A Grobman-Hartman Theorem for Control Systems

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Abstract

We consider the problem of locally linearizing a control system via topological transformations. According to [2, 3], there is no naive generalization of the classical Grobman-Hartman theorem for ODEs to control systems: a generic control system, when viewed as a set of underdetermined differential equations parametrized by the control, cannot be linearized using pointwise transformations on the state and the control values. However, if we allow the transformations to depend on the control at a functional level (open loop transformations), we are able to prove a version of the Grobman-Hartman theorem for control systems.

Keywords: Control systems, Linearization, Topological Equivalence, Grobman-Hartman Theorem.

1 Introduction

The classical Grobman-Hartman theorem states that, around a hyperbolic equilibrium, the flow of a nonlinear differential equation is conjugate via a (not necessarily differentiable) local homeomorphism to the flow of its tangent approximation [8]. Our point of departure will be a brief review of this classical result after fixing some notation. Consider the differential equation

$$\dot{x}(t) = f(x(t)), \tag{1.1}$$

where $f \in C^1(U, \mathbb{R}^n)$ and U is an open subset of $\mathbb{R}^n$. Assume that $x_0 \in U$ is an equilibrium, i.e. $f(x_0) = 0$. The linearized system associated to (1.1) near x_0 is

$$\dot{x}(t) = Ax(t) - Ax_0 \tag{1.2}$$

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where $A = Df(x_0)$ is the derivative of f at x_0. The equilibrium x_0 is said to be *hyperbolic* if the matrix A has no purely imaginary eigenvalue. Systems (1.1) and (1.2) are called *topologically conjugate* at x_0 if there exist neighborhoods V, W of x_0 in U and a homeomorphism $h : V \to W$ mapping the trajectories of (1.1) in V onto the trajectories of (1.2) in W in a time-preserving manner : for each $x \in V$, we should have

$$h \circ \phi_t(x) = e^{At}\big(h(x) - h(x_0)\big) + h(x_0) \tag{1.3}$$

provided that $\phi_\rho(x) \in V$ for $0 \leq \rho \leq t$, where ϕ_t denotes the flow of (1.1). The Grobman-Hartman theorem now goes as follows [8]:

Theorem 1.1 (Grobman-Hartman) *Under the assumption that x_0 is an hyperbolic equilibrium point,* (1.1) *is topologically conjugate to* (1.2) *at x_0.*

This theorem entails that the only invariant under local topological conjugacy around a hyperbolic equilibrium is the number of eigenvalues with positive real part in the Jacobian matrix, counting multiplicity. Indeed, it is well-known (*cf* [1]) that the linear system $\dot{x} = Ax$ where A has no pure imaginary eigenvalue is topologically conjugate to the linear system $\dot{x} = DX$ where D is diagonal with diagonal entries ± 1, the number of occurrences of $+1$ being the number of eigenvalues of A with positive real part, counting multiplicity.

When trying to extend this result to a control systems $\dot{x} = f(x, u)$, with state $x \in I\!R^n$ and control $u \in I\!R^m$, one has first to decide what the meaning of "topologically conjugate" should be, i.e. what kind of map should paly the role of the homeomorphism h in (1.3). The simplest idea is to ask for a pointwise transformation on the $n + m$ variables x, u, i.e. a local homeomorphism of $I\!R^{n+m}$. This is investigated in [2, 3] where it is proved that generic control systems are not topologically linearizable in that sense. This may not be too surprising, because the Grobman-Hartman theorem for differential equations is about conjugating *flows* whereas, since the control is an arbitrary function of time whose future values are not determined by past ones, control systems do not have flows, at least of finite dimension. In fact, this unpredictability of future control values forces a rather rigid triangular structure on conjugating homeomorphisms that ultimately results in the non-genericity of a linearizing homeomorphism: the latter would be very smooth and thereby should preserve too many special features of linear control systems [2].

The present paper is devoted to a different point of view on local linearization of control systems. Setting up a stage where a flow can be defined, either by restricting the input space or by enlarging the state space to an infinite dimensional one, we derive some analogs to the Grobman-Hartman theorem in this context. These do not contradict the above mentioned "negative" results because the notion of conjugacy is here much weaker: either the control itself is generated by a finite dimensional dynamical system, or else the linearizing transformation depends both on the past and on the future values of the control using an abstract representation of the system as a flow on some functional space in the style of [5]. These results will be derived from an abstract principle saying that if the controls are generated by a flow (*i.e.* a one parameter group of homeomorphism) on some topological space, then, under quantitative hyperbolicity assumptions, the system can be linearized *via* transformations that are continuously parameterized by elements of this topological space.

In Section 2, we state and prove a fairly general version of the abstract principle. Subsequently, in Section 3, we use it to obtain local linearizability in two more concrete sistuations as mentioned above.

2 An abstract Grobman-Hartman Theorem

We shall prove an abstract result on the linearization of dynamical systems which implies the local linearizability properties of control systems stated in sections 3.1 and 3.2. The proof closely follows that of the classical Grobman-Hartman theorem for ODEs as given by Hartman in [8, chap. IX, sect. 4, 7, 8, 9], and we tried to stick to his notations as much as possible. Nevertheless, we provide a detailed argument because the modifications needed to handle the dynamics of the control are not completely straightforward. Like [8], we state Theorem 2.1 below as a *global* linearizability property for a linear equation perturbed by a suitably normalized additive term. In sections 3.1 and 3.2, we shall use this result to derive local linearizability results for systems that locally coincide with a normalized one.

Let us mention in passing that the Grobman-Hartman Theorem for "random dynamical systems" given in [6] is similar in spirit to Theorem 2.1 : there, the set $\mathcal{E}$ of control parameters is a probability space instead of a topological space, and the conjugating transformation H is only required to be measurable with respect to $\zeta \in \mathcal{E}$ but need not be continuous. Both can be viewed as Grobman-Hartman Theorem "with parameters".

The setting is as follows. We consider a topological space $\mathcal{E}$ endowed with a one-parameter group of homeomorphisms $(\mathcal{S}_\tau)_{\tau\in\mathbb{R}}$. The space $\mathcal{E}$ is to be regarded as an abstract collection of input-producing events for a control system, these events being themselves subject to the dynamics of the flow $\mathcal{S}_\tau$. To describe the action of such an event on the system we simply let ζ enter as a parameter in the differential equation describing the evolution of the state variable x:

$$\dot{x} \;=\; Ax \,+\, G(x,\zeta,t)\;, \tag{2.4}$$

where the linear term at the origin Ax was singled out for convenience (but without loss of generality). Here, $G : \mathbb{R}^n\times\mathcal{E}\times\mathbb{R}\to\mathbb{R}^n$ is assumed to be measurable with respect to t for fixed x,ζ, and of class C^1 with respect to x for fixed ζ, t. To ensure the compatibility between the dynamics of ζ and that of x (see (2.7) below), we also require the condition

$$G(x,\mathcal{S}_\tau(\zeta),t) = G(x,\zeta,t+\tau) \tag{2.5}$$

to hold for all $(x,\zeta,\tau,t)\in\mathbb{R}^n\times\mathcal{E}\times\mathbb{R}\times\mathbb{R}$. Now, if we suppose that to each $(x,\zeta)\in\mathbb{R}^n\times\mathcal{E}$ there is a locally integrable function $\phi_{x,\zeta}:\mathbb{R}\to\mathbb{R}^+$ satisfying $G(x,\zeta,t)\le\phi_{x,\zeta}(t)$ for all $t\in\mathbb{R}$, and that to each $\zeta\in\mathcal{E}$ there is a locally integrable function $\psi_\zeta : \mathbb{R} \to \mathbb{R}^+$ satisfying $\partial G/\partial x\,(x,\zeta,t) \leq \psi_\zeta(t)$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, then for each $\zeta \in \mathcal{E}$ the solution to (2.4) with initial condition $x(0) = x_0 \in \mathbb{R}^n$ uniquely exists for all $t \in \mathbb{R}$, *cf.* [11, Theorem 54, Proposition C.3.4, Proposition C.3.8]. Subsequently, denoting by

$$\widehat{x}(\tau, x_0, \zeta) \tag{2.6}$$

the value of this solution at time $t = \tau$, it follows from (2.5) that

$$\widehat{x}(t+\tau, x_0, \zeta) = \widehat{x}(t, \widehat{x}(\tau, x_0, \zeta), \mathcal{S}_\tau(\zeta)) \tag{2.7}$$

and thus

$$\widehat{\Phi}_t(x_0, \zeta) \quad = \quad (\, \widehat{x}(t, x_0, \zeta)\, ,\, \mathcal{S}_t(\zeta)\,) \tag{2.8}$$

defines a flow on $\mathbb{R}^n \times \mathcal{E}$, the group property being a consequence of (2.7) and of the group property of $\mathcal{S}_\tau$. We call $(\widehat{\Phi}_t)_{t\in\mathbb{R}}$ the flow of system (2.4).

We also define the partially linear flow L_t by the formula:

$$L_t(x_0, \zeta) \quad = \quad (\, e^{tA} x_0\, ,\, \mathcal{S}_t(\zeta)\,)\ ; \tag{2.9}$$

it is the flow of (2.4) when $G = 0$, and the whole point in this subsection is to give conditions on G for $\widehat{\Phi}_t$ and L_t to be topologically conjugate over $\mathbb{R}^n \times \mathcal{E}$.

We will assume throughout that the $n \times n$ matrix A is hyperbolic, hence it is similar to a block diagonal one:

$$A \sim \begin{pmatrix} A_e & 0 \\ 0 & A_l \end{pmatrix} \tag{2.10}$$

where A_e and A_l are $e \times e$ and $l \times l$ real matrices, with $e + l = n$, whose eigenvalues have strictly negative and strictly positive real parts respectively. Now, there exist Euclidean norms on $\mathbb{R}^e$ and $\mathbb{R}^l$ for which e^{A_e} and e^{-A_l} are strict contractions, because their eigenvalues have modulus strictly less than 1 and any square complex matrix is similar to an upper triangular one having the eigenvalues of the original matrix as diagonal entries while the remaining entries are arbitrarily small, see *e.g.* [1, ch.3, sec.22.4, Lemma 4]. Therefore, combining (2.10) with a suitable linear change of variable on each factor in $\mathbb{R}^n = \mathbb{R}^e \times \mathbb{R}^l$, we can write

$$A = E^{-1} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} E, \tag{2.11}$$

where E is some nonsingular $n \times n$ real matrix while P and Q are $e \times e$ and $l \times l$ real matrices such that e^P and e^{-Q} are strict contractions for the standard Euclidean norm:

$$c \;\triangleq\; \|e^P\|_{\mathrm{O}} < 1 \quad \text{and} \quad \frac{1}{d} \;\triangleq\; \|e^{-Q}\|_{\mathrm{O}} < 1\ , \tag{2.12}$$

where $\|.\|_{\mathrm{O}}$ designates the familiar operator norm of a matrix. Subsequently, we define the real numbers

$$b_1 \;\triangleq\; \|e^{-P}\|_{\mathrm{O}} + \|e^{-Q}\|_{\mathrm{O}} \;=\; \frac{1}{d} + \|e^{-P}\|_{\mathrm{O}}\,, \tag{2.13}$$

$$c_1 \triangleq \left\|EAE^{-1}\right\|_{\mathrm{O}} = \max\{\|P\|_{\mathrm{O}}, \|Q\|_{\mathrm{O}}\}. \tag{2.14}$$

Besides the operator norm, we shall make use of another norm on real matrices, namely the Frobenius norm $\|.\|_{\mathrm{F}}$ which is the square root of the sum of the squares of the entries. Let us record the elementary inequalities, valid for any two real square matrices M, N :

$$\|M\|_{\mathrm{O}} \leq \|M\|_{\mathrm{F}}, \ \|MN\|_{\mathrm{F}} \leq \min\{\|M\|_{\mathrm{O}}\|N\|_{\mathrm{F}}, \|M\|_{\mathrm{F}}\|N\|_{\mathrm{O}}\}. \tag{2.15}$$

As usual, we keep the symbol $\|.\|$ to indicate the standard Euclidean norm on $I\!R^j$ irrespectively of j. Now, our main result is the following:

Theorem 2.1 *Let the hyperbolic matrix A and the numbers c, d, b_1 and c_1 be as in* (2.11)*,* (2.12)*,* (2.13) *and* (2.14)*. Assume that the topological space $\mathcal{E}$, its one-parameter group of homeomorphisms $(\mathcal{S}_\tau)$, and the map $G : I\!R^n \times \mathcal{E} \times I\!R \to I\!R^n$ satisfy the following conditions :*

- *Equation* (2.5) *holds for all $(x, \zeta, \tau, t) \in I\!R^n \times \mathcal{E} \times I\!R \times I\!R$.*
- *For fixed $\zeta \in \mathcal{E}$, the map $\tau \mapsto \mathcal{S}_\tau(\zeta)$ is Borel measurable $I\!R \to \mathcal{E}$, that is to say the inverse image of an open subset of $\mathcal{E}$ is measurable in $I\!R$.*
- *The map $x \mapsto G(x, \zeta, t)$ is continuously differentiable $I\!R^n \to I\!R^n$ for fixed $(\zeta, t) \in \mathcal{E} \times I\!R$, the map $t \mapsto G(x, \zeta, t)$ is measurable $I\!R \to I\!R^n$ for fixed $(x, \zeta) \in I\!R^n \times \mathcal{E}$, and to each $\zeta \in \mathcal{E}$ there are locally integrable functions $\phi_\zeta, \psi_\zeta : I\!R \to I\!R^+$ such that, for all $(x, t) \in I\!R^n \times I\!R$, one has :*

$$\|G(x, \zeta, t)\| \leq \phi_\zeta(t) , \qquad \|\frac{\partial G}{\partial x}(x, \zeta, t)\|_{\mathrm{F}} \leq \psi_\zeta(t) . \tag{2.16}$$

- *Defining the flow $\widehat{x}$ of* (2.4) *as in* (2.6)*, the map $(x_0, \zeta) \mapsto \widehat{x}(t, x_0, \zeta)$ is continuous $I\!R^n \times \mathcal{E} \to I\!R^n$ for fixed $t \in I\!R$.*
- *There are real numbers $M > 0$ and $\eta > 0$ such that*

$$\forall \zeta \in \mathcal{E} \quad , \quad \|\phi_\zeta\|_{L^1([0,1])} \leq M , \tag{2.17}$$
$$\|\psi_\zeta\|_{L^1([0,1])} \leq \eta ; \tag{2.18}$$

  Moreover, the number η in (2.18) *is so small that, putting*

$$\theta \triangleq \eta \, \|E\|_{\mathrm{O}} \, \|E^{-1}\|_{\mathrm{O}}$$

  and then

$$\alpha_1 \triangleq \theta e^{c_1} \left(1 + e^{\theta + c_1} (\theta + c_1)\right),$$

  one has

$$0 < b_1 \alpha_1 < 1 \quad \textit{and} \quad \alpha_1(1 + 1/d) + \max(c, 1/d) < 1 . \tag{2.19}$$

Then, there exists a homeomorphism

$$\mathcal{H} : \ I\!R^n \times \mathcal{E} \to I\!R^n \times \mathcal{E}$$

of the form

$$(x, \zeta) \mapsto \mathcal{H}(x, \zeta) = (H(x, \zeta), \zeta),$$

that conjugates $\widehat{\Phi}_t$ *defined in* (2.8) *to the partially linear flow* (2.9)*, namely* $\mathcal{H} \circ \widehat{\Phi}_t = L_t \circ \mathcal{H}$ *or, equivalently,*

$$H(\,\widehat{\Phi}_t(x,\zeta)\,) \quad = \quad e^{tA} H(x,\zeta) \tag{2.20}$$

for all $(t,x,\zeta) \in I\!R \times I\!R^n \times \mathcal{E}$*.*

To establish Theorem 2.1, we shall rely on two lemmas. The first one runs parallel to [8, chap. IX, lemma 8.3], and gives us sufficient conditions for perturbations of a map $(x,\zeta) \mapsto (Lx, \mathcal{S}_\tau(\zeta))$ to be topologically conjugate on $I\!R^n \times \mathcal{E}$, when τ is fixed and the linear map $L : I\!R^n \to I\!R^n$ is the product of a dilation and a contraction. This lemma is the mainspring of the proof, in that it will provide us with the desired conjugating $\mathcal{H}$ when applied to the flows (2.8) and (2.9) evaluated at $t = 1$ (this arbitrary value comes from the normalization of the constants c and d through (2.12)). The proof of the lemma is similar to that of [8, chap. IX, lemma 8.3], except that we need to keep track more carefully of uniqueness and continuity issues here; it uses the shrinking lemma on Lipschitz-small perturbations of hyperbolic linear maps, a classical device to build conjugating homeomorphisms that has many other applications, see [8, chap. IX, notes]. The reader will notice that the statement of the lemma redefines the constants c, d, b_1, and α_1 that were already fixed in the statement of Theorem 2.1. We allow ourself this minor incorrection, because we feel it helps following the argument since the lemma will be applied precisely with the previously defined constants.

