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COMPETITION BETWEEN GROWTHS GOVERNED BY BERNOULLI PERCOLATION

OLIVIER GARET AND RÉGINE MARCHAND

ABSTRACT. We study a competition model on $\mathbb{Z}^d$ where the two infections are driven by supercritical Bernoulli percolations with distinct parameters p and q. We prove that, for any q, there exist at most countably many values of $p < \min\{q, \overrightarrow{p_c}\}$ such that coexistence can occur.

1. INTRODUCTION

Consider two infections, say *blue* and *yellow*, which attempt to conquer, in discrete time, the space $\mathbb{Z}^d$. At time 0, all sites are empty but two: one is *active blue*, source of the blue infection, the other one is *active yellow*, source of the yellow infection. To evolve from time t to time $t+1$, the process is governed by the following rules. Each infection is only transmitted by *active* sites of its color to *empty* sites. Each active site tries to infect each of its empty neighbors, and succeeds with probability p_b or p_y, according to its color, blue or yellow. In case of success, the non-occupied site becomes an active site with the color of the infection; otherwise, it remains empty. In any case, the active site becomes a *passive* site of the same color, and can not transmit any infection anymore. Moreover, we make the following assumptions:

- the success of each attempt of contamination at a given time does not depend on the past,
- the successes of simultaneous attempts of contamination are independent.

The first point allows a modelization of this competition model by a homogeneous Markov chain while Markov chains satisfying the second point are sometimes called *Probabilistic Cellular Automata (PCA)*. Note that if the two initial sources are at an *odd* $\|.\|_1$-distance from each other, no empty site will be infected at the same time by the two distinct infections. To extend the definition of the model to more general initial configurations, we will add some extra rules in the next section.

Thus the two infections compete to invade space: once a site is colored, it keeps its color for ever and cannot be used by the other infection as a transmitter. As in other competition models, it is natural to ask whether coexistence, *i.e.* unbounded growth of the two infections, can occur. We propose the following conjecture:

Conjecture 1.1. *If $p_b = p_y > p_c$ then coexistence occurs with positive probability, while if $p_b \neq p_y$ and at least one them is strictly smaller than $\overrightarrow{p_c}$ then coexistence cannot occur.*

We will soon see that this competition model is closely linked to Bernoulli bond percolation on $\mathbb{Z}^d$, where, as usually, $p_c = p_c(d)$ denotes the critical probability

and $\overrightarrow{p_c} = \overrightarrow{p_c}(d)$ the critical probability in the oriented setting. This link will easily explain the fact that if $p_b < p_c$ – *resp.* $p_y < p_c$ – then with probability one the blue – *resp.* yellow – infection dies out. Thus the interesting case is when each infection has a parameter larger than p_c. In the same manner, it is not difficult to see, using properties of supercritical oriented Bernoulli percolation, that if both p_b and p_y exceed $\overrightarrow{p_c}$, then the blue and yellow infections can simultaneously grow unboundedly.

The coexistence statement of the conjecture has already been proved in a previous paper of the authors [9]. To precise the second part, and before stating the corresponding main result of this paper, we would like to recall the state of the art in competition problems of this type. A very natural way to obtain a competition model is to extend some well understood one-type interacting particle system in such a way that each infection behaves like the one-type model does in each region where only one of both types is present. Some famous one-type interacting particle have been considered: the contact process by Neuhauser [19], the Richardson model by Häggström and Pemantle [14, 15], or by another way by Kordzakhia and Lalley [17], Deijfen's continuous version of Richardson model [4] by Deijfen, Haggström and Bagley [3] and Deijfen and Häggström [5]. Each of these models actually corresponds to a family of stochastically comparable processes indexed by a continuous parameter and it is natural to ask if coexistence is possible when the two infections are governed by the same parameter – *resp.* by different parameters. Note that in all these models, the stochastically comparable processes are governed by exponential families. The following dichotomy seems to emerge.

Either the two infections have the same strength, or same speed of propagation. In this case, coexistence occurs with positive probability: it has been proved at first for the two-type Richardson model when $d = 2$ by Häggström and Pemantle [14] and then extended by Garet and Marchand [9] for a wide class of first-passage percolation models, including the percolation model that is studied here. An alternative proof is also given by Hoffman [16]. Similarly, Deijfen and Häggström [5] proved the possibility of coexistence for Deijfen's continuous version of Richardson model. The same result is also proved by Kordzakhia and Lalley [17] for their own extension of Richardson model. Nevertheless, their proof is conditioned by a difficult and reasonable conjecture on the curvature properties of the asymptotic shape for Richardson model.

Or one infection is stronger – or faster than the other one. It is then conjectured that coexistence is not possible. The first and famous result in this direction was done by Häggström and Pemantle [15]: they proved that for their model, coexistence is not possible, except perhaps for a denumerable set for the ratio of the speeds. The result of Deijfen, Häggström and Bagley [3] is submitted to the same irritating restriction rule.

For our model, we prove, in this paper, the similar following result:

Theorem 1.2. *Let $p_b > p_c$ be fixed: there exists a denumerable set $Bad \subset [p_c, \overrightarrow{p_c})$ such that for each $p_y \in [0, \min\{p_b, \overrightarrow{p_c}\}) \backslash Bad$, the probability that both infections infinitely grow is null.*

Before commenting this result, and to complete the survey, let us mention the recent paper by Deijfen and Häggström [6], where they exhibit graphs where coexistence occurs for several values for the ratio of the speeds. This should prevent

$$p_\mathrm{b} = p_\mathrm{y} = 0.6 \qquad\qquad p_\mathrm{b} = 0.7 \; p_\mathrm{y} = 0.6$$
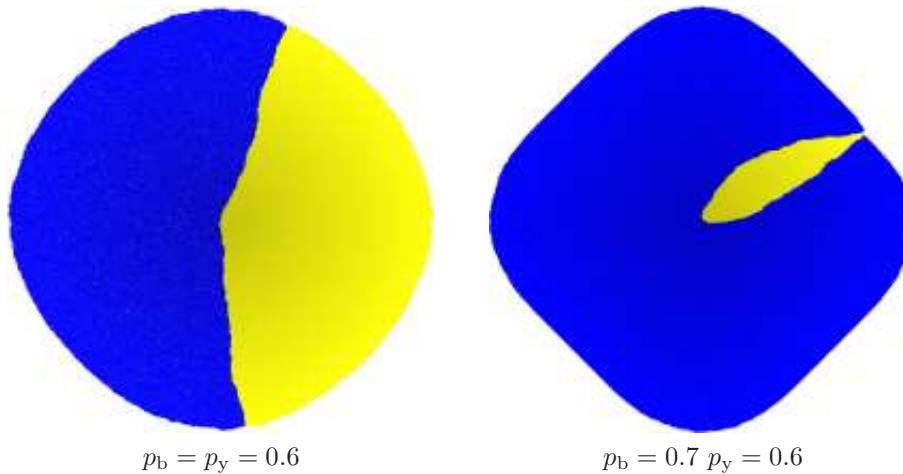
FIGURE 1. Bernoulli competition in a 4000×4000 grid

researchers from unsuccessful attempts to fill the gap with the only help of stochastic comparisons.

In its main lines, the present paper follows the strategy initiated by Häggström and Pemantle [15], but it has to overcome some extra difficulties. Our model also depends on one simple parameter – the parameter of the related Bernoulli percolation – which allows coupling and stochastic comparisons. However, note that:

- The memoryless properties of the exponential laws are lost: one active site tries to infect an empty neighbor only once.
- Scaling properties of the asymptotic shape in first-passage percolation with exponential times are lost: asymptotic shapes corresponding to different values of the parameter are not homothetic anymore.

The paper is organized as follows. First, in Section 2, we describe precisely the PCA underlying this competition process, exhibit its reformulation in terms of Bernoulli percolation, and give some related coupling properties and stochastic comparison results. Then, Section 3 gives a primer of results concerning Bernoulli percolation and the related chemical distance: we particularly recall there the convergence result of the chemical distance in Bernoulli percolation with supercritical parameter p to a norm $\|.\|_p$, and an associated large deviation result.

The first key point of the proof of the main result is the strict comparison of the norms associated to the asymptotic behavior of chemical distance in Bernoulli percolation with different parameters, which will replace the homothetic properties of asymptotic shapes in the case of exponential laws. Section 4 is devoted to the proof of this result:

Proposition 1.3. *Assume that $p_c < p < \overrightarrow{p_c}$ and $p < q \leq 1$. There exists a positive constant $C_{p,q} < 1$ such that*

$$\forall x \in \mathbb{R}^d \quad \|x\|_q \leq C_{p,q}\|x\|_p.$$

Although the large comparison $\|x\|_q \leq \|x\|_p$ is quite natural, the strict comparison will be necessary to ensure, roughly speaking, that in every direction, the

stronger infection can take a real advantage and grow strictly faster than the other one.

The second key step is to prove that when coexistence occurs, the global growth of the infected sites is governed by the norm of the weaker infection: denote by $\eta(t)$ is the set of already infected sites at time t and $|A|_p = \sup\{\|x\|_p : x \in A\}$. Then

Proposition 1.4. *Let p and q be such that $p_c < q \leq 1$ and $0 \leq p < \min\{q, \overrightarrow{p_c}\}$. On the event "the weak infection survives", we have almost surely:*

$$\overline{\lim_{t \to +\infty}} \ \frac{|\eta(t)|_p}{t} \leq 1.$$

The proof of this proposition – in fact the core of the paper – is given in Section 5. It relies both on the previous proposition and on the large deviation result on the set of infected points with respect to the asymptotic shape in the corresponding one infection model which is recalled in Section 3.

Finally, in Section 6, we collect all these results to prove the main theorem via coupling results that are in the spirit of Häggström and Pemantle's work [15].

2. The competition model

This section has several goals:

- to complete the progression rules exposed in the introduction and to define the model for general initial configurations. This will correspond to the artificial introduction of *green* sites.
- to define the PCA by describing the transition matrix of the homogeneous Markov chain in terms of local rules.
- to give an alternative description (3) in terms of Bernoulli percolation and chemical distance and to prove the equivalence between the two definitions in Lemma 2.1. This definition will be the one used in the next sections.
- Use this last definition to give monotonicity properties in Lemma 2.2 and comparisons properties between the one-type growth and the two-types growth model in Lemma 2.3.

Suppose from now on that $p_y \leq p_b$, which means that the blue infection is stronger than the yellow one. To complete the description of the model, let us first describe the interface between the two infections via the introduction of *green* sites. A green site is to be understood as a superposition of a blue site and a yellow site. To be coherent with the previous rules, we assume that an active green site transmits to each of its empty neighbors either both infections with probability p_y, or only the blue infection with probability $p_b - p_y$, or fails in its infection attempts with probability $1 - p_b$; it then becomes a passive green site. Note that this rule is quite arbitrary. The necessary part is that a green site transmits to one of its neighbor a yellow – *resp.* blue – infection with probability p_y – *resp.* p_b – and we choose the coupling between these two transmissions to simplify some coupling in the sequel, but it has no real influence on the behavior of the model.

To determine the state at time $t + 1$ of an empty site x at time t, we then check the types of infections that are transmitted to it: either they are all of the same color, blue or yellow, and x becomes an active site of this color, or they are of both colors, and x becomes an active green site, or no infection is transmitted to x, which then remains empty. We can now give the formal definition of the PCA.

2.1. Definition of the Probabilistic Cellular Automata (PCA).

Definition of the graph $\mathbb{L}^d$. We endow the set $\mathbb{Z}^d$ with the set of edges $\mathbb{E}^d$ between sites of $\mathbb{Z}^d$ that are at distance 1 for the Euclidean distance: the obtained graph is denoted $\mathbb{L}^d$. Two sites x and y that are linked by an edge are said to be *neighbors* and this relation is denoted: $x \sim y$.

State space. Let us introduce the set $S = \{0, \text{b}, \text{y}, \text{g}, \text{b}^*, \text{y}^*, \text{g}^*\}$ of possible states of a site: 0 is the state of an empty site, b, y, g – corresponding respectively to colors blue, yellow and green – the states of active sites, and $\text{b}^*, \text{y}^*, \text{g}^*$ the states of passive colored sites.

In the sequel, we will restrict our Markov chain to start from a configuration with a finite number of colored sites, whence the only configurations appearing during the whole process will also have a finite numbers of colored sites. Our Markov chain will thus live in the following denumerable state set:

$$S^{(\mathbb{Z}^d)} = \{\xi \in S^{\mathbb{Z}^d} : \exists \Lambda \text{ finite}, \xi_k = 0 \text{ for } k \in \mathbb{Z}^d \backslash \Lambda\}.$$

Local rules. To complete the definition of the Markov chain, it only remains to define its transition probabilities, via *local rules*, describing the evolutions of the infections exposed in the introduction. Define, for $c \in A = \{\text{b}, \text{y}, \text{g}\}$, the number $n_x^\xi(c)$ of active neighbors with color c of the site $x \in \mathbb{Z}^d$ in the configuration $\xi \in S^{(\mathbb{Z}^d)}$:

$$n_x^\xi(c) = |\{y \in \mathbb{Z}^d : x \sim y \text{ and } \xi_y = c\}|,$$

and define the probability $p_x^\xi(c, \tilde{c})$ that the site $x \in \mathbb{Z}^d$, in the configuration $\xi \in S^{(\mathbb{Z}^d)}$, swaps from color c to color $\tilde{c}$:

- If x is an empty site, *i.e.* if $\xi_x = 0$, set:

$$\begin{cases} p_x^\xi(0, 0) &= (1 - p_\text{b})^{n_x^\xi(\text{b}) + n_x^\xi(\text{g})}(1 - p_\text{y})^{n_x^\xi(\text{y})} \\ p_x^\xi(0, \text{y}) &= (1 - p_\text{b})^{n_x^\xi(\text{b}) + n_x^\xi(\text{g})}\left[1 - (1 - p_\text{y})^{n_x^\xi(\text{y})}\right] \\ p_x^\xi(0, \text{b}) &= \left[1 - (1 - p_\text{b})^{n_x^\xi(\text{b})}\right](1 - p_\text{y})^{n_x^\xi(\text{y}) + n_x^\xi(\text{g})} \\ &\quad + (1 - p_\text{b})^{n_x^\xi(\text{b})}(1 - p_\text{y})^{n_x^\xi(\text{y})}\left[1 - (1 - p_\text{b} + p_\text{y})^{n_x^\xi(\text{g})}\right] \\ p_x^\xi(0, \text{g}) &= 1 - p_x^\xi(0, 0) - p_x^\xi(0, \text{y}) - p_x^\xi(0, \text{b}). \end{cases}$$

- If x is an active site, it becomes passive: $\forall c \in A, \; p_x^\xi(c, c^*) = 1$.
- If x is an passive site, it remains passive: $\forall c \in A, \; p_x^\xi(c^*, c^*) = 1$.
- In any other case, the probability is null.

