



Optimal eigenvalues estimate for the Dirac operator on domains with boundary

Simon Raulot

► To cite this version:

Simon Raulot. Optimal eigenvalues estimate for the Dirac operator on domains with boundary. Letters in Mathematical Physics, Springer Verlag, 2005, 73, pp.135–145. hal-00021463

HAL Id: hal-00021463

<https://hal.archives-ouvertes.fr/hal-00021463>

Submitted on 21 Mar 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

OPTIMAL EIGENVALUES ESTIMATE FOR THE DIRAC OPERATOR ON DOMAINS WITH BOUNDARY

SIMON RAULOT

ABSTRACT. We give a lower bound for the eigenvalues of the Dirac operator on a compact domain of a Riemannian spin manifold under the MIT bag boundary condition. The limiting case is characterized by the existence of an imaginary Killing spinor.

1. INTRODUCTION

Let \emptyset be a compact domain in a n -dimensional Riemannian spin manifold (N^n, g) whose boundary is denoted by $\partial\emptyset$. In [HMR02], the authors studied four elliptic boundary conditions for the Dirac operator D of the domain \emptyset . More precisely, they prove a Friedrich-type inequality [Fri80] which relates the spectrum of the Dirac operator and the scalar curvature of the domain \emptyset . These boundary conditions are the following: the Atiyah-Patodi-Singer (APS) condition based on the spectral resolution of the boundary Dirac operator; a modified version of the APS condition, the mAPS condition; the boundary condition CHI associated with a chirality operator; and a Riemannian version of the MIT bag boundary condition. In fact, they show that, if the boundary $\partial\emptyset$ of \emptyset has non-negative mean curvature, then under the APS, CHI or mAPS boundary conditions, the spectrum of the classical Dirac operator of the domain \emptyset is a sequence of unbounded real numbers $\{\lambda_k : k \in \mathbb{Z}\}$ satisfying

$$\lambda_k^2 \geq \frac{n}{4(n-1)} R_0, \quad (1)$$

where R_0 is the infimum of the scalar curvature of the domain \emptyset . Moreover, equality holds only for the CHI and the mAPS conditions and in these cases, \emptyset is respectively isometric to a half-sphere or it carries a non-trivial real Killing spinor and has minimal boundary. In the case of the MIT boundary condition, they show that the spectrum of the Dirac operator on \emptyset is an unbounded discrete set of complex numbers λ^{MIT} with positive imaginary part satisfying

$$|\lambda^{\text{MIT}}|^2 > \frac{n}{4(n-1)} R_0, \quad (2)$$

if the mean curvature of the boundary is non-negative. This result leads to the following question: can one improve this inequality in order to obtain some boundary geometric

Date: March 21, 2006.

1991 *Mathematics Subject Classification.* Differential Geometry, Global Analysis, 53C27, 53C40, 53C80, 58G25, 83C60.

Key words and phrases. Dirac Operator, Spectrum, Boundary condition, Ellipticity, Constant mean curvature hypersurfaces.

invariants on the right hand side of (2)? We show in this paper that such a result can be obtained. More precisely, we prove the following theorem:

Theorem 1. *Let \mathcal{O} be a compact domain of an n -dimensional Riemannian spin manifold (N^n, g) whose boundary $\partial\mathcal{O}$ satisfies $H > 0$. Under the MIT boundary condition $\mathbb{B}_{\text{MIT}}^-$, the spectrum of the classical Dirac operator D on \mathcal{O} is an unbounded discrete set of complex numbers with positive imaginary part. Any eigenvalue λ^{MIT} satisfies*

$$|\lambda^{\text{MIT}}|^2 \geq \frac{n}{4(n-1)} R_0 + n \operatorname{Im}(\lambda^{\text{MIT}}) H_0, \quad (3)$$

where H_0 is the infimum of the mean curvature of the boundary. Moreover, equality holds if and only if the associated eigenspinor is an imaginary Killing spinor on \mathcal{O} and if the boundary $\partial\mathcal{O}$ is a totally umbilical hypersurface with constant mean curvature.

The proof of this theorem is based on a modification of the spinorial Levi-Civita connection which leads to a spinorial Reilly-type formula. This formula can be seen as a hyperbolic version of the Reilly inequality used in [HMR02].

