Structural Presburger-definable Digit Vector Automata

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Abstract: Digit Vector Automata (DVA) provide a natural symbolic representation for regular sets of integer vectors encoded as strings of digit vectors (least significant digit first). We prove that the minimal DVA that represents a Presburger-definable set is structurally Presburger-definable: that means, the DVA obtained by modifying the initial state and the set of final states represents a Presburger-definable set.

Key-words: Automata, Presburger arithmetic, Semi-linear set, Symbolic representation

(Résumé : tsvp)

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Structure des automates Presburger-définissables

Résumé : Les automates finis permettent de représenter symboliquement des ensembles infinis de vecteurs d'entiers, décomposés comme des mots de vecteurs de chiffres. On montre que l'automate minimal représentant un ensemble Presburger-définissable, est structurellement Presburger-définissable: c'est à dire, que les automates obtenus en changeant l'état initial et les états finaux représentent des ensembles Presburger-définissables.

Mots clés : Automate, Arithmétique de Presburger, Ensemble semilinéaire, Représentation symbolique

Presburger arithmetic [21] is a decidable logic used in a large range of applications. Different techniques [11] and tools have been developed for manipulating *the Presburger-definable sets* (the sets of integer vectors satisfying a Presburger formula): by working directly on the Presburger-formulas (implemented in OMEGA [20]), by using semi-linear sets [12] (implemented in BRAIN [22]), or by using Digit Vector Automata (DVA) that represent regular sets of integer vectors encoded as strings of digit vectors, least or most significant digit first [23, 7] (implemented in FAST [1], LASH [15] and CSL-ALV [2]). Presburger-formulas and semi-linear sets lack canonicity: there does not exist a natural way to canonically represent a set. As a direct consequence, a set that possesses a simple representation could unfortunately be represented in an unduly complicated way. Moreover, deciding if a given vector of integers is in a given set, is at least *NP-hard* [4, 12]. On the other hand, a minimization procedure for automata provides a canonical representation for *DVA-definable sets* (a set represented by a DVA). That means, the DVA that represents a given set only depends on the set and not on the way we have computed it. For this reason, DVA are well adapted for applications that require a lot of Boolean manipulations like model-checking.

Recently, the DVA obtained by modifying the set of final states, has provided some applications. First, we have proved that modifying the set of final states of a DVA, provides some simple sets that can be used for deciding in polynomial time if a DVA is Presburger-definable (that means, the DVA represents a Presburger-definable set) [17]. Recall that the previous algorithm for deciding this property, was given by Muchnik in 1991 [18, 19, 8], and works in *quadruply-exponential time*. Second, Bartzis and Bultan [3] provided a *widening operator* for DVA in order to enforce the convergence of the incrementally computed DVA, during the reachability state space exploration of an *infinite state system*. This operator is obtained by modifying the set of final states of Presburger-definable DVA, but they do not prove that the obtained DVA remain Presburger-definable.

However, from practical and theoretical point of view, working only with Presburger-definable DVA has some advantages. First the manipulation complexity (boolean operations and variable elimination) is at most 3-exponential time for Presburger-definable DVA (see [13, 17]) and non-elementary for general DVA (see [5]). Second, we can compute in polynomial time, a Presburger-formula that defines the set represented by a Presburger-definable DVA. Then this formula can be used in other tools like OMEGA.

In this paper, we introduce a new automata-based representation for regular subsets of $\mathbb{Z}^m$, called the *digit Vector automata (DVA)*. Even if DVA are very similar to other automata-based representations [6, 7, 8], it is the *first* automata-based representation for any regular subsets of $\mathbb{Z}^m$, that is both *canonical* (there exists a unique minimal DVA that represents a given set X) and *stable by modifying the initial state* (this stability provides a natural way for associating a subset of $\mathbb{Z}^m$ to any state of the DVA). Moreover, we prove that the minimal DVA that represents a Presburger-definable set is structurally Presburger-definable: that means, any DVA obtained by modifying the initial state and the set of final states, is Presburger-definable.

1 Notations

We denote by $\mathbb{Z}$ and $\mathbb{N}\backslash\{0\}$ respectively the set of integers and non-negative integers. The set X^m is called the set of vectors with $m \in \mathbb{N}$ components in a set X. Given an integer $i \in \{1, \dots, m\}$ and a vector $x \in X^m$, the i-th component of x is written $x[i] \in X$. We denote by $\mathbf{e}_0$ the vector $\mathbf{e}_0 = (0, \dots, 0)$. Vectors $x + y$ and $t.x$ are defined by $(x + y)[i] = (x[i]) + (y[i])$ and $(t.x)[i] = t.(x[i])$ for any $i \in \{1, \dots, m\}$, $x, y \in \mathbb{Q}^m$, $t \in \mathbb{Q}$. We denote by $\langle x, y\rangle = \sum_{i=1}^{m} x[i].y[i]$, the *dot product* of two vectors $x, y \in \mathbb{Q}^m$. Given a *functions* $f : X \to Y$, $A \subseteq X$ and $B \subseteq Y$, we define $f(A) = \{f(a);\ a \in A\}$ and $f^{-1}(B) = \{x \in X;\ f(x) \in B\}$.

Given a non-empty finite *alphabet* Σ, we denote by Σ^+ the set of non-empty *words* over Σ and we denote by ϵ the empty word. As usual Σ^* denotes the set of words $\Sigma^+ \cup \{\epsilon\}$. A subset $\mathcal{L} \subseteq \Sigma^*$ is called a *language*. The concatenation of two words σ_1 and σ_2 (resp. two languages $\mathcal{L}_1$ and $\mathcal{L}_2$) is denoted by $\sigma_1.\sigma_2$ (resp. $\mathcal{L}_1.\mathcal{L}_2 = \{\sigma_1.\sigma_2;\ (\sigma_1, \sigma_2) \in \mathcal{L}_1 \times \mathcal{L}_2\}$). Given a word $\sigma \in \Sigma^*$, we denote by $(\sigma^i)_{i\in\mathbb{N}}$ the sequence of words defined by the induction $\sigma^0 = \epsilon$ and $\sigma^{i+1} = \sigma^i.\sigma$. We denote by σ^* the language $\sigma^* = \{\sigma^i;\ i \in \mathbb{N}\}$. The *length* of a word σ is denoted by $|\sigma| \in \mathbb{N}$. For any non-empty word $\sigma \in \Sigma^+$, we denote by $\sigma[1]$, ..., $\sigma[|\sigma|]$ the elements in Σ such that $\sigma = \sigma[1]\dots\sigma[|\sigma|]$.

