DOUBLE–ANTIPRISM CENTRAL CONFIGURATIONS OF THE $3n$–BODY PROBLEM

ABSTRACT. In this paper we study numerically a new type of central configurations of the $3n$–body problem with equal masses which consist of three n–gons contained in three planes $z = 0$ and $z = \pm\beta \neq 0$. The n–gon on $z = 0$ is scaled by a factor α and it is rotated by an angle of π/n with respect to the ones on $z = \pm\beta$. In this kind of configurations, the masses on the planes $z = 0$ and $z = \beta$ are at the vertices of an antiprism with bases of different size. The same occurs with the masses on $z = 0$ and $z = -\beta$. We call this kind of central configurations *double–antiprism central configurations*. We will show the existence of central configurations of this type.

1. INTRODUCTION

We consider the spatial N–body problem

$$m_k\, \ddot{\mathbf{q}}_k = - \sum_{\substack{j=1 \\ j \neq k}}^{N} G\, m_k\, m_j\, \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} \ ,$$

$k = 1, \ldots, N$, where $\mathbf{q}_k \in \mathbb{R}^3$ is the position vector of the punctual mass m_k in an inertial coordinate system, and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. We fix the center of mass $\sum_{i=1}^{N} m_i\, \mathbf{q}_i / \sum_{i=1}^{N} m_i$ of the system at the origin of $\mathbb{R}^{3N}$. The *configuration space* of the N–body problem in $\mathbb{R}^3$ is

$$\mathcal{E} = \{(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathbb{R}^{3N} \ : \ \sum_{i=1}^{N} m_i\, \mathbf{q}_i = 0, \ \ \mathbf{q}_i \neq \mathbf{q}_j \text{ for } i \neq j\} \ .$$

Given $m_1, \ldots, m_N$ a configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_k = -\lambda\, \mathbf{q}_k \ ,$$

$k = 1, \ldots, N$. Thus a central configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ of the N–body problem with positive masses $m_1, \ldots, m_N$ is a solution of the system of the

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N vectorial equations

$$\sum_{\substack{j=1 \\ j \neq k}}^{N} m_j \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} = \lambda \, \mathbf{q}_k, \tag{1}$$

for $k = 1, \ldots, N$ and $N+1$ unknowns $\mathbf{q}_k$ for $k = 1, \ldots, N$ plus the unknown $\lambda > 0$.

The simplest spatial central configurations of the N–body problem are the ones with N equal masses at the vertices of a regular polyhedron of N vertices, see for instance [1]. There are also some papers studying central configurations consisting of nested regular polyhedra, see for instance [2, 3] and the references therein.

In [1] the authors study the spatial central configurations of the N–body problem with equal masses such that the set of positions $\{\mathbf{q}_1, \ldots, \mathbf{q}_N\}$ is an orbit by the action of a finite subgroup of $O(3)$. Under this condition, they find that the antiprism with $N = 2k > 6$ vertices, the prism with regular bases with $N = 2k > 4$ vertices are central configurations. Moreover they find some additional classes of symmetric central configurations for $N = 4, 6, 8, 12, 20, 24, 30, 48, 60, 120,$ between them the regular polyhedra.

In addition to regular polyhedron central configurations, the simplest spatial central configurations of the N–body problem are the ones known as *pyramidal central configurations*, which consists of $N = n + 1$ masses, n of which are coplanar and the $(n + 1)$–th being off the plane (see for instance [5] and [9]). The n positions of the coplanar masses are called the base of the pyramidal central configuration. There are also the central configurations known as *double pyramidal central configurations.* Such configurations consist of $N = n + 2$ masses, n of which are coplanar and the other two being off the plane and positioned symmetrically above and below that plane. In the literature we can find some papers related with double pyramidal central configurations with different shapes of bases. For instance, in [10] the authors studied for all $n \geq 4$ the double pyramidal central configurations such that the n equal coplanar masses are at the vertices of a regular n–gon, and the two masses outside the plane determined by the n–gon are equal. In [4] the authors also consider spatial central configurations consisting of $n \geq 2$ equal masses at the vertices of a regular n–gon and two masses outside the plane determined by the n–gon but they do not impose conditions on the two masses being outside the plane, neither on the positions, nor on the values of these masses (this problem has also been studied in [6, 7] for $n = 3$).

In [8] the authors find spatial central configurations that bifurcate from planar ones consisting of two regular n–gons, lying in horizontal planes, centered on a common vertical axis, and aligned so that corresponding vertices lie in the same vertical half–plane, i.e. the masses are at the vertices of a truncated pyramid.