Transition probabilities. We can then define the following transition probabilities on the state set $S^{(\mathbb{Z}^d)}$:

$$\forall (\xi^1, \xi^2) \in S^{(\mathbb{Z}^d)} \times S^{(\mathbb{Z}^d)} \quad p(\xi^1, \xi^2) = \prod_{x \in \mathbb{Z}^d} p_x^{\xi^1}(\xi_x^1, \xi_x^2). \tag{1}$$

Note that, as only a finite number of terms differ from 1, the previous product is convergent.

2.2. Realization of the Markov chain via Bernoulli percolation. The aim of this part is to link this PCA with some natural Bernoulli percolation structures on $\mathbb{Z}^d$, and to give an alternative description of the model in terms of random sets and of a specific first-passage percolation model. We begin with some classical notations of Bernoulli percolation on $\mathbb{Z}^d$.

Bernoulli percolation. On the set $\Omega = [0,1]^{\mathbb{E}^d}$ endowed with its Borel σ-algebra, consider the probability measure $\mathbb{P} = \mathrm{Unif}[0,1]^{\otimes \mathbb{E}^d}$. For each $p \in [0,1]$ and $\omega \in \Omega$, we denote by $\mathcal{G}_p(\omega)$ the subgraph of $\mathbb{L}^d$ whose bonds e are p-*open*, which means that they are are such that $\omega_e \le p$.

For $A \subset \mathbb{Z}^d$ and $p \in [0,1]$, we also note

$$(2) \qquad \partial_p A(\omega) = \{y \in \mathbb{Z}^d \backslash A : \ \exists x \in A \quad \{x,y\} \in \mathcal{G}_p(\omega)\}.$$

On this probability space, we now define a homogeneous Markov chain $(X_t)_{t \ge 0}$ with values in $S^{\mathbb{Z}^d}$ and with transition probabilities as in (1).

Definition of the process. Let $\xi^0 \in S^{\mathbb{Z}^d}$ be a fixed initial configuration. We define

$$A_0^{\mathrm{b}} = \{x \in \mathbb{Z}^d : \xi_x^0 \in \{\mathrm{b,g}\}\} \qquad \text{and} \quad A_0^{\mathrm{y}} = \{x \in \mathbb{Z}^d : \xi_x^0 \in \{\mathrm{y,g}\}\},$$
$$B_0^{\mathrm{b}} = \{x \in \mathbb{Z}^d : \xi_x^0 \in \{\mathrm{b,g,b^*,g^*}\}\} \quad \text{and} \quad B_0^{\mathrm{y}} = \{x \in \mathbb{Z}^d : \xi_x^0 \in \{\mathrm{y,g,y^*,g^*}\}\}.$$

Note that by intersection and difference, we can exactly recover through these four sets the whole configuration ξ^0.

Let $0 \le p_{\mathrm{y}} \le p_{\mathrm{b}} \le 1$, and consider a Bernoulli configuration $\omega \in \Omega$, which will give the evolution rules of the process. An infection can only travel from an active site of the corresponding color to an empty site, via an edge which is p-open in ω for the parameter p associated to this infection, *i.e.* either p_{y} or p_{b}. As before, an active green site is to be imagined as a superposition of an active yellow site and an active blue site. So, if e is an edge between an active green site and an empty site, then three cases arise: if $0 \le \omega_e \le p_{\mathrm{y}}$ then e will transmit to x both infections, if $p_{\mathrm{y}} \le \omega_e \le p_{\mathrm{b}}$ then e will only transmit to x the blue infection, while if $p_{\mathrm{b}} \le \omega_e$ then no infection will travel through e to x.

To determine the state at time $t+1$ of an empty site x at time t, we look simultaneously at all edges between active sites at time n and x: if all these edges transmit the same infection – blue or yellow – then x takes this color and becomes active, if these edges transmit infections of the two different types, then x becomes green and active, and otherwise, x remains empty. Active sites at time t become passive sites of the same color at time $t+1$. These rules are translated in the following recursive definitions:

$$\begin{cases} A_{t+1}^{\mathrm{y}}(\omega) &= \partial_{p_{\mathrm{y}}} A_t^{\mathrm{y}}(\omega) \backslash (B_t^{\mathrm{b}}(\omega) \cup B_t^{\mathrm{y}}(\omega)), \\ B_{t+1}^{\mathrm{y}}(\omega) &= B_t^{\mathrm{y}}(\omega) \cup A_{t+1}^{\mathrm{y}}(\omega) = B_t^{\mathrm{y}}(\omega) \cup (\partial_{p_{\mathrm{y}}} B_t^{\mathrm{y}}(\omega) \backslash B_t^{\mathrm{b}}(\omega)), \\ A_{t+1}^{\mathrm{b}}(\omega) &= \partial_{p_{\mathrm{b}}} A_t^{\mathrm{b}}(\omega) \backslash (B_t^{\mathrm{b}}(\omega) \cup B_t^{\mathrm{y}}(\omega)), \\ B_{t+1}^{\mathrm{b}}(\omega) &= B_t^{\mathrm{b}} \cup A_{t+1}^{\mathrm{b}}(\omega)) = B_t^{\mathrm{b}}(\omega) \cup (\partial_{p_{\mathrm{b}}} B_t^{\mathrm{b}}(\omega) \backslash B_t^{\mathrm{y}}(\omega)). \end{cases}$$

The set A_t^{b} (*resp.* A_t^{y}) is the set of active sites at time t in that are either blue or green (*resp.* yellow or green), while B_t^{b} (*resp.* B_t^{y}) is the set of sites at time t that are either blue or green (*resp.* yellow or green). Note that by these definitions, a given site can be active at one time at most. We define then, for every $t \ge 0$, the value of the process X_t at time t as the element of $S^{\mathbb{Z}^d}$ encoded by the four random sets A_t^{y}, B_t^{y}, A_t^{b} and B_t^{b}.

Lemma 2.1. *The process $(X_t)_{t \ge 0}$ is a homogeneous Markov chain governed by the transition probabilities defined in (1).*

Proof. Fix $\xi_0 \in S^{\mathbb{Z}^d}$, and define A_0^{b}, A_0^{y}, B_0^{b} and B_0^{y} as previously.

The only point is to prove that $(X_t)_{t\geq 0}$ is a homogeneous Markov chain, the identification of the transition probabilities is clear by construction. The ideas of the proof stay in the following easy remarks:

- During the process, any site can only be active at one time at most.
- Suppose that at time t, the process is in state ξ. To decide in which state it will switch at time $t+1$, the only edges that are to be examined are the ones between an active site and an empty site in ξ.
- Thus, during the process, each edge is examined only once at most.

So knowing the present, the past will not affect the future. In the rest of the proof, we try to turn this crude argument into a more rigorous one.

In order to define the four random sets at time $t+1$ from the four random sets at time t and ω, we introduce, for any subsets A, B, C of $\mathbb{Z}^d$, any Bernoulli configuration $\omega \in \Omega$ and any probability $0 \leq p \leq 1$, define the two following functions:

$$\begin{aligned} F(p, \omega, A, B, C) &= \partial_p A(\omega) \backslash (B \cup C), \\ G(p, \omega, A, B) &= A \cup (\partial_p A(\omega) \backslash B). \end{aligned}$$

Then, the previous definitions are equivalent to:

$$\begin{cases} A_{t+1}^{\mathrm{y}}(\omega) &= F(p_{\mathrm{y}}, \omega, A_t^{\mathrm{y}}(\omega), B_t^{\mathrm{y}}(\omega), B_t^{\mathrm{b}}(\omega)), \\ B_{t+1}^{\mathrm{y}}(\omega) &= G(p_{\mathrm{y}}, \omega, B_t^{\mathrm{y}}(\omega), B_t^{\mathrm{b}}(\omega)), \\ A_{t+1}^{\mathrm{b}}(\omega) &= F(p_{\mathrm{b}}, \omega, A_t^{\mathrm{b}}(\omega), B_t^{\mathrm{y}}(\omega), B_t^{\mathrm{b}}(\omega)), \\ B_{t+1}^{\mathrm{b}}(\omega) &= G(p_{\mathrm{b}}, \omega, B_t^{\mathrm{b}}(\omega), B_t^{\mathrm{y}}(\omega)), \end{cases}$$

which is equivalent to say that $(X_t)_{t\geq 0}$ satisfies a recurrence formula of the type $X_{t+1} = f(X_t, \omega)$, where the function f can be expressed in terms of the two functions F and G.

To obtain the canonical Markov Chain representation $X_{t+1} = f(X_t, \omega^{t+1})$, we are going to build a coupling between a random variable uniformly distributed on Ω and an independent and identically distributed sequence $(\omega^t)_{t\geq 1}$ with the same law. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space and let $(\omega^t)_{t\geq 1}$ be independent $[0,1]^{\mathbb{E}^d}$-valued random variables with $\mathrm{Unif}([0,1])^{\otimes \mathbb{E}^d}$ as common law. We define A_0^{b}, A_0^{y}, B_0^{b} and B_0^{y} exactly as previously. But now, we set recursively:

$$\begin{aligned} A_{t+1}^{\mathrm{y}} = F(p_{\mathrm{y}}, \omega^{t+1}, A_t^{\mathrm{y}}, B_t^{\mathrm{y}}, B_t^{\mathrm{b}}) &\quad \text{and} \quad B_{t+1}^{\mathrm{y}} = G(p_{\mathrm{y}}, \omega^{t+1}, B_t^{\mathrm{y}}, B_t^{\mathrm{b}}), \\ A_{t+1}^{\mathrm{b}} = F(p_{\mathrm{b}}, \omega^{t+1}, A_t^{\mathrm{b}}, B_t^{\mathrm{y}}, B_t^{\mathrm{b}}) &\quad \text{and} \quad B_{t+1}^{\mathrm{b}} = G(p_{\mathrm{b}}, \omega^{t+1}, B_t^{\mathrm{b}}, B_t^{\mathrm{y}}), \end{aligned}$$

Note that these four sets are measurable with respect to the σ-algebra generated by $(\omega^1, \ldots, \omega^{t+1})$. Let $\tilde{\omega}^0$ be a random variable defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, with law $\mathrm{Unif}([0,1])^{\otimes \mathbb{E}^d}$, and independent of the sequence $(\omega^t)_{t\geq 1}$, and define $(\tilde{\omega}^t)_{t\geq 0}$ recursively as follows: for any edge $e = \{x, y\} \in \mathbb{E}^d$, set

$$\tilde{\omega}_e^{t+1} = \begin{cases} \omega_e^{t+1} & \text{if } x \in (A_t^{\mathrm{y}} \cup A_t^{\mathrm{b}}) \text{ and } y \notin (B_t^{\mathrm{y}} \cup B_t^{\mathrm{b}}), \\ \tilde{\omega}_e^t & \text{otherwise.} \end{cases}$$

By natural induction, we prove that the law of $\tilde{\omega}^t$ under $\tilde{\mathbb{P}}$ is $\mathrm{Unif}([0,1])^{\otimes \mathbb{E}^d}$. By construction, each edge e writes $e = \{x, y\}$ with $x \in (A_t^1 \cup A_t^2)$ and $y \notin (B_t^1 \cup B_t^2)$ for at most one value of t. It follows that the sequence $(\tilde{\omega}^t)_{t\geq 0}$ converges in the product topology to a limit that we denote $\tilde{\omega}^\infty$. Since the law of $\tilde{\omega}^t$ under $\tilde{\mathbb{P}}$ is $\mathrm{Unif}([0,1])^{\otimes \mathbb{E}^d}$, it follows that the law of $\tilde{\omega}^\infty$ under $\tilde{\mathbb{P}}$ is also $\mathrm{Unif}([0,1])^{\otimes \mathbb{E}^d}$.

Now, it is not difficult to see that the sequence $(X_t)_{t \geq 0}$ defined from $\tilde{\omega}^{\infty}$ as previously, satisfies the recurrence formula $X_{t+1} = f(X_t, \tilde{\omega}^{\infty})$, but also $X_{t+1} = f(X_t, \omega^{t+1})$, which proves that $(X_t)_{t \geq 0}$ is an homogeneous Markov chain.   □

2.3. **Monotonicity properties and notations.** ¿From now on, we will denote by $(X_t^{\xi,p,q})_{t \geq 0}$ the competition process where

- $\xi \in S^{(\mathbb{Z}^d)}$ is the initial configuration: $X_0^{\xi,p,q} = \xi$,
- $0 \leq p \leq q \leq 1$: the weakest (also called yellow) infection uses parameter p while the strongest (also called blue) uses q.

The corresponding random sets are now denoted by:

$$\begin{cases} \eta_{\xi,p,q}^1(t) = \{x \in \mathbb{Z}^d : X_t^{\xi,p,q}(x) \in \{y, y^*, g, g^*\}\}, \\ \eta_{\xi,p,q}^2(t) = \{x \in \mathbb{Z}^d : X_t^{\xi,p,q}(x) \in \{b, b^*, g, g^*\}\}. \end{cases}$$

Thus for $t \geq 1$, they are also defined by the following recursive rules – remember that the notation ∂_p was defined in (2):

$$(3) \qquad \begin{cases} \eta_{\xi,p,q}^1(t) = \eta_{\xi,p,q}^1(t-1) \cup (\partial_p \eta_{\xi,p,q}^1(t-1) \backslash \eta_{\xi,p,q}^2(t-1)), \\ \eta_{\xi,p,q}^2(t) = \eta_{\xi,p,q}^2(t-1) \cup (\partial_q \eta_{\xi,p,q}^1(t-1) \backslash \eta_{\xi,p,q}^2(t-1)). \end{cases}$$

This particular realization of our competition process will be used in the sequel of the paper, because it presents the advantage to give an easy access to coupling and monotonicity properties. Note that the function

$$\begin{aligned} G : \quad [0,1] \times \Omega \times \mathcal{P}(\mathbb{Z}^d) \times \mathcal{P}(\mathbb{Z}^d) \quad &\longrightarrow \quad \mathcal{P}(\mathbb{Z}^d) \\ (p, \omega, A, B) \quad &\longmapsto \quad G(p, \omega, A, B) = A \cup (\partial_p A(\omega) \backslash B) \end{aligned}$$

introduced in the proof of Lemme 2.1 is non-decreasing in p and A, and non-increasing in B. As it defines the random sets at time $t+1$ from the random sets at time t, this implies in particular that:

Lemma 2.2.

- $\eta_{\xi,p,q}^1(t+1)$ *is non-decreasing in p and non-increasing in q,*
- $\eta_{\xi,p,q}^2(t+1)$ *is non-decreasing in q and non-increasing in p.*

The next Lemma is trivial, but it is an illustration of the fundamental role played by the chemical distance in Bernoulli percolation in our analysis of this competition model: it says that the set of sites infected by any of the two infections at time n can be compared with the single weaker infection.