2 Digit Vector Automata

In this section, the *Digit Vector Automata (DVA)* representation, a state-based representation of set of integer vectors, is presented. The sets obtained by *moving the initial state* and *modifying the set of final states* of a DVA are respectively characterized in sections 2.2 and 2.3.

2.1 Digit vector decomposition

Let us consider an integer $r \geq 2$ called the *basis of decomposition* and the *set of digits* $\Sigma_r = \{0, \dots, r-1\}$. In this section, we study the *least significant digit first decomposition* of an integer vector in $\mathbb{Z}^m$ into a word of *digit vectors* in $(\Sigma_r^m)^*$. This decomposition can be easily obtained by considering the sequence $(\gamma_{r,\sigma})_{\sigma \in (\Sigma_r^m)^*}$ of functions $\gamma_{r,\sigma} : \mathbb{Z}^m \to \mathbb{Z}^m$ uniquely defined by the following equalities [16]:

$$\begin{cases} \gamma_{r,b}(x) = r.x + b & (b,x) \in \Sigma_r^m \times \mathbb{Z}^m \\ \gamma_{r,\sigma_1.\sigma_2} = \gamma_{r,\sigma_1} \circ \gamma_{r,\sigma_2} & (\sigma_1,\sigma_2) \in (\Sigma_r^m)^* \times (\Sigma_r^m)^* \end{cases}$$

Assume that the dimension m is equal to 1 and consider a couple $(\sigma, s) \in \Sigma_r^* \times S_r$ where S_r is the set of *sign digits* $S_r = \{0, r-1\}$. The following equality is called the *least significant digit first decomposition with* 2*-complement*:

$$\gamma_{r,\sigma}\left(\frac{s}{1-r}\right) = \begin{cases} \sum_{i=1}^{|\sigma|} r^{i-1}\sigma[i] \in \mathbb{N} & \text{if } s = 0 \\ \sum_{i=1}^{|\sigma|} r^{i-1}\sigma[i] - r^{|\sigma|} \in \mathbb{Z}\backslash\mathbb{N} & \text{if } s = r-1 \end{cases}$$

The previous decomposition shows intuitively that $s = 0$ correspond to the *non-negative sign digit* whereas $s = r-1$ corresponds to the *negative one*.

For a general dimension $m \geq 1$, let us consider the function $\rho_r : (\Sigma_r^m)^* \times S_r^m \to \mathbb{Z}^m$ defined by the following equality:

$$\rho_r(\sigma, s) = \gamma_{r,\sigma}\left(\frac{s}{1-r}\right)$$

A couple $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$ such that $x = \rho_r(\sigma, s)$ is called a *r-decomposition* of $x \in \mathbb{Z}^m$. Remark that any $x \in \mathbb{Z}^m$ owns at least one r-decomposition.

Function ρ_r naturally associate to any language $\mathcal{L} \subseteq (\Sigma_r^m)^* \times S_r^m$ a subset $X = \rho_r(\mathcal{L})$ of $\mathbb{Z}^m$. Remark however that there exists some languages $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ such that $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}$ and such that $\rho_r(\mathcal{L}_1) \cap \rho_r(\mathcal{L}_2) \neq \rho_r(\mathcal{L})$. For instance, consider $\mathcal{L}_1 = \{(\epsilon, 0)\}$, $\mathcal{L}_2 = \{(0, 0)\}$ and $\mathcal{L} = \emptyset$. Such a side effect is due to the fact that an integer vector $x \in \mathbb{Z}^m$ does not have a unique r-decomposition. The following lemma characterizes r-decompositions associated to the same vector.

Lemma 1 *Two r-decompositions (σ_1, s_1) and (σ_2, s_2) are associated to the same vector if and only if $s_1 = s_2$ and $\sigma_1.s_1^* \cap \sigma_2.s_2^* \neq \emptyset$.*

Proof : Let us first remark that for any sign digit vector $s \in S_r^m$, we have $\gamma_{r,s}(\frac{s}{1-r}) = \frac{s}{1-r}$. In particular, we have $\rho_r(\sigma.s^k, s) = \rho_r(\sigma, s)$ for any word $\sigma \in (\Sigma_r^m)^*$ and for any $k \in \mathbb{N}$. This equality is well known when $s = 0$ and it just means that *adding extra zero digits* to the least significant digit first decomposition of a non-negative integer does not change its value.

Assume first that (σ_1, s_1) and (σ_2, s_2) are such that $s_1 = s_2$ and $\sigma_1.s_1^* \cap \sigma_2.s_2^* \neq \emptyset$, and let us prove that $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_2, s_2)$. There exist $k_1, k_2 \in \mathbb{N}$ such that $\sigma_1.s_1^{k_1} = \sigma_2.s_2^{k_2}$. In particular, from the previous paragraph we deduce $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_1.s_1^{k_1}, s_1) = \rho_r(\sigma_2.s_2^{k_2}, s_2) = \rho_r(\sigma_2, s_2)$.

Next, assume that $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_2, s_2)$ and let us prove that $s_1 = s_2$ and $\sigma_1.s_1^* \cap \sigma_2.s_2^* \neq \emptyset$. As the manipulated structures are defined component wise, we can assume without loss of generality that the dimension m is equal to 1. Remark that the sign digits s_1 and s_2 must be equal. In fact, otherwise, there exists $i_1, i_2 \in \{1, 2\}$ such that $s_{i_1} = 0$ and $s_{i_2} = r - 1$ and in this case we have shown that $\rho_r(\sigma_{i_1}, s_{i_1}) \in \mathbb{N}$ and $\rho_r(\sigma_{i_2}, s_{i_2}) \in \mathbb{Z}\backslash\mathbb{N}$ which is in contradiction with $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_2, s_2)$. Let us consider $k_1, k_2 \in \mathbb{N}$ such that the words $w_1 = \sigma_1.s_1^{k_1}$ and $w_2 = \sigma_2.s_2^{k_2}$ have the same length denoted by $k \in \mathbb{N}$. The first paragraph shows that $\rho_r(w_1, s_1) = \rho_r(w_2, s_2)$. As $s_1 = s_2$, we deduce the following equality:

$$\sum_{i=1}^{k} r^{i-1}.(w_1[i] - w_2[i]) = 0$$

Assume by contradiction that $w_1 \neq w_2$. In this case $k \in \mathbb{N}\backslash\{0\}$ and there exists a maximal (for $\leq$) $j \in \{1, \dots, k\}$ such that $w_1[j] \neq w_2[j]$. We have:

$$|w_1[i] - w_2[i]| \begin{cases} = 0 & \text{if } i > j \\ \geq 1 & \text{if } i = j \\ \leq r - 1 & \text{if } i < j \end{cases}$$

