Geometric Permutations of Disjoint Unit Spheres

Otfried Cheong, Xavier Goaoc, Na Hyeon-Suk

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Otfried Cheong[2] Xavier Goaoc[3] Hyeon-Suk Na[4]

Abstract

We show that a set of n disjoint unit spheres in $\mathbb{R}^d$ admits at most two distinct geometric permutations if $n \geq 9$, and at most three if $3 \leq n \leq 8$. This result improves a Helly-type theorem on line transversals for disjoint unit spheres in $\mathbb{R}^3$: if any subset of size at most 18 of a family of such spheres admits a line transversal, then there is a line transversal for the entire family.

1 Introduction

A *line transversal* for a set $\mathcal{F}$ of pairwise disjoint convex bodies in $\mathbb{R}^d$ is a line ℓ that intersects every element of $\mathcal{F}$. A line transversal induces two linear orders on $\mathcal{F}$, namely the orders in which the two possible orientations of ℓ intersect the elements of $\mathcal{F}$. Since the two orders are the reverse of each other, we consider them as a single *geometric permutation*.

Bounds on the maximum number of geometric permutations were established about a decade ago: a tight bound of $2n - 2$ is known for two dimensions [5], for higher dimension the number is in $\Omega(n^{d-1})$ [10] and in $O(n^{2d-2})$ [15]. The gap was closed for the special case of spheres by Smorodinsky et al. [14], who showed that n spheres in $\mathbb{R}^d$ admit $\Theta(n^{d-1})$ geometric permutations. This result can be generalized to "fat" convex objects [12].

The even more specialized case of congruent spheres was treated by Smorodinsky et al. [14] and independently by Asinowski [1]. They proved that n unit circles in $\mathbb{R}^2$ admit at most two geometric permutations if n is large enough (the proof by Asinowski holds for all $n \geq 4$). Zhou and Suri established an upper bound of 16 for all d, if n is sufficiently large, a result quickly improved by Katchalski, Suri, and Zhou [11] and independently by Huang, Xu, and Chen [8] to four.

Building on Katchalski et al.'s proof, we recently showed that there are in fact at most two geometric permutations [4]. As two geometric permutations are possible for any n, this bound is optimal. However, Katchalski et al.'s approach—and therefore our extension to it as well—relies strongly on the assumption that n is "sufficiently" large, which implies that any two line transversals of $\mathcal{F}$ are nearly parallel. The critical threshold has been estimated to be about 31 in three dimensions [7], but it increases exponentially with d. The proof gives no bound on the number of geometric permutations of n spheres if n is smaller than this threshold.

In the present paper we analyze line transversals for unit spheres in $\mathbb{R}^d$ in more detail. In particular, we prove that n disjoint unit spheres admit at most three geometric permutations, for any n, and at most two geometric permutations for $n \geq 9$.

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[2]Department of Mathematics and Computer Science, TU Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. Email: ocheong@win.tue.nl

[3]LORIA (INRIA Lorraine), 615, rue du Jardin Botanique, B.P. 101, 54602 Villers-les-Nancy, France. Email: xavier.goaoc@loria.fr

[4]School of Computing, Soongsil University, Seoul, South Korea. Email: hsnaa@computing.ssu.ac.kr

We prove these bounds by showing that some pairs of geometric permutations are *incompatible*. Let $\mathcal{F}$ be a family of disjoint convex objects (not necessarily spheres) in $\mathbb{R}^d$. A pair of geometric permutations, such as $(ABCD, BADC)$, is *incompatible* if no set of four objects $A, B, C, D \in \mathcal{F}$ admits both a line transversal realizing $ABCD$ and a line transversal realizing $BADC$.

Our first result is that if the pairs $(ABCD, BADC)$ and $(ABCD, ADCB)$ are both incompatible for a family $\mathcal{F}$, then $\mathcal{F}$ admits at most 3 geometric permutations. This fact was, in a sense, already used by Katchalski et al. [9, 10], but proven only for translates in the plane. We give a purely combinatorial proof. We then show that if the two additional pairs $(ABCD, ADBC)$ and $(ABCD, CADB)$ are incompatible as well, then $\mathcal{F}$ admits at most two geometric permutations that differ by the swapping of a single pair of adjacent objects.

To prove the incompatibility of $(ABCD, ADCB)$, we show that a line transversal that meets three unit spheres S, U, and T in that order makes an angle of less than 45° with the line through the centers of S and T. This bound is tight, and settles a problem posed by Holmsen et al. [7], who had conjectured the angle to be at most 60°.

Next, and maybe the cornerstone of this paper, we prove that the pair $(ABCD, BADC)$ is incompatible for disjoint unit spheres. This is nearly trivial in the plane, even for arbitrary convex objects, but takes considerable effort to prove for unit spheres in higher dimensions. The claim does not hold for general convex sets here, not even for spheres of different radia, or for unit spheres that are allowed to overlap somewhat. The bound of three geometric permutations for any family of disjoint unit spheres in any dimension follows.

We then establish that the pairs $(ABCD, ADBC)$ and $(ABCD, CADB)$ can be compatible only if the two line transversals make an angle of at least 45° with each other. We show that it is impossible for any set of nine unit spheres to admit two line transversals with such a large angle, and thus obtain the bound of two geometric permutations for at least nine unit spheres, with the two permutations differing only by the swapping of two adjacent spheres.

Incompatible pairs and triples of geometric permutations have been considered before, for instance by Asinowski et al. [2, 3], who give a complete characterization of the families of distinct geometric permutations that can appear for translates in the plane.

Surveys of geometric transversal theory are Goodman et al. [6] and Wenger [16]. The latter also discusses Helly-type theorems for line transversals. A recent result in that area by Holmsen et al. [7] proves the existance of a number n_0 such that the following holds: Let $\mathcal{F}$ be a set of disjoint unit spheres in $\mathbb{R}^3$. If every at most n_0 members of $\mathcal{F}$ have a line transversal, then $\mathcal{F}$ has a line transversal. Holmsen et al.'s proof implies $n_0 \leq 46$. Our results imply $n_0 \leq 18$.

The case of $4 \leq n \leq 8$ spheres is not completely resolved by our results: we prove that at most three geometric permutations exist, but no example realizing more than two is known. We conjecture that in fact there cannot be more than two geometric permutations for more than three unit spheres. One approach to proving this would be to show that the pairs $(ABCD, ADBC)$ and $(ABCD, CADB)$ are incompatible in general. Another approach might make use of our Lemma 1: if a set of $n \geq 4$ unit spheres had three distinct geometric permutations, then these permutations must realize all three geometric permutations of some subset of three spheres. Perhaps one can show that it is impossible to add a fourth sphere to such a configuration.

2 Incompatible pairs and geometric permutations

In this section we show that the incompatibility of certain pairs of geometric permutations implies a bound on the number of geometric permutations. Since these results can be proven purely

σ_1	σ_2	σ_3			
		ACBX	ACXB	AXCB	XACB
ABCX	ABXC	X,B,C	(II)σ_2/σ_3	(II)σ_1/σ_3	X,A,C
ABCX	AXBC	(II)σ_2/σ_3	C,X,B	(II)σ_1/σ_3	(I)σ_2/σ_3
ABCX	XABC	(II) σ_1/σ_2			
ABXC	AXBC	(II)σ_2/σ_3	(II)σ_1/σ_3	X,C,B	(I)σ_2/σ_3
ABXC	XABC	A,C,X	(II)σ_1/σ_3	(I)σ_2/σ_3	X,C,B
AXBC	XABC	(II)σ_1/σ_3	A,C,X	(I)σ_2/σ_3	(I)σ_1/σ_3

Table 1: Proof that $\sigma_1(ABCX) = \sigma_2(ABCX)$.

combinatorially, without referring to the geometry at all, we present them in a combinatorial setting.