Lemma 2.3. *Let us define, for any $0 \leq p \leq 1$ and any $s \in \mathbb{Z}^d$, the process $(B_p^s(t))_{t \in \mathbb{Z}_+}$ by :*

$$B_p^s(0) = \{s\} \text{ and } \forall t \geq 0, \; B_p^s(t+1) = B_p^s(t) \cup \partial_p B_p^s(t).$$

Let s_1 and s_2 be two distinct sites of $\mathbb{Z}^d$ and ξ be the element of $S^{\mathbb{Z}^d}$ where all sites are empty, but $\xi_{s_1} = y$ and $\xi_{s_2} = b$. Suppose that $0 \leq p \leq q \leq 1$. Then

$$\forall t \in \mathbb{Z}_+ \quad \begin{cases} B_p^{s_1}(t) \subset \eta_{\xi,p,q}^1(t) \cup \eta_{\xi,p,q}^2(t), \\ \eta_{\xi,p,q}^1(t) \subset B_p^{s_1}(t) \text{ and } \eta_{\xi,p,q}^2(t) \subset B_q^{s_2}(t). \end{cases}$$

It is easy to see that $B_p^s(t) = \{x \in \mathbb{Z}^d : D_p(s,x) \leq t\}$, where $D_p(x,y)$ is the cardinal of the shortest p-open path from x to y and is called the *chemical distance* between x and y. Note that the inclusion $\eta_{\xi,p,q}^1(t) \subset B_p^{s_1}(t)$ implies that if $p < p_c$, then the infection with parameter p almost surely dies out.

This description (3) of the competition model leads us to recall notations and results about chemical distance in Bernoulli percolation.

3. Chemical distance in Bernoulli percolation

In this section, we recall results concerning chemical distance in supercritical Bernoulli percolation:

- almost-sure convergence results (4) and (5) of the chemical distance to a deterministic norm,
- large deviations inequalities (6) and (7) associated to this convergence,
- classical estimates (8) and (9) on the geometry of clusters.

We first complete the notations introduced at the beginning of Subsection 2.2: the connected component of the site x in the random graph $\mathcal{G}_p$ is denoted C_p^x, and the event that two sites x and y are in the same connected component of this graph is denoted $x \overset{p}{\leftrightarrow} y$. Bernoulli percolation is in particular famous for its phase transition: there exists $0 < p_c = p_c(d) < 1$ such that

- if $p < p_c$ then with probability 1, the random graph $\mathcal{G}_p$ has only finite connected components,
- if $p > p_c$ then with probability 1, the random graph $\mathcal{G}_p$ has at least one infinite connected component, which is moreover almost surely unique and denoted C_p^∞.

See the reference book by Grimmett [12] for instance.

A *path* is a sequence $\gamma = (x_1, e_1, x_2, e_2, \ldots, x_n, e_n, x_{n+1})$ such that x_i and x_{i+1} are neighbors and e_i is the edge between x_i and x_{i+1}. We will also sometimes describe γ only by the vertices it visits $\gamma = (x_1, x_2, \ldots, x_n, x_{n+1})$ or by its edges $\gamma = (e_1, e_2, \ldots, e_n)$. The number n of edges in γ is called the *length* of γ and is denoted by $|\gamma|$. A path is said to be *p-open* in the configuration ω if all its edges are p-open in ω. The *chemical distance* D_p is the usual graph distance in $\mathcal{G}_p$:

$$\forall x, y \in \mathbb{Z}^d \quad D_p(x, y) = \inf\{|\gamma| : \gamma \text{ } p\text{-open path between } x \text{ and } y\}.$$

We also define the random balls associated to this random distance:

$$\forall x \in \mathbb{Z}^d, \forall t \geq 0 \quad B_p^x(t) = \{y \in \mathbb{Z}^d : D_p(x, y) \leq t\}.$$

The formulation in terms of random distance comes from classical first-passage percolation, and indeed, this model can be seen as i.i.d. first-passage percolation, where the passage-time of an edge takes value 1 with probability p and value ∞ with probability $1 - p$. An asymptotic shape result is also available for this model: in a previous paper [10], we proved the existence of a deterministic norm $\|.\|_p$ on $\mathbb{R}^d$ such that $B_p^0(t)/t$ converges to the unit ball for $\|.\|_p$ on the event $\{0 \overset{p}{\leftrightarrow} \infty\} = \{0 \in C_p^\infty\}$, for the Hausdorff distance between two non empty compact subsets of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $t \geq 0$, first define the deterministic balls associated to the norm $\|.\|_p$:

$$\mathcal{B}_p^x(t) = \{y \in \mathbb{R}^d : \|x - y\|_p \leq t\}.$$

The Hausdorff distance between two non empty compact subsets A and B of $\mathbb{R}^d$ is defined by

$$\mathcal{D}(A, B) = \inf\{t \geq 0 : A \subset B + \mathcal{B}_p^0(t) \text{ and } B \subset A + \mathcal{B}_p^0(t)\}.$$

Note that the equivalence of norms on $\mathbb{R}^d$ ensures that the topology induced by this Hausdorff distance does not depend on the choice of the norm $\|.\|_p$. The convergence result writes then: for every $p > p_c(d)$,

- Existence of an asymptotic speed (Lemma 5.7 in [10]).

$$(4) \qquad \overline{\lim_{\|x\|_p \to \infty}} \; \mathbb{1}_{\{0 \overset{p}{\leftrightarrow} x\}} \left(\frac{D_p(0,x)}{\|x\|_p} - 1 \right) = 0 \quad \mathbb{P} \text{ a.s.}$$

- Asymptotic shape result (Theorem 5.3 and Corollary 5.4 in [10]).
  If $\overline{\mathbb{P}}_p(A) = \mathbb{P}(A | 0 \overset{p}{\leftrightarrow} \infty)$, then

$$(5) \qquad \lim_{t \to +\infty} \mathcal{D}\left(\frac{B_p^0(t)}{t}, \mathcal{B}_p^0(1) \right) = 0 \quad \overline{\mathbb{P}}_p \text{ a.s.}$$

In the sequel, we will also use a corollary of these results. For $A \subset \mathbb{Z}^d$, we denote

$$|A|_p = \sup\{\|x\|_p : x \in A\} \quad \text{and} \quad |A|_{*,p} = \inf\{\|x\|_p : x \in C_p^\infty \backslash A\}.$$

Lemma 3.1. *Let $p > p_c(d)$. On the event $\{a \in C_p^\infty\}$, we have $\mathbb{P}$ almost surely:*

$$\frac{|B_p^a(t)|_p}{t} \to 1 \quad \text{and} \quad \frac{|B_p^a(t)|_{*,p}}{t} \to 1.$$

Proof. The identities $\overline{\lim}_{t \to +\infty} \frac{|B_p^a(t)|_p}{t} = 1$ and $\overline{\lim}_{t \to +\infty} \frac{|B_p^a(t)|_{*,p}}{t} \leq 1$ obviously follows from (5). It remains to show that for each $\delta > 0$,

$$\mathbb{P}\left(\frac{|B_p^a(t)|_{*,p}}{t} \leq 1 - \delta \;\; \text{i.o.} \right) = 0.$$

Suppose $\frac{|B_p^a(t)|_{*,p}}{t} \leq 1 - \delta$ i.o.: there exists sequences $(x_n)_{n \geq 1}$ and $(t_n)_{n \geq 1}$, with $x_n \in C_p^\infty$, $\|x_n\|_p \leq t_n(1 - \delta)$, $D_p(0, x_n) \geq t_n$ and $t_n \to +\infty$. The sequence $(x_n)_{n \geq 1}$ is necessary unbounded, otherwise there would exist a limiting value x, with $D_p(0, x) = +\infty$ and $x \in C_p^\infty$, which is not possible. It follows that there exist infinitely many $x \in C_p^\infty$ with $D_p(0, x) \geq (1 + \delta)\|x\|_p$. By (4), this happens with a null probability. $\qquad \square$

As a direct consequence of these convergence results and of the coupling identity

$$D_q(0, nx)\mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}} \leq D_p(0, nx)\mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}},$$

we obtain the natural large comparison between norms for different parameters. It will be improved in Section 4 to prove Proposition 1.3.

Lemma 3.2. *If $p_c(d) < p \leq q \leq 1$, then for every $x \in \mathbb{R}^d$, $\|x\|_q \leq \|x\|_p$.*

In another paper [8], we gave further information on the speed of convergence by establishing the following large deviation inequalities corresponding to the previous convergence results: for every $p > p_c(d)$, for every $\varepsilon > 0$, we have:

- Directional large deviation result.

$$(6) \qquad \overline{\lim_{\|x\|_1 \to +\infty}} \; \frac{1}{\|x\|_1} \ln \mathbb{P}\left(0 \overset{p}{\leftrightarrow} x, \; \frac{D_p(0,x)}{\|x\|_p} \notin (1 - \varepsilon, 1 + \varepsilon) \right) < 0.$$

- Shape large deviation result. There exist two strictly positive constants A and B such that

$$(7) \qquad \forall t > 0 \quad \overline{\mathbb{P}}_p \left(\mathcal{D} \left(\frac{B_p^0(t)}{t}, \mathcal{B}_p^0(1) \right) \geq \varepsilon \right) \leq A e^{-Bt}.$$

As a consequence, we obtain the next lemma, which enables the control of minimal paths:

Lemma 3.3. *Note $H_p(x, y, \varepsilon)$ the event: "There exists a p-open minimal path from x to y which is completely inside $\mathcal{B}_p^y((1+\varepsilon)\|x-y\|_p)$ and whose length is smaller than $(1+\varepsilon)\|x-y\|_p$".*

Then for every $p > p_c(d)$, for every $\varepsilon > 0$, there exist two strictly positive constants A and B such that:

$$\forall x, y \in \mathbb{Z}^d \quad \mathbb{P}(y \in C_p^\infty, \, x \in C_p^\infty, \, H_p(x, y, \varepsilon)^c) \leq A \exp(-B\|x-y\|_1).$$

Proof. Using translation invariance, we can assume that $y = 0$. Note that $H(x, 0, \varepsilon)^c$ contains the event $\{D_p(0, x) \leq \|x\|_p(1+\varepsilon/2)\} \cap \{B_p^0(t(1+\varepsilon/2)) \subset \mathcal{B}_p^0((1+\varepsilon)\|x\|_p)\}$ and apply the large deviation inequality for the chemical distance (6) and the large deviation inequality (7) for the asymptotic shape. $\qquad\square$

We also recall here some classical results concerning the geometry of clusters in supercritical percolation. Thanks to Chayes, Chayes, Grimmett, Kesten and Schonmann [2], we can control the radius of finite clusters: there exist two strictly positive constants A and A such that

$$(8) \qquad \forall r > 0 \quad \mathbb{P}\left(|C_p^0| < +\infty, \, 0 \overset{p}{\leftrightarrow} \partial_1 \mathcal{B}_1^0(r) \right) \leq A e^{-Br}.$$

The size of holes in the infinite cluster can also be controlled: there exist two strictly positive constants A and B such that

$$(9) \qquad \forall r > 0 \quad \mathbb{P}\left(C_p^\infty \cap \mathcal{B}_1^0(r) = \varnothing \right) \leq A e^{-Br}.$$

When $d = 2$, this result follows from large deviation estimates by Durrett and Schonmann [7]. Their methods can easily be transposed when $d \geq 3$. Nevertheless, when $d \geq 3$, the easiest way to obtain it seems to use Grimmett and Marstrand [11] slab's result.

Note that in Lemma 3.3, in (8) and in (9), thanks to the norm equivalence, the choice of the norm $\|.\|_1$ is of course irrelevant, but in the very values of the positive constants.

4. STRICT INCLUSION OF ASYMPTOTIC SHAPES FOR CHEMICAL DISTANCE

Inequalities on asymptotic shapes are already known for classical first-passage percolation – see the papers by Van den Berg and Kesten [20] and by Marchand [18]. The aim of this section is to prove Proposition 1.3, which is the analogous result in this context. We recall that the large inequality was easily established in Lemma 3.2, but that strict comparisons will be crucial to handle the competition problem.

The proof of Proposition 1.3 is based on renormalization techniques. We thus begin by stating an adapted renormalization lemma, which is the one used by Van den Berg and Kesten in [20].

4.1. **A renormalization lemma.**

The renormalization grid. Let N be a strictly positive integer. We introduce the following notations.

- C_N is the cube $[-1/2, N-1/2]^d$. We call N-cubes the cubes $C_N(k) = kN + C_N$ obtained by translating C_N according to Nk with $k \in \mathbb{Z}^d$. The coordinates of k are called the coordinates of the N-cube $C_N(k)$. Note that N-cubes induce a partition of $\mathbb{Z}^d$.
- L_N is the large cube $[-N-1/2, 2N-1/2]^d$, and the large cube $L_N(k)$ is obtained by translating L_N according to Nk with $k \in \mathbb{Z}^d$. The boundary of $L_N(k)$, denoted by $\partial L_N(k)$, is the set of sites outside $L_N(k)$ that have a neighbor in $L_N(k)$.
- R_N is the rectangular box $[-1/2, N-1/2]^{d-1} \times [-N-1/2, 2N-1/2]$. In the large cube $L_N(k)$, the N-cube $C_N(k)$ is surrounded by the $2d$ N-boxes, obtained by rotations and translations of R_N. For instance, in $L_N(0)$, the N-cube $C_N(0)$ is surrounded by the $2d$ following N-boxes: for $1 \le i \le d$, and for $\varepsilon \in \{-1, +1\}$, we define

$$R_N^{i,\varepsilon}(0) = \left[-N - \frac{1}{2}, 2N - \frac{1}{2}\right]^{i-1} \times \left[\varepsilon N - \frac{1}{2}, (1+\varepsilon)N - \frac{1}{2}\right] \times \left[-N - \frac{1}{2}, 2N - \frac{1}{2}\right]^{d-i}.$$

The set of all these surrounding boxes is denoted $\mathcal{R}_N$.