We deduce the following bound::

$$\begin{aligned} |\sum_{i=1}^{k} r^{i-1}.(w_1[i] - w_2[i])| &= |r^j.(w_1[j] - w_2[j]) + \sum_{i=1}^{j-1} r^{i-1}.(w_1[i] - w_2[i])| \\ &\geq |r^j.(w_1[j] - w_2[j])| - \sum_{i=1}^{j-1} |r^{i-1}.(w_1[i] - w_2[i])| \\ &\geq r^j - \sum_{i=1}^{j-1} r^{i-1}.(r-1) \\ &= 1 \end{aligned}$$

We obtain a contradiction. We deduce that $w_1 = w_2$ and in particular the word $w = w_1 = w_2$ is in $\sigma_1.s_1^* \cap \sigma_2.s_2^*$. Q.E.D

A *language* $\mathcal{L} \subseteq (\Sigma_r^m)^* \times S_r^m$ is said *saturated* [14] if for any $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$, we have $(\sigma, s) \in \mathcal{L}$ if and only if $(\sigma.s, s) \in \mathcal{L}$. Previous lemma 1 shows that a language $\mathcal{L}$ is saturated if and only if there exists $X \subseteq \mathbb{Z}^m$ such that $\mathcal{L} = \rho_r^{-1}(X)$. In particular, we deduce that the *side effect* $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}$ and $\rho_r(\mathcal{L}_1) \cap \rho_2(\mathcal{L}_2) \neq \rho_r(\mathcal{L})$ is no longer true for saturated language. In fact, for any saturated languages $\mathcal{L}_1, \mathcal{L}_2$ and for any $\# \in \{\cup, \cap, \backslash, \Delta\}$, the language $\mathcal{L}_1 \# \mathcal{L}_2$ is saturated and $\rho_r(\mathcal{L}_1) \# \rho_r(\mathcal{L}_2) = \rho_r(\mathcal{L}_1 \# \mathcal{L}_2)$.

We are interested in associating to a saturated language a *state-based symbolic representation*, called *Digit Vector Automata*.

Definition 1 (Digit Vector Automata) *A* Digit Vector Automaton (DVA) $\mathcal{A}$ *is a tuple* $\mathcal{A} = (Q, \Sigma_r^m, \delta, q_0, F_0)$ *where:*

- Q *is a non-empty finite set of* states.
- $\delta : Q \times \Sigma_r^m \to Q$ *is the* transition function.
- $q_0 \in Q$ *is the* initial state.
- $F_0 \subseteq Q \times S_r^m$ *is the set of* final states *such that* $(q, s) \in F_0$ *if and only if* $(q', s) \in F_0$ *for every* $q' = \delta(q, s)$.

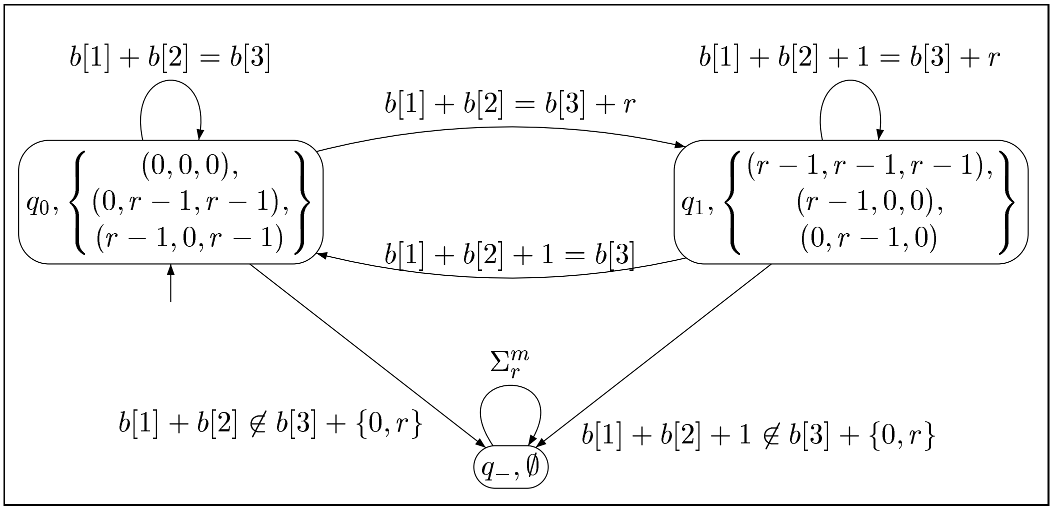

Figure 1: DVA $\mathcal{A}_X$ representing $X = \{x \in \mathbb{Z}^3;\ x[1] + x[2] = x[3]\}$

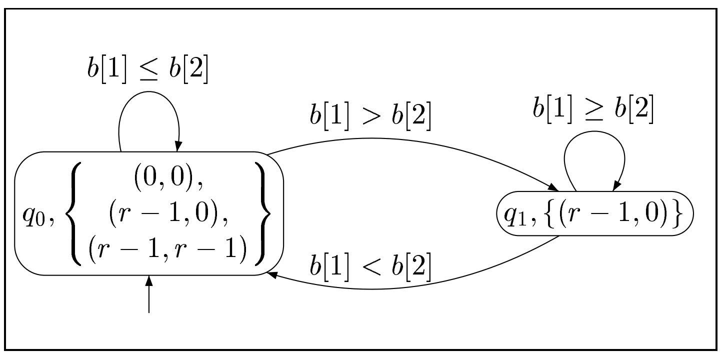

Figure 2: DVA $\mathcal{A}_X$ representing $X = \{x \in \mathbb{Z}^2;\ x[1] \leq x[2]\}$

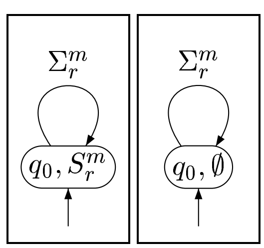

Figure 3: On the left, DVA $\mathcal{A}_{\mathbb{Z}^m}$. On the right, DVA $\mathcal{A}_{\emptyset}$

As usual, function δ is uniquely *extended* over $Q \times (\Sigma_r^m)^*$ by $\delta(q, \sigma_1.\sigma_2) = \delta(\delta(q, \sigma_1), \sigma_2)$. Moreover, a tuple (q, σ, q') such that $q' = \delta(q, \sigma)$ is denoted by $q \xrightarrow{\sigma} q'$ or just $q \to q'$, and called a *path* from q to q' *labeled* by σ. Such a state q' is said *reachable* from q (when $q = q_0$, we just say that q' is *reachable*).