Lemma 2.2 *Let us be given a homeomorphism* $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ *and two non-singular real matrices* C, D *of size* $e \times e$ *and* $l \times l$ *respectively, such that* $c = \|C\| < 1$ *and* $\frac{1}{d} = \|D^{-1}\| < 1$*.*

For $i = 1, 2$*, let* $Y_i : I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^e$ *and* $Z_i : I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^l$ *be two pairs of bounded continuous functions satisfying*

$$\max\{\|\Delta Y_i\|, \|\Delta Z_i\|\} \leq \alpha_1(\|\Delta y\| + \|\Delta z\|), \tag{2.21}$$

where ΔY_i *and* ΔZ_i *stand respectively for* $Y_i(y+\Delta y, z+\Delta z, \zeta) - Y_i(y,z,\zeta)$ *and* $Z_i(y+\Delta y, z+\Delta z, \zeta) - Z_i(y,z,\zeta)$*, and where* α_1 *is a constant such that, if we put* $a = \|C^{-1}\|$ *and* $b_1 = a + 1/d$*, then* $0 < b_1\alpha_1 < 1$ *and* $\alpha_1(1 + 1/d) + \max(c, 1/d) < 1$*. If we define for* $i = 1, 2$ *the maps*

$$\begin{array}{rccl} T_i : & I\!R^e \times I\!R^l \times \mathcal{E} & \to & I\!R^e \times I\!R^l \times \mathcal{E} \\ & (y,z,\zeta) & \mapsto & (Cy + Y_i(y,z,\zeta), Dz + Z_i(y,z,\zeta), \mathcal{T}(\zeta)), \end{array}$$

then there exists a unique map $R_0 : I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^e \times I\!R^l \times \mathcal{E}$ *of the form*

$$R_0(y,z,\zeta) = (H_0(y,z,\zeta), \zeta) \tag{2.22}$$

such that:

- $H_0(y,z,\zeta) - (y,z)$ *is bounded on* $I\!R^e \times I\!R^l \times \mathcal{E}$*,*
- *one has the commuting relation:*

$$R_0 T_1 = T_2 R_0. \tag{2.23}$$

Moreover, R_0 *is then necessarily a homeomorphism of* $I\!R^e \times I\!R^l \times \mathcal{E}$*.*

The second lemma that we need in order to prove Theorem 2.1 is of technical nature and ensures that, under the hypotheses stated in that proposition, we can indeed apply Lemma 2.2 to the flow (2.8) evaluated at $t = 1$. Recalling from (2.6) the definition of $\widehat{x}$, it will be convenient to define a map $\Xi : I\!\!R \times I\!\!R^n \times \mathcal{E} \to I\!\!R^n$ by the equation :

$$\widehat{x}(t, x_0, \zeta) \quad = \quad \exp(tA)\, x_0 + \Xi(t, x_0, \zeta)\ . \tag{2.24}$$

Thus the map Ξ capsulizes the deviation of the flow of (2.4) from the flow of the linearized equation $\dot{x} = Ax$.

Lemma 2.3 *Under the assumptions of Theorem 2.1, the map Ξ defined by (2.24) is bounded on $[0,1] \times I\!\!R^n \times \mathcal{E}$, it is of class C^1 with respect to x_0 for fixed t, ζ, and it satisfies, for all (t, x_0, ζ) in $[0,1] \times I\!\!R^n \times \mathcal{E}$, the inequality :*

$$\|\frac{\partial \Xi}{\partial x_0}(t, x_0, \zeta)\|_{\mathrm{F}} \ \leq\ \eta\, e^{\|A\|_{\mathrm{O}}} \left(1 + e^{\eta + \|A\|_{\mathrm{O}}}\, (\eta + \|A\|_{\mathrm{O}})\right). \tag{2.25}$$

Assuming Lemma 2.2 and Lemma 2.3 for a while, let us proceed immediately with the proof of Theorem 2.1.

Proof of Theorem 2.1. Performing on $I\!\!R^n$ the change of variables $x \mapsto E\,x$ and taking (2.15) into account, we may assume upon replacing M by $M\|E\|_{\mathrm{O}}$ in (2.17) and η by θ in (2.18) that $E = I_n$, the identity matrix of size n. Then $c_1 = \|A\|_{\mathrm{O}}$ and the right-hand side of (2.25) is just α_1. Moreover (2.11) expresses that A assumes a block-diagonal form, according to which we block-decompose the flow $\widehat{\Phi}_t(x_0, \zeta)$ defined by (2.8) into

$$\begin{pmatrix} y_0 \\ z_0 \\ \zeta \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} e^{tP} y_0 + Y(t, y_0, z_0, \zeta) \\ e^{tQ} z_0 + Z(t, y_0, z_0, \zeta) \\ \mathcal{S}_t(\zeta) \end{pmatrix} \tag{2.26}$$

where $(y_0^T, z_0^T)^T$ is the natural partition of $x_0 \in I\!\!R^n \sim I\!\!R^e \times I\!\!R^l$, and where Y and Z are respectively the first e and the last l components of the map Ξ defined in (2.24). Still taking into account the block decomposition induced by (2.11) where $E = I_n$, the partially linear flow L_t defined by (2.9) in turn splits into

$$\begin{array}{rlcl} L_t : & I\!\!R^e \times I\!\!R^d \times \mathcal{E} & \to & I\!\!R^e \times I\!\!R^d \times \mathcal{E} \\ & (y_0, z_0, \zeta) & \mapsto & (\exp(Pt)\, y_0, \exp(Qt)\, z_0, \mathcal{S}_t(\zeta)). \end{array}$$

We shall apply Lemma 2.2 with $\mathcal{T} = \mathcal{S}_1$ to $T_1 = \widehat{\Phi}_1$ and $T_2 = L_1$, that is to say we choose $C = e^P$, $D = e^Q$, $Y_2 = 0$, $Z_2 = 0$, and we define Y_1 and Z_1 by $Y_1(y, z, \zeta) = Y(1, y, z, \zeta)$ and $Z_1(y, z, \zeta) = Z(1, y, z, \zeta)$ where Y, Z are as in (2.26). The hypotheses on C and D are satisfied by (2.12), while the hypotheses on Y_2 and Z_2 are trivially met. As to Y_1 and Z_1, we observe that:

- their continuity, i.e. the continuity of $(x_0, \zeta) \mapsto \Xi(1, x_0, \zeta)$, follows *via* (2.24) from the continuity of $(x_0, \zeta) \mapsto \widehat{x}(1, x_0, \zeta)$ which is part of the hypotheses (see point 4 in the statement of the proposition);
- their boundedness, i.e. the boundedness of $(x_0, \zeta) \mapsto \Xi(1, x_0, \zeta)$, follows from Lemma 2.3;

- the inequalities on the Lipschitz constants of Y_1 and Z_1 required in Lemma 2.2 follow from the mean-value theorem and Lemma 2.3, equation (2.25), granted (2.19), (2.15), and the triangle inequality.

Therefore Lemma 2.2 does apply, providing us with a homeomorphism of $\mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} = \mathbb{R}^n \times \mathcal{E}$ of the form $R_0 = H_0 \times \mathrm{id}$, which is such that $H_0(x, \zeta) - x$ is bounded on $\mathbb{R}^n \times \mathcal{E}$ and, in addition, such that

$$R_0 \circ \widehat{\Phi}_1 \;=\; L_1 \circ R_0. \tag{2.27}$$

Equation (2.27) expresses that H_0 conjugates the flow $\widehat{\Phi}_t(x, \zeta))$ to the partially linear flow L_t at time $t = 1$, whereas we want these flows to be conjugate at any time t. For this, we use the same averaging trick (originally due to S. Sternberg) as in [8, chap. IX, sec. 9], namely we define $H : \mathbb{R}^n \times \mathcal{E} \to \mathbb{R}^n$ by the integral formula:

$$H(x, \zeta) = \int_0^1 e^{-rA} H_0(\widehat{\Phi}_r(x, \zeta))\, dr \tag{2.28}$$

where H_0, being the first factor of R_0, satisfies by virtue of (2.27):

$$H_0(\widehat{\Phi}_1(x, \zeta)) \;=\; e^A H_0(x, \zeta). \tag{2.29}$$

We need of course show that (2.28) is well-defined. Firstly, let us check that the integrand is a measurable function of r. As H_0 is continuous $\mathbb{R}^n \times \mathcal{E} \to \mathbb{R}^n$, this reduces to showing that the map

$$r \mapsto \widehat{\Phi}_r(x, \zeta) = (\, \widehat{x}(r, x, \zeta)\, ,\, \mathcal{S}_r(\zeta)\,) \tag{2.30}$$

is measurable $\mathbb{R} \mapsto \mathbb{R}^n \times \mathcal{E}$. Now, the map $r \mapsto \widehat{x}(r, x, \zeta)$ is *a fortiori* measurable since it is absolutely continuous, and the map $r \mapsto \mathcal{S}_r(\zeta)$ is also measurable by assumption (see point 2 in the statement of the proposition). Hence the inverse image under (2.30) of an open rectangle is measurable in $\mathbb{R}$. But any open subset of $\mathbb{R}^n \times \mathcal{E}$ is a countable union of open rectangles because $\mathbb{R}^n$ has a countable basis of open neighborhoods, and this establishes the measurability of (2.30). Secondly, the integrand in (2.28) is bounded, for $\|H_0(\widehat{\Phi}_r(x, \zeta)) - \widehat{x}(r, x, \zeta)\|$ is majorized uniformly with respect to r, x, and ζ since $H_0(x, \zeta) - x$ is bounded on $\mathbb{R}^n \times \mathcal{E}$ by the properties of R_0, while the continuous function $r \mapsto \widehat{x}(r, x, \zeta)$ is bounded for fixed x and ζ on the compact set $[0, 1]$. Therefore, the integral on the right-hand side of (2.28) indeed exists.

Observe now that $H(x, \zeta) - x$ is also bounded on $\mathbb{R}^n \times \mathcal{E}$. Indeed, by definition of $\widehat{\Phi}_r$ *via* (2.8) and of Ξ *via* (2.24), we can write

$$\begin{aligned} H(x, \zeta) - x \;&=\; \int_0^1 e^{-rA} \big(H_0(\, \widehat{x}(r, x, \zeta)\, ,\, \mathcal{S}_r(\zeta)\,) - \widehat{x}(r, x, \zeta)\, \big)\, dr \\ &+\; \int_0^1 e^{-rA}\, \Xi(r, x, \zeta)\, dr, \end{aligned} \tag{2.31}$$

and since both integrals on the right-hand side are bounded (the first because $H_0(x, \zeta) - x$ is bounded on $\mathbb{R}^n \times \mathcal{E}$ and the second because Ξ is bounded on $[0, 1] \times \mathbb{R}^n \times \mathcal{E}$ by Lemma 2.3), we get the desired boundedness of $H(x, \zeta) - x$. Next, *we claim that* (2.20) *holds*, and once we have proved this the proposition will follow because, specializing (2.20) to $t = 1$, we shall conclude by the uniqueness part of Lemma 2.2 that $H \times \mathrm{id} = R_0$ and therefore that R_0, which is a homeomorphism of $I\!R^n \times \mathcal{E}$ with the desired form, will meet $R_0 \circ \widehat{\Phi}_t = L_t \circ R_0$, not just for $t = 1$ as we knew already but in fact for all t. Thus it will be possible to take $\mathcal{H} = R_0$.

To establish the claim, we use the group property of the flow to write

$$e^{-tA} H(\,\widehat{\Phi}_t(x,\zeta)\,) \;=\; \int_0^1 e^{-(t+r)A} H_0(\widehat{\Phi}_{t+r}(x,\zeta)\,)\,dr,$$

and we set $t + r = \tau$ to convert the above integral into

$$\int_t^{t+1} e^{-\tau A} H_0(\widehat{\Phi}_\tau(x,\zeta)\,)\,\mathrm{d}\tau \;=\; \int_t^1 \ldots\ \mathrm{d}\tau + \int_1^{t+1} \ldots\ \mathrm{d}\tau, \tag{2.32}$$

where the dots indicate that the integrand is repeated in each integral. Now, putting $\lambda = \tau - 1$, the last integral in the right-hand side becomes

$$\int_0^t e^{-(\lambda+1)A} H_0(\widehat{\Phi}_{\lambda+1}(x,\zeta)\,)\,d\lambda \;=\; \int_0^t e^{-\lambda A} H_0(\widehat{\Phi}_\lambda(x,\zeta)\,)\,d\lambda,$$

where we have used the group property of the flow again together with (2.29). Plugging this into (2.32), we recover back $\int_0^1 e^{-tA} H_0(\widehat{\Phi}_t(x,\zeta)\,)dt$ on the right-hand side, so that finally $e^{-tA} H \circ \widehat{\Phi}_t \;=\; H$ *as claimed.* □

Let us now tie the loose ends in the proof of Theorem 2.1 by establishing Lemma 2.3 and Lemma 2.2.

Proof of Lemma 2.3 ¿From (2.4) and (2.24), we see that $t \mapsto \Xi(t, x_0, \zeta)$ is the solution to

$$\dot{\xi}(t) = A\xi(t) + G\left(\xi(t) + e^{tA}x_0\,,\,\zeta\,,\,t\right)$$

with initial condition $\xi(0) = 0$. Since $\|G(x,\zeta,t)\|$ is bounded by $\phi_\zeta(t)$ with $\|\phi_\zeta\|_{L^1([0,1])} \le M$ by (2.16) and (2.17), we get by integration that

$$\|\xi(t)\| \le M + \|A\|_{\mathrm{O}} \int_0^t \|\xi(s)\|\,d|s|, \qquad t \in [0,1],$$

so by the Bellman-Gronwall lemma (*cf* Lemma B.1 in Appendix (B)) :

$$\|\xi(t)\| \le M\left(1 + |t|\,\|A\|_{\mathrm{O}}\, e^{|t|\,\|A\|_{\mathrm{O}}}\right), \qquad t \in [0,1].$$

This entails that Ξ is bounded on $[0,1] \times I\!R^n \times \mathcal{E}$.

