and $m(\rho,R) = 0$. Moreover on the boundary $M(\rho,R) = 0$ and when $R \to +\infty$ the behaviour of $m(\rho,R)$ is equal to the one of the function $m_{II}(\rho)$ with $\rho \in (0, 0.152996\ldots)$. From the plot of $m(\rho,R)$ we see that its maximum value takes place either on $M(\rho,R) = 0$, or when $R \to \infty$. Thus on $\mathcal{D}_1$ we have that $m \in (0, \widetilde{m}_0)$ with $\widetilde{m}_0 = 0.000347182\ldots$. Clearly the function $M(\rho,R)$ is equal to zero on the boundary $M(\rho,R) = 0$ and the behaviour of $M(\rho,R)$ on the boundary $\rho = 0$ is equal to $m_I(R)$ which tends to infinity as $R \to \infty$. Therefore $M \in (0,+\infty)$ on $\mathcal{D}_1$.

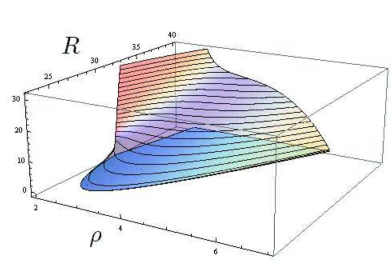


(c) The plot of $m(\rho, R)$ on the region $\mathcal{D}_2$. Here $m \in (0, \infty)$.

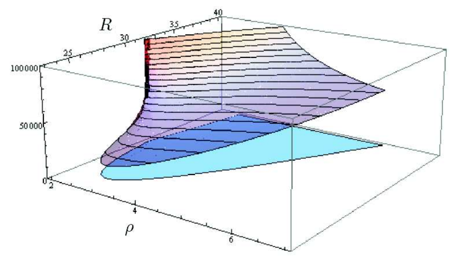


(d) The plot of $M(\rho, R)$ on the region $\mathcal{D}_2$. Here $M \in (25204.62\ldots, \infty)$.

Clearly the function $m(\rho, R)$ on $\mathcal{D}_2$ is equal to zero on the boundary $m(\rho, R) = 0$ and it tends to infinity when we approach to the boundary $\det(A) = 0$. Therefore on $\mathcal{D}_2$ we have that $m \in (0, +\infty)$. The function $M(\rho, R)$ tends also to infinity when we approach to the boundary $\det(A) = 0$. Moreover, from the plot of $M(\rho, R)$, we see that its minimum value takes place on the boundary $m(\rho, R) = 0$, in particular, it takes place on the point $\mathbf{q}$ and the minimum value is $M = M_0(\mathbf{q}) = 25204.620455\ldots$ (see Result 3 for the definitions of $\mathbf{q}$ and $M_0(\mathbf{q})$).

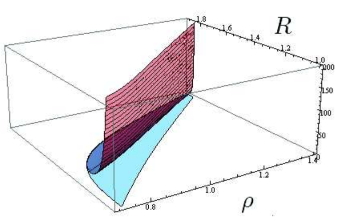


(e) The plot of $m(\rho, R)$ on the region $\mathcal{D}_3$. Here $m \in (0, \infty)$.

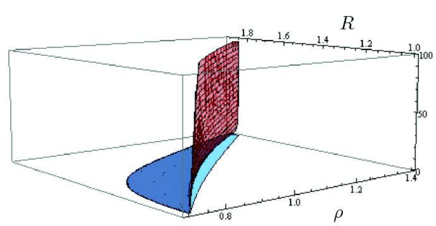


(f) The plot of $M(\rho, R)$ on the region $\mathcal{D}_3$. Here $M \in (11625.1167\ldots, \infty)$.

On $\mathcal{D}_3$ clearly the function $m(\rho, R)$ is equal to zero on the boundary $m(\rho, R) = 0$ and tends to infinity on the boundary $\det(A) = 0$, so $m \in (0, +\infty)$ on $\mathcal{D}_3$. The function $M(\rho, R)$ also tends to infinity on $\det(A) = 0$. Moreover, from the plot of $M(\rho, R)$, we see that its minimum value takes place on the boundary $m(\rho, R) = 0$, in particular, it takes place on the point $(\rho, R) = (2.7556\ldots, 24.7316\ldots)$ and the minimum value is $M = 11625.1167\ldots$.

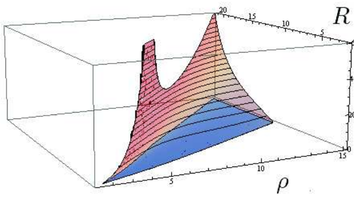


(g) The plot of $m(\rho, R)$ on the region $\mathcal{D}_4$. Here $m \in (0, \infty)$.

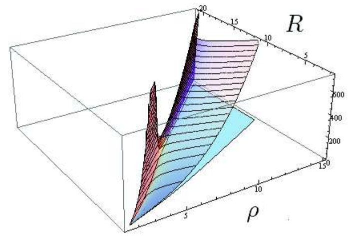


(h) The plot of $M(\rho, R)$ on the region $\mathcal{D}_4$. Here $M \in (0, \infty)$.

The set $\mathcal{D}_4$ is delimited by the curves $\det(A) = 0$, $M(\rho, R) = 0$, and finally, near $\mathbf{p}$, the curve $m(\rho, R) = 0$ (see Result 3 for the definition of $\mathbf{p}$). Clearly $m(\rho, R)$ (respectively, $M(\rho, R)$) is equal to zero on the curve $m(\rho, R) = 0$ (respectively, $M(\rho, R) = 0$) and tends to infinity when we approach to $\det(A) = 0$. Therefore $m \in (0, \infty)$ and $M \in (0, \infty)$ on $\mathcal{D}_4$.