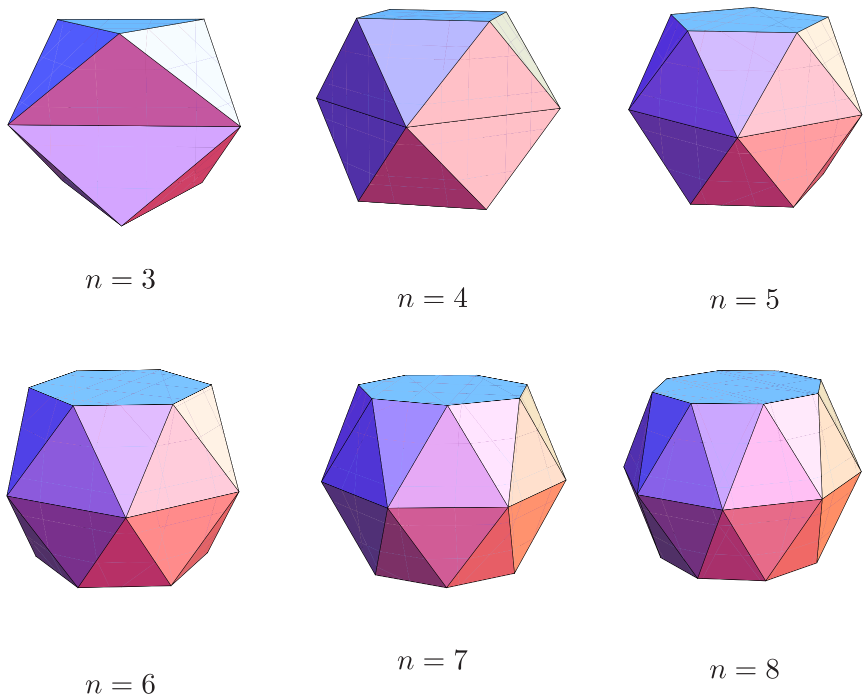


FIGURE 1. The double–antiprism central configurations for $n = 3, 4, 5, 6, 7, 8$.

In this paper we show the existence of spatial central configurations of the $N = 3n$ body problem with equal masses consisting of three regular n–gons with $n \geq 2$, lying in three horizontal equi–spaced planes centered on a common vertical axis, and such that the n–gons on the upper and the bottom plane are identical whereas the n–gon on the middle plane is scaled by a factor α and rotated by an angle π/n with respect to the other two, see Figure 1. Note that in these configurations the masses at the vertices located on the upper and the bottom plane form a prism, whereas the masses at the vertices located on the upper and the middle one (respectively, the bottom and the middle one) form an antiprism whose bases have different sizes. For this reason we call this kind of central configurations *double–antiprism central configurations*. Notice that these central configurations are neither a prism nor an antiprism, so in general they are not the ones obtained in [1].

As we shall see in this paper, we have found numerically double–antiprism central configurations for $n = 2, 3, \ldots, 100000$. We also provide some conjectures related to the existence and the uniqueness for each integer $n \geq 2$ of the double–antiprism central configurations.

2. Equations of the double–antiprism central configurations

We consider $3n$ equal masses $m_1 = \cdots = m_{3n} = 1$. Let $\alpha, \beta > 0$ and let $\theta_j = 2\pi\, j/n$ and $\varphi_j = \theta_j + \pi/n$ for all $j = 1, \ldots, n$. Assume that the masses $m_1, \ldots, m_n$ are at the vertices of a regular n–gon on the plane $z = 0$ and that their positions are $\mathbf{q_j} = (\alpha\, e^{i\theta_j}, 0) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$ with $\alpha > 0$. The masses $m_{n+1}, \ldots, m_{2n}$ are at the vertices of a regular n–gon on the plane $z = \beta > 0$ and their positions are $\mathbf{q_{n+j}} = (e^{i\varphi_j}, \beta)$. And finally the masses $m_{2n+1}, \ldots, m_{3n}$ are at the vertices of a regular n–gon on the plane $z = -\beta$ and their positions are $\mathbf{q_{2n+j}} = (e^{i\varphi_j}, -\beta)$.

It is easy to check that the center of mass of these configurations is at the origin of coordinates. Then equations (1) become

$$
\begin{aligned}
\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\mathbf{q}_k - \mathbf{q}_j}{r_{k,j}^3} + \sum_{j=1}^{n} \frac{\mathbf{q}_k - \mathbf{q}_{n+j}}{r_{k,n+j}^3} + \sum_{j=1}^{n} \frac{\mathbf{q}_k - \mathbf{q}_{2n+j}}{r_{k,2n+j}^3} &= \lambda\, \mathbf{q}_k,\\
\sum_{j=1}^{n} \frac{\mathbf{q}_{n+k} - \mathbf{q}_j}{r_{n+k,j}^3} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\mathbf{q}_{n+k} - \mathbf{q}_{n+j}}{r_{n+k,n+j}^3} + \sum_{j=1}^{n} \frac{\mathbf{q}_{n+k} - \mathbf{q}_{2n+j}}{r_{n+k,2n+j}^3} &= \lambda\, \mathbf{q}_{n+k},\\
\sum_{j=1}^{n} \frac{\mathbf{q}_{2n+k} - \mathbf{q}_j}{r_{2n+k,j}^3} + \sum_{j=1}^{n} \frac{\mathbf{q}_{2n+k} - \mathbf{q}_{n+j}}{r_{2n+k,n+j}^3} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\mathbf{q}_{2n+k} - \mathbf{q}_{2n+j}}{r_{2n+k,2n+j}^3} &= \lambda\, \mathbf{q}_{2n+k},
\end{aligned}
\tag{2}
$$

for $k = 1, \ldots, n$, where $r_{\ell,i} = |\mathbf{q}_\ell - \mathbf{q}_i|$.