To prove (2.25), we consider for fixed x_0, ζ the matrix-valued function $R(t) = \frac{\partial \widehat{x}}{\partial x_0}(t, x_0, \zeta)$, whose existence and continuity with respect to x_0 for fixed t, ζ depend on (2.16), (2.17) and (2.18) (*cf* Proposition B.2 in Appendix B), inducing in turn the existence and continuity with respect to x_0 of $Q(t) = \frac{\partial \Xi}{\partial x_0}(t, x_0, \zeta)$ *via* (2.24). The variational equation for $\frac{\partial \widehat{x}}{\partial x_0}$ (see again Proposition B.2 in Appendix B) yields :

$$\dot{R}(t) \;=\; \left[A + \frac{\partial G}{\partial x}(\widehat{x}(t,x_0,\zeta),\zeta,t)\right] R(t)\;, \quad R(0) \;=\; I_n\;,$$

and, since $R(t) = Q(t) + e^{tA}$ by (2.24), we have that

$$\dot{Q}(t) = \left[A + \frac{\partial G}{\partial x}\big(\widehat{x}(t, x_0, \zeta), \zeta, t\big)\right] Q(t) + \frac{\partial G}{\partial x}\big(\widehat{x}(t, x_0, \zeta), \zeta, t\big)\, e^{tA}\,,\ Q(0) = 0.$$

Put $\rho(t) = \|Q(t)\|_{\mathrm{F}}$. Due to the definition of the Frobenius norm, $\rho(t)$ is locally absolutely continuous and, by the Cauchy-Schwarz inequality, one has $\dot{\rho}(t) \leq \|\dot{Q}(t)\|_{\mathrm{F}}$. Thus, the differential equation satisfied by $Q(t)$ together with (2.16) yield :

$$\dot{\rho} \;\; \leq \;\; (\psi_\zeta(t) + \|A\|_{\mathrm{O}})\, \rho(t) \; + \; \psi_\zeta(t)\, e^{|t|\, \|A\|_{\mathrm{O}}}, \qquad \rho(0) = 0,$$

where we have used (2.15) and the elementary fact that $\|e^{tA}\|_{\mathrm{O}} \leq e^{|t|\,\|A\|_{\mathrm{O}}}$. Integrating this inequality and applying the Bellman-Gronwall lemma (*cf* Lemma B.1 in Appendix (B)) while taking (2.18) into account leads us to

$$\rho(t) \;\; \leq \;\; \eta\, e^{|t|\,\|A\|_{\mathrm{O}}} \left(1 + e^{\eta + |t|\,\|A\|_{\mathrm{O}}}\, (\eta + |t|\, \|A\|_{\mathrm{O}})\right), \qquad t \in [0, 1].$$

By definition of ρ, this implies (2.25). $\square$

Proof of Lemma 2.2 If we endow $I\!R^e \times I\!R^l$ with the norm $\|(y, z)\| = \|y\| + \|z\|$, it follows from (2.21) that, for fixed $(y, z, \zeta) \in I\!R^e \times I\!R^l \times \mathcal{E}$, the map $T_{y,z,\zeta} : I\!R^e \times I\!R^l \to I\!R^e \times I\!R^l$ defined by

$$T_{y,z,\zeta}(y', z') = (C^{-1}y, D^{-1}z) - \big(C^{-1}Y_1(y', z', \zeta)\,,\, D^{-1}Z_1(y', z', \zeta)\big)$$

is a shrinking map with shrinking constant $b_1\alpha_1 < 1$, whose fixed point is the unique $(\bar{y}, \bar{z}) \in I\!R^e \times I\!R^l$ satisfying $T_1(\bar{y}, \bar{z}, \zeta) = (y, z, \mathcal{T}(\zeta))$. In addition, it holds that $(\bar{y}, \bar{z}) = \lim_{k\to\infty} T^k_{y,z,\zeta}(y', z')$ for any (y', z'), and this classically implies that $(\bar{y}, \bar{z})$ is continuous with respect to y, z, and ζ. Indeed, the continuity of Y_1 and Z_1 entails that $T_{y,z,\zeta}(y', z')$ is continuous with respect to y, z and ζ for fixed y', z'. Therefore, if we write $\bar{y}(y, z, \zeta)$, $\bar{z}(y, z, \zeta)$ to emphasize the functional dependence, and if we choose y_0, z_0, ζ_0 together with $\varepsilon > 0$, there is a neighborhood $\mathcal{V}_0$ of (y_0, z_0, ζ_0) in $I\!R^e \times I\!R^l \times \mathcal{E}$ such that $(y, z, \zeta) \in \mathcal{V}_0$ implies :

$$\|T_{y,z,\zeta}(\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - (\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\| =$$

$$\|T_{y,z,\zeta}(\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - T_{y_0,z_0,\zeta_0}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\| < \varepsilon.$$

Consequently, for $(y, z, \zeta) \in \mathcal{V}_0$, we have by the shrinking property that

$$\big\|\big(\bar{y}(y, z, \zeta), \bar{z}(y, z, \zeta)\big) - \big(\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)\big)\big\|$$
$$= \|\lim_{k\to\infty} T^k_{y,z,\zeta}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - (\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\|$$
$$\leq \sum_{k=0}^{\infty} \|T^{k+1}_{y,z,\zeta}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - T^k_{y,z,\zeta}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\|$$
$$\leq \frac{\varepsilon}{1 - b_1\alpha_1}$$

which implies the desired continuity. Then, $(x, y) \mapsto (\bar{y}(y, z, \zeta), \bar{z}(y, z, \zeta))$ is, for fixed ζ, the inverse of the concatenation of the first two components of T_1, and it is continuous with respect to (x, y), and to ζ. Moreover, we see from the definition of $T_{y,z,\zeta}$ and the fixed point property of $\bar{y}, \bar{z}$ that

$$(\bar{y}, \bar{z}) = (C^{-1}y, D^{-1}z) - (C^{-1}Y_1(\bar{y}, \bar{z}, \zeta)\,,\, D^{-1}Z_1(\bar{y}, \bar{z}, \zeta))$$

and, since Y_1 and Z_1 are continuous and bounded, this makes for a relation of the form

$$(\bar{y}(y, z, \zeta), \bar{z}(y, z, \zeta)) = (C^{-1}y + \hat{Y}_1(y, z, \zeta), D^{-1}z + \hat{Z}_1(y, z, \zeta))$$

where $\hat{Y}_1$, $\hat{Z}_1$ are in turn continuous and bounded on $I\!R^e \times I\!R^l \times \mathcal{E}$ with values in $I\!R^e$ and $I\!R^l$ respectively. All this yields the existence of an inverse for the map T_1 itself, namely

$$T_1^{-1}(y, z, \zeta) = (C^{-1}y + \hat{Y}_1(y, z, \mathcal{T}^{-1}(\zeta)), D^{-1}z + \hat{Z}_1(y, z, \mathcal{T}^{-1}(\zeta)), \mathcal{T}^{-1}(\zeta)). \tag{2.33}$$

Let us now seek the map H_0 in (2.22) in the prescribed form, namely

$$H_0(y, z, \zeta) = (\, y + \Lambda(y, z, \zeta)\,,\, z + \Theta(y, z, \zeta)\,), \tag{2.34}$$

where the unknowns are bounded maps Λ and Θ with values in $I\!R^e$ and $I\!R^l$ respectively. Using (2.33), one checks easily that (2.23) is equivalent to the following pair of equations:

$$\begin{aligned} \Lambda \;&=\; C\left[\hat{Y}_1 + \Lambda(T_1^{-1})\right] \qquad (2.35)\\ &\quad + Y_2\left(C^{-1}y + \hat{Y}_1 + \Lambda(T_1^{-1}), D^{-1}z + \hat{Z}_1 + \Theta(T_1^{-1}), \mathcal{T}^{-1}(\zeta)\right), \end{aligned}$$

$$\Theta = D^{-1}[Z_1 + \Theta(Cy + Y_1, Dz + Z_1, \mathcal{T}(\zeta)) - Z_2(y + \Lambda, z + \Theta, \zeta)], \tag{2.36}$$

where the argument of $\Lambda, \Theta, Y_i, Z_i, \hat{Y}_i, \hat{Z}_i, T_1^{-1}$, when omitted, is always (y, z, ζ). The existence of Λ and Θ will follow from another application of the shrinking lemma, this time in the space $\mathcal{B}$ of bounded functions $I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^e \times I\!R^l$ endowed with a suitable norm. More precisely, letting (Λ_1, Θ_1) denote an arbitrary member of $\mathcal{B}$ acting coordinate-wise as $(y, z, \zeta) \mapsto (\Lambda_1(y, z, \zeta), \Theta_1(y, z, \zeta))$ where Λ_1 and Θ_1 are bounded $I\!R^e$ and $I\!R^l$-valued functions respectively, we define its norm to be

$$|||(\Lambda_1, \Theta_1)|||_+ = |||\Lambda_1||| + |||\Theta_1|||,$$

where $|||.|||$ indicates the *sup* norm of a map $I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^k$, irrespectively of k; this makes $(\mathcal{B}, |||.|||_+)$ into a a Banach space. Now, to each $(\Lambda_1, \Theta_1) \in \mathcal{B}$, we can associate another member (Λ_2, Θ_2) of $\mathcal{B}$ where $\Lambda_2 : I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^e$ and $\Theta_2 : I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^l$ are defined by

$$\begin{aligned} \Lambda_2 \;&=\; C\left[\hat{Y}_1 + \Lambda_1(T_1^{-1})\right] \qquad (2.37)\\ &\quad + Y_2\Big(C^{-1}y + \hat{Y}_1 + \Lambda_1(T_1^{-1}), D^{-1}z + \hat{Z}_1 + \Theta_1(T_1^{-1}), \mathcal{T}^{-1}(\zeta)\Big), \end{aligned}$$

$$\Theta_2 = D^{-1}\Big[Z_1 + \Theta_1(Cy + Y_1, Dz + Z_1, \mathcal{T}(\zeta)) - Z_2(y + \Lambda_1, z + \Theta_1, \zeta)\Big], \tag{2.38}$$

the argument (y, z, ζ) being omitted again for simplicity. The fact that (Λ_2, Θ_2) is indeed well-defined and belongs to $\mathcal{B}$ is a consequence of the preceding part of the proof. Consistently designating by a subscript 2 the effect of the right hand-side of (2.37) an (2.38) on some initial map, itself denoted with a subscript 1, we see from (2.21)) by inspection on (2.37) and (2.38) that, if (Λ_1, Θ_1) and (Λ_1', Θ_1') are two members of $\mathcal{B}$, then

$$|||\Lambda_2 - \Lambda_2'||| \leq c\,|||\Lambda_1 - \Lambda_1'||| + \alpha_1|||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+, \qquad (2.39)$$

$$|||\Theta_2 - \Theta_2'||| \leq \frac{1}{d}\left(|||\Theta_1 - \Theta_1'||| + \alpha_1|||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+\right). \qquad (2.40)$$

Adding up (2.39) and (2.40), we obtain

$$\begin{aligned} &|||(\Lambda_2 - \Lambda_2', \Theta_2 - \Theta_2')|||_+ \\ &\quad \leq \quad [\alpha_1(1+1/d) + \max(c, 1/d)]\ |||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+ \\ &\quad = \quad \alpha\,|||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+ \end{aligned}$$

where by assumption $\alpha < 1$. This means that $(\Lambda_1, \Theta_1) \mapsto (\Lambda_2, \Theta_2)$ is a shrinking map on $\mathcal{B}$ whose fixed point (Λ, Θ) provides us with the unique bounded solution to (2.35) and (2.36). Equivalently, if H_0 is defined through (2.34) and R_0 through (2.22), then R_0 is the unique map $I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^e \times I\!R^l \times \mathcal{E}$ of the form (H, id), where id is the identity map on $\mathcal{E}$, such that $H - (y, z) \in \mathcal{B}$ and such that the commuting relation (2.23) holds. It remains for us to show that R_0 is a homeomorphism. For this, notice first that R_0 is continuous, because H_0 turns out to be continuous: indeed, iterating the formulas (2.37) and (2.38) starting from any initial pair (Λ_1, Θ_1) yields a sequence of maps converging to (Λ, Θ) in $\mathcal{B}$, and if the initial pair is continuous (we may for instance choose the zero map) so is every member of the sequence hence also the limit since $|||.|||_+$ induces on $\mathcal{B}$ the topology of uniform convergence. Next, if we switch the roles of T_1 and T_2, the above argument provides us with a continuous map $R_0' : I\!R^e \times I\!R^l \times \mathcal{E} \to I\!R^e \times I\!R^l \times \mathcal{E}$ of the form (H', id)) with $H' - (y, z) \in \mathcal{B}$, satisfying $R_0' T_2 = T_1 R_0'$. Then, the composed map $R = R_0' R_0$ satisfies $R T_1 = T_1 R$, and since it is again of the form (H'', id)) with $H'' - (y, z) \in \mathcal{B}$, we get $R = \mathrm{id}$ by the uniqueness part of the previous proof. Similarly $R_0 R_0' = \mathrm{id}$, so that finally R_0 is invertible with continuous inverse R_0' hence a homeomorphism. □

3 Grobman-Hartman theorems for control systems

We consider a control system of the form:

$$\dot{x} = f(x, u)\ , \quad x \in I\!R^n\ , \quad u \in I\!R^m\ , \qquad (3.1)$$

and we suppose that $f(0, 0) = 0$, *i.e.* we work around an equilibrium point that we choose to be the origin without loss of generality. We assume that f is continuous, and throughout we also make the hypothesis that $\partial f/\partial x(x, u)$ exists and is jointly continuous with respect to (x, u). Subsequently, we single out the linear part of f by consistently setting

$A = \frac{\partial f}{\partial x}(0,0)$, so that (3.1) can be rewritten as

$$\begin{array}{rcl} \dot{x} & = & Ax \, + \, P(x,u) \\ \text{with} \quad P(0,0) & = & \dfrac{\partial P}{\partial x}(0,0) \; = \; 0 \; . \end{array} \tag{3.2}$$

If in addition f happens to be continuously differentiable with respect to u as well, we set $B = \frac{\partial f}{\partial u}(0,0)$ and we further expand (3.2) into

$$\begin{array}{rcl} \dot{x} & = & Ax \, + \, Bu \, + \, F(x,u) \\ \text{with} \quad F(0,0) & = & \dfrac{\partial F}{\partial x}(0,0) \; = \; \dfrac{\partial F}{\partial u}(0,0) \; = \; 0 \; . \end{array} \tag{3.3}$$

Since (3.3) is derived under the stronger hypothesis that f is of class C^1 with respect to both x and u, one would expect stronger results to hold in this case. We want to stress that, deceptively enough, local linearization of (3.3) will turn out to be a consequence of local linearization of (3.2) although the latter was derived without differentiability requirement with respect to u. This is due to the – even more surprising – fact that (3.2) will be locally conjugate to the *non controlled* system $\dot{x} = Ax$, that is to say the influence of the control can be entirely assigned to the linearizing homeomorphism. Compare Theorems 3.1 and 3.3, and see also Remark 3.8.