The author would like to thank the referee for helpful comments.

2. GEOMETRIC PRELIMINARIES

In this section, we give some standard facts about Riemannian spin manifolds with boundary. For more details, we refer to [BBW93] or [HMR02].

On a compact domain \mathcal{O} with smooth boundary $\partial\mathcal{O}$ in a n -dimensional Riemannian spin manifold (N^n, g) , denote by $\Sigma\mathcal{O}$ the complex spinor bundle corresponding to the metric g and by ∇ its Levi-Civita connection acting on $T\mathcal{O}$ as well as its lift to $\Sigma\mathcal{O}$. The map $\gamma : \mathcal{Cl}(\mathcal{O}) \rightarrow \operatorname{End}(\Sigma\mathcal{O})$ is the Clifford multiplication where $\mathcal{Cl}(\mathcal{O})$ is the Clifford bundle over \mathcal{O} . The spinor bundle is endowed with a natural Hermitian scalar product, denoted by $\langle \cdot, \cdot \rangle$, compatible with ∇ and γ . The Dirac operator is then the first order elliptic operator acting on sections of $\Sigma\mathcal{O}$ locally given by

$$\begin{aligned} D : \Gamma(\Sigma\mathcal{O}) &\longrightarrow \Gamma(\Sigma\mathcal{O}) \\ \psi &\longmapsto \sum_{i=1}^n \gamma(e_i) \nabla_{e_i} \psi, \end{aligned}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of $T\mathcal{O}$.

Consider now the boundary $\partial\mathcal{O}$ which is an oriented hypersurface of the domain \mathcal{O} with induced orientation and Riemannian structure. Since the normal bundle of $\partial\mathcal{O}$ is trivial, the boundary itself is a spin manifold. This spin structure on the boundary allows to construct an intrinsic spinor bundle $\Sigma(\partial\mathcal{O})$ over $\partial\mathcal{O}$ naturally endowed with a Hermitian metric, a Clifford multiplication $\gamma^{\partial\mathcal{O}}$ and a spinorial Levi-Civita connection $\nabla^{\partial\mathcal{O}}$. Moreover the restriction $\mathbf{S}(\partial\mathcal{O}) := \Sigma\mathcal{O}|_{\partial\mathcal{O}}$ to the boundary of the spinor bundle $\Sigma\mathcal{O}$ is a Dirac bundle, i.e. there exist on $\mathbf{S}(\partial\mathcal{O})$ a Hermitian metric denoted by $\langle \cdot, \cdot \rangle$ compatible with the Levi-Civita connection $\nabla^{\mathbf{S}}$ and the Clifford multiplication $\gamma^{\mathbf{S}}$. The Clifford multiplication $\gamma^{\mathbf{S}} : \mathcal{Cl}(\partial\mathcal{O}) \rightarrow \operatorname{End}(SS)$ is given by $\gamma^{\mathbf{S}}(X)\psi = \gamma(X)\gamma(\nu)\psi$ for all $X \in \Gamma(T\mathcal{O})$ and $\psi \in \Gamma(SS)$. Similarly we can relate the Levi-Civita connection acting on $\Sigma\mathcal{O}$ with that acting on $\mathbf{S}(\partial\mathcal{O})$ by the spinorial Gauss formula (see [Bär98]):

$$(\nabla_X \psi)|_{\partial\mathcal{O}} = \nabla_X^{\mathbf{S}} \psi|_{\partial\mathcal{O}} + \frac{1}{2} \gamma^{\mathbf{S}}(AX) \psi|_{\partial\mathcal{O}},$$

for all $X \in \Gamma(\mathbf{T}(\partial\mathcal{O}))$, $\psi \in \Gamma(\Sigma\mathcal{O})$ and where $AX = -\nabla_X\nu$ is the shape operator of the boundary $\partial\mathcal{O}$ with respect to the inner normal vector field ν . We can then define the boundary Dirac operator acting on $\mathbf{S}(\partial\mathcal{O})$ which is an elliptic first order differential operator locally given by

$$D^{\mathbf{S}} = \sum_{j=1}^{n-1} \gamma^{\mathbf{S}}(e_j) \nabla_{e_j}^{\mathbf{S}}. \quad (4)$$