Let $\mathcal{S}$ be a set of n symbols (which correspond to our spheres). We call a family of permutations $\mathcal{P}$ of $\mathcal{S}$ *reversible* if with every permutation $\sigma \in \mathcal{P}$ the reverse permutation σ^R is also in $\mathcal{P}$ (obviously, the family of permutations induced by line transversals is reversible). We will call a pair (σ, σ^R) a *geometric permutation* (corresponding to the two permutations realized by a line transversal). For a subset $S \subset \mathcal{S}$, we write $\sigma(S)$ for the restriction of σ to S, and for simplicity we will write $\sigma(ABC)$ for $\sigma(\{A, B, C\})$. A pair such as $(ABCD, BADC)$ is an *incompatible pair of* $\mathcal{P}$ if no four symbols $A, B, C, D \in \mathcal{S}$ and two permutations $\sigma_1, \sigma_2 \in \mathcal{P}$ exist with $\sigma_1(ABCD) = ABCD$ and $\sigma_2(ABCD) = BADC$ (generalizing the notion of incompatible pairs for line transversals).

The incompatible pairs we will consider are the following:

(I)	$(ABCD, BADC)$	(II)	$(ABCD, ADCB)$
(III)	$(ABCD, ADBC)$	(IV)	$(ABCD, CADB)$

Lemma 1 *Let $\mathcal{P}$ be a reversible family of permutations of $\mathcal{S}$ with incompatible pairs (I) and (II). If $\mathcal{P}$ contains at least six permutations (that is, at least three geometric permutations), then there are three symbols $A, B, C \in \mathcal{S}$ such that the restriction of $\mathcal{P}$ to $\{A, B, C\}$ consists of all six permutations of these symbols.*

Proof. We assume the contrary. There must then be three symbols $A, B, C \in \mathcal{S}$ and three permutations $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{P}$ such that $\sigma_1(ABC) = \sigma_2(ABC) = ABC$ and $\sigma_3(ABC) = ACB$.

Let $X \in \mathcal{S}$ be any other symbol. We claim that $\sigma_1(ABCX) = \sigma_2(ABCX)$. Indeed, assume this was not true. Then $\sigma_1(ABCX)$ and $\sigma_2(ABCX)$ must match one of the six rows of Table 1 (possibly after swapping σ_1 and σ_2). There are then four possibilities for $\sigma_3(ABCX)$, indicated by columns in the table. In each case, there are either three symbols that appear in three different geometric permutations in $\sigma_1, \sigma_2, \sigma_3$, or an incompatible pair appears as indicated in the table.

Since σ_1 and σ_2 are distinct permutations, there must then be two symbols $X, Y \in \mathcal{S} \setminus \{A, B, C\}$ such that $\sigma_1(ABCXY) \neq \sigma_2(ABCXY)$, and these two restrictions differ only by the swapping of the adjacent symbols X and Y. The four rows of Table 2 show the possible cases that can arise. If X and Y appear separated by a $Z \in \{A, B, C\}$ in $\sigma_3(ABCXY)$, then the three symbols X, Y, Z appear in three different geometric permutations in $\sigma_1, \sigma_2, \sigma_3$, a contradiction. So X, Y are consecutive in $\sigma_3(ABCXY)$, and by swapping X and Y we can assume that X appears before Y. This leaves four possible cases for $\sigma_3(ABCXY)$, shown in the columns of Table 2. In all 16 cases, either σ_1 and σ_3 or σ_2 and σ_3 contain an incompatible pair. Table 2 indicates the four symbols of the incompatible pair for each case. □

σ_1	σ_2	σ_3			
		ACBXY	ACXYB	AXYCB	XYACB
ABCXY	ABCYX	(I) CBXY σ_2	(II) CXYB σ_1	(II) AXYC σ_2	(I) XYAC σ_1
ABXYC	ABYXC	(II) CBXY σ_1	(II) AXYB σ_2	(II) AXYB σ_2	(II) XYAC σ_2
AXYBC	AYXBC	(II) ABXY σ_2	(II) CXYB σ_1	(I) XYCB σ_2	(I) XYCB σ_2
XYABC	YXABC	(I) ACXY σ_1	(II) AXYB σ_2	(I) AXCB σ_1	(I) XYCB σ_2

Table 2: All 16 cases involve an incompatible pair.

σ_1	σ_2	σ_3	σ_4			
			XBAC	BXAC	BAXC	BACX
XABC	AXBC	XBCA	(II)σ_2/σ_3			
		BXCA	(I)σ_3/σ_4	(II)σ_2/σ_4	(II)σ_1/σ_4	(II)σ_3/σ_4
		BCXA	(I)σ_1/σ_3			
		BCAX	(I)σ_2/σ_3			
XABC	ABXC	XBCA	(II)σ_2/σ_4	(I)σ_3/σ_4	(II)σ_1/σ_4	(I)σ_2/σ_4
		BXCA	(II)σ_2/σ_3			
		BCXA	(I) σ_1/σ_3			
		BCAX	(II)σ_2/σ_4	(II)σ_3/σ_4	(II)σ_1/σ_4	(I)σ_2/σ_4
AXBC	ABXC	XBCA	(II)σ_1/σ_3			
		BXCA	(II)σ_2/σ_3			
		BCXA	(II)σ_2/σ_4	(II)σ_1/σ_4	(II)σ_3/σ_4	(I)σ_2/σ_4
		BCAX	(I)σ_1/σ_3			

Table 3: Proof that $\sigma_1(ABCX) = \sigma_2(ABCX)$.

Lemma 2 *Let $\mathcal{P}$ be a reversible family of permutations of $\mathcal{S}$ with incompatible pairs (I) and (II). Then $\mathcal{P}$ contains at most six permutations (that is, at most three geometric permutations).*

Proof. Assume that $\mathcal{P}$ contains at least four geometric permutations. By Lemma 1, there are then three symbols $A, B, C \in \mathcal{S}$ and four permutations $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathcal{P}$ such that $\sigma_1(ABC) = \sigma_2(ABC) = ABC$, $\sigma_3(ABC) = BCA$, and $\sigma_4(ABC) = BAC$.

We claim that for any $X \in \mathcal{S} \setminus \{A, B, C\}$ we have $\sigma_1(ABCX) = \sigma_2(ABCX)$. Indeed, assume this is not true. As before, there are six possibilities for $\{\sigma_1(ABCX), \sigma_2(ABCX)\}$, shown in the six rows of Table 1. The case $\{XABC, ABCX\}$ (the third row of the table) is incompatible pair (II). In the other two cases involving $ABCX$ (the first and second row of Table 1), we reverse permutations σ_1 and σ_2, swap σ_3 and σ_4, and exchange the names A and C. This leaves us with the three cases in the bottom rows of Table 1, indicated again in Table 3. In each case, there are four possibilities for $\sigma_3(ABCX)$ and $\sigma_4(ABCX)$ each. As indicated in Table 3, each of the resulting 48 cases involves an incompatible pair.

Since $\sigma_1 \neq \sigma_2$, there must then be two symbols $X, Y \in \mathcal{S} \setminus \{A, B, C\}$ such that $\sigma_1(ABCXY) \neq \sigma_2(ABCXY)$, and these two restrictions differ only by the swapping of the adjacent symbols X and Y. We assume that X, Y appear before B in σ_1, σ_2 (otherwise we can again reverse σ_1 and σ_2 and swap σ_3 with σ_4 and A with C), and so there are the two cases indicated in the left and right half of Table 4. In the left half of the table, assume that X appears before Y in σ_3 (otherwise swap the names X and Y). There are ten possibilities for $\sigma_3(ABCXY)$, indicated in the table. In each case, either σ_1 and σ_3 or σ_2 and σ_3 contain an incompatible pair. Table 4 indicates the symbols of the incompatible pair. In the right half of Table 4, we similarly consider σ_4. Again an incompatible pair occurs in each case. $\square$

	σ_3	
σ_1 : XYABC	XYBCA	(I) XYCA σ_2
σ_2 : YXABC	XBYCA	
	XBCYA	(II) XBCY σ_2
	XBCAY	
	BXYCA	(II) BXYA σ_1
	BXCYA	
	BXCAY	(II) XCAY σ_1
	BCXYA	(I) BCXY σ_1
	BCXAY	
	BCAXY	

	σ_4	
σ_1 : AXYBC	XYBAC	(II) XYBA σ_1
σ_2 : AYXBC	XBYAC	(II) BYAC σ_1
	XBAYC	(I) XBAY σ_2
	XBACY	
	BXYAC	(II) XYAC σ_2
	BXAYC	(II) BXAC σ_1
	BXACY	(I) XACY σ_1
	BAXYC	(II) BXYC σ_2
	BAXCY	(II) BAXY σ_1
	BACXY	

Table 4: Proof of Lemma 2.