3.1 Prescribed dynamics for the control

We investigate in this subsection the situation where, in system (3.1), the control function $u(t)$ is itself the output of a dynamical system of the form:

$$\begin{array}{rcl} \dot{\zeta} & = & g(\zeta), \\ u & = & h(\zeta), \end{array} \tag{3.4}$$

where $\zeta(t) \in I\!\!R^q$, while $g : I\!\!R^q \to I\!\!R^q$ is locally Lipschitz continuous and $h : I\!\!R^q \to I\!\!R^m$ is continuous with, say, $h(0) = 0$. In particular, $u(t)$ is entirely determined by the finite-dimensional data $\zeta(0)$ and, from the control viewpoint, this is a particular instance of feed-forward on system (3.1) by system (3.4) where the input may only consist of Dirac delta functions.

Assume first that f is of class C^1 with respect to x and u so that (3.3) holds. Plugging (3.4) into the latter yields an ordinary differential equation in $I\!\!R^{n+q}$:

$$\begin{array}{rcl} \dot{x} & = & Ax \, + \, Bh(\zeta) \, + \, F(x, h(\zeta)), \\ \dot{\zeta} & = & g(\zeta). \end{array} \tag{3.5}$$

To motivate the developments to come, observe that if g is continuously differentiable with $g(0) = 0$, if A and $\partial g / \partial \zeta(0)$ are hyperbolic, and if h is continuously differentiable, then we can apply the standard Grobman-Hartman theorem on ordinary differential equations to conclude that the flow of (3.5) is topologically conjugate, *via* a local homeomorphism $(x, \zeta) \mapsto (z, \xi)$ around $(0,0)$, to that of

$$\begin{pmatrix} \dot{z} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & B\frac{\partial h}{\partial \zeta}(0) \\ 0 & \frac{\partial g}{\partial \zeta}(0) \end{pmatrix} \begin{pmatrix} z \\ \xi \end{pmatrix} .$$

However, the hyperbolicity requirement on $\partial g/\partial\zeta(0)$ is more stringent than it seems. Indeed, it is often desirable to study non-trivial steady behaviors, which usually entail oscillatory controls. This is why we rather seek a transformation of the form $(x,\zeta) \mapsto (H(x,\zeta),\zeta)$ that linearizes the first equation in (3.5) but preserves the second one. This can be done, as asserted by the following result which does not require hyperbolicity nor even continuous differentiability on g.

Theorem 3.1 *Suppose in system* (3.5) *that* $g : \mathbb{R}^q \to \mathbb{R}^q$ *is locally Lipschitz continuous, that* $h : \mathbb{R}^q \to \mathbb{R}^m$ *is continuous with* $h(0) = 0$*, that* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is continuously differentiable with* $F(0,0) = \partial F/\partial x(0,0) = 0$*, and that* A *is hyperbolic. Then, there exist two neighborhoods* V *and* W *of 0 in* $\mathbb{R}^n$ *and* $\mathbb{R}^q$ *respectively, and a map* $H : V \times W \to \mathbb{R}^n$ *with* $H(0,0) = 0$*, such that*

$$\begin{array}{rccc} H \times Id: & V \times W & \to & \mathbb{R}^n \times W \\ & (x,\zeta) & \mapsto & (H(x,\zeta),\zeta) \end{array}$$

is a homeomorphism from $V \times W$ *onto its image that conjugates* (3.5) *to*

$$\begin{array}{rcl} \dot{z} & = & Az \, + \, Bh(\zeta), \\ \dot{\zeta} & = & g(\zeta). \end{array} \tag{3.6}$$

Remark 3.2 *In Theorem 3.1 (resp. Theorem 3.3 to come), we assume for convenience that all the functions involved, namely* F *(resp.* P*),* g*, and* h*, are globally defined. However, since the conclusion is local with respect to* x *and* ζ*, the same holds when these functions are only defined locally on a neighborhood of the origin, as a partition of unity argument immediately reduces the local version to the present one.*

Although it looks natural, the above theorem deserves one word of caution for the homeomorphism H depends heavily on g and h, and in a rather intricate manner. In fact, it is possible to entirely incorporate the influence of the control into the change of variables, so as to obtain a statement in which the term $Bh(\zeta)$ does not even appear in the transformed system. This will follow from Theorem 3.3 to come, for which we no longer assume in (3.1) that f is differentiable with respect to u. Accordingly, we plug (3.4) into (3.2) rather than (3.3), and we obtain instead of (3.5) the following ordinary differential equation in $\mathbb{R}^{n+q}$:

$$\begin{array}{rcl} \dot{x} & = & = \, Ax \, + \, P(x,h(\zeta)), \\ \dot{\zeta} & = & g(\zeta), \end{array} \tag{3.7}$$

whose flow will be denoted by $(t,x_0,\zeta_0) \mapsto (x(t,x_0,\zeta_0),\zeta(t,\zeta_0))$.

Theorem 3.3 *Suppose in system* (3.7) *that* $g : \mathbb{R}^q \to \mathbb{R}^q$ *is locally Lipschitz continuous, that* $h : \mathbb{R}^q \to \mathbb{R}^m$ *is continuous with* $h(0) = 0$*, that* $P(x,u)$ *is continuous* $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *with* $P(0,0) = 0$*, that* $\partial P/\partial x$ *exists and is continuous* $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$ *with* $\partial P/\partial x(0,0) = 0$*, and that* A *is hyperbolic. Then, there exist two neighborhoods* V *and* W *of 0 in* $\mathbb{R}^n$ *and* $\mathbb{R}^q$ *respectively, and a map* $H : V \times W \to \mathbb{R}^n$ *with* $H(0,0) = 0$*, such that*

$$\begin{array}{rccc} H \times Id: & V \times W & \to & \mathbb{R}^n \times W \\ & (x,\zeta) & \mapsto & (H(x,\zeta),\zeta) \end{array}$$

is a homeomorphism from $V \times W$ onto its image that conjugates (3.7) *to*

$$\begin{array}{rcl} \dot{z} & = & Az, \\ \dot{\zeta} & = & g(\zeta), \end{array} \tag{3.8}$$

i.e. for all t, x_0, ζ_0 such that $(x(\tau, x_0, \zeta_0), \zeta(\tau, \zeta_0)) \in V \times W$ for all $\tau \in [0, t]$ (or $[t, 0]$ if $t < 0$), one has

$$H(x(t, x_0, \zeta_0), \zeta(t, \zeta_0)) = e^{tA} H(x_0, \zeta_0).$$

Theorem 3.1 is a consequence of Theorem 3.3 because the latter implies that (3.5) and (3.6) are both conjugate to (3.8). As to Theorem 3.3 itself, we will show that it is a consequence of Theorem 2.1. This will require an elementary lemma enabling us to normalize the original control system. To state the lemma, we fix, once and for all, a smooth function $\rho : [0, +\infty) \to [0, 1]$ such that

$$\left.\begin{array}{ccl} \forall t, & & |\dot{\rho}(t)| < 3, \\ 0 \leq t \leq \frac{1}{2} & \Rightarrow & \rho(t) = 1, \\ \frac{1}{2} < t < 1 & \Rightarrow & 0 < \rho(t) < 1, \\ 1 \leq t & \Rightarrow & \rho(t) = 0, \end{array}\right\} \tag{3.9}$$

and we associate to any map $\beta : I\!R^n \times I\!R^m \to I\!R^n$ a family of functions $G_s : I\!R^n \times I\!R^m \to I\!R^n$, indexed by a real number $s > 0$, using the formula:

$$G_s(x, u) \triangleq \rho\left(\frac{\|x\|^2}{s^2}\right) \beta(x, u). \tag{3.10}$$

Since the context will always make clear which β is involved, our notation does not explicitly indicate the dependency of G_s on the map β. The symbol $\|.\|$, in the statement of the lemma, denotes the norm, not only of a vector, but also of a matrix; the result does not depend on a specific choice of this norm. Also, $B(x, r)$ stands for the open ball of radius r, centered at x, in any Euclidean space.

Lemma 3.4 *Let $\beta(x, u)$ be continuous $I\!R^n \times I\!R^m \to I\!R^n$ and $\partial\beta/\partial x$ continuously exist $I\!R^n \times I\!R^m \to I\!R^{n \times n}$, with $\beta(0, 0) = \partial\beta/\partial x(0, 0) = 0$. Then $G_s(x, u)$ defined by* (3.10) *is in turn continuous and continuously differentiable with respect to x for every $s > 0$, and to each $\eta > 0$ there exist $\sigma > 0$ and $\theta > 0$ such that*

$$\forall (x, u) \in I\!R^n \times B(0, \theta), \quad \|\frac{\partial G_\sigma}{\partial x}(x, u)\| \leq \eta . \tag{3.11}$$

Proof. For the proof, we use the standard Euclidean norm on $I\!R^n$, $I\!R^m$, and the familiar operator norm on matrices. Clearly G_s is continuous and continuously differentiable with respect to x for every $s > 0$, and we have :

$$\frac{\partial G_s}{\partial x}(x, u) = \rho\left(\frac{\|x\|^2}{s^2}\right) \frac{\partial \beta}{\partial x}(x, u) + \frac{2}{s^2} \rho'\left(\frac{\|x\|^2}{s^2}\right) \beta(x, u)\, x^T, \tag{3.12}$$

where x^T is the transpose of x. Since β is continuously differentiable and $\partial\beta/\partial x\,(0, 0) = 0$, we get for $s > 0$ small enough that $\|\partial\beta/\partial x\,(x, u)\| < \eta/14$ as soon as $\|x\|, \|u\| < s$. Let σ be an s with this property. Since β

is continuous with $\beta(0,0)=0$, we can in turn pick θ with $0<\theta\leq\sigma$ such that $\|\beta(0,u)\|<\eta\sigma/12$ whenever $\|u\|<\theta$. Altogether, we get that

$$\left.\begin{array}{l}\|x\|<\sigma\\ \|u\|<\theta\end{array}\right\}\ \Rightarrow\ \left\{\begin{array}{l}\|\frac{\partial\beta}{\partial x}(x,u)\|<\frac{\eta}{14}\,,\\ \|\beta(0,u)\|<\frac{\eta\sigma}{12}\,.\end{array}\right. \tag{3.13}$$

Now, we need only check (3.11) when $\|x\|<\sigma$ for otherwise G_σ is identically zero; therefore we restrict ourselves to pairs (x,u) where $\|x\|<\sigma$ and $\|u\|<\theta$. On this domain, we get from (3.13) and the mean value theorem that

$$\|\beta(x,u)\|\ \leq\ \frac{\eta}{14}\,\sigma\ +\ \frac{\eta\sigma}{12}\ =\ \frac{13\eta\sigma}{84}.$$

Using this together with (3.13) and the inequalities $|\rho|\leq 1$, $\|\rho'\|\leq 3$, as well as $\|x^T\|<\sigma$, formula (3.12) with $s=\sigma$ yields :

$$\|\frac{\partial G_\sigma}{\partial x}(x,u)\|\ \leq\ \frac{\eta}{14}\ +\ \frac{6}{\sigma^2}\,\frac{13\eta\sigma}{84}\,\sigma=\eta. \qquad \square$$

Proof of Theorems 3.1 and 3.3. We already mentioned that Theorem 3.1 is a consequence of Theorem 3.3. To establish the latter, consider the following "renormalized" version of (3.7) :

$$\begin{array}{rcl}\dot{x}&=&Ax\ +\ \rho\left(\frac{\|x\|^2}{\sigma^2}\right)\ P\left(x,\rho\left(\frac{\|h(\zeta)\|}{\theta}\right)h(\zeta)\right)\right),\\ \dot{\zeta}&=&\rho(\|\zeta\|)\,g(\zeta),\end{array} \tag{3.14}$$

where ρ is as in (3.9) and where σ, θ are strictly positive real numbers to be adjusted shortly. Because the flows of (3.14) and (3.7) do coincide as long as $\|x\|<\sigma/\sqrt{2}$, $\|\zeta\|<1/2$, $\|h(\zeta)\|<\theta/2$, and since these inequalities define a neighborhood $(0,0)$ in $I\!\!R^n\times I\!\!R^m$ by the continuity of h and the fact that $h(0)=0$, it is enough to prove the theorem when (3.7) *gets replaced by* (3.14) for *some pair* of strictly positive σ,θ. To this effect, we shall apply Theorem 2.1 with $\mathcal{E}=I\!\!R^q$ endowed with the flow of $\rho(\|\zeta\|)\,g(\zeta)$, namely $\mathcal{S}_\tau(\zeta_0)$ is the value at $t=\tau$ of the solution to the second equation in (3.14) whose value at $t=0$ is ζ_0, and with

$$G(x,\zeta,t)=\rho\left(\frac{\|x\|^2}{\sigma^2}\right)\ P\left(x,\rho\left(\frac{\|h(\mathcal{S}_t(\zeta))\|}{\theta}\right)h(\mathcal{S}_t(\zeta))\right).$$

We now proceed to check that the assumptions of Theorem 2.1 are fulfilled if σ and θ are properly chosen. Firstly, since g is locally Lipschitz continuous while ρ is smooth with compact support on $[0,+\infty)$, we see that $\zeta\mapsto\rho(\|\zeta\|)\,g(\zeta)$ is a bounded Lipschitz continuous vector field on $I\!\!R^q$ hence it has a globally defined flow, which is continuous by Lemma A.1. This tells us that $(\tau,\zeta)\mapsto\mathcal{S}_\tau(\zeta)$ is continuous $I\!\!R\times I\!\!R^q\to I\!\!R^q$, so $\mathcal{S}_\tau$ is indeed a one-parameter group of homeomorphisms on $I\!\!R^q$ and $\tau\mapsto\mathcal{S}_\tau(\zeta)$ is certainly Borel measurable since it is even continuous. The continuity of $(\tau,\zeta)\mapsto\mathcal{S}_\tau(\zeta)$ also makes it clear that $G(x,\zeta,t)$ is continuous and continuously differentiable with respect to x granted the continuity of h, the smoothness of ρ, and the fact that P itself is continuous and continuously differentiable with respect to the first variable. *A fortiori* then, $x\mapsto G(x,\zeta,t)$ is continuously differentiable and $t\mapsto G(x,\zeta,t)$ is measurable.

Secondly, observe since ρ is bounded by 1 and vanishes outside $[0,1]$ that $\|\rho(\theta^{-1}\|u\|)u\| < \theta$ for all $u \in I\!\!R^m$, consequently G takes values in the smallest ball centered at 0 that contains $P(B(0,\sigma), B(0,\theta))$; this last set is relatively compact by the continuity of P hence G is bounded. The same argument shows that $\partial G/\partial x$ is also bounded, in other words we can choose ϕ_ζ and ψ_ζ to be suitable constant functions in (2.16), independently of ζ. In particular, (2.17) and (2.18) will hold. Moreover, if we set $\beta(x,u) = P(x,u)$, we have with the notations of (3.10) that

$$G(x,\zeta,t) = G_\sigma\left(x, \rho\left(\frac{\|h(\mathcal{S}_t(\zeta))\|}{\theta}\right) h(\mathcal{S}_t(\zeta))\right). \qquad (3.15)$$

Since $\rho(\theta^{-1}\|h(v)\|)h(v)$ lies in $B(0,\theta)$ for all $v \in I\!\!R^q$ so in particular for $v = \mathcal{S}_t(\zeta)$, we deduce from (3.15) and Lemma 3.4 that $\partial G/\partial x$ can be made uniformly small for suitable σ and θ. That is to say, the number η in (2.18) can be made arbitrarily small upon choosing σ and θ adequately, in particular we can meet (2.19).