Recall that there is a standard identification

$$\mathbf{S}(\partial\mathcal{O}) \equiv \begin{cases} \Sigma(\partial\mathcal{O}) & \text{if } n \text{ is odd} \\ \Sigma(\partial\mathcal{O}) \oplus \Sigma(\partial\mathcal{O}) & \text{if } n \text{ is even} \end{cases}$$

Taking into account the relation between the Hermitian bundle $\mathbf{S}(\partial\mathcal{O})$ and $\Sigma(\partial\mathcal{O})$, one can see that

$$\nabla^{\mathbf{S}} \equiv \begin{cases} \nabla^{\partial\mathcal{O}} & \text{if } n \text{ is odd} \\ \nabla^{\partial\mathcal{O}} \oplus \nabla^{\partial\mathcal{O}} & \text{if } n \text{ is even} \end{cases}$$

and

$$\gamma^{\mathbf{S}} \equiv \begin{cases} \gamma^{\partial\mathcal{O}} & \text{if } n \text{ is odd} \\ \gamma^{\partial\mathcal{O}} \oplus -\gamma^{\partial\mathcal{O}} & \text{if } n \text{ is even} \end{cases}$$

3. THE MIT BOUNDARY CONDITION

First, note that on a closed compact Riemannian spin manifold, the classical Dirac operator has exactly one self-adjoint L^2 extension, so it has real discrete spectrum. In the setting of manifolds with boundary, a defect of self-adjointness appears. It is given by the Green formula

$$\int_{\mathcal{O}} \langle D\varphi, \psi \rangle dv(g) - \int_{\mathcal{O}} \langle \varphi, D\psi \rangle dv(g) = - \int_{\partial\mathcal{O}} \langle \gamma(\nu)\varphi, \psi \rangle ds(g), \quad (5)$$

for all $\varphi, \psi \in \Gamma(\Sigma\mathcal{O})$. Furthermore, in this case, the Dirac operator has a closed range of finite codimension, but an infinite-dimensional kernel, which varies depending on the choice of the Sobolev space. We refer to [BBW93], [Lop53] or [HMR02] for a careful treatment of boundary conditions for elliptic operators.

The MIT bag boundary condition has first been introduced by physicists of the Massachusetts Institute of Technology in a Lorentzian setting (see [CJJ⁺74], [CJJT74] or [Joh75]). The Riemannian version of this condition has been studied in [HMR02] in order to get Friedrich estimates and in [HMZ02] because of its conformal covariance to give a conformal lower bound for the first eigenvalue of the intrinsic Dirac operator of hypersurfaces bounding a compact domain in a Riemannian spin manifold. Consider the pointwise endomorphism

$$i\gamma(\nu) : \Gamma(SS) \longrightarrow \Gamma(SS)$$

acting on the restriction to the boundary $\partial\mathcal{O}$ of the spinor bundle over \mathcal{O} and where i is the fundamental imaginary number. This map is an involution, and so the bundle SS

splits into two eigensubbundles V^\pm associated with the eigenvalues ± 1 . We then have two associated orthogonal projections given by

$$\begin{aligned} \mathbb{B}_{\text{MIT}}^\pm : \mathbf{L}^2(SS) &\longrightarrow \mathbf{L}^2(V^\pm) \\ \varphi &\longmapsto \frac{1}{2}(\text{Id} \pm i\gamma(\nu))\varphi. \end{aligned}$$

which define local elliptic boundary conditions for the Dirac operator D on the domain \emptyset . So under this boundary condition, the eigenvalue problem

$$\begin{cases} D\varphi = \lambda^{\text{MIT}}\varphi & \text{on } \emptyset \\ \mathbb{B}_{\text{MIT}}^\pm\varphi = 0 & \text{along } \partial\emptyset \end{cases} \quad (6)$$

has a discrete spectrum with finite dimensional eigenspaces consisting of smooth spinor fields.