Lemma 3 *Let $\mathcal{P}$ be a reversible family of permutations of $\mathcal{S}$ with incompatible pairs (I) to (IV). Then $\mathcal{P}$ contains at most four permutations, that is, at most two geometric permutations that differ only in the swapping of a single pair of adjacent symbols.*

Proof. Let $\sigma, \sigma' \in \mathcal{P}$. We first prove the following *claim (i)*: If two symbols A and D appear in consecutive positions in σ, then at most one other symbol can appear in between A and D in σ'. Indeed, assume A and D appear separated by two other symbols B and C in σ', so that $\sigma'(ABCD) = ABCD$. If B and C appear on opposite sides of the pair AD in σ, then $\sigma(ABCD)$ is either $BADC$ or $CADB$, a contradiction. If B and C appear on one side, we can assume (by renaming the symbols) that $\sigma(ABCD)$ is either $ADBC$ or $ADCB$, a contradiction.

We now number the symbols in the order in which they appear in σ, that is $\sigma = B_1 B_2 \dots B_n$. Let similarly $\sigma' = B'_1 B'_2 \dots B'_n$.

We prove the following *claim (ii)*: If, for some i, we have $\{B'_1, \dots, B'_i\} = \{B_1, \dots, B_i\}$ (note that this is *set equality*, not sequence equality) and $B'_i = B_i$, then either $B'_{i+1} = B_{i+1}$, or $B'_{i+1} = B_{i+2}$, $B'_{i+2} = B_{i+1}$, and $B'_{i+3} = B_{i+3}$. Indeed, if $B'_{i+1} = B_j$ with $j > i+2$, then B_i and B_j are adjacent in σ', but separated by B_{i+1} and B_{i+2} in σ, a contradiction to claim (i). If $B'_{i+1} = B_{i+1}$, we have the first case of the claim, so it rests to consider $B'_{i+1} = B_{i+2}$. Then B'_{i+2} must be B_{i+1} (otherwise, B_i and B_{i+1} are adjacent in σ but separated by two symbols in σ'), and finally $B'_{i+3} = B_{i+3}$ (otherwise B_{i+2} and B_{i+3} are adjacent in σ, but separated by two symbols in σ').

If $B'_1 = B_1$, we can repeatedly apply claim (ii) to observe that σ and σ' can differ only by the exchange of independent adjacent pairs. There cannot be more than one such pair since $(ABCD, BADC)$ is incompatible, and so the lemma follows.

It remains to consider the case $B'_1 \neq B_1$. Let $B'_j = B_1$, with $1 < j < n$ (if $B'_n = B_1$ we consider σ'^R instead of σ' and apply the previous argument). We observe that then $\{B'_{j-1}, B'_{j+1}\} = \{B_2, B_3\}$ since no other symbol can appear adjacent to B_1 in ℓ'. Without loss of generality, let $B'_{j-1} = B_2$, $B'_{j+1} = B_3$ (otherwise we again consider σ'^R instead of σ'). Now, B_4 cannot appear before B'_{j-1} (that is, as $B'_1, \dots, B'_{j-2}$), and inductively it follows that *no* symbol can appear before B'_{j-1}. This implies $j = 2$, and we have $\{B'_1, B'_2, B'_3\} = \{B_1, B_2, B_3\}$ with $B'_3 = B_3$. Once again we can use claim (ii) to prove the lemma. ◻

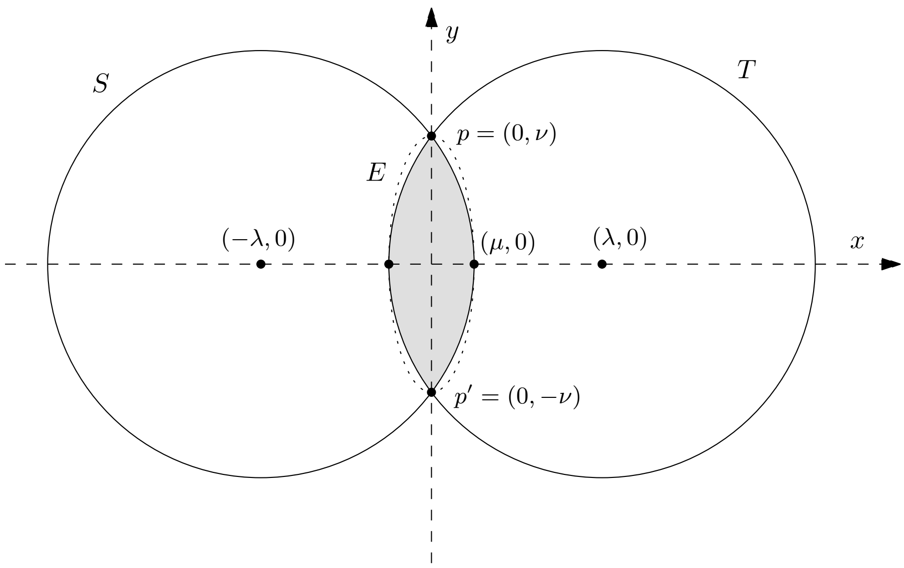

Figure 1: The intersection of two disks is contained in an ellipse.

3 Unit spheres and their transversals

A *unit sphere* is a sphere of radius one. We say that two unit spheres are *disjoint* if their interiors are (in other words, we allow the spheres to touch). A line *stabs* a sphere if it intersects the closed sphere (and so a tangent to a sphere stabs it). A *line transversal* for a set of disjoint unit spheres is a line that stabs all the spheres, with the restriction that it is not allowed to be tangent to two spheres in a common point (as such a line does not define a geometric permutation).

We will denote unit spheres by upper-case letters $A, B, \ldots$, and use the corresponding lower-case letters $a, b, \ldots$ for their centers. We make no distinction between points and vectors, so the vector from the center of sphere A to the center of sphere B is $b - a$.

Given two disjoint unit spheres A and B, let $\Pi(A, B)$ be their bisecting hyperplane. In other words, $\Pi(A, B)$ is the hyperplane through $(a + b)/2$ with normal $b - a$. We use $d(\cdot, \cdot)$ to denote the Euclidean distance of two points, that is $d(a, b)^2 = (b - a)^2$.

Let $u \cdot v$ denote the dot-product of two vectors u and v. The angle between two vectors u and v is $\arccos \frac{u \cdot v}{||u|| ||v||}$. The angle between a line ℓ with direction vector v and a hyperplane Π with normal n is $\pi/2 - \min(\angle(n, v), \angle(-n, v))$. Note that the angle does not change if the line is replaced by a parallel line, or the hyperplane by a parallel one.

We start with a warm-up lemma in two dimensions.