Thirdly, the condition (2.16) that we just proved to hold (actually with constant functions ϕ_ζ and ψ_ζ independent of ζ) entails that the first equation in (3.14) has a unique solution given initial conditions $x(0)$ and $\zeta(0)$ (*cf* for instance [11, Theorem 54, Proposition C.3.4, Proposition C.3.8]) and, since the same holds true for the second equation as was pointed out when we defined $\mathcal{S}_\tau(\zeta)$, we conclude that the whole vector field in the right hand-side of (3.14) has a flow on $I\!\!R^{n+q} = I\!\!R^n \times I\!\!R^q$, which is continuous by Lemma A.1. As $\widehat{x}$, defined in (2.6), is nothing but the projection of this flow onto the first factor $I\!\!R^n$, we conclude that $(\tau, x_0, \zeta) \mapsto \widehat{x}(\tau, x_0, \zeta)$ is continuous. Finally, notice that (2.5) is immediate from the group property of $\mathcal{S}_\tau$. Having verified all the hypotheses of Theorem 2.1, we apply the latter to conclude the proof of Theorem 3.3. □

3.2 Control systems viewed as flows

In [5], a general way of associating a flow to a control system is proposed, based on the action of the time shift on some functional space of inputs. Before giving the proper framework for our results, let us first carry out a few measure-theoretic preliminaries.

For arbitrary exponents $p \in [1,\infty]$, we denote by $\mathcal{L}^p(I\!\!R, I\!\!R^m)$, or simply by $\mathcal{L}^p$ for short, the space of measurable functions $\Upsilon : I\!\!R \to I\!\!R^m$ such that

$$\begin{aligned} \|\Upsilon\|_p &= \left(\int_{I\!\!R} \|\Upsilon(t)\|^p \, dt\right)^{1/p} < \infty \qquad \text{if } p < \infty, \\ \|\Upsilon\|_\infty &= \operatorname*{ess.\,sup.}_{t \in I\!\!R} \|\Upsilon(t)\| < \infty \qquad \text{if } p = \infty. \end{aligned}$$

In the above, measurability and summability were implicitly understood with respect to Lebesgue measure. The same definitions can of course be made for any positive measure. We only consider measures defined on the same σ-algebra as Lebesgue measure (namely the completion of the Borel σ-algebra with respect to sets of Lebesgue measure zero). We explicitly indicate the dependence on the measure μ of the corresponding functional spaces and norms by writing $\mathcal{L}^{p,\mu}$ and $\|.\|_{p,\mu}$.

Remark 3.5 *If μ is a positive measure on $I\!R$ as above, and if μ and Lebesgue measure are mutually absolutely continuous, then for any Lebesgue measurable (hence also μ-measurable) function Υ it holds that $\|\Upsilon\|_\infty = \|\Upsilon\|_{\infty,\mu}$. Indeed, we have that $\|\Upsilon\|_\infty \leq \alpha$ if, and only if, the set E_α of those $x \in I\!R$ for which $\|\Upsilon\|(x) > \alpha$ has Lebesgue measure zero. Since the latter holds if, and only if, $\mu(E_\alpha) = 0$, it is equivalent to require that $\|\Upsilon\|_{\infty,\mu} \leq \alpha$ as announced.*

For any $p \in [1, \infty]$ and $\tau \in I\!R$, we define the time shift $\Theta_\tau : \mathcal{L}^p \to \mathcal{L}^p$ by

$$\Theta_\tau(\Upsilon)(t) = \Upsilon(\tau + t) \ . \tag{3.16}$$

It is well known that, for fixed $\Upsilon \in \mathcal{L}^p$, the map $\tau \mapsto \Theta_\tau(\Upsilon)$ is continuous $I\!R \to \mathcal{L}^p$ if $1 \leq p < \infty$ [10, Theorem 9.5]. When $p = \infty$ it is no longer so, but the map is at least Borel measurable:

Lemma 3.6 *For fixed $\Upsilon \in \mathcal{L}^\infty$, consider the map $T_\Upsilon : I\!R \to \mathcal{L}^\infty$ defined by $T_\Upsilon(\tau) = \Theta_\tau(\Upsilon)$. If V is open in $\mathcal{L}^p$, then $T_\Upsilon^{-1}(V)$ is measurable in $I\!R$.*

Proof. Set for simplicity $T_\Upsilon(\tau) = \Upsilon_\tau$, and fix arbitrarily $v \in \mathcal{L}^\infty$ together with $\varepsilon > 0$. It is enough to show that the set

$$E = \{\tau \in I\!R; \ \ \|\Upsilon_\tau - v\|_\infty > \varepsilon\}$$

is measurable. Let μ be the measure on $I\!R$ such that $d\mu(t) = dt/(1 + t^2)$. In view of Remark 3.5, we can replace $\|.\|_\infty$ by $\|.\|_{\infty,\mu}$ in the definition of E. Now, since μ is finite, the functions Υ_τ and v belong to $\mathcal{L}^{1,\mu}$, which is to the effect that

$$\lim_{p\to\infty} \|\Upsilon_\tau - v\|_{p,\mu} = \|\Upsilon_\tau - v\|_{\infty,\mu}, \tag{3.17}$$

see *e.g.* [10, Chap. 3, Ex.4]. In particular, if we let

$$E_{p,\mu} = \{\tau \in I\!R; \ \ \|\Upsilon_\tau - v\|_{p,\mu} > \varepsilon\},$$

we deduce from (3.17) that

$$E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_{j,\mu}$$

where k and j assume integral values, so we are left to prove that $E_{j,\mu}$ is measurable. But since translating the argument is a continuous operation $I\!R \to \mathcal{L}^{p,\mu}$ when $p < \infty$ [10, Theorem 9.5][1], each $E_{j,\mu}$ is in fact open in $I\!R$ thereby proving the lemma. $\square$

Endowed with $\|.\|_p$-balls as neighborhoods of 0, the set $\mathcal{L}^p$ is a topological vector space but it is not Hausdorff; identifying functions that agree almost everywhere, we obtain the familiar Lebesgue space L^p of

[1]The proof is given there for Lebesgue measure only, but it does carry over *mutatis mutandis* to any complete regular Borel measure on $I\!R$, hence in particular to μ.

equivalence classes of $\mathcal{L}^p$-functions; it is a Banach space, whose norm, still denoted by $\|.\|_p$, is induced by $\|.\|_p$ defined in $\mathcal{L}^p$, and whose topology coincides with the quotient topology arising from the canonical map $\mathcal{L}^p \to L^p$. The time shift $\Theta_\tau : \mathcal{L}^p \to \mathcal{L}^p$ defined by (3.16) induces a well defined map $\Theta_\tau : L^p \to L^p$. In what follows, results are stated in terms of L^p, but we do make use of $\mathcal{L}^p$ for the proof because point-wise evaluation makes no sense in L^p.

Let us now come back to our control system, namely (3.2), which is obtained from (3.1) by singling out the linear term in x around the equilibrium $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$. This time, however, we emphasize the functional dependence on the control by writing

$$\dot{x} \;=\; Ax \;+\; P(x, \Upsilon(t)), \tag{3.18}$$

where, as in the preceding subsection, $P : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and has continuous derivative with respect to the first argument $\frac{\partial P}{\partial x}$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$. We fix some $p \in [1,\infty]$ and we consider controls $\Upsilon \in L^p(\mathbb{R}, \mathbb{R}^m)$. Thus, when $p < \infty$, we shall have to handle unbounded values for $\Upsilon(t)$, and this will necessitate an extra assumption. Namely, if $1 \leq p < \infty$, we assume that to each compact set $K \subset \mathbb{R}^n$, there are positive constants $c_1(K)$, $c_2(K)$ such that

$$\|P(x,u)\| + \|\frac{\partial P}{\partial x}(x,u)\| \leq c_1(K) + c_2(K)\,\|u\|^p, \quad (x,u) \in K \times \mathbb{R}^m, \tag{3.19}$$

where we agree, for definiteness, that the norm of a matrix is the operator norm. Classical results imply (see *e.g.* [11, Theorem 54, Proposition C.3.4]) that the solution to (3.18) uniquely exists on some maximal time interval once $x(0) = x_0$ and $\Upsilon \in L^p$ are chosen. This solution we denote by

$$t \mapsto x(t, x_0, \Upsilon)\ .$$

This allows one to define a flow on $\mathbb{R}^n \times \mathcal{L}^p$, or on $\mathbb{R}^n \times L^p$, the flow at time τ being given by

$$(x_0, \Upsilon) \;\mapsto\; (\,x(\tau, x_0, \Upsilon)\,,\, \Theta_\tau(\Upsilon)\,)\ . \tag{3.20}$$

The main result in this subsection is the theorem below. It is of purely *open loop* character, that is to say the linearizing transformation $(x, \Upsilon) \mapsto (z, \Upsilon)$ operates at a functional level where z depends not only on x, but also on the whole input function $\Upsilon : \mathbb{R} \mapsto \mathbb{R}^m$. That type of linearization is intriguing in the authors' opinion, but its usefulness in control is not clear unless the structure of the transformation is thoroughly understood. Unfortunately our method of proof does not reveal much in this direction, which may deserve further study.

Theorem 3.7 *Suppose in* (3.18) *that* $P(x,u)$ *is continuous* $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *with* $P(0,0) = 0$*, that* $\partial P/\partial x$ *exists and is continuous* $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$ *with* $\partial P/\partial x(0,0) = 0$*, and that* A *is hyperbolic. Let* $p \in [1,\infty]$*, and,if* $p < \infty$*, assume that, to each compact set* $K \subset \mathbb{R}^n$*, there are positive constants* $c_1(K)$*,* $c_2(K)$ *such that* (3.19) *holds. Then, there exist* *two neighborhoods V and W of 0 in $\mathbb{R}^n$ and $L^p(\mathbb{R},\mathbb{R}^m)$ respectively, and a map $H : V \times W \to \mathbb{R}^n$ with $H(0,0) = 0$, such that*

$$\begin{array}{rccc} H \times Id : & V \times W & \to & \mathbb{R}^n \times W \\ & (x,\Upsilon) & \mapsto & (H(x,\Upsilon),\Upsilon) \end{array} \tag{3.21}$$

is a homeomorphism from $V \times W$ onto its image that conjugates (3.18) to

$$\dot{z} = Az, \tag{3.22}$$

i.e. for all $(t, x_0, \Upsilon) \in \mathbb{R} \times \mathbb{R}^n \times L^p(\mathbb{R},\mathbb{R}^m)$ such that $(x(\tau, x_0, \Upsilon), \Upsilon) \in V \times W$ for all $\tau \in [0,t]$ (or $[t,0]$ if $t < 0$) one has

$$H(x(t, x_0, \Upsilon)) = e^{tA} H(x_0, \Upsilon). \tag{3.23}$$

Remark 3.8 *The above theorem parallels Theorem 3.3 of section 3.1, in that we initially wrote $\dot{x} = f(x,u)$ in the form (3.2), assuming that f is continuously differentiable with respect to x, to finally conclude, under suitable hypotheses, that (3.18) is locally conjugate in some appropriate sense to the non-controlled linear system (3.22). We might as well have stated an analog to Theorem 3.1 where, assuming this time that f is of class C^1, we write $\dot{x} = f(x,u)$ in the form (3.3) with hyperbolic A, assuming in addition if $p < \infty$ that for any compact $K \subset \mathbb{R}^n$ one has*

$$\|F(x,u)\| + \|\frac{\partial F}{\partial x}(x,u)\| \le c_1(K) + c_2(K)\,\|u\|^p, \quad (x,u) \in K \times \mathbb{R}^m, \tag{3.24}$$

to conclude that $\dot{x} = Ax + B\Upsilon(t) + F(x,\Upsilon(t))$ is conjugate via $z = H(x,\Upsilon)$ to $\dot{z} = Az + B\Upsilon(t)$, where $H \times Id$ is a local homeomorphism at 0×0 of $\mathbb{R}^n \times L^p$. Again, although the presence of the control term $B\Upsilon(t)$ in the linearized equation makes it look more natural, the result we just sketched is a logical consequence of Theorem 3.7 just like Theorem 3.1 was a consequence of Theorem 3.3.

To prove Theorem 3.7 we shall again apply Theorem 2.1 to a suitably normalized version of (3.18), the normalization step depending on the following lemma which stands analogous to Lemma 3.4 in the $\mathcal{L}^p$ context. For convenience, we denote below by $B_{\mathcal{L}^p}(v,r)$ the ball centered at v of radius r in $\mathcal{L}^p$, and by $\mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}^m)$ (or simply $\mathcal{L}^1_{loc}$ if no confusion can arise) the space of *locally integrable functions*, namely those whose restriction to any compact $K \subset \mathbb{R}$ belongs to $\mathcal{L}^1(K,\mathbb{R}^m)$.

Lemma 3.9 *Let $\beta(x,u)$ be continuous $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\partial\beta/\partial x$ continuously exist $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$, with $\beta(0,0) = \partial\beta/\partial x(0,0) = 0$. Assume for some $p \in [1,\infty)$ that, to each compact set $K \subset \mathbb{R}^n$, there are positive constants $c_1(K)$, $c_2(K)$ such that*

$$\|\beta(x,u)\| + \|\frac{\partial \beta}{\partial x}(x,u)\| \le c_1(K) + c_2(K)\,\|u\|^p, \quad (x,u) \in K \times \mathbb{R}^m. \tag{3.25}$$

Then, G_s being as in (3.10), it holds that for every $s > 0$ and any $\Upsilon \in \mathcal{L}^p(\mathbb{R},\mathbb{R}^m)$ we have $G_s(x,\Upsilon) \in \mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}^n)$ and $\partial G_s/\partial x(x,\Upsilon) \in \mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}^{n\times n})$ for fixed $x \in \mathbb{R}$. Moreover, to each $\eta > 0$ there exist $\sigma > 0$ and $\theta > 0$ such that G_σ satisfies :

$$\begin{array}{lll} \forall \Upsilon \in B_{\mathcal{L}^p}(0,\theta)\,, & \text{there exists} \;\; \psi_\Upsilon \in \mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}) \;\; \text{such that} & \\ \|\psi_\Upsilon\|_{L^1[0,1]} \le \eta & \text{and,} \;\; \forall x \in \mathbb{R}^n\,, \;\; \|\frac{\partial G_\sigma}{\partial x}(x,\Upsilon)\| \le \psi_\Upsilon. & \end{array} \tag{3.26}$$

Proof. For fixed $x \in I\!R$, it is clear from (3.25) that both $G_s(x,\Upsilon)$ and $\partial G_s/\partial x(x,\Upsilon)$ belong to $\mathcal{L}^1_{loc}(I\!R, I\!R^n)$ when $\Upsilon \in \mathcal{L}^p(I\!R, I\!R^m)$, measurability being ensured by the continuity of G_s and $\partial G_s/\partial x$. To prove (3.26), first apply Lemma 3.4 to find $\sigma > 0$ and $\theta_0 > 0$ such that

$$\forall (x,u) \in I\!R^n \times B(0,\theta_0)\,, \quad \|\frac{\partial G_\sigma}{\partial x}(x,u)\| \ \leq\ \eta/2\,. \tag{3.27}$$

Next, let $c_1 = c_1(\overline{B}(0,\sigma))$ and $c_2 = c_2(\overline{B}(0,\sigma))$ be defined after (3.25), and observe that

$$\forall (x,u) \in I\!R^n \times I\!R^m, \quad \|\partial G_\sigma/\partial x\,(x,u)\| \leq (1+6/\sigma)(c_1 + c_2\|u\|^p) \tag{3.28}$$

because when $\|x\| < \sigma$ this follows from (3.12), (3.25) and the fact that $|\rho'| < 3$, whereas G_σ vanishes anyway when $\|x\| \geq \sigma$. Introduce now the set

$$E_{\Upsilon,\theta_0} = \{t \in [0,1], \quad \|\Upsilon\| < \theta_0\}. \tag{3.29}$$

Letting $\psi_\Upsilon(t) = \eta/2$ for $t \in E_{\Upsilon,\theta_0}$ and $\psi_\Upsilon(t) = (1+6/\sigma)(c_1 + c_2\|\Upsilon(t)\|^p)$ otherwise, it is clear that $\psi_\Upsilon \in \mathcal{L}^1_{loc}(I\!R, I\!R)$ and it follows from (3.29), (3.27), and (3.28) that $\|\partial G_{\sigma_0}/\partial x(x,\Upsilon)\| \leq \psi_\Upsilon$ for any $x \in I\!R^n$. In another connection, let ν be the measure on $I\!R$ given by $d\nu(t) = |\Upsilon(t)|^p dt$. By absolute continuity of ν with respect to Lebesgue measure, there is $\varepsilon > 0$ such that

$$\int_E \|\Upsilon\|^p\,dt < \frac{\eta}{4c_2(1+6/\sigma)} \qquad \text{as soon as } |E| < \varepsilon, \tag{3.30}$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset I\!R$ [10, Theorem 6.11]. Pick $\theta > 0$ so small that

$$\frac{\theta}{\theta_0} < \max\left\{\varepsilon, \frac{\eta}{4(1+6/\sigma)c_1}\right\}. \tag{3.31}$$

Then, if $\|\Upsilon\|_p < \theta$, the set $[0,1]\setminus E_{\Upsilon,\theta_0}$ has measure at most θ/θ_0 hence, by definition of ψ_Υ, we get in view of (3.30) and (3.31) the estimate :

$$\|\psi_\Upsilon\|_{L^1[0,1]} \leq \frac{\eta}{2} + \frac{\theta}{\theta_0}(1+6/\sigma)c_1 + \frac{\eta}{4}$$

which is less that $\eta/2 + \eta/4 + \eta/4 = \eta$ by (3.31) again, as desired. $\square$

We are now in position to establish Theorem 3.7.