Remark 1. Under the MIT boundary condition $\mathbb{B}_{\text{MIT}}^-$, the spectrum of the Dirac operator D is contained in the upper half complex plane $\{z \in \mathbb{C} / \text{Im}(z) > 0\}$. Indeed, let λ^{MIT} be an eigenvalue of D under the MIT boundary condition and $\varphi \in \Gamma(\Sigma\emptyset)$ the associated spinor field, then taking $\psi = i\varphi$ in the Formula (5) leads to

$$2 \text{Im}(\lambda^{\text{MIT}}) \int_{\emptyset} |\varphi|^2 dv(g) = \int_{\partial\emptyset} |\varphi|^2 ds(g) \quad (7)$$

Two possibilities can occur: we have either $\text{Im}(\lambda^{\text{MIT}}) > 0$ or $\text{Im}(\lambda^{\text{MIT}}) = 0$. If $\text{Im}(\lambda^{\text{MIT}}) = 0$, then the spinor field φ should vanish along the boundary $\partial\emptyset$ and by the unique continuation principle (see [BBW93]), it should be identically zero on the manifold \emptyset . This is impossible because the spinor φ is supposed to be an eigenspinor, so a non trivial field. The first case is the only possibility, i.e. $\text{Im}(\lambda^{\text{MIT}}) > 0$. For the boundary condition $\mathbb{B}_{\text{MIT}}^+$, we can show that the imaginary part of all eigenvalues of the Dirac operator is negative.

4. THE HYPERBOLIC REILLY FORMULA

In this section, we give a spinorial Reilly formula based on a modification of the spinorial Levi-Civita connection. Let $\alpha \in \mathbb{R}$, then we define the connection ∇^α acting on $\Sigma\emptyset$ by

$$\nabla_X^\alpha \varphi := \nabla_X \varphi + i\alpha\gamma(X)\varphi, \quad (8)$$

for all $\varphi \in \Gamma(\Sigma\emptyset)$ and $X \in \Gamma(T\emptyset)$. We can now derive an integral version of the Schrödinger-Lichnerowicz formula using the modified connection ∇^α . Indeed, we have:

Proposition 2. *For all spinor fields $\varphi \in \Gamma(\Sigma\emptyset)$, we have:*

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{\mathbf{L}^2} = \langle D^2 \varphi, \varphi \rangle_{\mathbf{L}^2} - \langle \frac{\mathbf{R}}{4} \varphi, \varphi \rangle_{\mathbf{L}^2} + n\alpha^2 \|\varphi\|_{\mathbf{L}^2}^2 - \int_{\partial\emptyset} \langle \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g), \quad (9)$$

where \mathbf{R} is the scalar curvature of the domain \emptyset .

Proof: First note that the \mathbf{L}^2 -formal adjoint of the connection ∇^α is, by definition, given by

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{\mathbf{L}^2} = \|\nabla^\alpha \varphi\|_{\mathbf{L}^2}^2 = \sum_{j=1}^n \int_{\emptyset} \langle \nabla_{e_j}^\alpha \varphi, \nabla_{e_j}^\alpha \varphi \rangle dv(g),$$

for all $\varphi \in \Gamma(\Sigma\emptyset)$ and where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of $T\emptyset$. An easy calculation using the compatibility properties of the Hermitian metric with the spinorial connection and the Clifford multiplication gives

$$\sum_{j=1}^n \langle \nabla_{e_j}^\alpha \varphi, \nabla_{e_j}^\alpha \varphi \rangle = \sum_{j=1}^n \left(e_j \langle \nabla_{e_j}^\alpha \varphi, \varphi \rangle - \langle \nabla_{e_j}^{-\alpha} \nabla_{e_j}^\alpha \varphi, \varphi \rangle \right),$$

and Stokes theorem leads to

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{\mathbb{L}^2} = \left\langle - \sum_{j=1}^n \nabla_{e_j}^{-\alpha} \nabla_{e_j}^\alpha \varphi, \varphi \right\rangle_{\mathbb{L}^2} - \int_{\partial\emptyset} \langle \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g).$$