The *language* $\mathcal{L}(\mathcal{A})$ *recognized* by a DVA $\mathcal{A}$ is defined by $\mathcal{L}(\mathcal{A}) = \{(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m;\ (\delta(q_0, \sigma), s) \in F_0\}$. Thanks to the condition $(q, s) \in F_0$ if and only if $(q', s) \in F_0$ for every $q \xrightarrow{s} q'$, the language $\mathcal{L}(\mathcal{A})$ is saturated. The set $X = \rho_r(\mathcal{L}(\mathcal{A})) \subseteq \mathbb{Z}^m$ is called the set *represented* by the DVA $\mathcal{A}$.

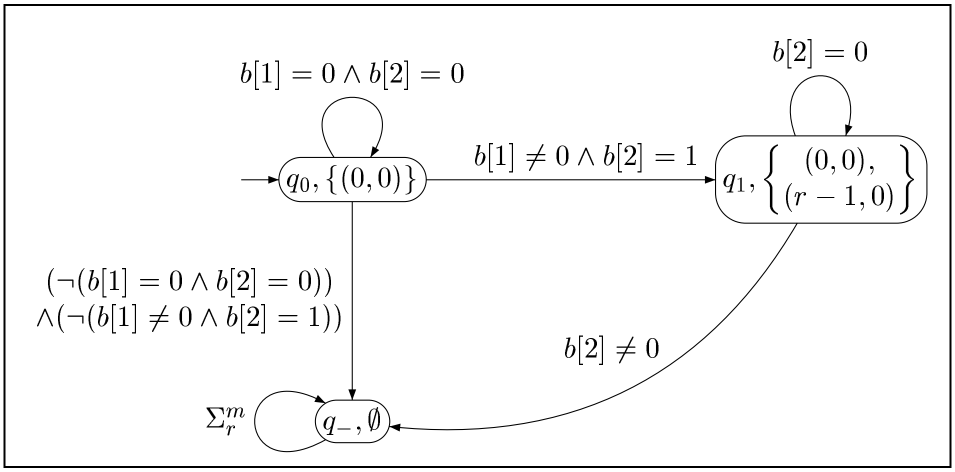

Figure 4: DVA $\mathcal{A}_X$ representing $X = \{x \in \mathbb{Z}^2;\ V_r(x[1]) = x[2]\}$

Sets represented by DVA correspond to the *r-definable sets*. Recall ([8]) that a set $X \subseteq \mathbb{Z}^m$ is said *r-definable* if it can be defined in the first order theory FO $(\mathbb{Z}, +, \leq, V_r)$ where $V_r : \mathbb{Z} \to \mathbb{Z}$ is the r-valuation function defined by $V_r(0) = 0$ and $V_r(x)$ is the greatest power of r that divides $x \in \mathbb{Z} \backslash \{0\}$ (figure 4). Recall also that a *Number Decision Diagram (NDD)* [6, 24] that represents a set $X \subseteq \mathbb{Z}^m$, is an automaton over Σ_r^m that recognizes the language $\{\sigma.s;\ (\sigma, s) \in \rho_r^{-1}(X)\}$. We do not consider NDD in this paper because the automaton obtained from a NDD by *replacing the initial state* by an other state is not a NDD in general (it does not recognizes a language of the form $\{\sigma.s;\ (\sigma, s) \in \rho_r^{-1}(X')\}$ where $X' \subseteq \mathbb{Z}^m$). However, DVA and NDD have slightly the same structure and we can easily compute a NDD from a DVA and conversely, that represents the same set X. In particular, we directly deduce from [8] and this remark, the following corollary 1

Corollary 1 *A set $X \subseteq \mathbb{Z}^m$ can be represented by a DVA if and only if it is r-definable.*

Remark 1 *As in the NDD case, DVA can be* efficiently manipulated *by reprensenting the set $\{b \in \Sigma_r^m;\ q \xrightarrow{b} q'\}$ and $\{s \in S_r^m;\ (q, s) \in F_0\}$ by some Binary Decision Diagrams (BDD) [9] over the alphabet Σ_r (and not the exponential one Σ_r^m).*

2.2 Moving the initial state

The DVA obtained from a DVA $\mathcal{A}$ by replacing the initial state q_0 by a state $q \in Q$ is denoted by $\mathcal{A}_q$. To simplify notations, when a set $X \subseteq \mathbb{Z}^m$ is implicitly represented by a DVA $\mathcal{A}$, we denote by $X_q \subseteq \mathbb{Z}^m$ the set represented by the DVA $\mathcal{A}_q$. We are going to characterize the set X_q in function of X. As an application, we show that any r-definable set $X \subseteq \mathbb{Z}^m$ is represented by a *unique minimal DVA*.