Proof of Theorem 3.7 For the proof we can replace L^p by $\mathcal{L}^p$, because if we find a local homeomorphism of $I\!R^n \times \mathcal{L}^p$ at 0×0, of the form $\widetilde{H} \times Id$, that conjugates (3.18) to (3.22), the fact that $x(\tau, x_0, \Upsilon)$ depends only on the equivalence class of Υ in L^p implies that the same holds true for $\widetilde{H}(x_0,\Upsilon)$, and therefore $\widetilde{H} \times Id$ will induce a quotient map $H \times Id$ around 0×0 in $I\!R^n \times L^p$ that is still a local homeomorphism by definition of the quotient topology. To prove the $\mathcal{L}^p$ version, we consider the following "re-normalization" of (3.18) :

$$\dot{x} \ =\ Ax \ +\ \rho\left(\frac{\|x\|^2}{\sigma^2}\right)\ P\left(x, \rho\left(\frac{\|\Upsilon\|_p}{\theta}\right)\Upsilon\right), \tag{3.32}$$

where ρ is as in (3.9) and σ, θ are strictly positive real numbers to be fixed. Because the right-hand sides of (3.32) and (3.18) agree as long as $\|x\| < \sigma/\sqrt{2}$ and $\|\Upsilon\|_p < \theta/2$ which defines a neighborhood $(0,0)$ in $\mathbb{R}^n \times \mathcal{L}^p$, it is enough to prove the theorem when (3.18) *gets replaced by* (3.32) for *some pair* σ, θ. To this effect, we shall apply Theorem 2.1 with $\mathcal{E} = \mathcal{L}^p$, endowed with the one-parameter group of transformations $\mathcal{S}_\tau = \Theta_\tau$ defined by (3.16), and

$$G(x,\zeta,t) = \rho\left(\frac{\|x\|^2}{\sigma^2}\right)\ P\left(x, \rho\left(\frac{\|\zeta\|_p}{\theta}\right)\zeta(t)\right).$$

Let us check that the assumptions of Theorem 2.1 are met if σ and θ are suitably chosen.

Firstly, it is obvious that $\mathcal{S}_\tau$ is continuous (hence a homeomorphism since $\mathcal{S}_\tau^{-1} = \mathcal{S}_{-\tau}$) because it is a linear isometry of $\mathcal{L}^p$. In addition, $\tau \mapsto \mathcal{S}_\tau(\zeta)$ is certainly Borel measurable, because it is even continuous when $p < \infty$ [10, Theorem 9.5] while Lemma 3.6 applies if $p = \infty$.

Secondly, it follows immediately from the assumptions on P and the smoothness of ρ that $G(x,\zeta,t)$ is continuously differentiable with respect to x for fixed ζ and t, while the measurability of $t \mapsto G(x,\zeta,t)$ follows from the continuity of P and the measurability of ζ. To prove the existence of ϕ_ζ and ψ_ζ in (2.16), we distinguish between $p < \infty$ and $p = \infty$. If $p < \infty$, by (3.19) and the fact that ρ is bounded by 1 and vanishes outside $[0,1]$, a valid choice for ϕ_ζ is

$$\phi_\zeta(t) = c_1(\,\overline{B}(0,\sigma)\,) + c_2(\,\overline{B}(0,\sigma)\,)\ \rho^p\left(\frac{\|\zeta\|_p}{\theta}\right)\|\zeta(t)\|^p$$

and, since by the properties of ρ we have that

$$\left\|\rho\left(\frac{\|\zeta\|_p}{\theta}\right)\zeta\right\|_p \leq \theta \quad \forall \zeta \in \mathcal{L}^p, \ \ 1 \leq p \leq \infty, \tag{3.33}$$

it follows that (2.17) is met with

$$M = c_1(\,\overline{B}(0,\sigma)\,) + c_2(\,\overline{B}(0,\sigma)\,)\ \theta^p.$$

As to ψ_ζ, observe if we set $\beta(x,u) = P(x,u)$ that, with the notations of (3.10), one has

$$G(x,\zeta,t) = G_\sigma\left(x, \rho\left(\frac{\|\zeta\|_p}{\theta}\right)\zeta(t)\right), \tag{3.34}$$

so Lemma 3.9 ensures the existence of ψ_ζ and also that the number η in (2.18) can be made arbitrarily small upon choosing σ and θ adequately; in particular we can meet (2.19). If $p = \infty$, we let

$$\phi_\zeta(t) = \sup_{x\in\overline{B}(0,\sigma)} \left\|P\left(x, \rho\left(\frac{\|\zeta\|_p}{\theta}\right)\zeta(t)\right)\right\|$$

so that the first half of (2.16) holds by the properties of ρ. By (3.33) we also have that

$$\|\phi_\zeta\|_\infty \ \leq \sup_{(x,u)\in\overline{B}(0,\sigma)\times\overline{B}(0,\theta)} \|P(x,u)\|, \tag{3.35}$$

so that $\phi_\zeta \in \mathcal{L}^\infty(I\!R, I\!R)$ hence it is locally summable, and the right-hand side of (3.35) may serve as M in (2.17). As to ψ_ζ, observe that (3.34) still holds for $p = \infty$, again with $\beta(x,u) = P(x,u)$, so we can set

$$\psi_\zeta(t) = \sup_{x\in\overline{B}(0,\sigma)} \left\| \frac{\partial G_\sigma}{\partial x} \left(x, \rho\left(\frac{\|\zeta\|_\infty}{\theta}\right)\zeta(t)\right)\right\|,$$

and using (3.33) once more we get

$$\|\psi_\zeta\|_\infty \leq \sup_{(x,u)\in\overline{B}(0,\sigma)\times\overline{B}(0,\theta)} \left\| \frac{G_\sigma}{\partial x}(x,u)\right\|. \tag{3.36}$$

Thus $\psi_\zeta \in \mathcal{L}^\infty(I\!R, I\!R)$ hence it is locally summable, and applying Lemma 3.4 to the right-hand side of (3.36) shows that $\|\psi_\zeta\|_\infty$ can be made arbitrarily small upon choosing σ and θ adequately. Consequently η in (2.18) can be as small as we wish and in particular we can meet (2.19).

Thirdly, $t \mapsto \widehat{x}(t, x_0, \zeta)$ defined in (2.6) is just the solution to (3.32) corresponding to $\Upsilon = \zeta$ and $x(0) = x_0$, which uniquely exists for all t by (2.16), see *e.g.* [11, Theorem 54, Proposition C.3.4, Proposition C.3.8]. The continuity $I\!R^n \times \mathcal{L}^p \to I\!R^n$ of $(x_0, \zeta) \mapsto \widehat{x}(t, x_0, \zeta)$ is now ascertained by Proposition C.1, once it is observed that $F(x,u) = Ax + \rho(\|x\|^2/\sigma^2)P(x,u)$ satisfies the hypotheses of that proposition by (3.19) and the properties of ρ, and also that $Ax + G(x,\zeta,t)$ is the composition of F with the continuous map on $I\!R^n \times \mathcal{L}^p$ given by $(x,\zeta) \mapsto (x, \rho(\|\zeta\|_p/\theta)\zeta)$ (Proposition C.1 was actually proved for L^p controls, but nothing is to be changed if we work in $\mathcal{L}^p$).

Finally, notice that (2.5) is immediate by the very definition of Θ_τ. Thus we can apply Theorem 2.1 to conclude the proof of Theorem 3.7. □

Remark 3.10 *It should be noted that, unlike Theorems 3.1 and 3.3, Theorem 3.7 cannot be localized with respect to u when $p < \infty$. However, using a partition of unity argument, the result carries over to the case where, in* (3.18)*, the map P is only defined on $\mathcal{V} \times I\!R^m$ where $\mathcal{V}$ is a neighborhood of* 0 *in $I\!R^n$.*

In [5], particular attention is payed to the weak-* topology on $\mathcal{L}^\infty$ for the control space, because it makes the flow $\tau \mapsto \Theta_\tau(\Upsilon)$ continuous for fixed Υ. Subsequently, this reference focuses on systems that are affine in the control : $\dot{x} = X_0(x) + C(x)u$, where X_0 is a C^1 vector field on $I\!R^n$ and $C : I\!R^n \to I\!R^{n\times m}$ a C^1 matrix-valued function; the reason for this affine restriction is that it ensures, in the weak-* context, the sequential continuity of $(x_0, \Upsilon) \mapsto x(\tau, x_0, \Upsilon)$ for fixed τ, whenever the flow makes sense : this is easily deduced from the Ascoli-Arzela theorem and the fact that weak-* convergent sequences are norm-bounded [9, Theorem 2.5]. Although the continuity of the flow Θ was never a concern to us (only Borel measurability was required), it is natural in this connection to ask what happens with Theorem 3.7 if we endow L^∞ with the weak-* topology inherited from the (L^1, L^∞) duality. On the one hand, *in case one restricts his attention, as is done in [5], to a balanced,* weak-* compact *time-shift invariant subset of L^∞ containing* 0*, e.g. a ball $\bar{B}_{L^\infty}(0,r)$, then the conclusions of the theorem still hold if we equip the subset in question* *with the weak-* topology.* Indeed, the weak-* topology is metrizable on any compact set E because L^1 is separable [9, Theorems 3.16] and, since weak-* convergent sequences are norm- bounded, it follows if E is balanced that one can find a neighborhood of 0 in E which is included in $\bar{B}_{L^\infty}(0,\theta)$ for arbitrary small θ. In particular we can embed this neighborhood in W of Theorem 3.7, and then it only remains to show that (3.21) remains continuous if W is equipped with the weak-* topology; this in turn reduces *via* (3.23) to the already mentioned fact that $(x_0,\Upsilon) \mapsto x(\tau,x_0,\Upsilon)$ is sequentially continuous for fixed τ when the topology on Υ is the weak-* one. On the other hand, working weak-* with unrestricted controls in L^∞ raises serious difficulties, for no weak-* neighborhood in L^∞ can be norm-bounded. This results in the fact that, although Θ is now continuous, the domain of definition of the flow (3.20) may fail to be open : for instance the equation $\dot{x} = x + x^2\Upsilon(t)$ with initial condition $x(0) = x_0$, where x and Υ are real-valued, cannot have a solution on a fixed interval $[0,t]$ for every $(x_0,\Upsilon) \in B(0,r) \times \mathcal{W}_0$ if $\mathcal{W}_0$ is a weak-* neighborhood of 0 in $L^\infty(\mathbb{R},\mathbb{R})$. Therefore it is hopeless to build a local homeomorphism by integrating the flow as is done in the proof of Theorem 2.1, and the authors do not know what analog to Theorem 3.7 could be carried out in this context.

Remark 3.11 *The paper [4] considers transformations $\mathbb{R}^n \times L^\infty \to \mathbb{R}^n \times L^\infty$, using for the input space a topology on L^∞ which is intermediate between the weak-* and the strong one. There the structure of conjugating homeomorphisms is not* (3.21) *but rather a triangular form:*

$$(x,\Upsilon) \quad \mapsto \quad (\, H(x)\, ,\, F(x,\Upsilon)\,)$$

that combines what is called in this reference "topological static state feedback equivalence" and "topological state equivalence"[4, Definition 5]. We refer the interested reader to the original paper for a result on topological linearization of systems with two states and one control, using this type of transformation, under some global hypotheses.

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APPENDIX

A Two basic lemmas on ODEs

Throughout this section, we let $\mathcal{U}$ be an open subset of $\mathbb{R}^d$. We say that a continuous vector field $X : \mathcal{U} \to \mathbb{R}^d$ *has a flow* if the Cauchy problem $\dot{x}(t) = X(x(t))$ with initial condition $x(0) = x_0$ has a unique solution, defined for $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon = \varepsilon(x_0) > 0$. The flow of X at time t is denoted by X_t, in other words we have with the preceding notations that $X_t(x_0) = x(t)$. It is easy to see that the domain of definition of $(t, x) \mapsto X(t, x)$ is open in $\mathbb{R} \times \mathcal{U}$.