The equations for the complex coordinate of the vectorial equations (2) are

$$
\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\alpha\, e^{i\theta_k} - \alpha\, e^{i\theta_j}}{r_{k,j}^3} + \sum_{j=1}^{n} \frac{\alpha\, e^{i\theta_k} - e^{i\varphi_j}}{r_{k,n+j}^3} + \sum_{j=1}^{n} \frac{\alpha\, e^{i\theta_k} - e^{i\varphi_j}}{r_{k,2n+j}^3} = \lambda\, \alpha\, e^{i\theta_k}, \tag{3}
$$

$$
\sum_{j=1}^{n} \frac{e^{i\varphi_k} - \alpha\, e^{i\theta_j}}{r_{n+k,j}^3} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{e^{i\varphi_k} - e^{i\varphi_j}}{r_{n+k,n+j}^3} + \sum_{j=1}^{n} \frac{e^{i\varphi_k} - e^{i\varphi_j}}{r_{n+k,2n+j}^3} = \lambda\, e^{i\varphi_k}, \tag{4}
$$

$$
\sum_{j=1}^{n} \frac{e^{i\varphi_k} - \alpha\, e^{i\theta_j}}{r_{2n+k,j}^3} + \sum_{j=1}^{n} \frac{e^{i\varphi_k} - e^{i\varphi_j}}{r_{2n+k,n+j}^3} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{e^{i\varphi_k} - e^{i\varphi_j}}{r_{2n+k,2n+j}^3} = \lambda\, e^{i\varphi_k}, \tag{5}
$$

for $k = 1, \ldots, n$. The equations for the real coordinate of the vectorial equations (2) are

$$
\begin{aligned}
& 0 = 0, \\
& \sum_{j=1}^{n} \frac{\beta}{r_{n+k,j}^3} + \sum_{j=1}^{n} \frac{2\beta}{r_{n+k,2n+j}^3} = \lambda\, \beta, \qquad (6) \\
& -\sum_{j=1}^{n} \frac{\beta}{r_{2n+k,j}^3} - \sum_{j=1}^{n} \frac{2\,\beta}{r_{2n+k,n+j}^3} = -\lambda\, \beta, \qquad (7)
\end{aligned}
$$

for $k = 1, \ldots, n$. The explicit expressions of the denominators that appear in (2) are

$$
\begin{aligned}
r_{k,j} &= 2\,\alpha \left|\sin\left(\frac{\theta_j - \theta_k}{2}\right)\right|, \\
r_{k,n+j} = r_{k,2n+j} &= \sqrt{1 + \alpha^2 + \beta^2 - 2\,\alpha\, \cos(\varphi_j - \theta_k)}, \\
r_{n+k,j} = r_{2n+k,j} &= \sqrt{1 + \alpha^2 + \beta^2 - 2\,\alpha\, \cos(\theta_j - \varphi_k)}, \\
r_{n+k,n+j} = r_{2n+k,2n+j} &= 2\left|\sin\left(\frac{\varphi_j - \varphi_k}{2}\right)\right|, \\
r_{n+k,2n+j} = r_{2n+k,n+j} &= \sqrt{2 + 4\,\beta^2 - 2\, \cos(\varphi_j - \varphi_k)},
\end{aligned}
$$

for $j, k = 1, \ldots, n$. Note that for all $k = 1, \ldots, n$ equations (4) and (5) are the same, and equations (6) and (7) only differ in the sign. Therefore system (2) is equivalent to the system formed by the equations (3), (4) and (6) for $k = 1, \ldots, n$.

By dividing the k–th equation of (3) by $e^{i\theta_k}$, the k–th equation of (4) by $e^{i\varphi_k}$, and the k–th equation of (6) by β we get that system (2) is equivalent to system

$$
\begin{aligned}
& \alpha \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 - e^{i(\theta_j - \theta_k)}}{r_{k,j}^3} + 2 \sum_{j=1}^{n} \frac{\alpha - e^{i(\varphi_j - \theta_k)}}{r_{k,n+j}^3} = \lambda\, \alpha, \\
& \sum_{j=1}^{n} \frac{1 - \alpha\, e^{i(\theta_j - \varphi_k)}}{r_{n+k,j}^3} + \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 - e^{i(\varphi_j - \varphi_k)}}{r_{n+k,n+j}^3} + \sum_{j=1}^{n} \frac{1 - e^{i(\varphi_j - \varphi_k)}}{r_{n+k,2n+j}^3} = \lambda, \qquad (8) \\
& \sum_{j=1}^{n} \frac{1}{r_{n+k,j}^3} + 2 \sum_{j=1}^{n} \frac{1}{r_{n+k,2n+j}^3} = \lambda,
\end{aligned}
$$

for $k = 1, \ldots, n$.

Playing with the real and imaginary part of the previous equations and by using the properties of the trigonometric functions we can see that all the summations in equations (8) are independent of k. Moreover, by choosing conveniently the value of k in each summation, after some tedious computations these equations can be written as

$$(9)\quad \begin{aligned}
\frac{1}{4\alpha^2}\sum_{j=1}^{n-1}\csc\left(\frac{\pi\, j}{n}\right)+2\sum_{j=1}^{n}\frac{\alpha+1-2\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)}{R_1^{3/2}} &= \lambda\,\alpha,\\
\sum_{j=1}^{n}\frac{\alpha+1-2\alpha\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)}{R_1^{3/2}}+\frac{1}{4}\sum_{j=1}^{n-1}\csc\left(\frac{\pi\, j}{n}\right)+ &\\
\frac{1}{4}\sum_{j=1}^{n}\frac{1-\cos^2\left(\frac{\pi\, j}{n}\right)}{R_2^{3/2}} &= \lambda,\\
\sum_{j=1}^{n}\frac{1}{R_1^{3/2}}+\frac{1}{4}\sum_{j=1}^{n}\frac{1}{R_2^{3/2}} &= \lambda,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= (\alpha+1)^2+\beta^2-4\,\alpha\,\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right),\\
R_2 &= 1+\beta^2-\cos^2\left(\frac{\pi\, j}{n}\right).
\end{aligned}$$

Substituting λ from the third equation of (9) into the first two equations of (9) we get the system of two equations

$$(10)\quad \begin{aligned}
&\frac{1}{4\alpha^2}\sum_{j=1}^{n-1}\csc\left(\frac{\pi\, j}{n}\right)+\sum_{j=1}^{n}\frac{\alpha+2-4\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)}{R_1^{3/2}}-\frac{1}{4}\sum_{j=1}^{n}\frac{\alpha}{R_2^{3/2}}=0,\\
&\frac{1}{4}\sum_{j=1}^{n-1}\csc\left(\frac{\pi\, j}{n}\right)+\alpha\sum_{j=1}^{n}\frac{1-2\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)}{R_1^{3/2}}-\frac{1}{4}\sum_{j=1}^{n}\frac{\cos^2\left(\frac{\pi\, j}{n}\right)}{R_2^{3/2}}=0,
\end{aligned}$$

with the two unknowns α and β.