We can now easily compute

$$\begin{aligned} \left\langle - \sum_{j=1}^n \nabla_{e_j}^{-\alpha} \nabla_{e_j}^\alpha \varphi, \varphi \right\rangle_{\mathbb{L}^2} &= \left\langle - \sum_{j=1}^n \nabla_{e_j} \nabla_{e_j} \varphi, \varphi \right\rangle_{\mathbb{L}^2} + n\alpha^2 \|\varphi\|_{\mathbb{L}^2}^2 \\ &= \langle \nabla^* \nabla \varphi, \varphi \rangle_{\mathbb{L}^2} + n\alpha^2 \|\varphi\|_{\mathbb{L}^2}^2, \end{aligned}$$

and then the classical Schrödinger-Lichnerowicz formula (see [LM89]) leads to Identity (9). \square

This formula is a first step to obtain Inequality (3). However, we have now to introduce the Dirac operator and the twistor operator associated with the connection ∇^α . The modified Dirac operator is locally defined by

$$D^\alpha \varphi = \sum_{j=1}^n \gamma(e_j) \nabla_{e_j}^\alpha \varphi, \quad (10)$$

and the associated twistor operator by

$$P_X^\alpha \varphi = \nabla_X^\alpha \varphi + \frac{1}{n} \gamma(X) D^\alpha \varphi, \quad (11)$$

for all $X \in \Gamma(T\emptyset)$ and $\varphi \in \Gamma(\Sigma\emptyset)$. Note that for $\alpha = 0$, the operators D^0 and P^0 are respectively the classical Dirac operator and the classical twistor operator which satisfy the relation (see [BHMM] or [Fri00] for example)

$$|\nabla \varphi|^2 = |P\varphi|^2 + \frac{1}{n} |D\varphi|^2$$

We can then check that the modified operators satisfy the same relation, i.e.

$$|\nabla^\alpha \varphi|^2 = |P^\alpha \varphi|^2 + \frac{1}{n} |D^\alpha \varphi|^2. \quad (12)$$

Indeed, if $\{e_1, \dots, e_n\}$ is a local orthonormal frame of $T\mathcal{O}$, we have

$$\begin{aligned} |\mathbb{P}^\alpha \varphi|^2 &= \sum_{j=1}^n \langle \nabla_{e_j}^\alpha \varphi + \frac{1}{n} \gamma(e_j) D^\alpha \varphi, \nabla_{e_j}^\alpha \varphi + \frac{1}{n} \gamma(e_j) D^\alpha \varphi \rangle \\ &= |\nabla^\alpha \varphi|^2 - \frac{2}{n} |D^\alpha \varphi|^2 + \frac{1}{n} |D^\alpha \varphi|^2 \\ &= |\nabla^\alpha \varphi|^2 - \frac{1}{n} |D^\alpha \varphi|^2, \end{aligned}$$

and so Identity (12) follows directly. We are now ready to establish the hyperbolic version of the spinorial Reilly formula given in [HMR02]. This formula can be seen as an analogous of the one used in [HMR03] to give a lower bound of the first eigenvalue of the intrinsic Dirac operator for hypersurfaces bounding a compact domain of a manifold with negative scalar curvature. More precisely, we prove:

Proposition 3. *For all $\varphi \in \Gamma(\Sigma\mathcal{O})$, we have:*

$$\begin{aligned} \|\mathbb{P}^\alpha \varphi\|_{L^2}^2 &= \frac{n-1}{n} \|D^\alpha \varphi\|_{L^2}^2 - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 \\ &\quad + \int_{\partial\mathcal{O}} \langle D^{\mathbb{S}} \varphi + \frac{n-1}{2} (2\alpha i\gamma(\nu)\varphi - H\varphi), \varphi \rangle ds(g), \end{aligned} \quad (13)$$

where H is the mean curvature of the boundary $\partial\mathcal{O}$ of \mathcal{O} .