Lemma A.1 *If $X : \mathcal{U} \to \mathbb{R}^d$ is a continuous vector field that has a flow, the map $(t, x) \mapsto X_t(x)$ is continuous on the open subset of $\mathbb{R} \times \mathcal{U}$ where it is defined.*

Proof. This is an easy consequence of the Ascoli-Arzela theorem, and actually a special case of [8, chap. V, Theorem 2.1]. $\square$

Lemma A.2 *Assume that the sequence of continuous vector fields $X^k : \mathcal{U} \to \mathbb{R}^d$ converges to X, uniformly on compact subsets of $\mathcal{U}$, and that all the X^k as well as X itself have a flow. Suppose that $X_t(x)$ is defined for all $(t, x) \in [0, T] \times K$ with $T > 0$ and $K \subset \mathcal{U}$ compact. Then $X_t^k(x)$ is also defined on $[0, T] \times K$ for k large enough, and the sequence of mappings $(t, x) \mapsto X_t^k(x)$ converges to $(t, x) \mapsto X_t(x)$, uniformly on $[0, T] \times K$.*

Proof. By assumption,

$$K_1 = \{X_t(x);\ (t, x) \in [0, T] \times K\}$$

is a well-defined subset of $\mathcal{U}$ that contains K, and it is compact by Lemma A.1. Let K_0 be another compact subset of $\mathcal{U}$ whose interior contains K_1, and put $d(K_1, \mathcal{U} \setminus K_0) = \eta > 0$ where $d(E_1, E_2)$ indicates the distance between two sets E_1, E_2. From the hypothesis there is $M > 0$ such that $\|X^k\| \leq M$ on K_0 for all k, hence the maximal solution to $\dot{x(t)} = X^k(x(t))$ with initial condition $x(0) = x_0 \in K$ remains in K_0 as long as $t \leq \eta/2M$. Consequently the flow $(t, x) \mapsto X_t^k(x)$ is defined on $[0, \eta/2M] \times K$ for all k, with values in K_0. *We claim* that it is a bounded equicontinuous sequence of functions there. Boundedness is clear since these functions are K_0-valued, so we must show that, to every $(t, x) \in [0, \eta/2M] \times K$ and every $\varepsilon > 0$, there is $\alpha > 0$ such that $\|X^k(t', x') - X^k(t, x)\| < \varepsilon$ for all k as soon as $|t - t'| + \|x - x'\| < \alpha$. By the mean-value theorem and the uniform majorization $\|X^k(X_t^k(x))\| \leq M$, it is sufficient to prove this when $t = t'$. Arguing by contradiction, assume for some subsequence k_l and some sequence x_l converging to x in K that

$$\|X_t^{k_l}(x) - X_t^{k_l}(x_l)\| \geq \varepsilon \quad \text{for all } l \in \mathbb{N}. \tag{A.1}$$

Then, by Lemma A.1, the index k_l tends to infinity with l. Next consider the sequence of maps $F_l : [0, \eta/2M] \to K_0$ defined by $F_l(t) = X_t^{k_l}(x_l)$. Again, by the mean value theorem, it is a bounded equicontinuous family of functions and, by the Ascoli-Arzela theorem, it is relatively compact in the topology of uniform convergence (compare [8, chap. II, Theorem

3.2]). But if $\Phi : [0, \eta/2M] \to K_0$ is the uniform limit of some subsequence F_{l_j}, and since $X^{k_{l_j}}$ converges uniformly to X on K_0 as $j \to \infty$, taking limits in the relation

$$X_t^{k_{l_j}}(x_{l_j}) = x_{l_j} + \int_0^t X^{k_{l_j}}(X_s^{k_{l_j}}(x_{l_j}))\, ds$$

gives us

$$\Phi(t) = x + \int_0^t X(\Phi(s))\, ds$$

so that $\Phi(t) = X_t(x)$ since X has a flow. Altogether $F_l(t)$ converges uniformly to $X_t(x)$ on $[0, \eta/2M]$ because this is the only accumulation point, and then (A.1) becomes absurd. *This proves the claim.* From the claim it follows, using the Ascoli-Arzela theorem again, that the family of functions $(t, x) \mapsto X_t^k(x)$ is relatively compact for the topology of uniform convergence $[0, \eta/2M] \times K \to K_0$, and in fact it converges to $(t, x) \mapsto X_t(x)$ because, by the same limiting argument as was used to prove the claim, every accumulation point $\Phi(t, x)$ must be a solution to

$$\Phi(t, x) = x + \int_0^t X(s, \Phi(s, x))\, ds$$

hence for fixed x is an integral curve of X with initial condition x. In particular, by definition of K_1, we shall have that $d(X_t^k(x), K_1) < \eta/2$ for all $(t, x) \in [0, \eta/2M] \times K$ as soon as k is large enough. For such k the flow $(t, x) \mapsto X_t^k(x)$ will be defined on $[0, \eta/M] \times K$ with values in K_0, and we can repeat the whole argument again to the effect that $X_t^k(x)$ converges uniformly to $X_t(x)$ there. Proceeding inductively, we obtain after $[2TM/\eta] + 1$ steps at most that $(t, x) \mapsto X_t^k(x)$ is defined on $[0, T] \times K$ with values in K_0 for k large enough, and converges uniformly to $(t, x) \mapsto X_t(x)$ there, as was to be shown. $\square$

B The variational equation

Our goal in this appendix is to give a version of the classical variational equation for ordinary differential equations, in the not-so-classical case where the dependence on time is L^1 but possibly unbounded. Let us first recall the Bellman-Gronwall lemma in a form which is suitable for us.

Lemma B.1 (The Bellman-Gronwall Lemma) *Let w, ϕ, ψ be non-negative real-valued measurable functions on real interval $[0, T]$, such that ψ, ψw and $\psi\phi$ are in $L^1([0, T])$. If it holds that*

$$w(t) \le \phi(t) + \int_0^t \psi(s) w(s)\, ds \quad \text{for } t \in [0, T],$$

then it also holds that

$$w(t) \le \phi(t) + \int_0^t \phi(s)\psi(s)\, \exp\left(\int_s^t \psi(\xi) d\xi\right) ds \quad \text{for } t \in [0, T]. \qquad \text{(B.1)}$$

Proof. By the hypotheses $y(t) = \int_0^t \psi(s)w(s)\,ds$ is an absolutely continuous function of t satisfying

$$\dot{y}(s) - \psi(s)y(s) \le \phi(s)\psi(s) \quad \text{a. e. } s \in [0,T],$$

therefore $z(t) = y(t)\exp(-\int_0^t \psi(s)\,ds)$ is also absolutely continuous and satisfies

$$\dot{z}(s) \le \phi(s)\psi(s)\exp\left(-\int_0^s \psi(\tau)\,\mathrm{d}\tau\right) \quad \text{a. e. } s \in [0,T]. \tag{B.2}$$

Integrating (B.2) from 0 to t and multiplying by $\exp(\int_0^t \psi(s)\,ds)$ yields

$$y(t) \le \int_0^t \phi(s)\psi(s)\,\exp\left(\int_s^t \psi(\xi)d\xi\right)ds \quad \text{for } t \in [0,T],$$

from which (B.1) follows since $w(t) \le \phi(t) + y(t)$ by hypothesis, compare for instance [7, sec. 10.5.1.3]. □

Let us now consider a differential equation of the form

$$\dot{x} = X(x,t) \tag{B.3}$$

where the time-dependent vector field $X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies the following properties:

(i) for fixed $t \in \mathbb{R}$, the map $x \to X(x,t)$ is continuously differentiable $\mathbb{R}^n \to \mathbb{R}^n$;

(ii) for fixed $x \in \mathbb{R}^n$, the map $t \to X(x,t)$ is measurable $\mathbb{R} \to \mathbb{R}$;

(iii) for some $x_1 \in \mathbb{R}^n$ there is a measurable and locally integrable function $\alpha_{x_1} : \mathbb{R} \to \mathbb{R}^+$ such that

$$\|X(x_1,t)\| \le \alpha_{x_1}(t), \quad \text{for all } t \in \mathbb{R};$$

(iv) there is a measurable and locally integrable function $\psi : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\left\|\frac{\partial X}{\partial x}(x,t)\right\|_{\mathrm{O}} \le \psi(t), \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $\|\ \|_{\mathrm{O}}$ denotes the familiar operator norm on $n \times n$ real matrices.

The choice of the operator norm in **(iv)** is only for definiteness since all norms are equivalent on $\mathbb{R}^{n\times n}$. Note also that, using **(iv)** and the mean-value theorem, property **(iii)** immediately strengthens to:

(iii)' to each $x \in \mathbb{R}^n$ there is a measurable and locally integrable function $\alpha_x : \mathbb{R} \to \mathbb{R}^+$ such that

$$\|X(x,t)\| \le \alpha_x(t), \quad \text{for all } t \in \mathbb{R}.$$

By **(i)**, **(ii)**, **(iii)'**, and **(iv)**, the solution to (B.3) with arbitrary initial condition $x(0) = x_0 \in I\!R^n$ uniquely exists for all $t \in I\!R$, *cf.* [11, Theorem 54, Proposition C.3.4, Proposition C.3.8], in the sense that there is a unique locally absolutely continuous function $x : I\!R \to I\!R^n$ satisfying (B.3) for almost every t and such that $x(0) = x_0$. We shall denote by $\widehat{x}(\tau, x_0)$ the value of this solution at time $t = \tau$, in other words we let $(t, x_0) \mapsto \widehat{x}(\tau, x_0)$ designate the flow of (B.3). By definition, the *variational equation* of (B.3) along the trajectory $t \mapsto \widehat{x}(t, x_0)$ is the linear differential equation:

$$\dot{R} = \frac{\partial X}{\partial x}\,(\widehat{x}(t, x_0), t)\, R \tag{B.4}$$

in the unknown matrix-valued function $R : I\!R \to I\!R^{n\times n}$. In view of **(iv)**, appealing again to [11, Theorem 54, Proposition C.3.4, Proposition C.3.8], we see that the solution to (B.4) uniquely exists for all t once some arbitrary initial condition $R(0) = R_0 \in I\!R^{n\times n}$ is prescribed. Accordingly, we let $\widehat{R}(t, R_0, x_0)$ denote the value at time t of that solution.

Proposition B.2 *If $X : I\!R^n \times I\!R \to I\!R^n$ satisfies properties* **(i)**-**(iv)** *above, and if $\widehat{x}$, $\widehat{R}$ are the respective flows of* (B.3), (B.4) *defined previously, then $\widehat{x}(t, x)$ is continuously differentiable with respect to x for fixed t and*

$$\frac{\partial \widehat{x}}{\partial x}(t, x) \;=\; \widehat{R}(t, I_n, x), \tag{B.5}$$

where I_n is the identity matrix of size n.

Proof. Upon changing $X(x, t)$ into $-X(x, -t)$ if necessary, we may assume throughout the proof that $t \geq 0$. We first show that $x \mapsto \widehat{x}(t, x)$ is continuous for fixed t. Indeed, setting for $x, h \in I\!R^n$

$$\delta(t, x, h) \stackrel{\Delta}{=} \widehat{x}(t, x + h) - \widehat{x}(t, x),$$

we get by definition of $\widehat{x}$ that $\delta(t, x, h)$ is locally absolutely continuous with respect to t for fixed x, h, with derivative given almost everywhere by

$$\begin{aligned} \dot{\delta}(t, x, h) \;&=\; X(\widehat{x}(t, x + h), t) - X(\widehat{x}(t, x), t) \qquad \text{(B.6)}\\ &=\; \left(\int_0^1 \frac{\partial X}{\partial x}\Big(\tau\widehat{x}(t, x + h) + (1 - \tau)\widehat{x}(t, x), t\Big)\, \mathrm{d}\tau\right) \delta(t, x, h), \end{aligned}$$

where we have used point **(i)** of our hypotheses. If we put for simplicity :

$$T(x, h, s) \stackrel{\Delta}{=} \int_0^1 \frac{\partial X}{\partial x}\Big(\tau\widehat{x}(s, x + h) + (1 - \tau)\widehat{x}(s, x)\,,\; s\Big)\, \mathrm{d}\tau \tag{B.7}$$

and if we notice by point **(iv)** of the hypotheses that

$$\|T(x, h, s)\|_{\mathrm{O}} \leq \psi(s), \tag{B.8}$$

we deduce from (B.6) and (B.8), since $\delta(0, x, h) = h$, that

$$\|\delta(t, x, h)\| \leq \|h\| + \int_0^t \psi(s)\, \|\delta(s, x, h)\|\, ds.$$

As ψ is locally L^1 while $s \mapsto \delta(s,x,h)$ is *a fortiori* continuous hence bounded on $[0,t]$, Lemma B.1 implies that

$$\|\delta(t,x,h)\| \leq \|h\| \left(1 + \exp\left(\int_0^t \psi(\xi)\, d\xi\right) \int_0^t \psi(s)\, ds\right). \tag{B.9}$$

Since the right-hand side of (B.9) can be made arbitrarily small with $\|h\|$, we get the announced continuity of $x \mapsto \widehat{x}(t,x)$.

Next, we put for $x, h \in \mathbb{R}^n$

$$y(t,x,h) \triangleq \widehat{x}(t,x+h) - \widehat{x}(t,x) - \widehat{R}(t,I_n,x)\, h, \tag{B.10}$$

and we need to show that $\|y(t,x,h)\|$ is little $o(\|h\|)$ for fixed t,x. Clearly $t \mapsto y(t,x,h)$ is locally absolutely continuous with $y(0,x,h) = 0$. Computing its derivative using (B.10), (B.3), and (B.4), we get

$$\begin{aligned} y(t,x,h) \quad = \quad & \int_0^t \Big(X(\widehat{x}(s,x+h),s) - X(\widehat{x}(s,x),s) \\ & \qquad - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s)\, \widehat{R}(s,I_n,x)\, h \Big)\, ds \end{aligned} \tag{B.11}$$

In view of (B.10), making use of the second equality in (B.6), we may rewrite (B.11) in the form :

$$\begin{aligned} y(t,x,h) \quad = \quad & \int_0^t T(x,h,s)\, y(s,x,h)\, ds \\ & + \int_0^t \left(T(x,h,s) - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s) \right) \widehat{R}(s,I_n,x)\, h\, ds \end{aligned} \tag{B.12}$$

where $T(x,h,s)$ was defined in (B.7). If we further define

$$\Phi(t,x,h) \triangleq \left\| \int_0^t \left(T(x,h,s) - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s) \right) \widehat{R}(s,I_n,x)\, ds \right\|_{\mathrm{O}} \tag{B.13}$$

we obtain from (B.12) and (B.8) the inequality :

$$\|y(t,x,h)\| \leq \Phi(t,x,h)\, \|h\| + \int_0^t \psi(s)\, \|y(s,x,h)\|\, ds.$$

Observe by (B.8) and point **(iv)** of the hypotheses that

$$\left\| T(x,h,s) - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s) \right\|_{\mathrm{O}} \leq 2\psi(s), \tag{B.14}$$

so that $t \mapsto \Phi(t,x,h)$ is locally bounded for fixed x, uniformly with respect to $h \in \mathbb{R}^n$, because ψ is locally L^1 and because $s \mapsto \|\widehat{R}(s,I_n,x)\|_{\mathrm{O}}$ is continuous hence locally bounded. Since $t \mapsto y(t,x,h)$ is also continuous hence locally bounded, Lemma B.1 yields :

$$\|y(t,x,h)\| \leq \Phi(t,x,h)\, \|h\| + \|h\| \exp\left(\int_0^t \psi(\xi)\, d\xi\right) \int_0^t \psi(s)\Phi(s,x,h)\, ds.$$

¿From this, appealing to the dominated convergence theorem, we shall deduce that $\|y(t,x,h)\|$ is little $o(\|h\|)$ for fixed t,x if only we can show that $s \mapsto \Phi(s,x,h)$ goes boundedly point-wise to zero with $\|h\|$ on $[0,t]$. In fact, we just pointed out that it is bounded there, independently of h. To see that it converges point-wise to zero when $\|h\| \to 0$, we return to the definition (B.13) of Φ and, taking into account (B.14) where ψ is locally L^1 and the already used boundedness of $s \mapsto \|\widehat{R}(s,I_n,x)\|_{\rm O}$ on $[0,t]$ for fixed x, we observe that it is enough by dominated convergence to establish the point-wise limit :

$$\lim_{\|h\|\to 0} \left\| T(x,h,s) - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s) \right\| = 0, \qquad x \in I\!R^n,\ s \in [0,t].$$

The latter in turn follows from another application of the dominated convergence theorem to the right-hand side of (B.7), considering points **(i)** and **(iv)** of the hypotheses together with the continuity of $x \mapsto \widehat{x}(t,x)$ proved earlier.