An edge is said to be *in a subset E* of $\mathbb{R}^d$ if at least one of its two extremities is in E. We now define the *inner* and *outer* boundaries of a N-box associated to a pair $(C_N(k), L_N(k))$ of cubes. Let's do this for $R_N^{1,+}(0)$ and extend the definition to other boxes by rotation and translation:

$$\begin{aligned}
\partial_{out} R_N^{1,+1}(0) &= \{(2N, y), \ y \in [-N, \dots, 2N-1]^{d-1}\}, \\
\partial_{in} R_N^{1,-1}(0) &= \{(N, y), \ y \in [-N, \dots, 2N-1]^{d-1}\}.
\end{aligned}$$

Note that $\partial L_N(k)$ is the disjoint union of the sets $(\partial_{out} R_N^{i,\varepsilon}(k))_{1 \le i \le d, \ \varepsilon \in \{+1,-1\}}$, and that a path entering in $C_N(k)$ and getting out of $L_N(k)$ has to cross one of the $2d$ N-boxes surrounding $C_N(k)$ in $L_N(k)$, from its inner boundary to its outer boundary. We can then define the crossing associated to a N-cube $C_N(k)$ – see also Figure 2:

Definition 4.1. *Let $\gamma = (x_0, \dots, x_l)$ be a path such that $x_0 \in C_N(k)$ and $x_l \notin L_N(k)$. We set $j_f = \min\{0 \le k \le l, \ x_k \in \partial L_N(k)\}$. There exists a unique (i, ε) such that $x_{j_f} \in R_N^{i,\varepsilon}(k)$. Let then $j_0 = \max\{0 \le k \le j_f, \ x_k \notin R_N^{i,\varepsilon}(k)\}$. The portion $(x_{j_0+1}, \dots, x_{j_f})$ of γ is the crossing of γ associated to $C_N(k)$.*

Main crossings of a path. Let N be a strictly positive integer, x be a point in $\mathbb{Z}^d$ and γ be a path without any double point from 0 to x. We want to associate to γ a sequence of crossings of N-boxes (the main crossings of γ), in a way that two different crossings are edge-disjoint. Consider first the sequence $\sigma_0 = (k_1, \dots, k_{\tau_0})$ made of the coordinates of the N-cubes successively visited by γ. As the N-cubes induce a partition of $\mathbb{Z}^d$, this sequence is well defined, and has the following properties:

$$(\mathrm{P}_0) \begin{cases} 0 \in C_N(k_1), \ x \in C_N(k_{\tau_0}), \\ \forall 1 \le i \le \tau_0 - 1, \ \|k_{i+1} - k_i\|_1 = 1. \end{cases}$$

But σ_0 can have doubles points; we remove them by the classical loop-removal process described in [13]. We thus obtain a sequence $\sigma_1 = (k_{\varphi_1(1)}, \dots, k_{\varphi_1(\tau_1)})$
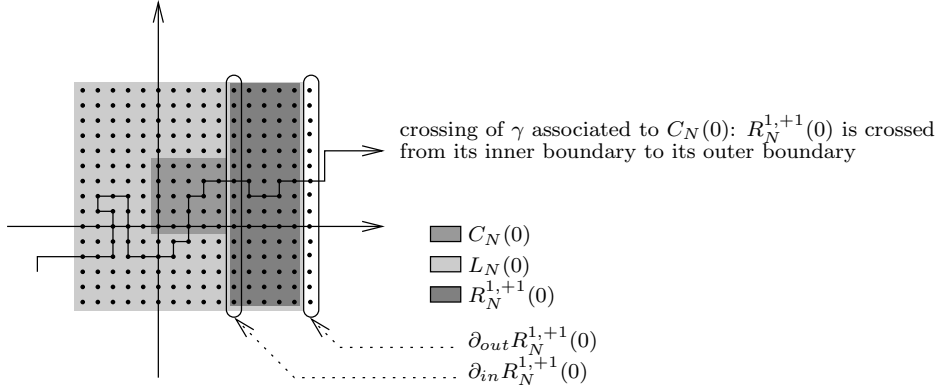
FIGURE 2. Elements of the renormalization grid for $N = 6$ in dimension $d = 2$.

extracted from σ_0 , with the following properties:

$$(P_1) \begin{cases} 0 \in C_N(k_{\varphi_1(1)}), \;\; x \in C_N(k_{\varphi_1(\tau_1)}), \\ \forall\, 1 \le i \le \tau_1 - 1, \; \|k_{\varphi_1(i+1)} - k_{\varphi_1(i)}\|_1 = 1, \\ \sigma_1 \text{ has no double point.} \end{cases}$$

To every cube $C_N(k)$ in this sequence such that γ gets out of $L_N(k)$, that means for every N-cube in σ_1 with the possible exception of the $2d$ last, we associate a crossing of a N-box in the following way: let z be the first point of γ to be in $C_N(k)$, and let z_2 be the first point of γ after z to be in $\partial L_N(k)$. Then the crossing associated to the N-cube $C_N(k)$ is the crossing of the portion of γ between z and z_2 associated to $C_N(k)$ in Definition 4.1.

The problem now is that two distinct cubes in σ_1 can have the same associated crossing. We have to extract a subsequence once again in order to obtain edge-disjoint crossings. Set $\varphi_2(1) = 1$, and define φ_2 by induction:

$$\varphi_2(i+1) = \inf\{j > \varphi_2(i) \text{ such that } \|k_{\varphi_1(j)} - k_{\varphi_1 \circ \varphi_2(i)}\|_\infty > 1\} - 1$$

if the infimum exists, and let τ be the smallest index i for which $\varphi_2(i+1)$ is not defined. Set $\varphi = \varphi_1 \circ \varphi_2$; the elements of $\sigma = (k_{\varphi(i)})_{1 \le i \le \tau}$ are called the main cubes of γ, and their associated crossings the *main crossings of γ*. This sequence has the following properties (see [20]):

$$(P) \begin{cases} 0 \in C_N(k_{\varphi(1)}), \\ \|k_{\varphi(\tau)} - k_{\tau_0}\|_\infty \le 1, \\ \forall\, 1 \le i \le \tau - 1, \; \|k_{\varphi(i+1)} - k_{\varphi(i)}\|_\infty = 1, \\ \text{the main crossings of } \gamma \text{ are edge-disjoint.} \end{cases}$$

¿From Properties (P) we can deduce that for every x in $\mathbb{Z}^d$, the number τ of main N-cubes of a path with no double point from 0 to x satisfies the following inequality:

$$(10) \qquad\qquad \tau \ge \frac{\|x\|_\infty}{N}.$$

A renormalization lemma. The following lemma is an adaptation of Lemma (5.2) in [20], and its proof is a standard Peierl's argument (see proof of (3.12) in [13]). We thus just state it without any proof.

Lemma 4.2. *For each $N \in \mathbb{Z}_+$, we give to the N-cubes a random color, black or white, according to the states of the edges in the initial model, such that:*

- *For each $N \in \mathbb{Z}_+^*$, the colors of the N-cubes are identically distributed.*
- *For each $N \in \mathbb{Z}_+^*$, for each $k \in \mathbb{Z}^d$, the color of the N-cube $C_N(k)$ depends only on the states of the edges in $L_N(k)$.*
- $\lim\limits_{N \to +\infty} \mathbb{P}(C_N(k) is black) = 1.$

Then for every $\rho \in]0,1[$, there exists N_ρ such that for all $N \geq N_\rho$, there exist two strictly positive constants A and B such that for every $x \in \mathbb{R}^d$:
(11)
$$\mathbb{P}\left(\begin{array}{c} \text{There exists a path } \gamma \text{ from } 0 \text{ to } x \text{ that, among} \\ \text{its } \tau \text{ main } N\text{-cubes, has less than } \rho\tau \text{ black cubes} \end{array} \right) \leq A \exp(-B\|x\|_\infty).$$

4.2. **Proof of the strict comparison result Proposition 1.3.**

Fix p and q such that $p_c(d) < p < \overrightarrow{p_c}(d)$ and $p < q \leq 1$. Roughly speaking, as $p < \overrightarrow{p_c}(d)$, we can find along a p-minimal path from 0 to nx a certain number of crossing of rectangular boxes such that:

- the restriction of the p-minimal path of one box is not direct,
- by adding q-open edges, as $q > p$, we can find in this box a direct q-minimal path with the same extremities, which is thus an improvement of the p-minimal path.

By using these improvements, we can exhibit a significant discrepancy, *i.e.* of order n, between $D_p(0, nx)$ and $D_q(0, nx)$. The proof consists in giving estimates to these crude arguments.

Proof. Consider the space $\Omega = \{0,1\}^{\mathbb{E}^d} \times \{0,1\}^{\mathbb{E}^d}$, endowed with the classical Borel σ-algebra on Ω and the probability measure

$$\mathbb{P} = \text{Ber}(p)^{\otimes \mathbb{E}^d} \otimes \text{Ber}\left(\frac{q-p}{1-p}\right)^{\otimes \mathbb{E}^d}.$$

Write points of Ω in the following manner

$$\omega = (\omega^1, \omega^2) \text{ with } \omega^1 = (\omega_e^1)_{e \in \mathbb{E}^d} \in \{0,1\}^{\mathbb{E}^d} \text{ and } \omega^2 = (\omega_e^2)_{e \in \mathbb{E}^d} \in \{0,1\}^{\mathbb{E}^d}.$$

Define then, for every $e \in \mathbb{E}^d$, $\omega_e^3 = \omega_e^1 \vee \omega_e^2$. Clearly, the law of $(\omega_e^1)_{e \in \mathbb{E}^d}$ under $\mathbb{P}$ is $\text{Ber}(p)^{\otimes \mathbb{E}^d}$ whereas the law of $(\omega_e^3)_{e \in \mathbb{E}^d}$ under $\mathbb{P}$ is $\text{Ber}(q)^{\otimes \mathbb{E}^d}$. We denote by $\mathcal{G}_p$ – resp. $\mathcal{G}_q$ – the corresponding random graphs and by $D_p(x,y)$ – resp. $D_q(x,y)$ – the random distance from x to y in $\mathcal{G}_p$ – resp. $\mathcal{G}_q$. Note that in this special coupling,

$$\mathcal{G}_p \subset \mathcal{G}_q \text{ and } D_q(x,y) \leq D_p(x,y).$$

For each $N \in \mathbb{Z}_+$, we consider the same renormalization grid as previously and give to each N-box $R_N^{i,\varepsilon}(k)$ a random color:

Definition 4.3. *The box $R_N^{i,\varepsilon}(k)$, with $k \in \mathbb{Z}^d$, is said to be black if and only if it satisfies the following property:*

$$\forall y \in \partial_{in} R_N^1(k), \ \forall z \in \partial_{out} R_N^1(k), \ \forall \gamma \ p\text{-open path from } y \text{ to } z \text{ included in } R_N^1(k),$$
$$|\gamma| \geq \|z - y\|_1 + 1.$$

It is said to be white otherwise. This definition is naturally extended to other boxes by translation and rotation.

Thus a box is black if and only if it can not be directly crossed from its inner boundary to its outer boundary by a p-open path. Let us verify that this coloring satisfies the conditions of renormalization Lemma 4.2. It is clear that the colors of the different cubes are identically distributed, and that the color of $C_N(k)$ only depends on the states of the edges in $L_N(k)$. Let us now estimate the probability p_N for $C_N(0)$ to be white. It is clear by translation invariance that $p_N \leq 2d\mathbb{P}\left(R_N^{1;+1}(0) \text{ is white}\right)$ and that the probability for $R_N^{1;+1}(0)$ to be white is bounded by

$$\mathbb{P}\left(R_N^{1;+1}(0) \text{ is white}\right) \leq (3N+1)^{d-1}2^{d-1}\mathbb{P}\left(\max\{\|x\|_1 : x \in \overrightarrow{C_p}0\} \geq N\right),$$

where $\overrightarrow{C_p}0$ is the cluster containing 0 in *oriented* percolation with parameter p. The term $(3N+1)^{d-1}$ counts the possible starting points of the oriented open path, while the term 2^{d-1} counts its possible orientations. As in the non-oriented case, when $p < \overrightarrow{p_c}(d)$, the probability in the left-hand side member decreases exponentially fast with N – see the paper by Aizenman and Barsky [1] – which proves that:

$$\lim_{N \to +\infty} p_N = 0.$$

We can then apply the renormalization Lemma 4.2 with a fixed parameter ρ satisfying $0 < \rho < 1$. Let N be large enough to have (11) with positive A and B. These ρ and N are now fixed for the sequel of the proof.

For each $n \geq 1$ and every $x \in \mathbb{Z}^d\setminus\{0\}$, if the event $\{0 \overset{p}{\leftrightarrow} nx\}$ occurs, we denote by $\gamma_{n,x}$ be a p-open path from 0 to nx whose length is equal to $D_p(x,y)$. Let $\sigma_{n,x} = (k_1,\ldots,k_{\tau_{n,x}})$ be the sequence of its main cubes and denote by $A_{n,x}$ the event that among these $\tau_{n,x}$ main cubes, at most $\rho\tau_{n,x}$ cubes are black. With Lemma 4.2 we have:

$$\mathbb{P}(A_{n,x} \cap \{0 \overset{p}{\leftrightarrow} nx\}) = \mathbb{P}\left(\begin{array}{c} \text{there exists a } p\text{-open path } \gamma \text{ from 0 to } nx \\ \text{that, among its } \tau \text{ main } N\text{-cubes,} \\ \text{has less than } \rho\tau \text{ black cubes} \end{array}\right)$$

$$(12) \qquad \leq A\exp(-Bn\|x\|_\infty).$$

We define now the notion of *good rectangular boxes*

Definition 4.4. *A rectangular box R is good if it is black and if, moreover, for every $e \in R$, $\omega^3(e) = 1$.*

In other words, in a good box, edges that are not p-open are q-open. Let n be large enough and let $R \in \mathcal{R}_N$ be a good box. Suppose that the path $\gamma_{n,x}$ crosses R and that this crossing, denoted by $\gamma_{n,x|R}$, is a main crossing of $\gamma_{n,x}$. Denote by y and z the extremities of the restriction $\gamma_{n,x|R}$ of the path $\gamma_{n,x}$ to the box R. Then, by definition of *black* and *good* boxes,

$$(13) \qquad \|z - y\|_1 = D_q(y,z) \leq D_p(y,z) - 1.$$

Note that moreover, in this case, any q-open path between y and z with length $\|z - y\|_1 = D_q(y,z)$ is completely inside R. Choose one and call it an *improvement* for D_q of $\gamma_{n,x}$ in R.