3. The existence of double–antiprism central configurations

Let

$$
\begin{aligned}
A &= \frac{1}{4}\sum_{j=1}^{n-1} \csc\left(\frac{\pi j}{n}\right) > 0,\\
f_1(\alpha,\beta) &= \sum_{j=1}^{n} \frac{\alpha^3 + 2\alpha^2 - 4\alpha^2\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)}{\left((\alpha+1)^2+\beta^2-4\alpha\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)\right)^{3/2}},\\
f_2(\alpha,\beta) &= \frac{1}{4}\sum_{j=1}^{n} \frac{\alpha^3}{\left(1+\beta^2-\cos^2\left(\frac{\pi j}{n}\right)\right)^{3/2}},\\
g_1(\alpha,\beta) &= \alpha\sum_{j=1}^{n} \frac{1-2\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)}{\left((\alpha+1)^2+\beta^2-4\alpha\cos^2\left(\frac{\pi}{n}\left(j-\frac{1}{2}\right)\right)\right)^{3/2}},\\
g_2(\beta) &= \frac{1}{4}\sum_{j=1}^{n} \frac{\cos^2\left(\frac{\pi j}{n}\right)}{\left(1+\beta^2-\cos^2\left(\frac{\pi j}{n}\right)\right)^{3/2}}.
\end{aligned}
$$

Note that all the denominators in the previous functions are positive because $\beta > 0$ and they are distances between two different bodies. By using this notation the system (10) can be written as

$$
\begin{aligned}
F(\alpha,\beta) &= A + f_1(\alpha,\beta) - f_2(\alpha,\beta) = 0,\\
G(\alpha,\beta) &= A + g_1(\alpha,\beta) - g_2(\beta) = 0.
\end{aligned}
\tag{11}
$$

3.1. Analysis of the curve $G(\alpha,\beta) = 0$.

When $\alpha = 0$ we have that

$$G(0,\beta) = A - g_2(\beta).$$

We claim that for $\alpha = 0$ equation $G(0,\beta) = 0$ has a unique solution with $\beta > 0$. Now we prove the claim, clearly

$$\lim_{\beta\to 0^+} g_2(\beta) = +\infty \qquad \text{and} \qquad \lim_{\beta\to+\infty} g_2(\beta) = 0.$$

Moreover

$$g_2'(\beta) = -\frac{3\beta}{4}\sum_{j=1}^{n} \frac{\cos^2\left(\frac{\pi j}{n}\right)}{\left(1+\beta^2-\cos^2\left(\frac{\pi j}{n}\right)\right)^{5/2}} < 0,$$

for all $\beta > 0$. In short, $G(0,\beta)$ is an increasing function of β such that

$$\lim_{\beta\to 0^+} G(0,\beta) = -\infty, \qquad \lim_{\beta\to+\infty} G(0,\beta) = A > 0,$$

and consequently $G(0,\beta) = 0$ has a unique solution $\beta = h_0 > 0$. So the claim is proved.

When $\alpha \to +\infty$, then

$$\lim_{\alpha\to+\infty} G(\alpha,\beta) = G(0,\beta).$$

Therefore when $\alpha \to +\infty$, equation $G(\alpha,\beta)=0$ has a unique solution which coincides with the solution for $\alpha = 0$.

If the derivative $dG/d\beta \neq 0$ for all $\alpha, \beta > 0$ then we can assure that there exist a curve $\beta = h(\alpha)$, defined for all $\alpha \geq 0$, such that $h(0) = h_0 > 0$, where $\beta = h_0$ is the unique solution $G(0,\beta) = 0$, and $\lim_{\alpha\to+\infty} h(\alpha) = h_0$ that satisfies $G(\alpha,\beta) = 0$. Moreover if $dG/d\beta$ evaluated at $(\alpha,\beta) = (0,h_0)$ is different from zero, then from the implicit function theorem the solution of $G(\alpha,\beta) = 0$ for $\alpha = 0$ can be continued to a unique solution $G(\alpha,\beta) = 0$ for $\alpha > 0$ small. Therefore the curve $\beta = h(\alpha)$ is the unique curve solution of $G(\alpha,\beta) = 0$ such that $\lim_{\alpha\to 0^+} h(\alpha) = h_0$. After some simplifications, we get

$$\frac{d}{d\beta} G(\alpha,\beta) = 3\,\alpha\beta \sum_{j=1}^{n} a_j - g_2'(\beta), \tag{12}$$

where

$$a_j = \frac{\cos\psi_j}{(1+\alpha^2+\beta^2-2\,\alpha\,\cos\psi_j)^{5/2}},$$

and $\psi_j = \frac{\pi}{n}(2j-1)$. We have seen that $g_2'(\beta) < 0$ for all $\beta > 0$. So, from (12), if $\sum_{j=1}^{n} a_j > 0$ for all $\alpha,\beta > 0$, then $dG/d\beta > 0$ for all $\alpha,\beta > 0$. Moreover, from (12) again, $dG/d\beta$ evaluated at $\alpha = 0$ is different from zero. Therefore if $\sum_{j=1}^{n} a_j > 0$ for all $\alpha,\beta > 0$, then there exists a unique curve solution of $G(\alpha,\beta) = 0$, $\beta = h(\alpha)$, defined for all $\alpha > 0$, satisfying that $\lim_{\alpha\to 0} h(\alpha) = \lim_{\alpha\to+\infty} h(\alpha) > 0$.