Lemma 4 *Let S and T be two unit-radius disks in $\mathbb{R}^2$ with centers $(-\lambda, 0)$ and $(\lambda, 0)$, where $\lambda \geq \cos \beta$ for some angle β with $0 < \beta \leq \pi/2$. Then $S \cap T$ is contained in the ellipse*

$$\left(\frac{x}{\sin^2 \beta}\right)^2 + \left(\frac{y}{\sin \beta}\right)^2 \leq 1.$$

Proof. Let $(\mu, 0)$ and $(0, \nu)$ be the rightmost and topmost point of $S \cap T$ (see Figure 1). Consider the ellipse E defined as

$$(\frac{x}{\mu})^2 + (\frac{y}{\nu})^2 \leq 1.$$

E intersects the boundary of S in $p = (0, \nu)$ and $p' = (0, -\nu)$, and is tangent to it in $(\mu, 0)$. An ellipse can intersect a circle in at most four points and the tangency counts as two intersections, and so the intersections at p and p' are proper and there is no further intersection between the two curves. This implies that the boundary of E is divided into two pieces by p and p', with one piece inside S and one outside S. Since $(-\mu, 0)$ lies inside S, the right hand side of E lies outside S. Symmetrically, the left hand side of E lies outside T, and so $S \cap T$ is contained in E. It remains to observe that

$$\nu^2 = 1 - \lambda^2 \leq 1 - \cos^2\beta = \sin^2\beta,$$

so $\nu \leq \sin\beta$, and

$$\mu = 1 - \lambda \leq 1 - \cos\beta \leq 1 - \cos^2\beta = \sin^2\beta,$$

which proves the lemma. □

We now show that a transversal for two spheres cannot pass too far from their common center of gravity.

Lemma 5 *Given two disjoint unit spheres A and B in $\mathbb{R}^d$ and a line ℓ stabbing both spheres, let p be the point of intersection of ℓ and $\Pi(A, B)$, and let β be the angle between ℓ and $\Pi(A, B)$. Then*

$$d(p, (a + b)/2) \leq \sin\beta.$$

Proof. Let ν be the direction vector of ℓ, that is, ℓ can be written as $\{p + \lambda\nu \mid \lambda \in \mathbb{R}\}$. We first argue that proving the lemma for $d = 3$ is sufficient. Indeed, assume $d > 3$ and consider the 3-dimensional subspace Γ containing ℓ, a, and b. Since we have $d(a, \ell) \leq 1$ and $d(b, \ell) \leq 1$, the line ℓ stabs the 3-dimensional unit spheres $A \cap \Gamma$ and $B \cap \Gamma$. And since $\pi/2 - \beta$ is the angle between two vectors in Γ, namely ν and $b - a$, β is also the angle between ℓ and the two-dimensional plane $\Pi(A, B) \cap \Gamma$. So if the lemma holds in Γ, then it also holds in $\mathbb{R}^d$.

In the rest of the proof we can therefore assume that $d = 3$. We choose a coordinate system where $a = (0, 0, -\rho)$, $b = (0, 0, \rho)$ with $\rho \geq 1$, and $\nu = (\cos\beta, 0, \sin\beta)$. Then $\Pi := \Pi(A, B)$ is the xy-plane and $g := (a + b)/2 = (0, 0, 0)$. Consider the cylinders $\mathcal{C}_A := \{u + \lambda\nu \mid u \in A, \lambda \in \mathbb{R}\}$ and $\mathcal{C}_B := \{u + \lambda\nu \mid u \in B, \lambda \in \mathbb{R}\}$. Since ℓ stabs A and B, we have $p \in \mathcal{C}_A \cap \mathcal{C}_B \cap \Pi$.

The intersection $B' := \mathcal{C}_B \cap \Pi$ is the ellipse (see Figure 2)

$$\sin^2\beta(x + \frac{\rho}{\tan\beta})^2 + y^2 \leq 1,$$

and symmetrically $A' := \mathcal{C}_A \cap \Pi$ is

$$\sin^2\beta(x - \frac{\rho}{\tan\beta})^2 + y^2 \leq 1.$$

If we let τ be the linear transformation

$$\tau : (x, y) \mapsto (x \sin\beta, y),$$

then $\tau(A')$ and $\tau(B')$ are unit-radius disks with centers $(\rho\cos\beta, 0)$ and $(-\rho\cos\beta, 0)$. By Lemma 4, the intersection $\tau(A' \cap B')$ is contained in the ellipse

$$\left(\frac{x}{\sin^2\beta}\right)^2 + \left(\frac{y}{\sin\beta}\right)^2 \leq 1.$$

Applying τ^{-1} we find that $A' \cap B'$ is contained in the circle with radius $\sin\beta$ around g. Since $p \in A' \cap B'$, the lemma follows. □

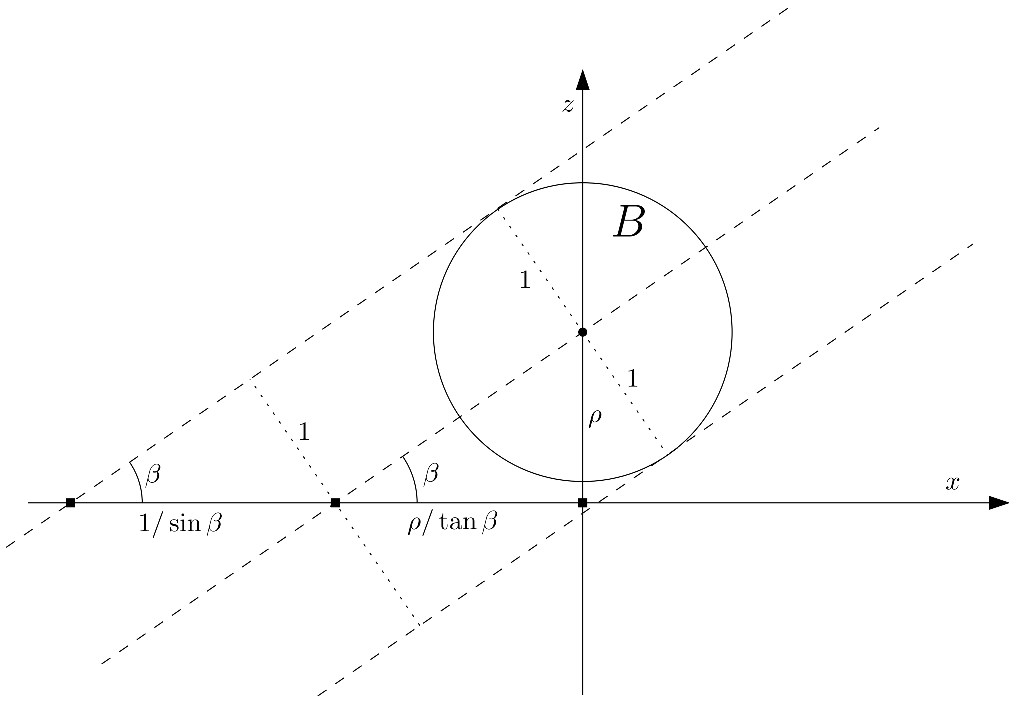

Figure 2: The intersection of the cylinder with the xy-plane is an ellipse.

Let ℓ be a line transversal for a family $\mathcal{S}$ of n disjoint unit spheres in $\mathbb{R}^d$. This implies that the center of any sphere in $\mathcal{S}$ lies inside a cylinder of radius one around ℓ. A volume argument [11] shows that the distance between the first and the last sphere met by ℓ is $\Omega(n)$, with a constant depending exponentially on the dimension d. The following lemma improves this to the absolute constant $\sqrt{2}$, which is easily seen to be tight in any dimension.