Now, on the event $\{0 \overset{p}{\leftrightarrow} nx\}$, replace in $\gamma_{n,x}$ all the restrictions associated to main crossings of $\gamma_{n,x}$ by their improvements for D_q, to obtain a modified path $\hat{\gamma}_{n,x}$ from 0 to nx: this is possible, because by definition, main crossings are in non-intersecting boxes. Then

$$\mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}}(D_p(0, nx) - D_q(0, nx))$$

$$\geq \quad \mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}} \sum_{R \in \mathcal{R}_N} \mathbb{1}_{\{R \text{ is good}\}} \mathbb{1}_{\left\{ \substack{\gamma_{n,x} \text{ crosses } R, \text{ and} \\ \text{this is a main crossing of } \gamma_{n,x}} \right\}}$$

$$\geq \quad \mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}} \sum_{R \in \mathcal{R}_N} \left(\prod_{e \in R} \mathbb{1}_{\omega_e^3 = 1} \right) \mathbb{1}_{\{R \text{ is black}\}} \mathbb{1}_{\left\{ \substack{\gamma_{n,x} \text{ crosses } R, \text{ and} \\ \text{this is a main crossing of } \gamma_{n,x}} \right\}}$$

$$\geq \quad \mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}} \sum_{R \in \mathcal{R}_N} \left(\prod_{e \in R} \mathbb{1}_{\omega_e^2 = 1} \right) \mathbb{1}_{\{R \text{ is black}\}} \mathbb{1}_{\left\{ \substack{\gamma_{n,x} \text{ crosses } R, \text{ and} \\ \text{this is a main crossing of } \gamma_{n,x}} \right\}}.$$

Note $G(R)$ the event $\{\forall e \in R \quad \omega_e^2 = 1\}$. As ω_1 and ω_2 are independent, the conditional law of the random variable

$$Y_{n,x} \overset{\text{def}}{=} \sum_{R \in \mathcal{R}_N} \mathbb{1}_{G(R)} \mathbb{1}_{\{R \text{ is black}\}} \mathbb{1}_{\left\{ \substack{\gamma_{n,x} \text{ crosses } R, \text{ and} \\ \text{this is a main crossing of } \gamma_{n,x}} \right\}}$$

knowing ω^1 is a binomial law $\text{Bin}(Z_{n,x}, r)$ with parameters

$$Z_{n,x} \overset{\text{def}}{=} \sum_{R \in \mathcal{R}_N} \mathbb{1}_{\{R \text{ is black}\}} \mathbb{1}_{\left\{ \substack{\gamma_{n,x} \text{ crosses } R, \text{ and} \\ \text{this is a main crossing of } \gamma_{n,x}} \right\}},$$

$$r = \mathbb{P}(G(R)) \geq \left(\frac{q-p}{1-p} \right)^{c_d N^d} > 0.$$

We have then, using Estimate (10) on the event $A_{n,x}^c$:

$$\mathbb{1}_{A_{n,x}^c} \mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}} Z_{n,x} \geq \rho \tau_{n,x} \mathbb{1}_{A_{n,x}^c} \mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}} \geq \rho \frac{n\|x\|_\infty}{N} \mathbb{1}_{A_{n,x}^c} \mathbb{1}_{\{0 \overset{p}{\leftrightarrow} nx\}}.$$

Thus, if $\delta > 0$, we have

$$\mathbb{P}\left(0 \overset{p}{\leftrightarrow} nx, \ Y_{n,x} \leq \frac{\rho n\|x\|_\infty}{N} r(1-\delta) \right)$$

$$\leq \quad \mathbb{P}(\{0 \overset{p}{\leftrightarrow} nx\} \cap A_{n,x}) + \sum_{k = \frac{\rho n\|x\|_\infty}{N}}^{\infty} \mathbb{P}\left(Z_{n,x} = k, \ Y_{n,x} \leq \frac{\rho n\|x\|_\infty}{N} r(1-\delta) \right),$$

$$\leq \quad \mathbb{P}(\{0 \overset{p}{\leftrightarrow} nx\} \cap A_{n,x}) + \sum_{k \geq \frac{\rho n\|x\|_\infty}{N}} \mathbb{P}\left(Z_{n,x} = k, \ Y_{n,x} \leq kr(1-\delta) \right),$$

$$\leq \quad \mathbb{P}(\{0 \overset{p}{\leftrightarrow} nx\} \cap A_{n,x}) + \sum_{k \geq \frac{\rho n\|x\|_\infty}{N}} \mathbb{P}\left(Z_{n,x} = k \right) 2 \exp\left(-\frac{k\delta^2}{4r(1-r)} \right)$$

by Chernov inequality,

$$\leq \quad \mathbb{P}(\{0 \overset{p}{\leftrightarrow} nx\} \cap A_{n,x}) + 2 \exp\left(-\frac{\rho n\|x\|_\infty \delta^2}{4Nr(1-r)} \right).$$

By (12) and Borel-Cantelli Lemma, this leads to

$$\mathbb{P}\left(0 \overset{p}{\leftrightarrow} nx, \ \frac{D_p(0, nx)}{n} - \frac{D_q(0, nx)}{n} \leq \frac{\rho\|x\|_\infty(1-\delta)r}{N} \ \text{ i.o.} \right) = 0.$$

On the event $\{0 \overset{p}{\leftrightarrow} \infty\} \subset \{0 \overset{q}{\leftrightarrow} \infty\}$, by the convergence result (4), we obtain $\|x\|_p - \|x\|_q \geq \frac{\rho\|x\|_\infty(1-\delta)r}{N}$, and finally, by letting δ going to 0,

$$\forall x \in \mathbb{Z}^d \quad \|x\|_p - \|x\|_q \geq \frac{\rho r}{N}\|x\|_\infty.$$

Since norms are homogeneous and continuous, this ends the proof. $\square$

5. Coexistence can only happen at slow speed

We tackle in this section the core of the paper: the proof of Proposition 1.4. For p and q larger than p_c, we define

$$C_{p,q} = \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{\|x\|_q}{\|x\|_p}.$$

We fix here $p_1 < p_2$ and two distinct sites s_1 and s_2 of $\mathbb{Z}^d$: the initial state ξ is the configuration where every site is empty, but s_1, which is active yellow, and s_2, which is active blue. In the sequel, to lighten notations, we omit the subscripts p_1, p_2, ξ: for instance,

$$\eta^2(t) = \eta^2_{p_1,p_2,\xi}(t).$$

By Proposition 1.3, we know that $C_{p_1,p_2} < 1$.

In fact, Proposition 1.4 will appear as a by-product of the following theorem, which ensures that if the p_1-infection survives, then the time of infection of x by the p_2-infection, when it is finite, should be of order $\|x\|_{p_1}$ rather than $\|x\|_{p_2}$, expected time of infection for one simple p_2-infection.

Define, for $x \in \mathbb{Z}^d$:

$$t(x) = \inf\{t \geq 0 : x \in \eta^2(t)\},$$
$$\mathcal{G}^i = \left\{ \sup_{t \geq 0} |\eta^i(t)| = +\infty \right\} \text{ for } i = 1, 2.$$

Theorem 5.1. *Let $\delta > 0$. Then there exist $A, B > 0$ such that*

$$\forall x \in \mathbb{Z}^d \quad \mathbb{P}\left(\mathcal{G}^1 \cap \{t(x) \leq (1-\delta)\|x\|_{p_1}\}\right) \leq A\exp(-B\|x\|).$$

At first, let us see how Theorem 5.1 implies Proposition 1.4:

Proof. Let $\delta > 0$. We must prove that $\mathbb{P}(\mathcal{G}^1, |\eta^2(t)|_{p_1} \geq (1+\delta)t$ i.o.$) = 0$. Obviously, it is equivalent to prove that

$$\mathbb{P}(\mathcal{G}^1, (1+\delta)t(x) \leq \|x\|_{p_1} \text{ for infinitely many } x) = 0.$$

This comes from Theorem 5.1, with the help of Borel-Cantelli's lemma. $\square$

We still need some extra notations and lemmas.

Definitions.
We note $\mathcal{S} = \{x \in \mathbb{R}^d : \|x\|_{p_2} = 1\}$ and define the *shells*: for each $A \subset \mathcal{S}$, and every $0 < r < R$, we set

$$\hat{x} = x/\|x\|_{p_2},$$
$$\text{Shell}(A, r, R) = \{x \in \mathbb{Z}^d : \hat{x} \in A \text{ and } r \leq \|x\|_{p_2} \leq R\}.$$

So roughly speaking, A is to think about as the set of possible directions for the points in the shell, while $[r, R]$ is the set of radii.

For $A \subset \mathcal{S}$ and $\varphi > 0$, define the following enlargement of A:

$$A \oplus \varphi = (A + \mathcal{B}_{p_2}^0(\varphi)) \cap \mathcal{S}.$$

Lemma 5.2. *For any norm $|.|$ on $\mathbb{R}^d$, one has*

$$\forall x, y \in \mathbb{R}^d \backslash \{0\} \quad \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq \frac{2|x-y|}{\max\{|x|, |y|\}}.$$

Lemma 5.3. *For every $\rho > 0$, there exists $\theta > 0$ such that*

$$\forall x, y \in \mathbb{R}^d \backslash \{0\} \quad \|\hat{x} - \hat{y}\|_{p_2} \leq \theta \implies (1-\rho) \frac{\|x\|_{p_1}}{\|x\|_{p_2}} \leq \frac{\|y\|_{p_1}}{\|y\|_{p_2}} \leq (1+\rho) \frac{\|x\|_{p_1}}{\|x\|_{p_2}}.$$

Proof. Note $F(x) = \frac{\|x\|_{p_1}}{\|x\|_{p_2}}$. Then, $\left| \frac{F(y)}{F(x)} - 1 \right| = \frac{|F(y) - F(x)|}{F(x)} \leq C_{p_1,p_2} |F(x) - F(y)|$. Now we have

$$|F(x) - F(y)| = |F(\hat{x}) - F(\hat{y})| = |\, \|\hat{x}\|_{p_1} - \|\hat{y}\|_{p_1}| \leq \|\hat{x} - \hat{y}\|_{p_1} \leq C_{p_2,p_1} \|\hat{x} - \hat{y}\|_{p_2}.$$

Thus, we can take $\theta = \frac{\rho}{C_{p_1,p_2} C_{p_2,p_1}} > 0$. $\qquad\square$

We can now begin the proof of Theorem 5.1, which is cut into three main steps.

5.1. **Initialization of the spread.** The aim of the next lemma is to see that if the event $\{t(x) \leq (1-\delta)\|x\|_{p_1}\}$ is realized, then with high probability, at the slightly largest time $(1-\delta')\|x\|_{p_1}$, the p_2-infection has colonized a small shell, and this will provide it a strategic advantage for the next steps of the spread.

Lemma 5.4. *Let $\delta > 0$ and choose any $0 < \delta' < \delta$.*

For any $x \in \mathbb{Z}^d \backslash \{0\}$, any $1 < \gamma < \gamma'$ and any $\theta > 0$, we define the following events, depending on x, γ, γ' and θ:

$$\begin{aligned}
E_0 &= \{x \in C_{p_2}^\infty\}, \\
E_1 &= \{\eta^1((1-\delta')\|x\|_{p_1}) \subset \mathcal{B}_{p_1}^0(\|x\|_{p_1})\}, \\
E_2 &= \{\eta^2((1-\delta')\|x\|_{p_1}) \supset C_{p_2}^\infty \cap Shell(\{\hat{x}\} \oplus \theta, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2})\}, \\
E &= E_0 \cap E_1 \cap E_2.
\end{aligned}$$

Then there exist $\gamma_0' > 1$ and $\theta_0 > 0$ such that for any $1 < \gamma < \gamma' < \gamma_0'$ and any $0 < \theta < \theta_0$, there exist two strictly positive constants A and B such that

$$\forall x \in \mathbb{Z}^d \quad \mathbb{P}(\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \backslash E) \leq A \exp(-B\|x\|).$$

Proof. Let $\delta > 0$ and choose any $0 < \delta' < \delta$.

We first need to introduce a certain number of parameters: Let

$$(14) \qquad 0 < \rho < \frac{1}{1 - \delta'/2} - 1$$

By Lemma 5.3, we can then choose θ_1 such that

$$(15) \qquad \|\hat{x} - \hat{y}\|_{p_2} \leq \theta_1 \implies (1-\rho) \frac{\|x\|_{p_2}}{\|x\|_{p_1}} \leq \frac{\|y\|_{p_2}}{\|y\|_{p_1}} \leq (1+\rho) \frac{\|x\|_{p_2}}{\|x\|_{p_1}}.$$

Choose now $\gamma_0' > 1$ and $\theta_0 > 0$ small enough to fulfill the three following conditions:

$$\begin{aligned}
2(\gamma_0' - 1) + 3\theta_0 &< \theta_1 \\
1 - \theta_0 &> (1 - \delta'/2)(1 + \rho) \\
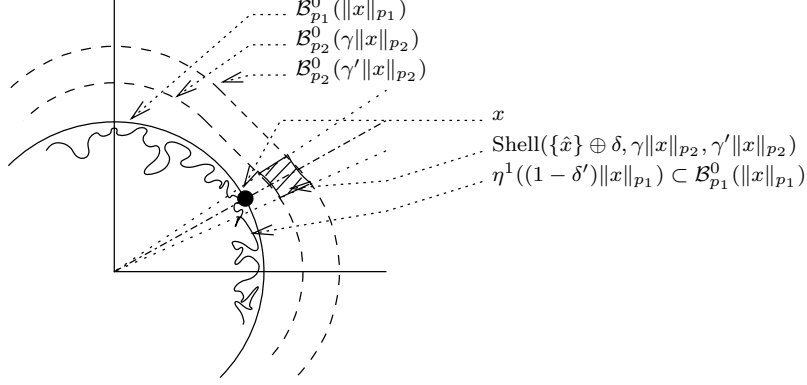\gamma_0' - 1 + \theta_0 &< \frac{\delta - \delta'}{C_{p_1,p_2}}.
\end{aligned}$$

FIGURE 3. Initialization of the spread.

Note that the second condition is allowed by the choice (14) for ρ. As these conditions are monotone, they are still fulfilled for any $\gamma' \in (1, \gamma'_0)$ and any $\theta \in (0, \theta_0)$. Choose then such a θ and such a γ', and choose $\alpha > 0$ small enough to have:

$$(16) \qquad 2(1+\alpha)(\gamma' - 1 + \theta) + \theta \quad < \quad \theta_1$$

$$(17) \qquad (1+\alpha)(1-\theta) - \alpha\gamma' \quad > \quad (1 - \delta'/2)(1 + \rho)$$

$$(18) \qquad (1+\alpha)(\gamma' - 1 + \theta) \quad < \quad \frac{\delta - \delta'}{C_{p_1,p_2}}.$$

Note that these conditions are allowed by the three previous ones. Finally, choose any $1 < \gamma < \gamma'$.

Step 0: Suppose that $t(x) \leq (1 - \delta)\|x\|_{p_1}$. This implies that there exists a p_2-open finite path from the source s_2 to x, and by the classical estimate (8) on the radius of finite open clusters in supercritical percolation, there exist two strictly positive constants A_0 and B_0 such that

$$(19) \qquad \forall x \in \mathbb{Z}^d \quad \mathbb{P}(\{t(x) \leq (1 - \delta)\|x\|_{p_1}\} \backslash E_0) \leq A_0 \exp(-B_0\|x\|).$$

Step 1: In this step, we use the typical spread of first-passage percolation with parameter p_1 in a amount of time of $(1 - \delta')\|x\|_{p_1}$. Note

$$E'_1 = \left\{ \eta^1((1 - \delta')\|x\|_{p_1}) \subset \mathcal{B}^0_{p_1}((1 - \delta'/2)\|x\|_{p_1}) \right\} \subset E_1.$$

The large deviations result associated to the shape theorem (7) ensures that there exist two strictly positive constants A_1 and B_1 such that

$$(20) \qquad \mathbb{P}((E'_1)^c) \quad \leq \quad A_1 \exp(-B_1\|x\|).$$

Step 2: In this step, we control the spread of first-passage percolation with parameter p_2. Let us first prove the geometrical fact:
$$(21)$$
$$\left(\bigcup_{y \in \text{Shell}(\{\hat{x}\} \oplus \theta, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2})} \mathcal{B}^y_{p_2}((1+\alpha)\|y - x\|_{p_2}) \right) \cap \mathcal{B}^0_{p_1}((1 - \delta'/2)\|x\|_{p_1}) = \varnothing$$

Note that since $y - x = (\|y\|_{p_2} - \|x\|_{p_2})\hat{y} + \|x\|_{p_2}(\hat{y} - \hat{x})$, we have, for every $y \in \mathrm{Shell}(\{\hat{x}\} \oplus \theta, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2})$,

$$(22) \qquad \|y - x\|_{p_2} \leq \|y\|_{p_2} - \|x\|_{p_2} + \theta\|x\|_{p_2} \leq (\gamma' - 1 + \theta)\|x\|_{p_2}.$$

Then, if $y \in \mathrm{Shell}(\{\hat{x}\} \oplus \theta, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2})$ and $z \in \mathcal{B}_{p_2}^y((1+\alpha)\|y - x\|_{p_2})$, we obtain first:

$$
\begin{aligned}
\|z\|_{p_2} &\geq \|y\|_{p_2} - \|z - y\|_{p_2} \geq \|y\|_{p_2} - (1+\alpha)\|y - x\|_{p_2} \\
&\geq \|y\|_{p_2} - (1+\alpha)\left(\|y\|_{p_2} - \|x\|_{p_2} + \theta\|x\|_{p_2}\right) \quad \text{with (22)} \\
&\geq (1+\alpha)(1-\theta)\|x\|_{p_2} - \alpha\|y\|_{p_2} \\
(23) \qquad &\geq \left((1+\alpha)(1-\theta) - \alpha\gamma'\right)\|x\|_{p_2},
\end{aligned}
$$

and then:

$$
\begin{aligned}
\|\hat{z} - \hat{x}\|_{p_2} &\leq \|\hat{z} - \hat{y}\|_{p_2} + \|\hat{y} - \hat{x}\|_{p_2} \\
&\leq \frac{2\|z - y\|_{p_2}}{\|y\|_{p_2}} + \theta \quad \text{with Lemma 5.2} \\
&\leq \frac{2(1+\alpha)\|x - y\|_{p_2}}{\|y\|_{p_2}} + \theta \leq \frac{2(1+\alpha)(\gamma' - 1 + \theta)}{\gamma} + \theta \quad \text{with (22)} \\
&\leq 2(1+\alpha)(\gamma' - 1 + \theta) + \theta < \theta_1 \text{ with assumption (16).}
\end{aligned}
$$

Thus, by definition (15) of θ_1, we have:

$$
\begin{aligned}
\|z\|_{p_1} &\geq \left(\frac{1}{1+\rho}\right)\frac{\|z\|_{p_2}}{\|x\|_{p_2}}\|x\|_{p_1} \geq \frac{(1+\alpha)(1-\theta) - \alpha\gamma'}{1+\rho}\|x\|_{p_1} \quad \text{with (23)} \\
&> (1 - \delta'/2)\|x\|_{p_1} \quad \text{with assumption (17),}
\end{aligned}
$$

which proves inclusion (21).