Now we analyze the sign of $\sum_{j=1}^{n} a_j$ when $\alpha > 0$. By using the properties of the trigonometric functions we see that

$$\cos\psi_{n-j+1} = \cos\psi_j \qquad \text{for all } j = 1,\dots,[n/2]. \tag{13}$$

Assume that n is even, that is $n = 2k$ for some $k \in \mathbb{N}$. From (13) we have that

$$\sum_{j=1}^{n} a_j = 2\sum_{j=1}^{k} a_j.$$

Moreover it is easy to check that

$$\cos\psi_{k-j+1} = -\cos\psi_j \qquad \text{for all } j = 1,\dots,[k/2].$$

So if k is even, that is, $k = 2\ell$ for some $\ell \in \mathbb{N}$, then $\sum_{j=1}^{k} a_j$ is given by

$$\sum_{j=1}^{\ell} \cos\psi_j \left(\frac{1}{(1+\alpha^2+\beta^2-2\,\alpha\,\cos\psi_j)^{5/2}} - \frac{1}{(1+\alpha^2+\beta^2+2\,\alpha\,\cos\psi_j)^{5/2}} \right).$$

Since $\cos\psi_j > 0$ for all $j = 1,\dots,\ell$, then

$$\frac{1}{(1+\alpha^2+\beta^2-2\,\alpha\,\cos\psi_j)^{5/2}} > \frac{1}{(1+\alpha^2+\beta^2+2\,\alpha\,\cos\psi_j)^{5/2}}.$$

Therefore if $n = 4\ell$, then $\sum_{j=1}^{n} a_j > 0$ for all $\alpha,\beta > 0$.

If k is odd, that is $k = 2\ell + 1$ for some $\ell \in \mathbb{N}$, then $\sum_{j=1}^{k} a_j$ is given by

$$\sum_{j=1}^{\ell} \cos\psi_j \left(\frac{1}{(1+\alpha^2+\beta^2-2\,\alpha\,\cos\psi_j)^{5/2}} - \frac{1}{(1+\alpha^2+\beta^2+2\,\alpha\,\cos\psi_j)^{5/2}} \right) + \frac{\cos\psi_{\ell+1}}{(1+\alpha^2+\beta^2-2\,\alpha\,\cos\psi_{\ell+1})^{5/2}}.$$

Since $\cos\psi_{\ell+1} = \cos\pi/2 = 0$, we have that $\sum_{j=1}^{n} a_j > 0$ for all $\alpha, \beta > 0$ when $n = 4\ell + 2$.

In short we have proved the following result.

Lemma 1. *If n is even, then there exists a curve $\beta = h(\alpha)$ solution of $G(\alpha, \beta) = 0$, defined for all $\alpha > 0$, such that $\lim_{\alpha\to 0} h(\alpha) = \lim_{\alpha\to+\infty} h(\alpha) > 0$. Moreover this is the unique curve solution of $G(\alpha, \beta) = 0$ satisfying these properties.*

Assume now that n is odd, that is $n = 2k + 1$ for some $k \in \mathbb{N}$,

$$\sum_{j=1}^{n} a_j = 2\sum_{j=1}^{k} a_j + a_{k+1}.$$

Clearly

$$\cos\psi_{k+1} = \cos\pi = -1.$$

Since

$$\cos\psi_{k-j+1} = -\cos(2\pi\, j/n) \qquad \text{for all } j = 1, \ldots, [k/2],$$

it is not difficult to see that $\sum_{j=1}^{k} a_j$ can be written as $\sum_{j=1}^{k} b_j$ where

$$b_j = \frac{(-1)^{j+1}\cos\left(\frac{\pi\, j}{n}\right)}{\left(1+\alpha^2+\beta^2-2(-1)^{j+1}\alpha\,\cos\left(\frac{\pi\, j}{n}\right)\right)^{5/2}}.$$

Since $\cos(\pi\, j/n) > \cos(\pi\,(j+1)/n) > 0$ for all $j = 1, \ldots, k-1$, we have that

$$\frac{\cos\left(\frac{\pi}{n}(j+1)\right)}{\left(1+\alpha^2+\beta^2+2\,\alpha\,\cos\left(\frac{\pi}{n}(j+1)\right)\right)^{5/2}} < \frac{\cos\left(\frac{\pi\, j}{n}\right)}{\left(1+\alpha^2+\beta^2-2\,\alpha\,\cos\left(\frac{\pi\, j}{n}\right)\right)^{5/2}}.$$

So $b_1 > |b_2|$, $b_3 > |b_4|$, and so on. Therefore $\sum_{j=1}^{k} b_j > 0$ for all $\alpha, \beta > 0$.

We have numerical evidences that

$$2\sum_{j=1}^{k} b_j > -a_{k+1},$$

for all $k \geq 1$. So, under this assumption $\sum_{j=1}^{n} a_j > 0$ for all $\alpha, \beta > 0$. Therefore we do the following conjecture.

Conjecture 1. *Lemma 1 holds for all positive integer $n \geq 2$.*

3.2. **Analysis of the curve $F(\alpha,\beta)=0$.** Notice that the n–th term of the summation $f_2(\alpha,\beta)$, which is given by $\alpha^3/(4\,\beta^3)$, is not defined when $\beta=0$. All the other terms of the summations in the expression of $F(\alpha,\beta)$ are defined for all $\alpha,\beta\ge 0$.