Proposition 1 *For any path $q \xrightarrow{\sigma} q'$ in a DVA $\mathcal{A}$ that represents a set X, we have $X_{q'} = \gamma_{r,\sigma}^{-1}(X_q)$.*

Proof : Without loss of generality, we can restrict our proof to a path $q_0 \xrightarrow{\sigma} q$ in a DVA $\mathcal{A}$ that represents a set X. Let us consider an integer vector $x \in X_q$. There exists a path $q \xrightarrow{w} q'$ and $s \in S_r^m$ such that $x = \rho_r(w, s)$ and $(q', s) \in F_0$. We deduce that we have a path $q_0 \xrightarrow{\sigma.w} q'$ with $(q', s) \in F_0$. Therefore $\rho_r(\sigma.w, s) \in X$. From $\rho_r(\sigma.w, s) = \gamma_{r,\sigma}(\rho_r(w, s)) = \gamma_{r,\sigma}(x)$, we deduce that $x \in \gamma_{r,\sigma}^{-1}(X)$ and we have proved the inclusion $X_q \subseteq \gamma_{r,\sigma}^{-1}(X)$. For the converse inclusion, consider an integer vector $x \in \gamma_{r,\sigma}^{-1}(X)$. As $\gamma_{r,\sigma}(x) \in X$, there exists a path $q_0 \xrightarrow{w} q'$ and $s \in S_r^m$ such that $\gamma_{r,\sigma}(x) = \rho_r(w, s)$ and $(q, s) \in F_0$. Moreover, as $x \in \mathbb{Z}^m$, there exists $(w', s') \in (\Sigma_r^m)^* \times S_r^m$ such that $x = \rho_r(w', s')$. From the equality $\gamma_{r,\sigma}(x) = \rho_r(w, s)$, we deduce that $\rho_r(\sigma.w', s') = \rho_r(w, s)$. Lemma 1 shows that $s' = s$ and there exists $k_1, k_2 \in \mathbb{N}$ such that $\sigma.w'.s^{k_1} = w.s^{k_2}$. As we have a path $q_0 \xrightarrow{w} q'$ with $(q', s) \in F_0$ and $\mathcal{A}$ is a DVA, we deduce that $q'' = \delta(q', s^{k_2})$ is such that $(q'', s) \in F_0$. From $\sigma.w'.s^{k_1} = w.s^{k_2}$, we get that $q_0 \xrightarrow{\sigma.w'.s^{k_1}} q''$. In particular we have a path $q \xrightarrow{w'.s^{k_1}} q''$ with $(q'', s) \in F_0$. We deduce that $x = \rho_r(w'.s^{k_1}, s) \in X_q$ and we have proved $\gamma_{r,\sigma}^{-1}(X) \subseteq X_q$. Q.E.D

The previous proposition 1 proves in particular that the set $Q_X = \{\gamma_{r,\sigma}^{-1}(X);\ \sigma \in (\Sigma_r^m)^*\}$ is finite when X is r-definable. The *minimal (for the number of states) DVA* that represents a r-definable set $X \subseteq \mathbb{Z}^m$ can be easily characterized by introducing the DVA $\mathcal{A}_X$ defined by the set of states Q_X, the transition function δ_X defined by a $\delta_X(X', b) = \gamma_{r,b}^{-1}(X')$ for any $X' \in Q_X$, the initial state $q_{0,X} = X$, the set of final states $F_{0,X} = \{(X', s) \in Q_X \times S_r^m;\ \frac{s}{1-r} \in X'\}$.

A DVA $\mathcal{A}$ is said *minimal* if for any DVA $\mathcal{A}'$ that represents the same set than $\mathcal{A}$, the number of states $|Q|$ of $\mathcal{A}$ is less than or equal to the number of states $|Q'|$ of $\mathcal{A}'$. Two DVA $\mathcal{A}_1 = (Q_1, \Sigma_r^m, \delta_1, q_{0,1}, F_{0,1})$ and $\mathcal{A}_2 = (Q_2, \Sigma_r^m, \delta_2, q_{0,2}, F_{0,2})$ are said *isomorph* if there exists a *one-to-one relation* $\sim \subseteq Q_1 \times Q_2$ such that $\delta_1(q_1, b) \sim \delta_2(q_2, b)$ and $\{s \in S_r^m;\ (q_1, s) \in F_{0,1}\} = \{s \in S_r^m;\ (q_2, s) \in F_{0,2}\}$ for any $q_1 \sim q_2$, and such that $q_{0,1} \sim q_{0,2}$.

Theorem 1 *For any r-definable set $X \subseteq \mathbb{Z}^m$, the DVA $\mathcal{A}_X$ is the unique (up to isomorphism) minimal DVA that represents X.*

Proof : First remark that $\mathcal{A}_X$ is a DVA that represents X. Next, let us consider a minimal DVA $\mathcal{A} = (Q, \Sigma_r^m, \delta, q_0, F_0)$ that represents X. Proposition 1 proves that there exists a function $f : Q_X \to Q$ such that $X_{f(X')} = X'$ for any $X' \in Q_X$. In particular $|Q_X| \leq |Q|$ and as $\mathcal{A}$ is minimal, we have $|Q_X| = |Q|$ and in particular $\mathcal{A}_X$ is also minimal. Moreover, we deduce that f is a one-to-one function. Just remark that $\mathcal{A}$ and $\mathcal{A}_X$ are isomorph for the one-to-one relation $\sim = \{(X', f(X'));\ X' \in Q_X\}$. Q.E.D

From the previous theorem 1 and corollary 1, we deduce that a set $X \subseteq \mathbb{Z}^m$ is r-definable if and only if $Q_X = \{\gamma_{r,\sigma}^{-1}(X);\ \sigma \in (\Sigma_r^m)^*\}$ is finite.

2.3 Replacing the set of final states

Given a DVA $\mathcal{A}$, the class of subsets $F \subseteq Q \times S_r^m$ such that $(q, s) \in F$ if and only if $(q', s) \in F$ for any transition $q \xrightarrow{s} q'$, is denoted by $\mathcal{F}_{\mathcal{A}}$. The DVA obtained from a DVA $\mathcal{A}$ be replacing the set of final states F_0 by a set $F \in \mathcal{F}_{\mathcal{A}}$ is denoted by $\mathcal{A}^F$. To simplify notions, when a set $X \subseteq \mathbb{Z}^m$ is implicitly represented by a DVA $\mathcal{A}$, we denote by X^F the set represented by the DVA $\mathcal{A}^F$. In this section, the set $\mathcal{F}_{\mathcal{A}}$ is geometrically characterized by introducing the notion of *eyes*, *semi-eyes* and *kernel*.

Let us consider the *equivalence relation* $\sim_{\mathcal{A}}$ over $Q \times S_{r^m}$ defined by $(q_1, s_1) \sim_{\mathcal{A}} (q_2, s_2)$ if and only if $s_1 = s_2$ and $\delta(q_1, s_1^*) \cap \delta(q_2, s_2^*) \neq \emptyset$.

An *eye* Y is an *equivalence class* for the relation $\sim_{\mathcal{A}}$ (see figure 5). A *semi-eye* is a finite union of eyes. Remark that the class of semi-eyes is exactly $\mathcal{F}_{\mathcal{A}}$.