Now, if we denote

$$
E_2' = \bigcap_{y \in \mathrm{Shell}(\{\hat{x}\} \oplus \theta, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2}) \cap C_{p_2}^\infty} \left\{ \begin{array}{c} x \overset{p_2}{\leftrightarrow} y \text{ in } \mathcal{B}_{p_2}^y((1+\alpha)\|y - x\|_{p_2}) \\ \text{by a path of length smaller} \\ \text{than } (1+\alpha)\|y - x\|_{p_2} \end{array} \right\}
$$

Lemma 3.3 ensures that there exist two strictly positive constants A_2, B_2 such that

$$(24) \qquad \forall x \in \mathbb{Z}^d \quad \mathbb{P}\left(\{x \in C_{p_2}^\infty\} \setminus E_2'\right) \leq C_d(\gamma'\|x\|_{p_2})^d A_2 \exp(-B_2(\gamma - 1)\|x\|).$$

<u>Step 3</u>: To conclude, it only remains to see that:

$$(25) \qquad \forall y \in \mathrm{Shell}(\{\hat{x}\} \oplus \theta, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2}) \quad (1+\alpha)\|y - x\|_{p_2} \leq (\delta - \delta')\|x\|_{p_1}.$$

Indeed, we have, for any $y \in \mathrm{Shell}(\{\hat{x}\} \oplus \theta, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2})$:

$$
\begin{aligned}
(1+\alpha)\|y - x\|_{p_2} &\leq (1+\alpha)(\gamma' - 1 + \theta)\|x\|_{p_2} \quad \text{with (22)} \\
&\leq (1+\alpha)(\gamma' - 1 + \theta)C_{p_1,p_2}\|x\|_{p_1} \\
&\leq (\delta - \delta')\|x\|_{p_1} \quad \text{with assumption (18).}
\end{aligned}
$$

Now, if $E' = E_0 \cap E_1' \cap E_2'$, then Equation (21) and inclusion (25) imply that $\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \cap E' \subset \{t(x) \leq (1-\delta)\|x\|_{p_1}\} \cap E$, and thus

$$
\begin{aligned}
\mathbb{P}(\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \setminus E) &\leq \mathbb{P}(\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \setminus E_0) \\
&\quad + \mathbb{P}(\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \setminus E_1') \\
&\quad + \mathbb{P}(\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \setminus E_2' \cap E_0).
\end{aligned}
$$

Equations (19), (20), (24) and the fact that if $t(x) \leq (1-\delta)\|x\|_{p_1}$, then $t(x) + (\delta - \delta')\|x\|_{p_1} \leq (1-\delta')\|x\|_{p_1}$ give the desired result. $\qquad\square$

5.2. **Typical progression of the stronger infection from one shell to the next one.**

In this subsection, we forget for a moment the competition model, and study the progression of one infection with parameter p_2. For simplicity, we omit, only in this subsection, the subscript p_2. In the next lemma, we want to bound the minimal time needed for the infection to colonize the big $Shell(T, (1+h)r, (1+h)^2r)$ from the small $Shell(S, r, (1+h)r)$.

Lemma 5.5. *Let $\varphi \in (0, 2]$, $h \in (0, 1)$ and $\alpha \in (1, 2)$ be fixed parameters such that*

$$(26) \qquad (1+h)^2(1+\varphi) - (1+h) < \alpha h < 2.$$

For any S and T subsets of $\mathcal{S}$ and for any $r > 0$, we define the following event $E = E(S, T, r)$: "Any point in the big $Shell(T, (1+h)r, (1+h)^2r) \cap C^\infty$ is linked to a point in the small $Shell(S, r, (1+h)r)$ by an open path whose length is less than αhr." Two subsets S and T of $\mathcal{S}$ are said to be "good" if

$$\forall \hat{z} \in T \quad \exists \hat{v}_z \in \mathcal{S} \text{ such that } \{\hat{v}_z\} \oplus \frac{\varphi}{2} \subset S \text{ and } \|\hat{z} - \hat{v}_z\| \leq \varphi.$$

Then there exist two strictly positive constants A and B, only depending on φ, h, α, such that for any $r > 0$ and for any two "good" subsets S and T of $\mathcal{S}$, we have $\mathbb{P}(E^c) \leq A \exp(-Br)$.

Moreover, we can assume that all the infection paths needed in E are completely included in the bigger $Shell(T \oplus (2\alpha h), [1 - 3\varphi](1+h)r, \infty)$.
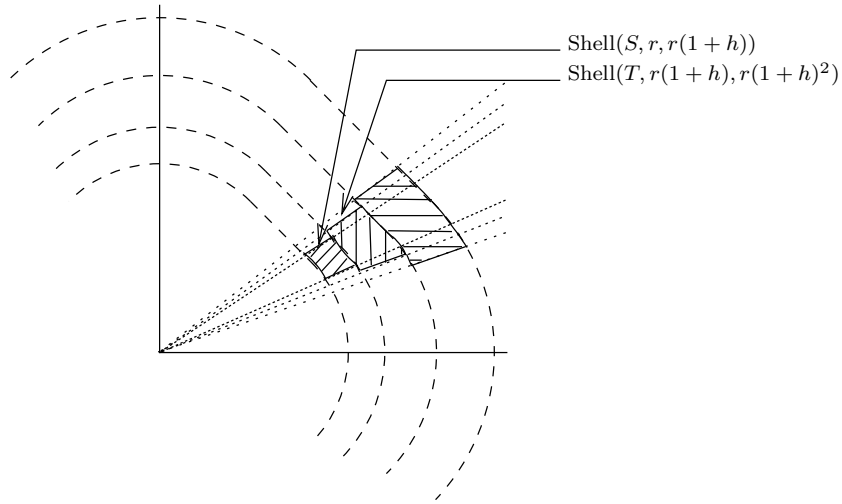

FIGURE 4. Scheme of progression of a single infection from one shell to the other. A possible third larger shell is also drawn.

Proof. Let $\varphi \in (0,2]$, $h \in (0,1)$ and $\alpha \in (1,2)$ be fixed parameters satisfying Equation (26) and choose, in this order, $\alpha' > 1$, $\varepsilon > 0$ and $\rho > 0$ such that

$$(27) \qquad (1+h)^2(1+\varphi) - (1+h-2\rho) \;\leq\; \alpha'h < \alpha h,$$

$$(28) \qquad \frac{2\rho\alpha'h}{1+h-2\rho} \;\leq\; \frac{\varphi}{2},$$

$$(29) \qquad h - 2\rho - \rho\alpha'h \;>\; 0,$$

$$(30) \qquad (1+\varepsilon)(1+\rho)\alpha' \;\leq\; \alpha.$$

Take any two "good" subsets S and T of $\mathcal{S}$. For any $z \in \text{Shell}(T, (1+h)r, (1+h)^2r)$, we can choose $\hat{v}_z \in \mathcal{S}$ such that

$$\{\hat{v}_z\} \oplus \tfrac{\varphi}{2} \subset S \quad \text{and} \quad \|\hat{z} - \hat{v}_z\| \leq \varphi,$$

and we set $v_z = (1+h-2\rho)r\hat{v}_z$. Let us first estimate $\|z - v_z\|$: on the one hand,

$$\|z - v_z\| \;\leq\; \|z - \|z\|\hat{v}_z\| + |\,\|z\| - (1+h-2\rho)r|$$

$$\leq\; \|z\|\varphi + \|z\| - (1+h-2\rho)r$$

$$(31) \qquad \leq\; \|z\|(1+\varphi) - (1+h-2\rho)r$$

$$(32) \qquad \leq\; [(1+h)^2(1+\varphi) - (1+h-2\rho)]r \leq \alpha'hr \text{ thanks to (27)},$$

and, on the other hand,

$$(33) \qquad \|z - v_z\| \;\geq\; \|z\| - \|v_z\| \geq 2\rho r.$$

<u>Idea of the proof:</u> The idea of the proof is the following. Take z in $C^\infty_{p_2}$ and in $\text{Shell}(T, (1+h)r, (1+h)^2r)$. The ball $\mathcal{B}^{v_z}(\rho\|z - v_z\|)$ is included in the $\text{Shell}(S, r, (1+h)r)$ and in the ball $\mathcal{B}^z((1+\rho)\|z - v_z\|)$. As it is of radius of order r, it contains with high probability some point of the infinite cluster, and this point should be with high probability, thanks to Lemma 3.3, linked to z by an open path inside $\mathcal{B}^z((1+\varepsilon)(1+\rho)\|z - v_z\|)$ with length less than $(1+\varepsilon)(1+\rho)\|z - v_z\|$. We chose the parameters to ensure that $(1+\varepsilon)(1+\rho)\|z - v_z\| \leq \alpha hr$. It will then only remain to control the positions of the points in the union of the $\mathcal{B}^z((1+\varepsilon)(1+\rho)\|z - v_z\|)$. Let us make all this more precise.

<u>Geometrical facts:</u> Let us first note that, by the triangular inequality,

$$(34) \quad \forall z \in \text{Shell}(T, (1+h)r, (1+h)^2r) \quad \mathcal{B}^{v_z}(\rho\|z - v_z\|) \subset \mathcal{B}^z((1+\rho)\|z - v_z\|).$$

Let us see now that

$$(35) \quad \forall z \in \text{Shell}(T, (1+h)r, (1+h)^2r) \quad \mathcal{B}^{v_z}(\rho\|z - v_z\|) \subset \text{Shell}(S, r, (1+h)r).$$

Let $u \in \mathcal{B}^{v_z}(\rho\|z - v_z\|)$, then, by Lemma 5.2,

$$\|\hat{u} - \hat{v}_z\| \;\leq\; \frac{2\rho\|z - v_z\|}{\|v_z\|} \leq \frac{2\rho\alpha'h}{1+h-2\rho} \text{ by Equation (32) and definition of } v_z$$

$$\leq\; \frac{\varphi}{2} \text{ thanks to Equation (28)}$$

and thus $\hat{u} \in S$. For the norm of u, by definition of v_z and Equation (32), we have:

$$\|v_z\| - \rho\|z - v_z\| \leq \;\|u\|\; \leq \|v_z\| + \rho\|z - v_z\|$$

$$(1+h-2\rho)r - \rho\alpha'hr \leq \;\|u\|\; \leq (1+h-2\rho)r + \rho\alpha'hr$$

$$r \leq \;\|u\|\; \leq (1+h)r,$$

thanks to Equations (29) and (27). This proves the second geometrical fact (35).

<u>Probabilistic estimates:</u> We can then estimate the probability of E. Note first

$$E_1 = \bigcup_{z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r)} \left\{ \mathcal{B}^{v_z}\left(\rho \| z - v_z \| \right) \cap C_{p_2}^\infty = \varnothing \right\}.$$

By estimate (33), we know that $\|z - v_z\| \geq 2\rho r$; moreover, for each $z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r)$, the point v_z is in $\mathrm{Shell}(S,r,(1+h)r)$. Thus, using the estimate on the holes of the infinite cluster (9), there exist two strictly positive constants A_1 and B_1 such that for every "good" S and T, for every $r > 0$,

$$
\begin{aligned}
\mathbb{P}(E_1) &\leq \mathbb{P}\left(\bigcup_{v \in \mathrm{Shell}(S,r,(1+h)r)} \left\{ \mathcal{B}^v\left(2\rho^2 r\right) \cap C_{p_2}^\infty = \varnothing \right\} \right) \\
&\leq C_d[(1+h)r]^d A_1 \exp(-B_1 2\rho^2 r).
\end{aligned}
$$

Then, note

$$
E_2 = \bigcup_{\substack{z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r) \\ u \in \mathcal{B}^{v_z}(\rho\|z-v_z\|)}} \left\{
\begin{array}{c}
u \in C_{p_2}^\infty, \ z \in C_{p_2}^\infty, \ \text{and } u \text{ is not linked to} \\
z \text{ by an open path of length smaller} \\
\text{than } (1+\varepsilon)(1+\rho)\|z - v_z\| \\
\text{inside } \mathcal{B}^z((1+\varepsilon)(1+\rho)\|z-v_z\|)
\end{array}
\right\}.
$$

By Lemma 3.3, Equations (33) and (32), there exist two strictly positive constants A_2 and B_2 such that for every "good" S and T, for every $r > 0$,

$$
\begin{aligned}
\mathbb{P}(E_2) &\leq \sum_{\substack{z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r) \\ u \in \mathcal{B}^{v_z}(\rho\|z-v_z\|)}} \mathbb{P}\left(H(u,z,(1+\varepsilon)(1+\rho)-1)^c\right) \\
&\leq \sum_{\substack{z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r) \\ u \in \mathcal{B}^{v_z}(\rho\|z-v_z\|)}} A_2 \exp(-B_2(1+\varepsilon)(1+\rho)\|z-v_z\|) \\
&\leq C_d[(1+h)^2 r]^d \times C_d(\rho\alpha' h r)^d \times A_2 \exp(-B_2(1+\rho)2\rho r).
\end{aligned}
$$

<u>Conclusion:</u> For every $z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r)$, thanks to (32) and (30), one has $(1+\varepsilon)(1+\rho)\|z - v_z\|_{p_2} \leq \alpha h r$. This, combined with geometrical facts (34) and (35), implies that $E^c \subset E_1 \cup E_2$, which proves the exponential estimate of the lemma.