Lemma 6 *Let $\mathcal{C}$ be a cylinder of radius one and length less than $s\sqrt{2}$, for some $s \in \mathbb{N}$. Then $\mathcal{C}$ contains at most $2s$ points with pairwise distance at least 2.*

Proof. Let the axis of $\mathcal{C}$ be the x_1-axis, assume $\mathcal{C}$ contains at least $2s+1$ points, and partition it into s pieces of length less than $\sqrt{2}$. One of these pieces must contain at least three points a, b, c. We can assume $0 = a_1 \le b_1 \le c_1 < \sqrt{2}$. We increase c_1 to $\sqrt{2}$—this will increase $d(a, c)$ and $d(b, c)$ so that we have $d(b, c) > 2$. Let a', b', c' be the projection of the points on the hyperplane $x_1 = 0$. These points are contained in a unit sphere S with center in the origin. Let Π be the two-dimensional plane containing a', b', c'. It intersects S in a disk of radius at most 1. Let p be the center of this disk. The pairwise distance of the points a', b', c' is at least $\sqrt{2}$, as the pairwise difference of a_1, b_1, c_1 is at most $\sqrt{2}$. It follows that the angles $\angle a'pb', \angle b'pc', \angle c'pa'$ are all at least $\pi/2$. This implies that moving all three points away from p can only increase their pairwise distances, and so we can assume $d(p, a') = d(p, b') = d(p, c') = 1$. Furthermore, we can rotate c' around p towards a' until $\angle a'pc' = \pi/2$, as this can only increase $d(b', c')$. We have

$$\begin{aligned} 4 &\le d(a, b)^2 = d(a', b')^2 + b_1^2, \\ 4 &< d(b, c)^2 = d(b', c')^2 + (\sqrt{2} - b_1)^2, \end{aligned}$$

Let now $a'' = p + (p - a')$ and $c'' = p + (p - c')$. The point b' lies somewhere on the quarter circle around p between a'' and c''. By Thales' theorem, the angles $\angle a''b'a'$ and $\angle c''b'c'$ are right angles, so we have

$$\begin{aligned} d(b',a'')^2 &= d(a',a'')^2 - d(a',b')^2 = 4 - d(a',b')^2 \le b_1^2, \\ d(b',c'')^2 &= d(c',c'')^2 - d(c',b')^2 = 4 - d(c',b')^2 < (\sqrt{2} - b_1)^2. \end{aligned}$$

This implies $d(b',a'') \le b_1$ and $d(b',c'') < \sqrt{2} - b_1$. By the triangle inequality, however, we have

$$\sqrt{2} = d(a'',c'') \le d(a'',b') + d(b',c'') < b_1 + (\sqrt{2} - b_1) = \sqrt{2},$$

a contradiction. □

The following lemma is our first major geometric result. It settles a conjecture by Holmsen et al. [7].

Lemma 7 *Given three disjoint unit spheres* A, B, *and* C *in* $\mathbb{R}^d$, *and a directed line* ℓ *with direction vector* v *stabbing them in the order* ABC. *Then*

$$\angle(v, c - a) < \pi/4.$$

The bound $\pi/4$ is tight, as can be seen by chosing abc to be a nearly rectangular triangle. If one wishes to bound the angle between v and the *plane* spanned by a, b, c, then the maximal angle ϑ is given by $\cos\vartheta = 3/\sqrt{9 + 6\sqrt{3}}$, which is roughly $43°$ [13].

Proof. We first argue that it is sufficient to prove the result in three dimensions. Indeed, let Π be the two-dimensional plane through a, b, and c. If ℓ is a line with direction vector v stabbing ABC in that order, then there is a parallel line ℓ' in a three-dimensional subspace Λ containing Π and stabbing the spheres (in order ABC). This is obvious if ℓ is parallel to Π (take Λ as the affine hull of ℓ and Π). Otherwise, let v be the direction vector of ℓ, and let Λ be the subspace spanned by Π and v. Let Π' be a hyperplane orthogonal to ℓ, and let a', b', and c' be the orthogonal projection of a, b, c on Π'. We have $a' = a + \lambda v$ for some $\lambda \in \mathbb{R}$, so from $a, v \in \Lambda$ follows $a' \in \Lambda$, and analogously $b', c' \in \Lambda$. The points a', b', and c' lie in the unit sphere with center $\ell \cap \Pi'$. That implies that the circumcircle of the triangle $a'b'c'$ has radius at most one. Let p be the center of this circumcircle. The line $\ell' = \{p + \lambda v \mid \lambda \in \mathbb{R}\}$ is parallel to ℓ and intersects ABC in this order.

Let now $\mathcal{K}(ABC)$ be the set of vectors $v \in \mathbb{R}^3$ such that there is an oriented line with direction vector v that intersects the spheres in the order ABC. Holmsen et al. [7, Lemma 1] have shown that the set $\mathcal{K}(ABC)$ is convex. This implies that if there is a transversal with direction vector v and $\angle(v, c - a) \ge \pi/4$, then there is also a line transversal with angle exactly $\pi/4$ (since clearly there is a transversal with direction $c - a$).

In the following, we therefore assume that a line transversal with $\angle(v, c - a) = \pi/4$ exists. We choose a coordinate system where ℓ is the line $\{(-\lambda, -\lambda, 0) \mid \lambda \in \mathbb{R}\}$ (that is, $v = (-1, -1, 0)$), and the line ca is the line $\ell_1 = \{(\lambda, 0, -\rho) \mid \lambda \in \mathbb{R}\}$. Let $\mathcal{C}$ be the cylinder of radius 1 around ℓ. Since B lies in the convex hull of A and C, we can translate a in direction $(1, 0, 0)$ and c in direction $(-1, 0, 0)$ up to the points of intersection of ℓ_1 and $\mathcal{C}$ (this means that ℓ is now tangent to A and C). This implies $c = (-\sqrt{2 - 2\rho^2}, 0, -\rho)$, $a = (\sqrt{2 - 2\rho^2}, 0, -\rho)$. Without loss of generality, we can assume $b_1 \ge 0$ (otherwise we exchange the role of a and c).

We will now show that it is impossible to have $d(a, b) \ge 2$, a contradiction to the disjointness of A and B. We observe first that we can translate b in direction v until $b_1 = 0$ (if $b_1 > 0$, this strictly increases $d(a, b)$). The intersection of $\mathcal{C}$ and the plane $x_1 = 0$ is an ellipse E with half-axes 1 and $\sqrt{2}$, and we now have $b \in E$. On the other hand, the sphere with center a and radius 2 intersects the plane $x_1 = 0$ in a circle C with center $p = (0, 0, -\rho)$ and radius $\sqrt{2 + 2\rho^2}$.

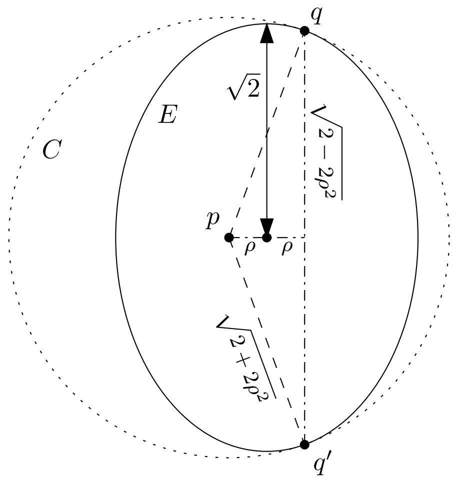

Figure 3: Illustration for Lemma 7.

Let $q = (0, \sqrt{2-2\rho^2}, \rho)$, and $q' = (0, -\sqrt{2-2\rho^2}, \rho)$. The points q and q' are points of tangency of E and C , and so there cannot be any other intersection points between E and C, see Figure 3. It follows that E lies entirely inside C, with the exception of the two shared points q and q'. The points q, q' are therefore the only possible candidates for the location of the point b. However, $(q - a) \cdot v = 0 = (q' - c) \cdot v$, so neither of these is admissible. □

The previous angular inequality yields a first incompatible pair:

Lemma 8 *The geometric permutations ABCD and ADCB are incompatible for disjoint unit spheres.*

Proof. Let ℓ be a transversal with direction vector v stabbing four spheres in the order $ABCD$, and let ℓ' be a transversal with direction vector v' stabbing them in the order $ADCB$. By Lemma 7, it follows that $\angle(v, d-b) < \pi/4$, and $\angle(v', b-d) < \pi/4$, and therefore $\angle(v, v') > \pi/2$. On the other hand, $\angle(v, c-a) < \pi/4$ and $\angle(v', c-a) < \pi/4$, a contradiction. □

4 The geometric permutations $ABCD$ and $BADC$ are incompatible

We start with a somewhat technical lemma.