To complete the proof, it remains for us to show that $x \mapsto \widehat{R}(t,I_n,x)$ is continuous for fixed t. In other words, if we put for $x,h \in I\!R^n$:

$$\Delta(t,x,h) \triangleq \widehat{R}(t,I_n,x+h) - \widehat{R}(t,I_n,x),$$

we need to show that $\|\Delta(t,x,h)\|_{\rm O}$ is little $o(\|h\|)$ as $\|h\| \to 0$ for fixed t and x. To this effect, using (B.4), we write

$$\begin{aligned}
\Delta(t,x,h) &= \int_0^t \left(\frac{\partial X}{\partial x}(\widehat{x}(s,x+h),s)\, \widehat{R}(s,I_n,x+h) \right. \\
&\qquad\qquad \left. - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s)\, \widehat{R}(s,I_n,x) \right) ds \\
&= \int_0^t \left(\frac{\partial X}{\partial x}(\widehat{x}(s,x+h),s)\, \Delta(s,x,h) \right. \qquad\qquad \text{(B.15)} \\
&\quad + \left. \left(\frac{\partial X}{\partial x}(\widehat{x}(s,x+h),s) - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s) \right) \widehat{R}(s,I_n,x) \right) ds.
\end{aligned}$$

Setting

$$\Theta(t,x,h) \triangleq \left\| \int_0^t \left(\frac{\partial X}{\partial x}(\widehat{x}(s,x+h),s) - \frac{\partial X}{\partial x}(\widehat{x}(s,x),s) \right) \widehat{R}(s,I_n,x)\, ds \right\|_{\rm O},$$

we obtain from (B.15) and point **(iv)** of the hypotheses that

$$\|\Delta(t,x,h)\|_{\rm O} \le \int_0^t \psi(s) \|\Delta(s,x,h)\|_{\rm O}\, ds + \Theta(t,x,h). \qquad \text{(B.16)}$$

Since $t \to \Theta(t,x,h)$ is locally bounded for fixed x independently of $h \in I\!R^n$, as follows from point **(iv)** again and the fact that $s \mapsto \widehat{R}(s,I_n,x)$ is continuous hence bounded on $[0,t]$, Lemma B.1 now yields :

$$\|\Delta(t,x,h)\|_{\rm O} \le \Theta(t,x,h) + \exp\left(\int_0^t \psi(\xi)\, d\xi \right) \int_0^t \psi(s) \Theta(s,x,h)\, ds.$$

¿From this, appealing to the dominated convergence theorem, we shall deduce that $\|\Delta(t,x,h)\|_{\rm O}$ is little $o(\|h\|)$ for fixed t,x if only we can show that $s \mapsto \Theta(s, x, h)$ goes boundedly point-wise to zero with $\|h\|$ on $[0, t]$. But we already proved its boundedness, and the desired limit :

$$\lim_{\|h\| \to 0} \Theta(t, x, h) = 0, \qquad x \in I\!\!R^n, \; t \in I\!\!R,$$

follows from yet another application of the dominated convergence theorem in the equation defining Θ, granted points **(i)** and **(iv)** of the hypotheses together with the continuity of $x \mapsto \widehat{x}(t, x)$ already established. □

C Continuity of the flow with L^p controls

In this appendix, we deal with a differential equation of the form

$$\dot{x} = F(x, \Upsilon(t)) \tag{C.1}$$

where $x \in I\!\!R^n$ while $\Upsilon \in L^p = L^p(I\!\!R, I\!\!R^m)$, the familiar Lebesgue space of (equivalence classes of) functions $I\!\!R \to I\!\!R^m$ whose p-th power is integrable in case $p < \infty$ and whose norm is essentially bounded if $p = \infty$; we endow L^p with the usual norm, namely $\|\Upsilon\|_p = (\int_{I\!\!R} \|\Upsilon\|^p dt)^{1/p}$ if $p < \infty$ and $\|\Upsilon\|_\infty = \text{ess.sup.}_{I\!\!R} \|\Upsilon\|$, where $\|.\|$ denotes the Euclidean norm. Of course, a solution to the differential equation is understood here in the sense that $x(t)$ is absolutely continuous, and that its derivative is a locally summable function whose value is given by the right-hand side of (C.1) for almost every t. Classically, even if $F : I\!\!R^n \times I\!\!R^m \to I\!\!R^n$ is very smooth, the existence of solutions to (C.1) when $1 \leq p < \infty$ requires some restrictions on the growth of F at infinity. Even then however, the continuity of that solution with respect to $\Upsilon \in L^p$ is difficult to ferret out in the literature. We propose below a set of conditions that ensures such a continuity property, this result being used in the proof of Theorem 3.7. For definiteness, we agree in the statement that $\|.\|$ refers to the operator norm when applied to a matrix.

Proposition C.1 *Let $F(x, u)$ be continuous $I\!\!R^n \times I\!\!R^m \to I\!\!R^n$, and the partial derivative $\partial F / \partial x$ exist continuously $I\!\!R^n \times I\!\!R^m \to I\!\!R^{n \times n}$. Let $p \in [1, \infty]$ and assume if $p < \infty$ that, to each compact $K \subset I\!\!R^n$, there are constants $c_1(K)$, $c_2(K)$, such that :*

$$\|F(x, u)\| + \|\frac{\partial F}{\partial x}(x, u)\| \leq c_1(K) + c_2(K)\, \|u\|^p, \quad (x, u) \in K \times I\!\!R^m. \tag{C.2}$$

Then, for any $\Upsilon \in L^p(I\!\!R, I\!\!R^m)$, the solution $t \mapsto x(t, x_0, \Upsilon)$ to (C.1) with initial condition $x(0) = x_0$ uniquely exists on some maximal time interval $\mathcal{I}_{x_0, \Upsilon}$ containing 0. Moreover, if $\mathcal{K}$ is a compact subinterval of $\mathcal{I}_{x_0, \Upsilon}$, there is a neighborhood $\mathcal{V}$ of (x_0, Υ) in $I\!\!R^n \times L^p(I\!\!R, I\!\!R^m)$ such that $\mathcal{K} \subset \mathcal{I}_{x_0', \Upsilon'}$ whenever $(x_0', \Upsilon') \in \mathcal{V}$; within this neighborhood, it further holds that

$$\lim_{(x_0', \Upsilon') \to (x_0, \Upsilon)} x(t, x_0', \Upsilon') = x(t, x_0, \Upsilon), \tag{C.3}$$

uniformly with respect to $t \in \mathcal{K}$.

Proof. If $\Upsilon \in L^p$, and provided (C.2) holds in case $p < \infty$, it follows immediately from classical existence and uniqueness results (see *e.g.* [11, Theorem 54, Proposition C.3.4])[2] that $x(t, x_0, \Upsilon)$ is uniquely defined on some maximal time interval containing 0, say $\mathcal{I}_{x_0,\Upsilon}$. Next, let us replace $F(x, u)$ by $F_1(x, u) = \varphi(x)F(x, u)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is smooth with compact support and assumes the value 1 on a neighborhood of the compact set $x(\mathcal{K}, x_0, \Upsilon)$. Note that F_1 again satisfies an estimate of the form (C.2) if F does, and that it vanishes for x outside the support of φ. Therefore, if F gets replaced by F_1, (C.2) will hold when $p < \infty$ for some constants c_1, c_2 that are in fact independent of K, whereas if $p = \infty$ $\partial F_1/\partial x\,(x, \Upsilon(t))$ is bounded by a constant a.e. in t for fixed $\Upsilon \in L^\infty$. This is to the effect that, if we deal with F_1 instead of F, the solution to (C.1) exists for all $t \in \mathbb{R}$ [11, Proposition C.3.8]. This entails that if we prove the proposition for F_1, then we get it for F as well, because the property for system (C.1) that $\mathcal{K} \subset \mathcal{I}_{x_0',\Upsilon'}$ whenever (x_0', Υ') is sufficiently close to (x_0, Υ) in $\mathbb{R}^n \times L^p$ will be a mechanical consequence of property (C.3) for the system $\dot{x} = F_1(x, \Upsilon(t))$, granted that $F(x, u)$ and $F_1(x, u)$ coincide for x in a neighborhood of $x(\mathcal{K}, x_0, \Upsilon)$. To recap, we are left to prove (C.3) under the stronger assumption that $F(x, u)$ hence also $\partial F/\partial x$ vanishes for x outside some compact set, in which case $c_1(K)$ and $c_2(K)$ in (C.2) are taken to be absolute constants c_1 and c_2, while $\mathcal{I}_{x_0,\Upsilon} = \mathbb{R}$ for all $(x_0, \Upsilon) \in \mathbb{R}^n \times L^p$.

Pick $(x_0', \Upsilon') \in \mathbb{R}^n \times L^p$ and set for simplicity $x(t) = x(t, x_0, \Upsilon)$ and $x'(t) = x(t, x_0', \Upsilon')$. ¿From the definitions, we get that

$$\begin{aligned} x(t) - x'(t) \quad = \quad & x_0 - x_0' + \int_0^t \big(F\left(x(\tau), \Upsilon(\tau)\right) - F\left(x(\tau), \Upsilon'(\tau)\right)\big)\, \mathrm{d}\tau \\ & + \int_0^t \big(F\left(x(\tau), \Upsilon'(\tau)\right) - F\left(x'(\tau), \Upsilon'(\tau)\right)\big)\, \mathrm{d}\tau. \end{aligned}$$

If $p = \infty$, we obtain at once from the mean-value theorem :

$$\|x(t) - x'(t)\| \le$$

$$\begin{aligned} & \|x_0 - x_0'\| + \int_0^t \big\|F(x(\tau), \Upsilon(\tau)) - F((x(\tau), \Upsilon'(\tau))\big\|\, \mathrm{d}\tau \\ & \quad + \sup_{\substack{(x,u)\in\mathbb{R}^n\times\mathbb{R}^m \\ \|u\|\le\|\Upsilon'\|_\infty}} \left\|\frac{\partial F}{\partial x}(x, u)\right\| \int_0^t \|x(\tau) - x'(\tau)\|\, \mathrm{d}\tau, \end{aligned} \tag{C.4}$$

and if $1 \le p < \infty$ we additionally take (C.2) into account to get :

$$\|x(t) - x'(t)\| \le$$

$$\begin{aligned} & \|x_0 - x_0'\| + \int_0^t \big\|F(x(\tau), \Upsilon(\tau)) - F((x(\tau), \Upsilon'(\tau))\big\|\, \mathrm{d}\tau \\ & \quad + \int_0^t \big(c_1 + c_2\|\Upsilon'(\tau)\|^p\big)\, \|x(\tau) - x'(\tau)\|\, \mathrm{d}\tau. \end{aligned} \tag{C.5}$$

[2]Strictly speaking, to apply Theorem 54 of that reference, we need to choose a specific representative of Υ which is defined *everywhere*; this causes no difficulty because the solution of course does not depend on this representative.

To establish (C.3), we may of course assume that $\|\Upsilon'\|_p$ remains bounded and therefore, by the Bellman-Gronwall lemma as applied to (C.4) or (C.5) according whether $p=\infty$ or $p<\infty$ (see Lemma B.1), we shall be done if only we can show that

$$\phi_{\Upsilon'}(t)=\int_0^t \big\|F(x(\tau),\Upsilon(\tau))-F((x(\tau),\Upsilon'(\tau))\big\|\,\mathrm{d}\tau, \tag{C.6}$$

can be made small with $\|\Upsilon'-\Upsilon\|_p$ for fixed $t\in\mathbb{R}$ (compare [11, Theorem 55]). This is obvious if $p=\infty$ by the uniform continuity of F relatively to the compact set $x([0,t])\times\overline{B}(0,\|\Upsilon\|_\infty)$, thus we assume in the remaining of the proof that $p<\infty$. Choose Υ' such that $\|\Upsilon'-\Upsilon\|_p<\varepsilon$. Since both $\|\Upsilon\|^p$ and $F(x(\tau),\Upsilon(\tau))$ are summable using (C.2), there is by absolute continuity an $\eta>0$ such that

$$\max\left\{\int_E \|\Upsilon(\tau)\|^p\,\mathrm{d}\tau\ ,\ \int_E \|F(x(\tau),\Upsilon(\tau))\|\,\mathrm{d}\tau\right\}<\varepsilon \quad \text{whenever } |E|<\eta,$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E\subset\mathbb{R}$ [10, Theorem 6.11]. Then, again from (C.2), we have that

$$\begin{aligned}
&\int_E \|F(x(\tau),\Upsilon'(\tau))\|\,\mathrm{d}\tau\\
\leq\ & c_1|E|+c_2\int_E \|\Upsilon'(\tau)\|^p\,\mathrm{d}\tau\\
\leq\ & c_1|E|+c_2'\int_E \big(\|\Upsilon'(\tau)-\Upsilon(\tau)\|^p+\|\Upsilon(\tau)\|^p\big)\,\mathrm{d}\tau
\end{aligned}$$

for some constant c_2' ($c_2 2^{p/q}$ will do if $1/p+1/q=1$).

Using the triangle inequality and collecting terms, we find that

$$\int_E \big\|F(x(\tau),\Upsilon(\tau))-F((x(\tau),\Upsilon'(\tau))\big\|\,\mathrm{d}\tau\leq c_1\eta+\varepsilon(1+c'2+c_2'\varepsilon^{p-1}),$$

and if we further impose, without loss of generality, that $\eta\leq\varepsilon<1$ while putting $c_3=1+c_1+2c_2'$, we obtain :

$$\int_E \big\|F(x(\tau),\Upsilon(\tau))-F((x(\tau),\Upsilon'(\tau))\big\|\,\mathrm{d}\tau\leq c_3\,\varepsilon \quad \text{if } |E|<\eta. \tag{C.7}$$

Now, pick $M>0$ so large that

$$E_M=\{\tau\in\mathbb{R};\ \|\Upsilon(\tau)\|>M\} \tag{C.8}$$

has Lebesgue measure less than $<\eta$. By uniform continuity of F relatively to $x([0,t])\times\overline{B}(0,M)$, there is $\alpha>0$ such that

$$\|F(x(\tau),u')-F(x(\tau),u)\|<\varepsilon \ \text{ for } \tau\in[0,t],\ \|u\|\leq M,\ \|u'-u\|<\alpha.$$

Let us further define

$$E_{\alpha,\Upsilon'}=\{\tau\in\mathbb{R};\ \|\Upsilon'(\tau)-\Upsilon(\tau)\|\geq\alpha\}. \tag{C.9}$$

By (C.8), (C.9), and the definition of α, we get that

$$\|F(x(\tau),\Upsilon(\tau))-F(x(\tau),\Upsilon'(\tau))\|<\varepsilon \ \text{ for } \tau\in[0,t]\backslash(E_M\cup E_{\alpha,\Upsilon'}). \tag{C.10}$$

Finally, since $|E_{\alpha,\Upsilon'}| \leq \|\Upsilon - \Upsilon'\|_p/\alpha$, we can make it less than η by requiring that $\|\Upsilon - \Upsilon'\|_p < \eta\alpha$. Altogether, starting from $0 < \varepsilon < 1$, we have found $\eta > 0$ and $\alpha > 0$ such that, if

$$\|\Upsilon - \Upsilon'\|_p < \max\{\eta\alpha, \varepsilon\},$$

then both E_M defined by (C.8) and $E_{\alpha,\Upsilon'}$ defined by (C.9) have Lebesgue measure less than η while (C.7) and (C.10) hold. When these conditions are satisfied, we get upon decomposing

$$\int_{[0,t]} = \int_{E_M} + \int_{E_{\alpha,\Upsilon'}} + \int_{[0,t]\setminus(E_M \cup E_{\alpha,\Upsilon'})}$$

that $\phi_{\Upsilon'}(t)$ defined in (C.6) is less than $\varepsilon(|t| + 2c_3)$ which is arbitrarily small, as announced. $\square$