<u>Control of the infection paths:</u> It remains to estimate the minimal room needed to perform this infection, or in other words to control

$$\bigcup_{z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r)} \mathcal{B}^z((1+\varepsilon)(1+\rho)\|z - v_z\|).$$

Let $z \in \mathrm{Shell}(T,(1+h)r,(1+h)^2 r)$ and $u \in \mathcal{B}^z((1+\varepsilon)(1+\rho)\|z-v_z\|)$. We have:

$$
\begin{aligned}
\|u\| &\geq \|z\| - (1+\varepsilon)(1+\rho)\|z - v_z\| \\
&\geq (1+\varepsilon)(1+\rho)(1+h-2\rho)r - [(1+\varepsilon)(1+\rho)(1+\varphi)-1]\|z\| \\
&\qquad \text{thanks to (31)} \\
&\geq (1+\varepsilon)(1+\rho)(1+h-2\rho)r - [(1+\varepsilon)(1+\rho)(1+\varphi)-1](1+h)^2 r \\
&\geq [1 - 3\varphi](1+h)r.
\end{aligned}
$$

The last inequality is obtained by looking at the the limit of the right-hand side term, when ε and ρ tend to 0, and by decreasing if necessary ε and ρ. Finally, by applying Lemma 5.2 and then Inequality (32), we have

$$\|\hat{u} - \hat{z}\| \leq \frac{2\|u - z\|}{\|z\|} \leq \frac{2(1 + \varepsilon)(1 + \rho)\alpha'h}{(1 + h)^2} \leq 2\alpha h.$$

Thus $u \in \text{Shell}(T \oplus (2\alpha h), r_{min}, \infty)$, which ends the proof of the lemma. $\square$

5.3. **Final step: proof of Theorem 5.1.** We come back now to the competition context, with a weaker infection with parameter p_1 and a stronger infection with parameter $p_2 > p_1$.

Proof. Let $\delta > 0$.

Idea of the proof: The idea is quite natural: start the progression by the initialization Lemma 5.4, and apply recursively the progression Lemma 5.5 until the stronger infection surrounds the weaker one. The point is to ensure that this progression is not disturbed by the spread of the weaker infection.

Step 0. Choice of constants: Remember that $C_{p_1,p_2} < 1$ and choose:

$$(36) \qquad \delta' > 0 \quad \text{such that} \quad \delta' < \delta \text{ and } \delta' < 1 - C_{p_1,p_2},$$

$$(37) \qquad \rho > 0 \quad \text{such that} \quad (1 + \rho)(1 - \delta') < 1.$$

By Lemma 5.3, there exists $\theta > 0$ such that for any $x, y \in \mathbb{Z}^d \backslash \{0\}$, we have:

$$(38) \qquad \|\hat{x} - \hat{y}\| \leq \theta \Rightarrow (1 - \rho)\frac{\|x\|_{p_2}}{\|x\|_{p_1}} \leq \frac{\|y\|_{p_2}}{\|y\|_{p_1}} \leq (1 + \rho)\frac{\|x\|_{p_2}}{\|x\|_{p_1}}.$$

Choose then h and α such that:

$$\qquad 0 < h < 1 \quad \text{such that} \quad (1 + h)C_{p_1,p_2} < 1 - \delta',$$

$$(39) \qquad 1 < \alpha < 2 \quad \text{such that} \quad \alpha > 1 + h \text{ and } \alpha C_{p_1,p_2} < 1 - \delta',$$

$$(40) \qquad 2\alpha h < \theta.$$

The first condition is allowed by condition (36) on δ', and allows itself the choice (39) for α. We obtain (40) by decreasing h if necessary. Let $\gamma'_0 > 1$ and $\theta_0 > 0$ be given by Lemma 5.4. Choose $\gamma', \gamma, \varepsilon$ and φ in the following manner:

$$(41) \quad 1 < \gamma' < \gamma'_0 \quad \text{such that} \quad \alpha\gamma'C_{p_1,p_2} < 1 - \delta',$$

$$(42) \quad 1 < \gamma < \gamma' \quad \text{such that} \quad \gamma = \frac{\gamma'}{1 + h},$$

$$(43) \qquad \varepsilon > 0 \quad \text{such that} \quad \begin{cases} \alpha C_{p_1,p_2}(1 + \varepsilon) < 1, \\ \text{and} \quad (1 + \varepsilon)(1 + \rho)(1 - \delta') < \gamma, \end{cases}$$

$$(44) \quad 0 < \varphi < \theta_0 \quad \text{such that} \quad \begin{cases} \alpha C_{p_1,p_2}(1 + \varepsilon) < 1 - 3\varphi, \\ \text{and} \quad (1 + \varepsilon)(1 + \rho)(1 - \delta') < \gamma(1 - 3\varphi), \\ \text{and} \quad (1 + h)^2(1 + \varphi) - (1 + h) < \alpha h. \end{cases}$$

Note that condition (41) is allowed by the choice (39), and condition (42) is obtained by decreasing h if necessary. Conditions (43) are respectively permitted by (39) and (37), and allow the first two conditions on φ. The last condition in (44) is allowed by (39) and (40). Choose now $K \geq 2$ large enough to have for every $k \geq K$

$$(45) \quad C_{p_1,p_2}(1 + \varepsilon)\left[(1 - \delta')C_{p_2,p_1} + \alpha\gamma((1 + h)^{k-1} - 1)\right] < \gamma[1 - 3\varphi](1 + h)^{k-1},$$

which is allowed by (44). By decreasing φ if necessary, we can assume, thanks to (40), that

$$(46) \qquad (1+K)\frac{\varphi}{2} + 2\alpha h < \theta.$$

Step 1. Initialization of the spread: By Lemma 5.4, there exist two strictly positive constants A_1 and B_1 such that for every $x \in \mathbb{Z}^d \backslash \{0\}$, we have

$$(47) \qquad \mathbb{P}(\{t(x) \le (1-\delta)\|x\|_{p_1}\} \backslash \{E_1(x) \cap \{x \in C_{p_2}^\infty\}\}) \le A_1 \exp(-B_1\|x\|),$$

where we use the following notations:

$$
\begin{aligned}
E_1^1(x) &= \{\eta^1((1-\delta')\|x\|_{p_1}) \subset \mathcal{B}_{p_1}^0(\|x\|_{p_1})\}, \\
E_1^2(x) &= \left\{\eta^2((1-\delta')\|x\|_{p_1}) \supset \left(\mathrm{Shell}\left(\{\hat{x}\} \oplus \frac{\varphi}{2}, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2}\right) \cap C_{p_2}^\infty\right)\right\}, \\
E_1(x) &= E_1^1(x) \cap E_1^2(x).
\end{aligned}
$$

Thus, if $t(x) \le (1-\delta)\|x\|_{p_1}$, then at the slightly larger time $t_1(x) = (1-\delta')\|x\|_{p_1}$, the first shell

$$S_1(x) = C_{p_2}^\infty \cap \mathrm{Shell}\left(\{\hat{x}\} \oplus \frac{\varphi}{2}, \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2}\right)$$

is with high probability colonized by the p_2-infection.

We want now to extend this colonization to larger and larger shells by applying recursively Lemma 5.5.

Notations: We still need to introduce a certain number of notations, inspired by Lemma 5.5:

$k = 1$
$r_1 = \gamma$ $A_1(x) = \{\hat{x}\} \oplus \frac{\varphi}{2}$ $S_1(x) = C_{p_2}^\infty \cap \mathrm{Shell}(A_1(x), \gamma\|x\|_{p_2}, \gamma'\|x\|_{p_2})$ $t_1(x) = (1-\delta')\|x\|_{p_1}$
$k \ge 2$
$r_k = (1+h)^{k-1}r_1 \quad$ and $\quad r_k^{min} = [1-3\varphi](1+h)r_{k-1}$ $A_k(x) = A_{k-1}(x) \oplus \frac{\varphi}{2}$ $S_k(x) = C_{p_2}^\infty \cap \mathrm{Shell}(A_k(x), r_k\|x\|_{p_2}, r_{k+1}\|x\|_{p_2})$ $t_k(x) = t_{k-1}(x) + h\alpha\gamma r_{k-1}\|x\|_{p_2} = (1-\delta')\|x\|_{p_1} + \alpha[(1+h)^{k-1}-1]\|x\|_{p_2}$ $\quad = t_1(x) + \alpha(r_k - r_1)\|x\|_{p_2}$

Define also the following events, for $k \ge 2$ and $x \in \mathbb{Z}^d \backslash \{0\}$:

$$
\begin{aligned}
E_k^1(x) &= \{\eta^1(t_k(x)) \subset \mathcal{B}_{p_1}^0((1+\varepsilon)t_k(x))\}, \\
E_k^2(x) &= \{\eta^2(t_k(x)) \supset S_k(x)\}, \\
E_k(x) &= E_k^1(x) \cap E_k^2(x).
\end{aligned}
$$

The aim is the following: we want to apply Lemma 5.5 to prove that if $E_k^2(x)$ is realized, then with high probability $E_{k+1}^2(x)$ is also realized. But we need first to control the spread of the slow p_1-infection, and to see that it will not disturb the spread of the fast p_2-infection from $S_k(x)$ to $S_{k+1}(x)$.

Step 2. Rough control of the slow p_1-infection: Let us prove that there exist two strictly positive constants A_2 and B_2 such that

$$(48) \qquad \forall x \in \mathbb{Z}^d \backslash \{0\} \quad \mathbb{P}\left(\complement \left(\bigcap_{k \geq 2} E_k^1(x) \right) \right) \leq A_2 \exp(-B_2 \|x\|).$$

Indeed, by the large deviation result (7), for any $x \in \mathbb{Z}^d \backslash \{0\}$, we have:

$$\mathbb{P}\left(\complement \left(\bigcap_{k \geq 2} E_k^1(x) \right) \right)$$
$$\leq \sum_{k \geq 2} \mathbb{P}\left(\eta^1(t_k(x)) \not\subset \mathcal{B}_{p_1}((1+\varepsilon)t_k(x))) \right) \leq \sum_{k \geq 2} A \exp\left(-B t_k(x) \right)$$
$$\leq \sum_{k \geq 2} A \exp\left(-B \left[(1 - \delta')\|x\|_{p_1} + \alpha((1+h)^{k-1} - 1)\gamma\|x\|_{p_2} \right] \right)$$
$$\leq A \exp\left(-B \left[(1 - \delta')\|x\|_{p_1} - \alpha\gamma\|x\|_{p_2} \right] \right) \sum_{k \geq 2} \exp\left(-B\alpha(1+h)^{k-1}\gamma\|x\|_{p_2} \right).$$

1. As there exists $B' > 0$ such that $\forall k \geq 2$, $B\alpha(1+h)^{k-1}\gamma \geq B'k$, the last sum is bounded by

$$\sum_{k \geq 2} \exp\left(-B\alpha(1+h)^{k-1}\gamma\|x\|_{p_2} \right) \leq \sum_{k \geq 2} \exp\left(-B'k\|x\|_{p_2} \right)$$
$$\leq \frac{\exp\left(-2B'\|x\|_{p_2} \right)}{1 - \exp\left(-B'\|x\|_{p_2} \right)} \leq A' \exp\left(-2B'\|x\|_{p_2} \right)$$

with $A' > 0$ because $\inf\{\|x\|_{p_2} : x \in \mathbb{Z}^d \backslash \{0\}\} > 0$.

2. For the first factor, we have

$$(1 - \delta')\|x\|_{p_1} - \alpha\gamma\|x\|_{p_2} = (1 - \delta' - \alpha\gamma C_{p_1,p_2})\|x\|_{p_1} + \alpha\gamma(C_{p_1,p_2}\|x\|_{p_1} - \|x\|_{p_2})$$
$$\geq (1 - \delta' - \alpha\gamma C_{p_1,p_2})\|x\|_{p_1} \text{ by definition of } C_{p_1,p_2}$$
$$\geq B''\|x\|_{p_1},$$

with $B'' > 0$ thanks to conditions (41) and (42). This proves (48).

But we will also need a more precise control of this slow infection in order to prevent it from bothering the fast one while applying Lemma 5.5.

Step 3. More precise control of the slow p_1-infection for large times: Remember that K was defined in (45). Let us prove the following geometrical fact:

$$(49) \qquad \forall k \geq K \quad \forall x \in \mathbb{Z}^d \backslash \{0\} \quad \mathcal{B}_{p_1}^0((1+\varepsilon)t_k(x)) \subset \mathcal{B}_{p_2}^0(r_k^{min}\|x\|_{p_2}).$$

Let $k \geq K$, $x \in \mathbb{Z}^d \backslash \{0\}$ and $u \in \mathcal{B}_{p_1}^0((1+\varepsilon)t_k(x))$. Then:

$$\|u\|_{p_2} \leq C_{p_1,p_2}\|u\|_{p_1} \leq C_{p_1,p_2}(1+\varepsilon)t_k(x)$$
$$\leq C_{p_1,p_2}(1+\varepsilon)[(1 - \delta')\|x\|_{p_1} + \alpha\gamma((1+h)^{k-1} - 1)\|x\|_{p_2}]$$
$$\leq C_{p_1,p_2}(1+\varepsilon)\left[(1 - \delta')C_{p_2,p_1} + \alpha\gamma((1+h)^{k-1} - 1) \right]\|x\|_{p_2}$$
$$\leq [1 - 3\varphi](1+h)^{k-1}\gamma\|x\|_{p_2} = r_k^{min}\|x\|_{p_2},$$

thanks to (45), which proves (49).