Let us consider the function $\delta_e : Q \times S_r^m \to Q \times S_r^m$ defined by $\delta_e(q, s) = (\delta(q, s), s)$.

The *kernel* $\ker(Y)$ of a subset $Y \subseteq Q \times S_r^m$ is defined as $\ker(Y) = \bigcap_{n \in \mathbb{N}} \delta_e^n(Y)$ and corresponds to the greatest (for $\subseteq$) fix-point for δ_e included in Y. Remark that the kernel of any eye Y is a non

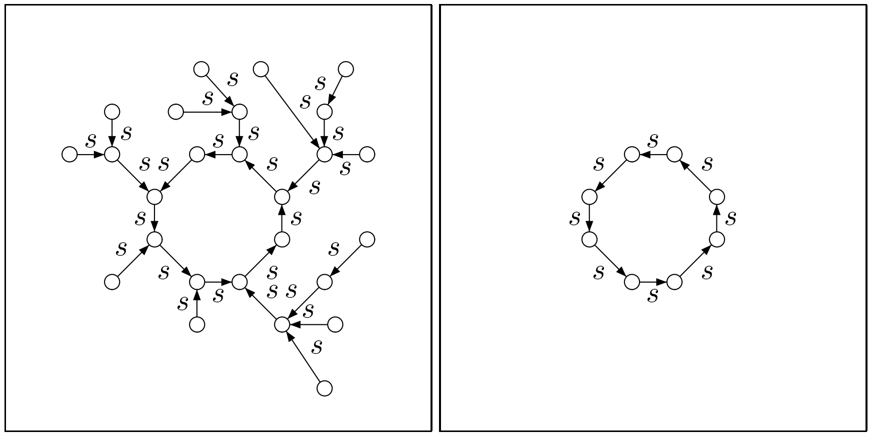

Figure 5: On the left an eye. On the right its kernel.

empty set of the form $\ker(Y) = \{(q_0, s), \ldots, (q_{n-1}, s), (q_n, s) = (q_0, s)\}$ such that $\delta(q_i, s) = q_{i+1}$ for any $i \in \{0, \ldots, n-1\}$ (see figure 5).

Example 1 *Let $\mathcal{A}_X$ be the minimal DVA representing $X = \{x \in \mathbb{Z}^3;\ x[1] + x[2] = x[3]\}$ given in figure 1. The eyes of $\mathcal{A}$ are $\{(q_0, (0,0))\}$, $\{(q_0, (r-1, r-1))\}$, $\{(q_1, (0,0))\}$, $\{(q_1, (r-1, r-1))\}$, $\{(q_0, (0, r-1)), (q_1, (0, r-1))\}$, and $\{(q_0, (r-1, 0)), (q_1, (r-1, 0))\}$.*

3 Presburger-definable DVA

A subset $X \subseteq \mathbb{Z}^m$ is said *Presburger-definable* if it can be defined by a formula in the first order theory $\mathrm{FO}(\mathbb{Z}, +, \leq)$ (see figure 6). A DVA $\mathcal{A}$ is said *Presburger-definable* if the set represented by $\mathcal{A}$ is Presburger-definable. A set X is said *structurally Presburger-definable* if the minimal DVA $\mathcal{A}$ that represents X, is such that $\mathcal{A}_q^F$ is Presburger-definable for any state $q \in Q$ and for any semi-eyes $F \in \mathcal{F}_{\mathcal{A}}$. Naturally, as $\mathcal{A}_{q_0}^{F_0}$ represents X, a structurally Presburger-definable set is Presburger-definable. In this section, we prove the converse.

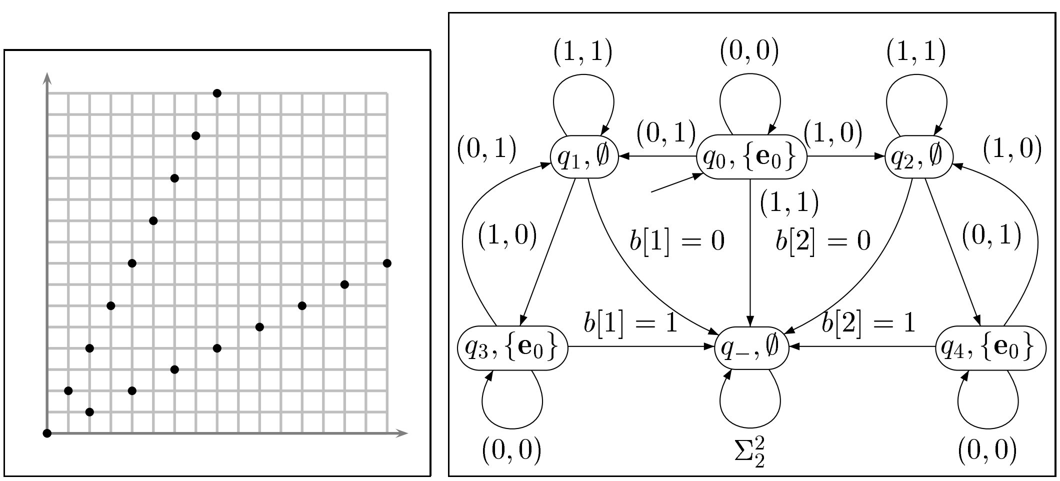

Figure 6: A Presburger-definable set $\{x \in \mathbb{N}^2;\ (x[1] = 2.x[2]) \vee (2.x[1] = x[2])\}$ and its minimal DVA $\mathcal{A}_X$ in basis $r = 2$.