Lemma 2. *The function F satisfies the following properties:*

(a) *fixed a value of $\alpha>0$, $\lim_{\beta\to 0^+} F(\alpha,\beta)=-\infty$,*
(b) *if $\alpha\to+\infty$ and $\beta\to 0^+$, then $F(\alpha,\beta)\to-\infty$,*
(c) *fixed a value of $\alpha>0$, $\lim_{\beta\to+\infty} F(\alpha,\beta)=A>0$,*
(d) *the limit of F when $(\alpha,\beta)\to(+\infty,+\infty)$ on straight lines of the form $\beta=a\,\alpha$ is*

$$\lim_{\alpha\to+\infty} F(\alpha,a\,\alpha)=A+\frac{n}{(a^2+1)^{3/2}}-\frac{n}{4a^3}=L(a),$$

(e) *for all $\beta>0$, $F(0,\beta)=A>0$,*
(f) $\lim_{\beta\to+\infty} F(0,\beta)=A>0$,
(g) *fixed a value of $\beta>0$, $\lim_{\alpha\to+\infty} F(\alpha,\beta)=-\infty$,*
(h) *the limit of F when $(\alpha,\beta)\to(0,0)$ on straight lines of the form $\beta=b\,\alpha$ is*

$$\lim_{\alpha\to 0} F(\alpha,b\,\alpha)=A-\frac{1}{4b^3}.$$

Proof. The proof follows immediately from simple computations. □

Lemma 3. *For each $\alpha>0$, there exists at least one $\beta=H(\alpha)>0$ such that $F(\alpha,H(\alpha))=0$.*

Proof. The proof follows immediately from Lemma 2(a) and (c). □

Lemma 4. *Let $\widetilde{H}$ be a curve solution of $F(\alpha,\beta)=0$. Then $\widetilde{H}$ must satisfy one of the following properties.*

(a) *$\widetilde{H}$ is a curve that starts and ends at the point $(0,0)$ tangent to the straight line $\beta=(4\,A)^{-1/3}\,\alpha$,*
(b) *$\widetilde{H}$ is a curve that starts and ends at $(+\infty,+\infty)$ tangent to the straight line $\beta=\overline{a}\,\alpha$ where $\overline{a}$ is the unique real solution of equation $L(a)=0$ and L is defined in Lemma 2 d),*
(c) *$\widetilde{H}$ is a curve, defined for all $\alpha>0$, that starts at $(0,0)$ tangent to the straight line $\beta=(4\,A)^{-1/3}\,\alpha$ and ends at $(+\infty,+\infty)$ tangent to the straight line $\beta=\overline{a}\,\alpha$. Here $\overline{a}$ is defined as in statement b).*
(d) *$\widetilde{H}$ is a closed bounded curve that does not approach the coordinate axes.*

Proof. Assume that $\beta=H(\alpha)$ is a solution of $F(\alpha,H(\alpha))=0$ defined for all $\alpha>0$ sufficiently small (the existence of this solution follows from Lemma 3), and let $\widetilde{H}$ be the curve defined by this solution.

From Lemma 2(e) the curve $\widetilde{H}$ does not intersect the β axis with $\beta>0$, and from Lemma 2(f) it cannot go to $\beta=+\infty$ when $\alpha\to 0$. Therefore

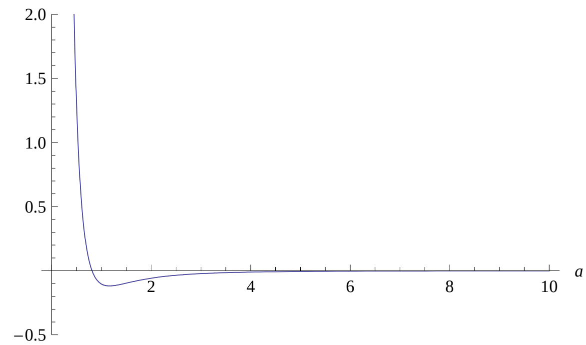


FIGURE 2. The plot of the function ℓ.

$\widetilde{H}$ must tend to the origin when $\alpha \to 0$. Moreover, since the limit when $(\alpha, \beta) \to (0,0)$ on the straight lines of the form $\beta = b\,\alpha$ (see Lemma 2(h)) equals zero when $b = (4A)^{-1/3}$, we have that the curve $\widetilde{H}$ tends to the origin tangent to the straight line $\beta = (4A)^{-1/3}\,\alpha$ when $\alpha \to 0$.

From Lemma 2(c), the curve $\widetilde{H}$ cannot go to $\beta = +\infty$ with a finite value of $\alpha > 0$, from Lemma 2(a) it cannot cross the α axis with $\alpha > 0$, from Lemma 2(b) it cannot tend to $\alpha = +\infty$ with $\beta \to 0$, and from Lemma 2(g) it cannot go to $\alpha = +\infty$ for a finite value of $\beta > 0$. Therefore, either the curve $\widetilde{H}$ is defined for all $\alpha > 0$ and tends to $\beta = +\infty$ when $\alpha \to +\infty$, or it is a curve that starts and ends at the point $(0,0)$ tangent to the straight line $\beta = (4\,A)^{-1/3}\,\alpha$ (i.e. $\widetilde{H}$ satisfies statement (a)).