Proof: Observe first that the modified Dirac operator D^α is not formally self-adjoint. Indeed an easy calculation using (5) gives

$$\int_{\mathcal{O}} \langle D^\alpha \varphi, \psi \rangle dv(g) = \int_{\mathcal{O}} \langle \varphi, D^{-\alpha} \psi \rangle dv(g) - \int_{\partial\mathcal{O}} \langle \gamma(\nu)\varphi, \psi \rangle ds(g), \quad (14)$$

for all $\varphi, \psi \in \Gamma(\Sigma\mathcal{O})$. However, we have:

$$D^2 \varphi = D^{-\alpha} D^\alpha \varphi - n^2 \alpha^2 \varphi,$$

and so substituting in Formula (9) gives

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} = \langle D^{-\alpha} D^\alpha \varphi, \varphi \rangle_{L^2} - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 - \int_{\partial\mathcal{O}} \langle \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g).$$

The integration by parts formula (14) leads to

$$\begin{aligned} \langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} &= \|D^\alpha \varphi\|_{L^2}^2 - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 \\ &\quad - \int_{\partial\mathcal{O}} \langle \gamma(\nu) D^\alpha \varphi + \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g). \end{aligned}$$

With the help of Identity (12), we have

$$\begin{aligned} \|\mathbb{P}^\alpha \varphi\|_{L^2}^2 &= \frac{n-1}{n} \|D^\alpha \varphi\|_{L^2}^2 - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 \\ &\quad - \int_{\partial\mathcal{O}} \langle \gamma(\nu) D^\alpha \varphi + \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g). \end{aligned}$$

However the boundary term can be written

$$-\gamma(\nu)D^\alpha\varphi - \nabla_\nu^\alpha\varphi = -\gamma(\nu)D\varphi - \nabla_\nu\varphi + (n-1)\alpha i\gamma(\nu)\varphi,$$

and using the identity

$$-\gamma(\nu)D\varphi - \nabla_\nu\varphi = D^{\mathbf{S}}\varphi - \frac{n-1}{2}H\varphi,$$

Formula (13) follows directly. \square

We are now ready to prove Theorem 1.

5. THE ESTIMATE

Proof of Theorem 1: Consider now a compact domain \mathcal{O} of a Riemannian spin manifold such that the mean curvature H of the boundary satisfies $H \geq 2\alpha$, for $\alpha > 0$. By ellipticity of the MIT boundary condition $\mathbb{B}_{\text{MIT}}^-$, consider a smooth spinor field $\varphi \in \Gamma(\Sigma\mathcal{O})$ solution of the eigenvalue boundary problem (6), i.e. φ satisfies

$$\begin{cases} D\varphi = \lambda^{\text{MIT}}\varphi & \text{on } \mathcal{O} \\ \mathbb{B}_{\text{MIT}}^-\varphi = 0 & \text{along } \partial\mathcal{O} \end{cases} \quad (15)$$

with $\text{Im}(\lambda^{\text{MIT}}) > 0$ by Remark 1. We now apply the hyperbolic Reilly formula (13) to the spinor field φ to get

$$\begin{aligned} \|\mathbb{P}^\alpha\varphi\|_{\mathbf{L}^2}^2 &= \left(\frac{n-1}{n} |\lambda^{\text{MIT}} - n\alpha i|^2 - n(n-1)\alpha^2 \right) \|\varphi\|_{\mathbf{L}^2}^2 - \left\langle \frac{R}{4}\varphi, \varphi \right\rangle_{\mathbf{L}^2} \\ &\quad + \int_{\partial\mathcal{O}} \left\langle D^{\mathbf{S}}\varphi + \frac{n-1}{2}(2\alpha i\gamma(\nu)\varphi - H\varphi), \varphi \right\rangle ds(g). \end{aligned}$$

Note that since $i\gamma(\nu)\varphi = \varphi$ along the boundary, we can compute

$$\langle D^{\mathbf{S}}\varphi, \varphi \rangle = \langle D^{\mathbf{S}}\varphi, i\gamma(\nu)\varphi \rangle = \langle i\gamma(\nu)D^{\mathbf{S}}\varphi, \varphi \rangle = -\langle D^{\mathbf{S}}(i\gamma(\nu)\varphi), \varphi \rangle = -\langle D^{\mathbf{S}}\varphi, \varphi \rangle,$$

and so the preceding formula gives

$$\begin{aligned} \|\mathbb{P}^\alpha\varphi\|_{\mathbf{L}^2}^2 + \frac{n-1}{2} \int_{\partial\mathcal{O}} (H - 2\alpha)|\varphi|^2 ds(g) &= \\ \frac{n-1}{n} (|\lambda^{\text{MIT}}|^2 - 2n\alpha \text{Im}(\lambda^{\text{MIT}})) \|\varphi\|_{\mathbf{L}^2}^2 - \left\langle \frac{R}{4}\varphi, \varphi \right\rangle_{\mathbf{L}^2} \end{aligned} \quad (16)$$