Lemma 9 *Let A and B be two disjoint unit spheres with centers a and b in $\mathbb{R}^d$, and let ℓ be a line with direction vector v stabbing both spheres. Let p be the point of intersection of ℓ and $\Pi(A, B)$, and let q be the point on ℓ closest to b. Let $b - a = u + \lambda v$ be the unique factorization of $b - a$ with $u \cdot v = 0$, and let $\delta := \angle(b - q, u)$. Then $\delta \le \pi/2$ and $d(p, q) \ge \sin\delta$.*

Note that when ℓ is parallel to ab, we have $u = 0$ and δ is not defined. In that case, $d(p, q) \ge 1$, and the lemma holds for any angle δ.

Proof. We choose a coordinate system where $a = (-\rho, 0, \dots, 0)$, $b = (\rho, 0, \dots, 0)$, where $\rho \ge 1$, and ℓ is the line $(\lambda \sin\beta, p_2 + \lambda\cos\beta, p_3, \dots, p_d)$. Then $\Pi(A, B)$ is the hyperplane $x_1 = 0$, $g(A, B)$ is the origin, $v = (\sin\beta, \cos\beta, 0, \dots, 0)$, and u is a multiple of $u' := (\cos\beta, -\sin\beta, 0, \dots, 0)$.

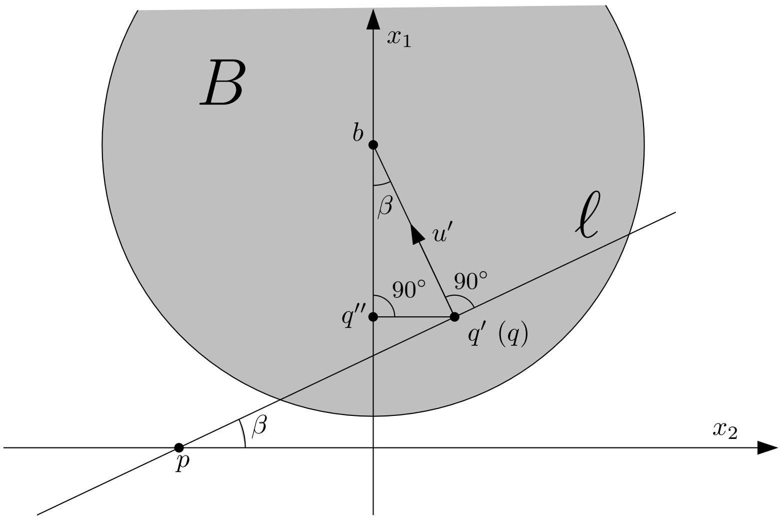

Figure 4: The configuration projected on the x_1x_2-plane

Let q' be the orthogonal projection of q on the x_1x_2-plane, and consider the rectangular triangle $bq'q$. We have $\angle q'bq = \delta$, as $b - q'$ is a multiple of u', and therefore

$$d(b, q') = d(b, q) \cos \delta \leq \cos \delta.$$

Figure 4 shows the projection of the configuration on the x_1x_2-plane. Since ℓ intersects A, clearly b lies above the projection of ℓ on the x_1x_2-plane, and therefore $\delta \leq \pi/2$. Consider now the projection q'' of q' on the x_1-axis. We have $\angle q'bq'' = \beta$, and so

$$d(b, q'') = d(b, q') \cos \beta \leq \cos \delta \cos \beta.$$

It follows that

$$d(q, \Pi(A, B)) = d(q'', \Pi(A, B)) = \rho - d(b, q'') \geq 1 - \cos \delta \cos \beta.$$

Since the angle between ℓ and $\Pi(A, B)$ is β, we have

$$d(p, q) = \frac{d(q, \Pi(A, B))}{\sin \beta} \geq \frac{1 - \cos \delta \cos \beta}{\sin \beta}.$$

Finally, we observe that

$$1 \geq \cos(\beta - \delta) = \sin \delta \sin \beta + \cos \delta \cos \beta,$$

and so $1 - \cos \delta \cos \beta \geq \sin \delta \sin \beta$, and we obtain

$$d(p, q) \geq \frac{\sin \delta \sin \beta}{\sin \beta} = \sin \delta.$$

□

We also need the following trigonometric inequality.

Lemma 10 *Let α, β be angles. Then*

$$2 \cos(\alpha + \beta) \geq (\sin \alpha - \sin \beta)^2 - 2.$$

Proof. We have

$$0 \le (\cos\alpha + \cos\beta)^2 = \cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta = 1 - \sin^2\alpha + 2\cos\alpha\cos\beta + 1 - \sin^2\beta,$$

and since $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, that implies

$$0 \le 2 - (\sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta) + 2\cos(\alpha+\beta) = 2 - (\sin\alpha - \sin\beta)^2 + 2\cos(\alpha+\beta),$$

and the inequality follows. □

We now fix four disjoint unit spheres A, B, C, D in $\mathbb{R}^d$. Let $\Pi_1 := \Pi(A,B)$, $\Pi_2 = \Pi(C,D)$, $g_1 := (a+b)/2$, and $g_2 := (c+d)/2$. Also let φ be the angle between the normals of Π_1 and Π_2.

Note that since we will be working with only four spheres, we could restrict our arguments to $\mathbb{R}^3$: after all, if a line ℓ stabs A, B, C, D in $\mathbb{R}^d$, then the orthogonal projection of ℓ into the three-dimensional subspace spanned by a, b, c, d does so as well. We will nevertheless prove the following lemma in $\mathbb{R}^d$, as the stronger result takes no additional effort.

A line transversal ℓ for the four spheres must intersect Π_1 and Π_2. We define $t(\ell)$ to be the finite segment on ℓ between the two intersection points.

Lemma 11 *Given four disjoint unit spheres A, B, C, D in $\mathbb{R}^d$ as above. Assume there is a line transversal ℓ intersecting the four spheres in the order ABCD, and a line transversal ℓ' intersecting them in the order BADC. Then*

$$\min\{|t(\ell)|, |t(\ell')|\} \le \sin\varphi.$$

Proof. We choose a coordinate system where Π_1 is the hyperplane $x_1 = 0$, Π_2 is the hyperplane $x_1\cos\varphi - x_2\sin\varphi = 0$, and so the intersection $\Pi_1 \cap \Pi_2$ is the subspace $x_1 = x_2 = 0$. We can make this choice such that the x_1-coordinate of a is negative, and that the x_2-coordinate of c is less than the x_2-coordinate of d. We can also assume that the x_2-coordinate of g_1 is non-negative (otherwise we swap A with B, C with D, and ℓ with ℓ'). Figure 5 shows the projection of the configuration on the x_1x_2-plane.

Since ℓ stabs A before B and C before D, it intersects Π_1 from bottom to top, and Π_2 from left to right. The segment $t(\ell)$ therefore lies in the top-left quadrant of Figure 5. On the other hand, ℓ' stabs B before A and D before C, so it intersects Π_1 from top to bottom, and Π_2 from right to left, and so the segment $t(\ell')$ lies in the bottom-right quadrant of the figure.

We introduce some further notation: Let $t := |t(\ell)|$, $t' := |t(\ell')|$, let $p_i := \ell \cap \Pi_i$, $p'_i := \ell' \cap \Pi_i$, let β_i be the angle between ℓ and Π_i, and let β'_i be the angle between ℓ' and Π_i. Let u_1 (u'_1) be the orthogonal projection of p_1 (p'_1) on Π_2, u_2 (u'_2) the orthogonal projection of p_2 (p'_2) on Π_1. Consider the rectangular triangle $p_1u_2p_2$. We have $\angle u_2p_1p_2 = \beta_1$, and so

$$t\sin\beta_1 = d(p_2, u_2) = d(p_2, \Pi_1). \tag{1}$$

Similarly, we can consider the rectangular triangles $p_2u_1p_1$, $p'_1u'_2p'_2$, and $p'_2u'_1p'_1$ to obtain

$$\begin{aligned} t\sin\beta_2 &= d(p_1, u_1) = d(p_1, \Pi_2), && (2)\\ t'\sin\beta'_1 &= d(p'_2, u'_2) = d(p'_2, \Pi_1), && (3)\\ t'\sin\beta'_2 &= d(p'_1, u'_1) = d(p'_1, \Pi_2). && (4) \end{aligned}$$

We now distinguish between two cases.