Applying Lemma 3 again we know the existence of $\beta = H_1(\beta)$ that is a solution of $F(\alpha, H_1(\alpha)) = 0$ for α sufficiently large. Let $\widetilde{H}_1$ the curve defined by this solution. By using the previous arguments this curve must tend to $\beta = +\infty$ when $\alpha \to +\infty$. Therefore, either the curve $\widetilde{H}_1$ coincides with the curve $\widetilde{H}$, or it is a curve that starts and ends at $(+\infty, +\infty)$. Now we analyze how the curve $\widetilde{H}_1$ tends to $\beta = +\infty$ when $\alpha \to +\infty$. By equating the limit when $(\alpha, \beta) \to (+\infty, +\infty)$ on the straight lines of the form $\beta = a\,\alpha$ to zero, we get that equation $L(a) = 0$ is equivalent to the equation

$$\frac{A}{n} = \frac{1}{4a^3} - \frac{1}{(a^2+1)^{3/2}} = \ell(a),$$

(see Lemma 2(d)). We can see easily that ℓ is defined for all $a > 0$, $\lim_{a\to 0^+} \ell(a) = +\infty$, $\lim_{a\to+\infty} \ell(a) = 0$, and $\ell'(a) = 0$ has a unique real solution $a = a_0 \sim 1.16161$ which correspond to a minimum of ℓ with $\ell(a_0) < 0$, see Figure 2. Since A/n is positive for all integer $n \geq 2$, then equation $L(a) = 0$ has a unique solution $\overline{a}$. Therefore the curve $\widetilde{H}_1$ tends to $(+\infty, +\infty)$ tangent to the line $\beta = \overline{a}\,\alpha$.

In short, either $\widetilde{H}_1$ starts and ends at $(+\infty, +\infty)$ tangent to the line $\beta = \overline{a}\,\alpha$ (i.e. it satisfies statement (b)), or $\widetilde{H}_1$ coincides with $\widetilde{H}$ and consequently

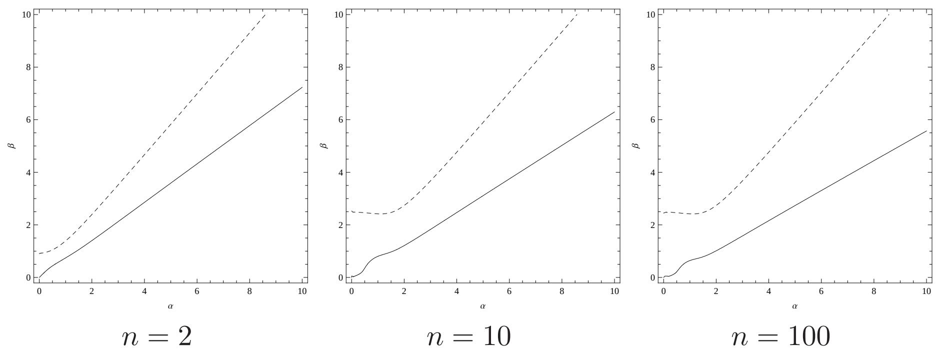


FIGURE 3. The plot of the curve $F(\alpha, \beta) = 0$ (continuous line) and the plot of $dF/d\beta(\alpha, \beta) = 0$ (dashed line) for some values of n.

it is a curve defined for all $\alpha > 0$ that tends to the origin tangent to the straight line $\beta = b\,\alpha$ when $\alpha \to 0$ and tends to $\beta = +\infty$ when $\alpha \to +\infty$ tangent to the line $\beta = \overline{a}\,\alpha$ (i.e. it satisfies statement (c)).

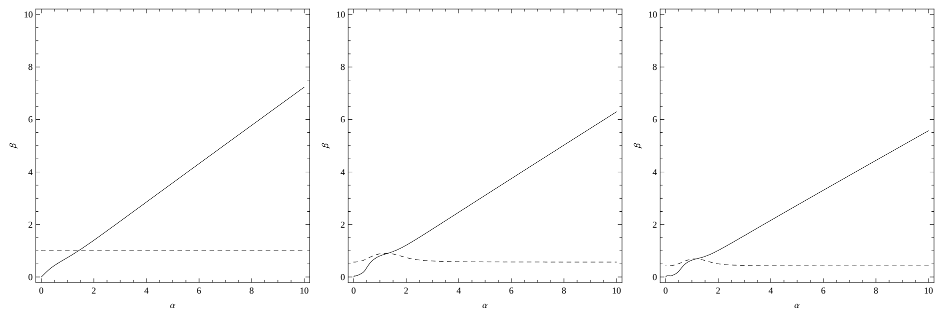
Finally equation $F(\alpha, \beta) = 0$ could define a bounded closed curve that does not approach the coordinate axes (i.e. a curve satisfying statement (d)). This completes the proof. □

Note that if the derivative $dF/d\beta$ evaluated at the solutions of $F(\alpha, \beta) = 0$ is different from zero, then the curves of statements (a), (b) and (d) of Lemma 4 are not possible.

We have plotted, with the help of Mathematica, the curves defined implicitly by the equations $F(\alpha, \beta) = 0$ and $(dF/d\beta)(\alpha, \beta) = 0$ for a large number of values of n (in Figure 3 there is a sample of them). We observe that for all these values of n the curve $(dF/d\beta)(\alpha, \beta) = 0$ is upper the curve $F(\alpha, \beta) = 0$, thus the derivative $dF/d\beta$ is different from zero on the curve solution of $F(\alpha, \beta) = 0$. Moreover we see that the behaviour of these curves is essentially the same for all those values of n, so we conjecture that the derivative $dF/d\beta$ evaluated at the solutions of $F(\alpha, \beta) = 0$ is different from zero for all $n \geq 2$. In short, we have the following.