Step 4. More precise control of the slow p_1-infection in the early stage of the process: To look at the p_1-infection in the early stage of the process, we need to focus on a small cone around $\hat{x}$ in order to control more precisely the discrepancy between

the two norms $\|.\|_{p_1}$ and $\|.\|_{p_2}$. Let us see that for every k, for every $x \in \mathbb{Z}^d \backslash \{0\}$ and for every $z \in \mathbb{Z}^d \backslash \{0\}$

$$(\|\hat{z} - \hat{x}\|_{p_2} \leq \theta \text{ and } \|z\|_{p_1} \leq (1+\varepsilon) t_k(x)) \implies \|z\|_{p_2} \leq r_k^{min} \|x\|_{p_2}.$$

We recall that θ was defined in (38). Then,

$$r_k^{min} \|x\|_{p_2} - \|z\|_{p_2} \geq r_k^{min} \|x\|_{p_2} - (1+\rho)\|z\|_{p_1} \frac{\|x\|_{p_2}}{\|x\|_{p_1}}$$

$$\geq (1 - 3\varphi) r_k \|x\|_{p_2} - (1+\rho)(1+\varepsilon)\frac{\|x\|_{p_2}}{\|x\|_{p_1}} t_k(x)$$

$$\geq (1 - 3\varphi) r_k \|x\|_{p_2} - (1+\rho)(1+\varepsilon)\frac{\|x\|_{p_2}}{\|x\|_{p_1}} (t_1(x) + \alpha(r_k - r_1)\|x\|_{p_2}).$$

As $(r_k)_k$ is increasing, the worst case is for $k = 1$:

$$r_k^{min} \|x\|_{p_2} - \|z\|_{p_2} \geq (1 - 3\varphi) r_1 \|x\|_{p_2} - (1+\rho)(1+\varepsilon)\frac{\|x\|_{p_2}}{\|x\|_{p_1}}(1 - \delta')\|x\|_{p_1}$$

$$\geq ((1 - 3\varphi)\gamma - (1+\rho)(1+\varepsilon)(1 - \delta'))\|x\|_{p_2} > 0$$

thanks to Conditions (44). Thus, thanks to Equation (46), we obtain that for every $k \leq K$, for every $x \in \mathbb{Z}^d \backslash \{0\}$ and for every $z \in \mathbb{Z}^d \backslash \{0\}$

(50) $(\hat{z} \in A_k \oplus (2\alpha h) \quad \text{and} \quad \|z\|_{p_1} \leq (1+\varepsilon) t_k(x)) \quad \implies \quad \|z\|_{p_2} \leq r_k^{min} \|x\|_{p_2}.$

Step 5. Control of the fast p_2-infection: Equations (49) and (50) ensure that for every $k \geq 2$, for every $x \in \mathbb{Z}^d \backslash \{0\}$, we have

(51) $\qquad \mathcal{B}_{p_1}^0((1+\varepsilon) t_k(x)) \cap \text{Shell}(A_k(x) \oplus (2\alpha h), r_k^{min} \|x\|_{p_2}, \infty) = \varnothing.$

Thus, the spread of the (single) fast p_2-infection from $S_{k-1}(x)$ to $S_k(x)$, ensured by Lemma 5.5, is not disturbed by the slow p_1-infection on the event $E_k^1(x) \cap E_{k-1}^2(x)$. Let A_3 and B_3 be the two strictly positive constants given by Lemma 5.5; we apply the lemma with

$$\begin{aligned} S &= A_{k-1}(x), \\ T &= A_k(x) = A_{k-1}(x) \oplus \frac{\varphi}{2}, \\ r &= r_{k-1}\|x\|_{p_2}. \end{aligned}$$

But we must first be sure that S and T are "good" subsets of $\mathcal{S}$, in the sense

$$\forall z \in T \quad \exists v \in S \text{ such that } v \oplus \frac{\varphi}{2} \subset S \text{ and } \|z - v\|_{p_2} \leq \varphi.$$

Indeed, let $k \geq 2$ and $z \in A_k(x) = A_{k-1}(x) \oplus \frac{\varphi}{2}$: by definition, there exist $w \in A_{k-1}(x)$ and $u_1 \in \mathcal{B}_{p_2}^0(\varphi/2)$ such that $z = w + u_1$. But $A_{k-1}(x) = A_{k-2}(x) \oplus \frac{\varphi}{2}$, where, for $k = 2$, we set $A_0(x) = \{\hat{x}\}$. So there exist $v \in A_{k-2}(x)$ and $u_2 \in \mathcal{B}_{p_2}^0(\varphi/2)$ such that $w = v + u_2$. Now, $z = v + u_1 + u_2$ and

- as $v \in A_{k-2}(x) \subset S$ and $S = A_{k-1}(x) = A_{k-2}(x) \oplus \frac{\varphi}{2}$, we have $v \oplus \frac{\varphi}{2} \subset S$,
- as $u_1 \in \mathcal{B}_{p_2}^0(\varphi/2)$ and $u_2 \in \mathcal{B}_{p_2}^0(\varphi/2)$, we have $\|z - v\|_{p_2} = \|u_1 + u_2\|_{p_2} \leq \|u_1\|_{p_2} + \|u_2\|_{p_2} \leq \varphi.$

Thus any point in $S_k(x)$ can be infected by the p_2-infection from $S_{k-1}(x)$ in a time less than $\alpha h r_{k-1}\|x\|_{p_2} = t_k(x) - t_{k-1}(x)$ using only paths inside $\text{Shell}(A_k(x) \oplus (2\alpha h), r_k^{min}\|x\|_{p_2}, \infty)$, if it is not bothered by the slow p_1-infection.

But on the event $E_k^1(x)$, this is ensured by Equation (51). Thus, the application of Lemma 5.5 implies that for any $x \in \mathbb{Z}^d \backslash \{0\}$, for every $k \geq 2$,

$$\mathbb{P}((E_k^2(x))^c \backslash (E_k^1(x) \cap E_{k-1}^2(x)) \leq A_3 \exp(-B_3 r_{k-1} \|x\|_{p_2}).$$

Thus,

$$
\begin{aligned}
\sum_{k \geq 2} \mathbb{P}\left((E_k^2(x))^c \cap (E_k^1(x) \cap E_{k-1}^2(x))\right) &\leq A_3 \sum_{k \geq 2} \exp(-B_3 r_{k-1} \|x\|_{p_2}) \\
&\leq A_3 \sum_{k \geq 2} \exp\left(-B_3 (1+h)^{k-2} \gamma \|x\|_{p_2}\right) \\
&\leq A_4 \exp(-B_4 \|x\|)
\end{aligned}
$$

(52)

where A_4 and B_4 are two strictly positive constants.

<u>Conclusion:</u> For k large enough, the set $S_k(x)$ disconnects 0 from infinity, and thus the event $\bigcap_{k \geq 1} E_k$ implies that the slow p_1-infection is surrounded by the fast p_2-infection and thus dies out. So, using (47), (48) and (52), we obtain:

$$
\begin{aligned}
&\mathbb{P}(\mathcal{G}^1 \cap \{t(x) \leq (1-\delta)\|x\|_{p_1}\}) \\
\leq\ & \mathbb{P}\left(\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \cap \bigcup_{k \geq 1} E_k(x)^c\right) \\
\leq\ & \mathbb{P}\left(\{t(x) \leq (1-\delta)\|x\|_{p_1}\} \cap (E_1(x) \cap \{x \in C_{p_2}^\infty\})\right) + \mathbb{P}\left(\bigcup_{k \geq 1} (E_k^1(x))^c\right) \\
& + \sum_{k \geq 2} \mathbb{P}\left((E_k^2(x))^c \cap (E_k^1(x) \cap E_{k-1}^2(x))\right) \\
\leq\ & A \exp(-B\|x\|),
\end{aligned}
$$

which completes the proof. $\qquad \square$

6. Proof of the main Theorem 1.2

In all this section, s_1 and s_2 are two distinct sites in $\mathbb{Z}^d$ and ξ is the element of $S^{\mathbb{Z}^d}$ where all sites are empty, but $\xi_{s_1} = \mathrm{y}$ and $\xi_{s_2} = \mathrm{b}$. This initial configuration is now fixed. We will thus, in the following, omit the explicit dependence in ξ.

Suppose that $0 \leq p \leq q \leq 1$. In our competition process, the survival of the weaker – *resp.* stronger – infection is represented by the event $\mathcal{G}_{p,q}^1$ – *resp.* $\mathcal{G}_{p,q}^2$ – where, for $i = 1, 2$:

$$\mathcal{G}_{p,q}^i = \left\{\sup_{t \geq 0} |\eta_{p,q}^i(t)| = +\infty\right\}.$$

The main Theorem 1.2 can be reformulated now in the following form:

Theorem 6.1. *Let $q > p_c$. The set of parameters p such that $p < \min(q, \overrightarrow{p_c})$ and $\mathbb{P}(\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2) > 0$ is at most denumerable.*

The corresponding Conjecture 1.1 can be formulated as follows:

Conjecture 6.2. *Let $q > p_c$ and $p < \min(q, \overrightarrow{p_c})$. Then $\mathbb{P}\left(\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2\right) = 0$.*

Proof of Theorem 6.1. It strongly relies on Propositions 1.3 and 1.4 and the coupling arguments that are also used are widely inspired by the proof of Häggström and Pemantle [15].

Step 1. Let us prove that if $p < q < \min(r, \overrightarrow{p_c})$, then $\mathbb{P}(\mathcal{G}_{p,r}^1 \cap \mathcal{G}_{q,r}^2) = 0$.

Since, by the coupling Lemma 2.2, $\mathcal{G}_{p,r}^2 \subset \mathcal{G}_{q,r}^2$, we have $\mathcal{G}_{p,r}^1 \cap \mathcal{G}_{q,r}^2 = (\mathcal{G}_{p,r}^1 \cap \mathcal{G}_{p,r}^2) \cap \mathcal{G}_{q,r}^2$. So, we can assume that $\mathcal{G}_{p,r}^1 \cap \mathcal{G}_{p,r}^2$ occurs and prove that $\mathcal{G}_{q,r}^2$ can not happen. By Proposition 1.4, we have

$$\varlimsup_{t \to +\infty} \frac{|\eta_{p,r}^2(t)|_p}{t} \leq 1, \text{ which implies } \varlimsup_{t \to +\infty} \frac{|\eta_{q,r}^2(t)|_p}{t} \leq 1.$$

Indeed, by the coupling Lemma 2.2, $\eta_{q,r}^2(t) \subset \eta_{p,r}^2(t)$. Now, by Proposition 1.3, we have

$$\varlimsup_{t \to +\infty} \frac{|\eta_{q,r}^2(t)|_q}{t} \leq C_{p,q}.$$

On the other hand, $\mathcal{G}_{q,r}^1 \subset \{s_1 \in C_q^\infty\}$, so using Lemma 2.3 and Lemma 3.1 together, we get

$$\varliminf_{t \to +\infty} \frac{|\eta_{q,r}^1(t) \cup \eta_{q,r}^2(t)|_{*,q}}{t} \geq 1.$$

Now, let t be large enough to ensure that

$$\frac{|\eta_{q,r}^2(t)|_p}{t} \leq \frac{C_{p,q} + 2}{3} = \alpha \text{ and } \frac{|\eta_{q,r}^1(t) \cup \eta_{q,r}^2(t)|_{*,q}}{t} \geq \frac{2C_{p,q} + 1}{3} = \beta.$$

Then every point x such that $x \in C_q^\infty$ and $\alpha < \|x\|_q < \beta$ belongs to $\eta_{q,r}^1(t) \backslash \eta_{q,r}^2(t)$, which prevents the occurrence of the event $\mathcal{G}_{q,r}^2$.

Step 2. Let $q > 0$. Let us prove that $\mathbb{P}$ almost surely, there exists at most one value $p \leq \min(q, \overrightarrow{p_c})$ such that $\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2$ occurs.

Assume that there exist p and p' with $p < p' \leq q$ and such that $\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2$ and $\mathcal{G}_{p',q}^1 \cap \mathcal{G}_{p',q}^2$ are satisfied. Denote by A this event. Let r and s be two rational numbers such that $p < r < s < p'$.

By the coupling Lemma 2.2, $\mathcal{G}_{p,q}^1 \subset \mathcal{G}_{r,q}^1$ and $\mathcal{G}_{p',q}^2 \subset \mathcal{G}_{s,q}^2$, whence

$$A \subset \bigcup_{\substack{0 \leq r < s \leq q \\ (r,s) \in \mathbb{Q}^2}} \mathcal{G}_{r,q}^1 \cap \mathcal{G}_{s,q}^2.$$

Then, it follows from the previous step that A has probability 0.

Step 3. Proof of Theorem 6.1. Let $q > 0$.

Let $n \geq 1$, and let E a finite subset of the set of real numbers $p \in [0, \min(q, \overrightarrow{p_c}))$ such that $\mathbb{P}(\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2) \geq 1/n$. By the previous step

$$\sum_{p \in E} \mathbb{1}_{\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2} \leq 1, \text{ which implies that } \sum_{p \in E} \mathbb{P}(\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2) \leq 1.$$

Thus the set of p such that $p \leq q$ and $\mathbb{P}(\mathcal{G}_{p,q}^1 \cap \mathcal{G}_{p,q}^2) \geq 1/n$ contains at most n points, which proves the theorem.

References

[1] Michael Aizenman and David J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987.

[2] J. T. Chayes, L. Chayes, G. R. Grimmett, H. Kesten, and R. H. Schonmann. The correlation length for the high-density phase of Bernoulli percolation. *Ann. Probab.*, 17(4):1277–1302, 1989.

[3] M. Deijfen, O. Häggström, and J. Bagley. A stochastic model for competing growth on $\mathbf{R}^d$. *Markov Process. Related Fields*, 10(2):217–248, 2004.

[4] Maria Deijfen. Asymptotic shape in a continuum growth model. *Adv. in Appl. Probab.*, 35(2):303–318, 2003.

[5] Maria Deijfen and Olle Häggström. Coexistence in a two-type continuum growth model. *Adv. in Appl. Probab.*, 36(4):973–980, 2004.

[6] Maria Deijfen and Olle Häggström. Nonmonotonic coexistence regions for the two-type richardson model on graphs. *preprint*, 2005.

[7] R. Durrett and R. H. Schonmann. Large deviations for the contact process and two-dimensional percolation. *Probab. Theory Related Fields*, 77(4):583–603, 1988.

[8] Olivier Garet and Régine Marchand. Large deviations for the chemical distance in super-critical bernoulli percolation. *Preprint, available at* `http://arxiv.org/abs/math.PR/0409317`, 2004.

[9] Olivier Garet and Régine Marchand. Coexistence in two-type first-passage percolation models. *Ann. Appl. Probab.*, 15(1A):298–330, 2005.

[10] Olivier Garet and Régine Marchand. Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster. *ESAIM Probab. Statist.*, 8:169–199, 2004.

[11] G. R. Grimmett and J. M. Marstrand. The supercritical phase of percolation is well behaved. *Proc. Roy. Soc. London Ser. A*, 430(1879):439–457, 1990.

[12] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[13] Geoffrey Grimmett and Harry Kesten. First-passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete*, 66(3):335–366, 1984.

[14] Olle Häggström and Robin Pemantle. First passage percolation and a model for competing spatial growth. *J. Appl. Probab.*, 35(3):683–692, 1998.

[15] Olle Häggström and Robin Pemantle. Absence of mutual unbounded growth for almost all parameter values in the two-type Richardson model. *Stochastic Process. Appl.*, 90(2):207–222, 2000.

[16] Christopher Hoffman. Coexistence for Richardson type competing spatial growth models. *Ann. Appl. Probab.*, 15(1B):739–747, 2005.

[17] George Kordzakhia and Steven P. Lalley. A two-species competition model on $\mathcal{Z}^d$. *Stochastic Process. Appl.*, 115(5):781–796, 2005.

[18] R. Marchand. Strict inequalities for the time constant in first passage percolation. *Ann. Appl. Probab.*, 12(3):1001–1038, 2002.

[19] Claudia Neuhauser. Ergodic theorems for the multitype contact process. *Probab. Theory Related Fields*, 91(3-4):467–506, 1992.

[20] J. van den Berg and H. Kesten. Inequalities for the time constant in first-passage percolation. *Ann. Appl. Probab.*, 3(1):56–80, 1993.

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