The *first case* occurs if, as in the figure, the x_1-coordinate of g_2 is negative or zero. By Lemma 5 we have $d(p_2, g_2) \le \sin\beta_2$. Since p_2 and g_2 lie on opposite sides of Π_1, we have

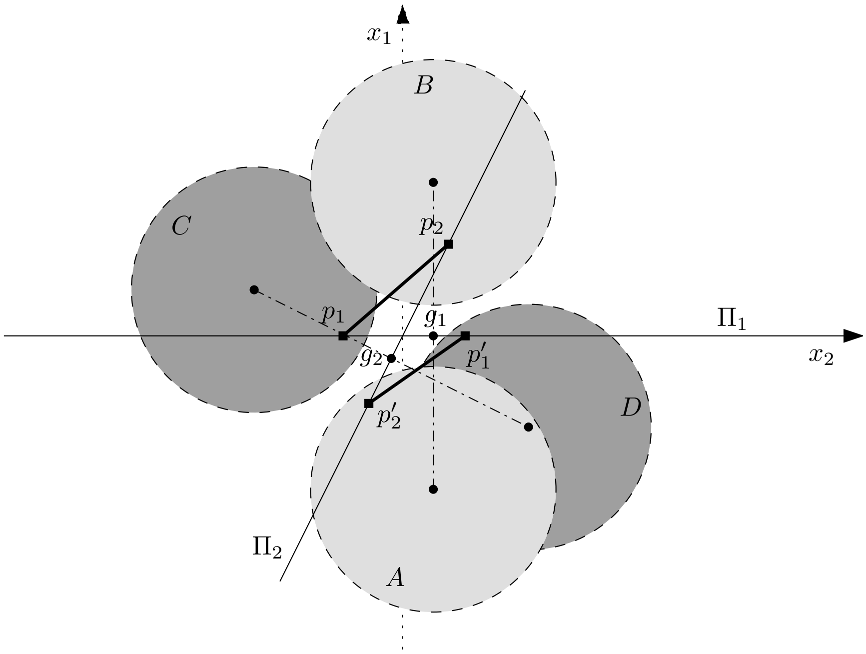

Figure 5: The two hyperplanes define four quadrants

$d(p_2, \Pi_1) \le \sin\beta_2 \sin\varphi$. Similarly, we have $d(p_1, g_1) \le \sin\beta_1$, and p_1 and g_1 lie on opposite sides of Π_2, implying $d(p_1, \Pi_2) \le \sin\beta_1 \sin\varphi$. Plugging into Eq. (1) and (2), we obtain

$$t \le \min\{\frac{\sin\beta_2}{\sin\beta_1}, \frac{\sin\beta_1}{\sin\beta_2}\}\sin\varphi \le \sin\varphi,$$

which proves the lemma for this case.

The *second case* occurs if the x_1-coordinate of g_2 is positive. We let $s_1 := d(g_1, \Pi_2)$, and $s_2 := d(g_2, \Pi_1)$. Applying Lemma 5 , we then have

$$\begin{aligned} d(p_2, \Pi_1) &\le d(p_2, g_2)\sin\varphi + s_2 \le \sin\beta_2 \sin\varphi + s_2, & (5)\\ d(p_1, \Pi_2) &\le d(p_1, g_1)\sin\varphi - s_1 \le \sin\beta_1 \sin\varphi - s_1, & (6)\\ d(p'_2, \Pi_1) &\le d(p'_2, g_2)\sin\varphi - s_2 \le \sin\beta'_2 \sin\varphi - s_2, & (7)\\ d(p'_1, \Pi_2) &\le d(p'_1, g_1)\sin\varphi + s_1 \le \sin\beta'_1 \sin\varphi + s_1. & (8)\end{aligned}$$

Plugging Ineqs. (5) to (8) into (1) to (4), we obtain

$$t \le \frac{\sin\beta_2 \sin\varphi + s_2}{\sin\beta_1}, \quad (9)$$

$$t \le \frac{\sin\beta_1 \sin\varphi - s_1}{\sin\beta_2}, \quad (10)$$

$$t' \le \frac{\sin\beta'_2 \sin\varphi - s_2}{\sin\beta'_1}, \quad (11)$$

$$t' \le \frac{\sin\beta'_1 \sin\varphi + s_1}{\sin\beta'_2}. \quad (12)$$

We want to prove that $\min(t, t') \le \sin\varphi$. We assume the contrary. From $t > \sin\varphi$ and Ineq. (10) we obtain

$$\sin\beta_2 \sin\varphi < \sin\beta_1 \sin\varphi - s_1,$$

and from $t' > \sin\varphi$ and Ineq. (11) we get

$$\sin\beta_1' \sin\varphi < \sin\beta_2' \sin\varphi - s_2.$$

Plugging this into Ineq. (9) and (12) results in

$$\begin{aligned} t &\le \frac{\sin\beta_2 \sin\varphi + s_2}{\sin\beta_1} < \frac{\sin\beta_1 \sin\varphi - s_1 + s_2}{\sin\beta_1} = \sin\varphi + \frac{s_2 - s_1}{\sin\beta_1}, \\ t' &\le \frac{\sin\beta_1' \sin\varphi + s_1}{\sin\beta_2'} < \frac{\sin\beta_2' \sin\varphi - s_2 + s_1}{\sin\beta_2'} = \sin\varphi + \frac{s_1 - s_2}{\sin\beta_2'}. \end{aligned}$$

It follows that if $s_2 < s_1$ then $t < \sin\varphi$, otherwise $t' < \sin\varphi$. In either case the lemma follows. □

Theorem 12 *The geometric permutations $ABCD$ and $BADC$ are incompatible for disjoint unit spheres in $\mathbb{R}^d$.*

Proof. Assume two line transversals ℓ and ℓ' exist, realizing the geometric permutations $ABCD$ and $BADC$. By Lemma 11 we have $\min\{|t(\ell)|, |t(\ell')|\} \le \sin\varphi$. Without loss of generality, we can assume that $|t(\ell)| \le \sin\varphi$.

Let n_i be the unit normal vector of Π_i pointing into the halfspace containing $t(\ell)$, for $i = 1, 2$. We can express n_i uniquely as $n_i = u_i + \lambda_i v$, where v is the direction vector of ℓ and $u_i v = 0$. Notice that $\|u_i\| \le \|v_i\| = 1$. Since ℓ stabs A before B, we have $n_1 v > 0$. Since it stabs C before D, we have $n_2 v < 0$. This implies $\lambda_1 > 0$, $\lambda_2 < 0$, and therefore $\lambda_1\lambda_2 < 0$. Recall that $\varphi = \angle(n_1, n_2)$, and let $\vartheta = \angle(u_1, u_2)$. We have

$$\cos\varphi = n_1 n_2 = (u_1 + \lambda_1 v)(u_2 + \lambda_2 v) = u_1 u_2 + \lambda_1\lambda_2 v^2 < u_1 u_2 < \frac{u_1 u_2}{\|u_1\|\|u_2\|} = \cos\vartheta,$$

and so $\vartheta < \varphi$.

Let $p_i = \ell \cap \Pi_i$, for $i = 1, 2$, let $q_1 \in \ell$ be the point closest to b, and let $q_2 \in \ell$ be the point closest to c. The points q_1 and q_2 lie between p_1 and p_2, that is, in the segment $t(\ell)$, and so we have

$$d(p_1, q_1) + d(q_1, q_2) + d(q_2, p_2) = d(p_1, p_2) = |t(\ell)| \le \sin\varphi, \tag{13}$$

the last inequality stemming from Lemma 11.