Remark 2 *A* linear set X *of* $\mathbb{Z}^m$ *is a set of the form* $X = b + \sum_{p \in P} \mathbb{N}.p$ *where* $b \in \mathbb{Z}^m$ *is called the* basis *and* $P \subseteq \mathbb{Z}^m$ *is a finite subset of* $\mathbb{Z}^m$ *called the set of* periods. *A* semi-linear set *of* $\mathbb{Z}^m$ *is a finite union of linear sets of* $\mathbb{Z}^m$*. Recall that a set* X *is Presburger-definable if and only if it is* semi-linear *[12].*

Example 2 *The Presburger-definable set* $X = \{x \in \mathbb{N}^2;\ (x[1] = 2.x[2]) \vee (2.x[1] = x[2])\}$ *and its minimal DVA* $\mathcal{A}_X$ *in basis* $r = 2$ *are given in figure 6. Remark that the set of final states* F_0 *can be decomposed into 3 eyes* $Y_0 = \{(q_0, \boldsymbol{e}_0)\}$, $Y_3 = \{(q_3, \boldsymbol{e}_0)\}$ *and* $Y_4 = \{(q_4, \boldsymbol{e}_0)\}$*. The DVA* $\mathcal{A}_X^{Y_0}$, $\mathcal{A}_X^{Y_3}$ *and* $\mathcal{A}_X^{Y_4}$ *respectively represent* $X^{Y_0} = \{\boldsymbol{e}_0\}$, $X^{Y_3} = \{x \in \mathbb{N}^2 \backslash \{\boldsymbol{e}_0\};\ x[1] = 2.x[2]\}$ *and* $X^{Y_4} = \{x \in \mathbb{N}^2 \backslash \{\boldsymbol{e}_0\};\ 2.x[1] = x[2]\}$*.*

From proposition 1, we get the following corollary.

Corollary 2 *For any reachable state* q *of a Presburger-definable DVA* $\mathcal{A}$*, the DVA* $\mathcal{A}_q$ *is Presburger-definable.*

Proof : Let $\mathcal{A}$ be a DVA that represents a Presburger-definable set X and consider a reachable state q of $\mathcal{A}$. There exists a path $q_0 \xrightarrow{\sigma} q$. Proposition 1 proves that $X_q = \gamma_{r,\sigma}^{-1}(X)$. As X is Presburger-definable, there exists a Presburger-formula ϕ that defines X. Now, just remark that X_q is defined by the Presburger formula $\phi_\sigma(x) := \exists x'\ (x' = r^{|\sigma|}.x + \gamma_{r,\sigma}(\mathbf{e}_0) \wedge \phi(x'))$. Hence $\mathcal{A}_q$ is Presburger-definable. Q.E.D

A quantification elimination shows that a Presburger-definable set X is a boolean combination in $\mathbb{Z}^m$ of sets of the form $X = \{x \in \mathbb{Z}^m;\ x[i] \in c + n.\mathbb{Z}\}$ where $(i, c, n) \in \{1, \dots, m\} \times \mathbb{Z} \times (\mathbb{N} \backslash \{0\})$, and sets of the form $X = \{x \in \mathbb{Z}^m;\ \langle \alpha, x \rangle \leq c\}$ where $(\alpha, c) \in (\mathbb{Z}^m \backslash \{0\}) \times \mathbb{Z}$. The following technical lemmas 2 and 3 prove that these sets are structurally Presburger-definable.

Lemma 2 *The set* $X = \{x \in \mathbb{Z}^m;\ x[i] \in c + n.\mathbb{Z}\}$ *where* $(i, c, n) \in \{1, \dots, m\} \times \mathbb{Z} \times (\mathbb{N} \backslash \{0\})$ *is structurally Presburger-definable.*

Proof : Let $\mathcal{A}$ be the minimal DVA that represents $X = \{x \in \mathbb{Z}^m;\ x[i] \in c + n.\mathbb{Z}\}$. There exists a unique integer $k \in \mathbb{N}$ such that $n_0 = \frac{n}{r^k}$ is a r-prime integer (an integer relatively prime with r). Let us consider the set $\mathfrak{L}$ of words $\sigma \in (\Sigma_r^m)^k$ such that $\gamma_{r,\sigma}^{-1}(X) \neq \emptyset$. Remark that for any word $\sigma \in \mathfrak{L}$, we have $\gamma_{r,\sigma}^{-1}(X) = \{x \in \mathbb{Z}^m;\ r^k.x[i] \in c - \gamma_{r,\sigma}(\mathbf{e}_0)[i] + n.\mathbb{Z}\}$. As $\gamma_{r,\sigma}^{-1}(X) \neq \emptyset$, we deduce that $c_\sigma = \frac{c - \gamma_{r,\sigma}(\mathbf{e}_0)[i]}{r^k}$ is an integer, and in particular we get $\gamma_{r,\sigma}^{-1}(X) = \{x \in \mathbb{Z}^m;\ x[i] \in c_\sigma + n_0.\mathbb{Z}\}$. As n_0 is r-prime, there exists an integer $k_0 \in \mathbb{N}$ such that $r^{k_0} \in 1 + n_0.\mathbb{Z}$. For any $\sigma \in \mathfrak{L}$ and for any $(w, s) \in ((\Sigma_r^m)^{k_0})^* \times S_r^m$, we have:

$$\begin{aligned}
\gamma_{r,\sigma.w}^{-1}(X) &= \gamma_{r,w}^{-1}(\{x \in \mathbb{Z}^m;\ x[i] \in c_\sigma + n_0.\mathbb{Z}^m\}) \\
&= \{x \in \mathbb{Z}^m;\ r^{|w|}.x[i] + \gamma_{r,w}(\mathbf{e}_0)[i] \in c_\sigma + n_0.\mathbb{Z}\} \\
&= \{x \in \mathbb{Z}^m;\ x[i] \in c_\sigma + \frac{s}{1-r} - \rho_r(w, s)[i] + n_0.\mathbb{Z}\}
\end{aligned}$$

Let us consider an eye Y of $\mathcal{A}$, let $s \in S_r^m$ be the unique sign vector such that $Y \subseteq Q \times \{s\}$. Let us consider the Presburger-definable set $Z_s = \{\rho_r(\sigma, s);\ \sigma \in (\Sigma_r^m)^*\}$ of vectors with the same sign s.

We first assume that $X_q \neq \emptyset$ for any $(q, s) \in \ker(Y)$. We denote by P the set of $p \in \mathbb{Z}$ such that $\{x \in \mathbb{Z}^m;\ x[i] \in -p + n_0.\mathbb{Z}\} \in \{X_q;\ (q, s) \in \ker(Y)\}$. Remark that P is Presburger-definable because

$P = (P \cap \{0, \ldots, n_0 - 1\}) + n_0.\mathbb{Z}$. Moreover, we have:

$$\begin{aligned} x \in X^Y &\iff \exists \sigma \in (\Sigma_r^m)^* \; x = \rho_r(\sigma, s) \wedge (\delta(q_0, \sigma), s) \in Y \\ &\iff \exists \sigma \in \mathcal{L} \; \exists w \in ((\Sigma_r^m)^{k_0})^* \; x = \rho_r(\sigma.w, s) \wedge (\delta(q_0, \sigma.w), s) \in \ker(Y) \\ &\iff \exists \sigma \in \mathcal{L} \; \exists w \in ((\Sigma_r^m)^{k_0})^* \; \begin{cases} x = \gamma_{r,\sigma}(\rho_r(w, s)) \\ \wedge \rho_r(w, s)[i] \in c_\sigma + \frac{s}{1-r} + P \end{cases} \\ &\iff \exists \sigma \in \mathcal{L} \; \exists z \in Z_s \; x = \gamma_{r,\sigma}(z) \wedge z[i] \in c_\sigma + \frac{s}{1-r} + P \end{aligned}$$

We have proved that X^Y is Presburger-definable.