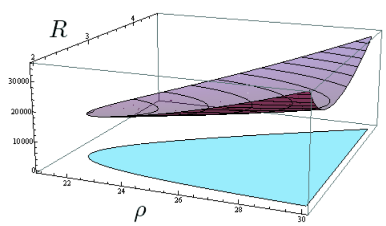


(i) The plot of $m(\rho, R)$ on the region $\mathcal{D}_5$. Here $m \in (0, \infty)$.

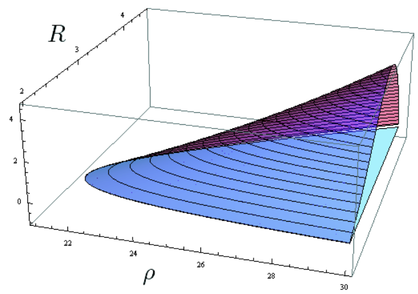


(j) The plot of $M(\rho, R)$ on the region $\mathcal{D}_5$. Here $M \in (0, \infty)$.

The set $\mathcal{D}_5$ is also delimited by the curves $\det(A) = 0$, $M(\rho, R) = 0$, and $m(\rho, R) = 0$. So $m \in (0, \infty)$ and $M \in (0, \infty)$ on $\mathcal{D}_5$.



(k) The plot of $m(\rho, R)$ on the region $\mathcal{D}_6$. Here $m \in (14997.0524\ldots, \infty)$.



(l) The plot of $M(\rho, R)$ on the region $\mathcal{D}_6$. Here $M \in (0, \infty)$

From the plot of $m(\rho, R)$, we see that the minimum value of $m(\rho, R)$ takes place on the boundary $M(\rho, R) = 0$, in particular, it takes place on the point $(\rho, R) = (21.62574\ldots, 2.56200\ldots)$ and the minimum value is $m = 14997.0524\ldots$. Moreover $m(\rho, R) \to +\infty$ when $\rho \to +\infty$. Therefore $m \in (14997.0524\ldots, \infty)$ on $\mathcal{D}_6$. On the other hand clearly $M(\rho, R)$ is equal to zero on $M(\rho, R) = 0$ and $M(\rho, R) \to m_I(R)$ as $\rho \to +\infty$. So $M \in (0, \infty)$.

Appendix

In this appendix we analyze the solutions of the equations of the form $F(x) = 0$ when F is a rational function containing radicals. These type of equations are solved by following the next steps.

(1) We eliminate the denominators of the fractions which appear in $F(x)$ by multiplying equation $F(x) = 0$ by the least common multiple of all the denominators.
(2) We eliminate the radicals of the resulting equation by isolating in a convenient way one or more radicals on one side of the equation and squaring both sides of the equation. If the resulting equation still contains radicals, then we repeat the process again. At the end we obtain a polynomial equation.
(3) We find numerically all the solutions of the polynomial equation obtained in step 2.
(4) Finally we cheek which of these solutions are really solutions of the initial equation $F(x) = 0$.

Now we detail how to group the radicals in step 2 for each type of equations that appear in this work after applying step 1.

(a) Equations with one radical: $\alpha_1\sqrt{a} + \alpha_2 = 0$. We eliminate the radicals by applying step 2 in the following way

$$(\alpha_1\sqrt{a})^2 = (-\alpha_2)^2 .$$

(b) Equations of the form: $\alpha_1\sqrt{a} + \alpha_2\sqrt{b} + \alpha_3\sqrt{a}\sqrt{b} + \alpha_4 = 0$. Applying step 2 in the following way

$$(\alpha_1\sqrt{a} + \alpha_2\sqrt{b})^2 = (-\alpha_3\sqrt{a}\sqrt{b} - \alpha_4)^2 .$$

We obtain an equation with one radical of the form $\beta_1\sqrt{a}\sqrt{b} + \beta_2 = 0$.
(c) Equations of the form: $\alpha_1\sqrt{a}\sqrt{b}+\alpha_2\sqrt{a}\sqrt{c}+\alpha_3\sqrt{b}\sqrt{c} = 0$. Applying step 2 in the following way

$$(\alpha_1\sqrt{a}\sqrt{b})^2 = (-\alpha_2\sqrt{a}\sqrt{c} - \alpha_3\sqrt{b}\sqrt{c})^2 .$$

We obtain an equation with one radical of the form $\beta_1\sqrt{a}\sqrt{b} + \beta_2 = 0$.

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References

[1] F. Cedó and J. Llibre, *Symmetric central configurations of the spatial n–body problem*, J. of Geometry and Physics **6** (1989), 367–394.

[2] M. Corbera and J. Llibre, *Central configurations for the spatial $2n$–body problem of nested regular polyhedra*, J. of Geometry and Physics **58** (2008), 1241–1252.

[3] M. Corbera and J. Llibre, *Central configurations of three nested regular polyhedra for the spatial $3n$–body problem*, preprint 2008.

[4] H. Goldstein, *The Euler angles and Euler angles in alternative conventions*, Section 4.4 and Appendix B in Celestial Mechanics, 2nd ed. Reading, MA, Addison-Wesley, pp 143–148 and 606–610, 1980.

[5] X. Liu, T. Feng, and X. Chen, *Eight-body central configuration for conjugate-nested regular tetrahedrons*, J. of Chongqing University (English Edition), **6** (2007), 294–298.

[6] C. Zhu, *Central configurations of nested regular tetrahedrons*, J. Math. Anal. Appl. **312** (2005), 83–92.