Let $\delta_1 := \angle(u_1, b - q_1)$, $\delta_2 := \angle(u_2, c - q_2)$. By Lemma 9, this implies $d(p_1, q_1) \ge \sin\delta_1$ and $d(p_2, q_2) \ge \sin\delta_2$. Applying Ineq. (13) results in

$$\sin\delta_1 + \sin\delta_2 + d(q_1, q_2) \le \sin\varphi. \tag{14}$$

Consider the hyperplane Γ orthogonal to ℓ in q_1. It contains the points q_1 and b, and its normal is v. Let c' be the orthogonal projection of c on Γ, so that we have $c - q_2 = c' - q_1$. Let $\psi := \angle c' q_1 b$. Since B and C are disjoint, we have

$$4 \le d(b, c)^2 = d(q_1, q_2)^2 + d(b, c')^2 \tag{15}$$

Consider now the triangle bq_1c'. By the cosine-theorem, we have

$$\begin{aligned} d(b, c')^2 &= d(b, q_1)^2 + d(c', q_1)^2 - 2d(b, q_1)d(c', q_1)\cos\psi \\ &= d(b, q_1)^2 + d(c, q_2)^2 - 2d(b, q_1)d(c, q_2)\cos\psi \\ &\le 2 - 2d(b, q_1)d(c, q_2)\cos\psi. \end{aligned}$$

Ineq. (13) implies $d(q_1, q_2) \le 1$. Combining with Ineq. (15) results in $d(b, c')^2 \ge 3$, which implies $\cos\psi < 0$. We can therefore apply the upper bounds $d(b, q_1) \le 1$ and $d(c, q_2) \le 1$ again to obtain $d(b, c')^2 \le 2 - 2\cos\psi$. Together with Ineq. (15) this gives $4 \le d(q_1, q_2)^2 + 2 - 2\cos\psi$, or

$$2\cos\psi \le d(q_1, q_2)^2 - 2. \tag{16}$$

By Lemma 9, we have $0 \le \delta_1, \delta_2 \le \pi/2$. Let $\delta := \delta_1 + \delta_2$. We claim that $\delta \le \pi/2$. Indeed, assume that $\delta > \pi/2$. By Ineq. (14), we have

$$\sin\delta_1 + \sin(\delta - \delta_1) = \sin\delta_1 + \sin\delta_2 \le \sin\varphi \le 1.$$

The function $\delta_1 \mapsto \sin\delta_1 + \sin(\delta - \delta_1)$ over the interval $[\delta - \pi/2, \pi/2]$ is minimized for $\delta_1 = \pi/2$ or $\delta_1 = \delta - \pi/2$, where its value is $\sin\pi/2 + \sin(\delta - \pi/2) > 1$, a contradiction.

We now argue that $\varphi + \delta \le \pi$. This is true if $\varphi \le \pi/2$. Otherwise, $\pi - \varphi < \pi/2$. By Ineq. (14) we have

$$\sin\delta \le \sin\delta_1 + \sin\delta_2 \le \sin\varphi = \sin(\pi - \varphi),$$

which implies $\delta \le \pi - \varphi$ and therefore $\delta + \varphi \le \pi$. Since $\vartheta < \varphi$, this also implies $\vartheta + \delta < \pi$.

Consider now the angle $\psi = \angle b q_1 c'$. We can write it as the sum of the three *oriented* angles $\angle(b - q_1, u_1)$, $\angle(u_1, u_2)$, and $\angle(u_2, c' - q_1)$. Since $\vartheta + \delta_1 + \delta_2 < \pi$, this implies $0 \le \psi \le \vartheta + \delta_1 + \delta_2 = \vartheta + \delta < \varphi + \delta \le \pi$. We apply Lemma 10 and obtain

$$2\cos\psi > 2\cos(\varphi + \delta) \ge (\sin\varphi - \sin\delta)^2 - 2.$$

Together with Ineq. (16) we get $(\sin\varphi - \sin\delta)^2 < d(q_1, q_2)^2$, so $d(q_1, q_2) > \sin\varphi - \sin\delta$. Combining with Ineq. (14), we obtain

$$\sin\varphi = \sin\delta + \sin\varphi - \sin\delta < \sin\delta_1 + \sin\delta_2 + d(q_1, q_2) \le \sin\varphi,$$

a contradiction. □

5 Putting it all together

We now apply the combinatorial results of Section 2 to our geometric results. Lemma 2 immediately implies the following theorem, using Lemma 8 and Theorem 12.

Theorem 13 *Let $\mathcal{S}$ be a family of disjoint unit spheres in $\mathbb{R}^d$. Then $\mathcal{S}$ admits at most three distinct geometric permutations.*

This is the first bound valid for a small number of spheres in dimension greater than two. To improve the bound to the optimal two, we need the two additional incompatible pairs (III) and (IV). Our proof of incompatibility of these pairs, however, uses the additional assumption that n is at least nine. Note that this threshold is independant of the dimension.

Lemma 14 *Let $\mathcal{S}$ be a family of $n \ge 9$ disjoint unit spheres in $\mathbb{R}^d$. Then any two line transversals for $\mathcal{S}$ make an angle of less than $\pi/4$.*

Proof. Let ℓ and ℓ' be two line transversals for $\mathcal{S}$, and let $\mathcal{C}$ and $\mathcal{C}'$ be cylinders of radius one with axis ℓ and ℓ', respectively. The centers of all spheres in $\mathcal{S}$ are contained in $\mathcal{C} \cap \mathcal{C}'$. If ℓ and ℓ' make an angle of at least $\pi/4$, then $\mathcal{C} \cap \mathcal{C}'$ is contained in a section of $\mathcal{C}$ of length at most $2 + 2\sqrt{2} < 4\sqrt{2}$. By Lemma 6, this implies $n \le 8$, a contradiction. □

The threshold nine can probably be lowered by analyzing the shape of $\mathcal{C} \cap \mathcal{C}'$ more carefully. We do not pursue this, as we cannot close the gap entirely: values of n remain where our best bound on the number of geometric permutations is three.

We can now prove that $(ABCD, ADBC)$ and $(ABCD, CADB)$ are incompatible pairs.

Lemma 15 *Let $\mathcal{S}$ be a family of $n \geq 9$ disjoint unit spheres in $\mathbb{R}^d$. Then the pairs $(ABCD, ADBC)$ and $(ABCD, CADB)$ are incompatible for $\mathcal{S}$.*

Proof. Let v be the direction vector of a line transversal realizing $ABCD$, and let v' be the direction vector of a transversal realizing either $ADBC$ or $CADB$. By Lemma 7, $\angle(v, d-b) < \pi/4$. On the other hand, $\angle(v', b-d) < \pi/2$, and so $\angle(v, v') > \pi/4$, a contradiction with Lemma 14. □

The final theorem now follows from Lemma 3, using Lemmas 8 and 15 and Theorem 12.

Theorem 16 *Let $\mathcal{S}$ be a family of $n \geq 9$ disjoint unit spheres in $\mathbb{R}^d$. Then $\mathcal{S}$ admits at most two distinct geometric permutations, which differ only in the swapping of two adjacent spheres.*

Our results also improve the constants involved in recent results by Holmsen et al. [7]. First, Lemma 7 implies the following improvement to Holmsen et al.'s Theorem 2, a Hadwiger-type theorem (their constant is 12).

Theorem 17 *Let $\mathcal{S}$ be a family of at least nine disjoint unit spheres in $\mathbb{R}^3$. If there is a linear ordering on $\mathcal{S}$ such that every nine members are met by a directed line consistent with that ordering, then $\mathcal{S}$ admits a line transversal.*

This improvement, combined with Theorem 16, reduces the constant in their Helly-type Theorem 1 from 46 to 18. (The justification for both improvements can be found in Holmsen et al.'s paper [7], in the first remark of their Section 4.)

Theorem 18 *Let $\mathcal{S}$ be a family of n disjoint unit spheres in $\mathbb{R}^3$. There exists an integer $n_0 \leq 18$ such that if any subset $\mathcal{S}' \subset \mathcal{S}$ of size at most n_0 admits a line transversal, then $\mathcal{S}$ admits a line transversal.*

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