Finally, assume that $X_q = \emptyset$ for at least one $(q, s) \in \ker(Y)$. We have $X^Y = Z_s \backslash \bigcup_{Y' \in \mathcal{C} \backslash \{Y\}} X^{Y'}$ where $\mathcal{C}$ is the set of eyes $Y' \subseteq Q \times \{s\}$. Remark that if there exists an eye $Y' \in \mathcal{C} \backslash \{Y\}$ and $(q', s) \in \ker(Y')$ such that $X_{q'} = \emptyset$, as $\mathcal{A}$ is minimal, we get $q = q'$ and in particular $Y = Y'$ which is impossible. From the previous paragraph, we deduce that $X^{Y'}$ is Presburger-definable for any $Y' \in \mathcal{C} \backslash \{Y\}$. Therefore X^Y is Presburger-definable. Q.E.D

Lemma 3 *The set $X = \{x \in \mathbb{Z}^m;\; \langle \alpha, x \rangle \leq c\}$ where $(\alpha, c) \in (\mathbb{Z}^m \backslash \{0\}) \times \mathbb{Z}$ is structurally Presburger-definable.*

Proof : Let $\mathcal{A}$ be the minimal DVA that represents $X = \{x \in \mathbb{Z}^m;\; \langle \alpha, x \rangle \leq c\}$. For any $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$, and for any $k \in \mathbb{N}$, we have:

$$\gamma_{r,\sigma.s^k}^{-1}(X) = \left\{ x \in \mathbb{Z}^m;\; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle \leq \frac{c - \langle \alpha, \rho_r(\sigma, s) \rangle}{r^{|\sigma| + k}} \right\}$$

In particular, for any $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$, there exists $k_0 \in \mathbb{N}$ such that for any integer $k \geq k_0$, we have:

$$\gamma_{r,\sigma.s^k}^{-1}(X) = \begin{cases} \{x \in \mathbb{Z}^m;\; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle \leq 0\} & \text{if } \langle \alpha, \rho_r(\sigma, s) \rangle \leq c \\ \{x \in \mathbb{Z}^m;\; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle < 0\} & \text{if } \langle \alpha, \rho_r(\sigma, s) \rangle > c \end{cases}$$

Let us consider an eye Y and the unique sign digit vector $s \in S_r^m$ such that $Y \subseteq Q \times \{s\}$. Let us consider the Presburger-definable set $Z_s = \{\rho_r(\sigma, s);\; \sigma \in (\Sigma_r^m)^*\}$ of vectors with the same sign s.

From the previous equality, we deduce that there exists $\# \in \{<, \leq\}$ such that for any $(q, s) \in \ker(Y)$ we have $X_q = \{x \in \mathbb{Z}^m;\; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle \# 0\}$. In particular $\ker(Y)$ is reduced to $\ker(Y) = \{(q, s)\}$. Let us consider $\#' \in \{\leq, >\}$ such that $(\#, \#') \in \{(\leq, \leq), (<, >)\}$. We have:

$$\begin{aligned} x \in X^Y &\iff \exists \sigma \in (\Sigma_r^m)^* \; (\delta(q, \sigma.s^*), s) \cap \ker(Y) \neq \emptyset \\ &\iff x \in Z_s \wedge \langle \alpha, x \rangle \#' c \end{aligned}$$

Therefore X^Y is Presburger-definable. Q.E.D

Theorem 2 *A set X is structurally Presburger-definable if and only if it is Presburger-definable.*

Proof : Recall that a quantification elimination shows that a Presburger-definable set is a boolean combination in $\mathbb{Z}^m$ of sets of the form $X = \{x \in \mathbb{Z}^m;\; x[i] \in c + n.\mathbb{Z}\}$ and sets of the form $X = \{x \in \mathbb{Z}^m;\; \langle \alpha, x \rangle \leq c\}$. Lemmas 2 and 3 prove that these sets are structurally Presburger-definable. Moreover, as the complement of a structurally Presburger-definable set remains structurally Presburger-definable, it is sufficient to prove that the intersection $X = X_1 \cap X_2$ of two structurally Presburger-definable sets X_1 and X_2 remains structurally Presburger definable. Let $\mathcal{A}_1$, $\mathcal{A}_2$

and $\mathcal{A}'$ be the minimal DVA that represent respectively X_1, X_2 and X. Remark that X is represented by the *Cartesian product* $\mathcal{A} = (Q_1 \times Q_2, \Sigma_r^m, \delta, q_0, F_0)$ where $\delta((q_1, q_2), b) = (\delta_1(q_1, b), \delta_2(q_2, b))$, $q_0 = (q_{1,0}, q_{2,0})$, and $F_0 = F_{1,0} \times F_{2,0}$. Remark that for any eye Y of the DVA $\mathcal{A}'$, there exists a finite sequence $(Y_{1,i}, Y_{2,i})_{i \in I}$ where $Y_{1,i}$ and $Y_{2,i}$ are some eyes of respectivelly $\mathcal{A}_1$ and $\mathcal{A}_2$, such that X^Y is represented by the DVA $\mathcal{A}^{\bigcup_{i \in I} Y_{1,i} \times Y_{2,i}}$. Therefore $X^Y = \bigcup_{i \in I} X_1^{Y_1} \cap X_2^{Y_2}$ is Presburger-definable. In particular X is structurally Presburger-definable. We are done. Q.E.D

4 Future work

We have proved that any Presburger-definable set is structurally Presburger-definable. In particular, the widening operator for DVA introduced by Bartzis and Bultan provides Presburger-definable DVA from the widening of two Presburger-definable DVA. We are interested in extending the geometrical widening operators known for the *closed convex polyhedrons* [10], to the Presburger-definable DVA.

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