The assumption on the mean curvature gives:

$$|\lambda^{\text{MIT}}|^2 - 2n\alpha \text{Im}(\lambda^{\text{MIT}}) \geq \frac{n}{4(n-1)} R_0.$$

For $\alpha_0 = \frac{1}{2}H_0$, where $H_0 = \inf_{\partial\mathcal{O}}(H)$, we get Inequality (16). Suppose now that equality is achieved, thus

$$\|\mathbb{P}^{\alpha_0}\varphi\|_{\mathbf{L}^2}^2 = 0 \quad \text{and} \quad \frac{n-1}{2} \int_{\partial\mathcal{O}} (H - 2\alpha_0)|\varphi|^2 ds(g) = 0.$$

Moreover the spinor field φ is a solution of (15), so it satisfies the Killing equation

$$\nabla_X\varphi = -\frac{\lambda^{\text{MIT}}}{n}\gamma(X)\varphi, \quad \text{for all } X \in \Gamma(\text{T}\mathcal{O}).$$

Since such a spinor field has no zeroes (see [Fri00]), the mean curvature of the boundary is constant with $H = 2\alpha_0$. Furthermore, it is a well-known result [BFGK90] that, in this case, the eigenvalue λ^{MIT} has to be either real or purely imaginary. Here we have $\text{Im}(\lambda^{\text{MIT}}) > 0$, then $\lambda^{\text{MIT}} \in i\mathbb{R}_*^+$. The domain \mathcal{O} is in particular an Einstein manifold. We now show that the boundary has to be totally umbilical. Indeed, note that we have for all $X \in \Gamma(T(\partial\mathcal{O}))$:

$$\begin{aligned} \nabla_X(i\gamma(\nu)\varphi) &= i\gamma(\nabla_X\nu)\varphi + i\gamma(\nu)\nabla_X\varphi \\ &= i\gamma(\nabla_X\nu)\varphi + \alpha_0\gamma(\nu)\gamma(X)\varphi \\ &= i\gamma(\nabla_X\nu)\varphi - \alpha_0\gamma(X)\gamma(\nu)\varphi \\ &= i\gamma(\nabla_X\nu)\varphi + i\alpha_0\gamma(X)\varphi. \end{aligned}$$

However along the boundary we have $i\gamma(\nu)\varphi = \varphi$, so we obtain

$$\gamma(\nabla_X\nu)\varphi = -2\alpha_0\gamma(X)\varphi.$$

Since the spinor field φ has no zeros, we have $A(X) = -\nabla_X\nu = 2\alpha X$ and the boundary is totally umbilical. We can again show that in the equality case, we have $\text{Im}(\lambda^{\text{MIT}}) = n\alpha_0$. In fact, just note that the boundary term can be rewritten as

$$\int_{\partial\mathcal{O}} \langle D^S\varphi - \frac{n-1}{2}H\varphi + (n-1)\alpha_0\varphi, \varphi \rangle ds(g) = - \int_{\partial\mathcal{O}} \langle \nabla_\nu\varphi + \gamma(\nu)D\varphi - (n-1)\alpha_0\varphi, \varphi \rangle ds(g).$$

This term is zero since we have equality in (16). Now using that the spinor field φ is an imaginary Killing spinor satisfying (6) gives

$$\nabla_\nu\varphi + \gamma(\nu)D\varphi = \frac{n-1}{n}\text{Im}(\lambda^{\text{MIT}})\varphi.$$

Substituting in the preceding identity gives

$$(n-1) \int_{\partial\mathcal{O}} \left(\alpha_0 - \frac{\text{Im}(\lambda^{\text{MIT}})}{n} \right) |\varphi|^2 ds(g) = 0,$$

and since φ has no zeroes, $\text{Im}(\lambda^{\text{MIT}}) = n\alpha_0 = \frac{nH_0}{2}$. \square

Remark 2.