Conjecture 2. *For any integer $n \geq 2$, there exists a curve $\beta = H(\alpha)$ solution of $F(\alpha, \beta) = 0$, defined for all $\alpha > 0$, such that* $\lim_{\alpha \to 0} H(\alpha) = 0$ *and* $\lim_{\alpha \to +\infty} H(\alpha) = +\infty$.

3.3. **The existence of solutions of system (11).** From Lemma 1 and Conjecture 1, we have that for all $n \geq 2$ the solution of equation $G(\alpha, \beta) = 0$ consists of a curve $\beta = h(\alpha)$, defined for all $\alpha > 0$, such that $\lim_{\alpha \to 0} h(\alpha) = \lim_{\alpha \to +\infty} h(\alpha) > 0$. On the other hand, from Conjecture 2, we have that for all $n \geq 2$ there exists a curve $\beta = H(\alpha)$ defined for all $\alpha > 0$, such that $\lim_{\alpha \to 0} H(\alpha) = 0$ and $\lim_{\alpha \to +\infty} H(\alpha) = +\infty$ and satisfying equation



$n = 2$ $n = 10$ $n = 100$

FIGURE 4. The plot of $F(\alpha, \beta) = 0$ (continuous line) and $G(\alpha, \beta) = 0$ (dashed line) for some values of n.

$F(\alpha, \beta) = 0$. Therefore the curves $\beta = h(\alpha)$ and $\beta = H(\alpha)$ intersect at least at one point.

For $n \geq 2$ even if Conjecture 2 holds, then it follows the existence of double–antiprism central configurations of the $3n$–body problem.

For $n \geq 2$ odd if Conjectures 1 and 2 hold, it follows the existence of double–antiprism central configurations of the $3n$–body problem.

We note that in both previous paragraphs we only get the existence but not the uniqueness of the double–antiprism central configurations of the $3n$–body problem for $n \geq 2$.

We have plotted, with the help of Mathematica, the curves defined implicitly by $F(\alpha, \beta) = 0$ and $G(\alpha, \beta) = 0$ for a large number of values of n (see Figure 4 for a sample of them). We observe that the behaviour of these curves the curves $\beta = h(\alpha)$ and $\beta = H(\alpha)$ when we vary n is essentially the same, and that they intersect at a unique point. In short, we can do the following conjecture.

Conjecture 3. *For all integer $n \geq 2$ there exist a unique double–antiprism central configuration of the $3n$–body problem.*

3.4. **Some numerical values of the solutions of system (11).** With the help of Mathematica, we have computed numerically the solution of system (11) for all $n = 2, \ldots, 100000$. We have seen that for all these values of n system (11) has a unique solution, which gives more numerical evidence that Conjecture 3 holds. In Table 1 we show the solutions of this system for some values of n. We note that the solution for $n = 2$ can also be found analytically.

We note that for some values of n the central configurations provided by the solutions of Table 1 are already known. For instance, if $n = 2$, then the configuration corresponds to the octahedron, if $n = 4$, then it corresponds to the cuboctahedron (both studied in [1]). But in general they are new.

n	α	β
2	$\sqrt{2}$	1
3	1.42229	0.989129
4	$\sqrt{2}$	1
5	1.38904	0.990101
6	1.36331	0.97153
7	1.34136	0.951556
8	1.32334	0.93281
9	1.30851	0.916005
10	1.29616	0.901161

n	α	β
20	1.23345	0.816412
30	1.20746	0.777097
40	1.19222	0.752651
50	1.18182	0.735311
60	1.17411	0.72206
70	1.16808	0.711436
80	1.16317	0.702628
90	1.15908	0.695143
100	1.15558	0.688659

n	α	β
1000	1.10482	0.58228
10000	1.07884	0.51492
100000	1.06309	0.467138

TABLE 1. The solutions of system (11) for some values of n.

REFERENCES

[1] F. CEDÓ AND J. LLIBRE, *Symmetric central configurations of the spatial n–body problem*, J. of Geometry and Physics **6** (1989), 367–394.

[2] M. CORBERA AND J. LLIBRE, *Central configurations of nested regular polyhedra for the spatial $2n$–body problem*, J. of Geometry and Physics **58** (2008), 1241–1252.

[3] M. CORBERA AND J. LLIBRE, *Central configurations of three nested regular polyhedra for the spatial $3n$–body problem*, J. of Geometry and Physics **59** (2009), 321–339.

[4] M. CORBERA AND J. LLIBRE, *On the existence of bi–pyramidal central configurations of the $n+2$–body problem with an n–gon base*, Preprint 2011.

[5] N. FAYCAL, *On the classification of pyramidal central configurations*, Proc. Amer. Math. Soc. **124** (1996), 249-258.

[6] E.S.G. LEANDRO, *Finiteness and bifurcations of some symmetrical classes of central configurations*, Arch. Ration. Mech. Anal. **167** (2003), 147–177.

[7] L.F. MELLO AND A.C. FERNANDES, *New spatial central configurations in the 5–body problem*, An. Acad. Bras. Ciênc. **83** (2011), 763–774.

[8] R. MOECKEL AND C. SIMÓ, *Bifurcation of spatial central configurations from planar ones*, SIAM J. Math. Anal. **26** (1995), 978–998.

[9] T. OUYANG, Z. XIE AND S. ZHANG, *Pyramidal central configurations and perverse solutions*, Electron. J. Differential Equations, **106** (2004), 1–9.

[10] S. ZHANG AND Q. ZHOU, *Double pyramidal central configurations*, Phys. Lett. A **281** (2001), 240–248.