- (1) The orthogonal projection $\mathbb{B}_{\text{MIT}}^+$ defines a local elliptic boundary condition for the Dirac operator D of \mathcal{O} . We can easily check that in this case, the imaginary part of an eigenvalue λ^{MIT} of D satisfies $\text{Im}(\lambda^{\text{MIT}}) < 0$. Inequality (3) is then given by

$$|\lambda^{\text{MIT}}|^2 \geq \frac{n}{4(n-1)} R_0 - n \text{Im}(\lambda^{\text{MIT}}) H_0.$$

- (2) For $H_0 = 0$, we obtain Inequality (2). In fact, if we suppose that equality is achieved, Theorem 1 implies $\text{Im}(\lambda^{\text{MIT}}) = \frac{nH_0}{2} = 0$ which is impossible by Remark 1.
- (3) Note that the Riemannian spin manifolds with an imaginary Killing spinor with Killing number $i\alpha$ have been classified by H. Baum in [Bau89a] and [Bau89b]. Such manifolds are called pseudo-hyperbolic and they are given by

$$(\mathbb{R} \times_{\text{exp}} M_0, g) = (\mathbb{R} \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0}),$$

where (M_0, g_{M_0}) is a complete Riemannian spin manifold carrying a non-trivial parallel spinor. After suitable rescaling of the metric, we can assume that the Killing number is either $i/2$ or $-i/2$, i.e. we have

$$\nabla_X \phi = \pm \frac{i}{2} \gamma(X) \phi.$$

Moreover, constant mean curvature hypersurfaces in pseudo-hyperbolic manifolds are classified by the Hyperbolic Alexandrov Theorem proved in [Mon99] (see also [HMR03] for a proof using spinors). Indeed, such a hypersurface is either a round geodesic hypersphere (and, in this case, M_0 is flat and $H > 1$) or a slice $\{s\} \times M_0$ (and, in this case, M_0 is compact and $H = 1$).

We can then prove the following corollary:

Corollary 4. *If the boundary of the compact domain \mathcal{O} is connected, there is no manifold satisfying the equality case in Inequality (3).*

Proof: If \mathcal{O} is a compact domain with connected boundary achieving equality in (3), then there exists an imaginary Killing spinor on \mathcal{O} and the boundary $\partial\mathcal{O}$ is a totally umbilical constant mean curvature hypersurface with $H = 2\alpha$. However, using Remark (2).3, \mathcal{O} is a domain in a pseudo-hyperbolic space whose connected boundary is a slice $\{s\} \times M_0$ and then \mathcal{O} is non-compact. \square

Remark 3. With a slight modification of the boundary condition, we give a domain \mathcal{O} whose boundary has two connected components carrying an imaginary Killing spinor field $\varphi \in \Gamma(\Sigma\mathcal{O})$ which satisfy

$$i\gamma(\nu_1)\varphi|_{\partial\mathcal{O}_1} = \varphi|_{\partial\mathcal{O}_1} \quad \text{and} \quad i\gamma(\nu_2)\varphi|_{\partial\mathcal{O}_2} = -\varphi|_{\partial\mathcal{O}_2}, \quad (17)$$

where ν_1 (resp. ν_2) is an inner unit vector field normal to $\partial\mathcal{O}_1$ (resp. $\partial\mathcal{O}_2$). First recall that one distinguishes two types of imaginary Killing spinors (see [Bau89a] and [Bau89b]). Indeed, if $\varphi \in \Gamma(\Sigma\mathcal{O})$ is an imaginary Killing spinor, denote by f its length function, then the function

$$q_\varphi(x) := f(x)^2 - \frac{1}{4\alpha^2} \|\nabla f\|^2$$

satisfies q_φ is constant and $q_\varphi \geq 0$. If $q_\varphi = 0$, φ is a Killing spinor of type I whereas if $q_\varphi > 0$, φ is a Killing spinor of type II. If (N^n, g) is a complete connected Riemannian spin manifold with an imaginary Killing spinor of type II associated with the Killing number $i\alpha$, then (N^n, g) is isometric to the hyperbolic space $\mathbb{H}_{-4\alpha^2}^n$. If (N^n, g) admits an imaginary Killing spinor of type I, then (N^n, g) is isometric to the warped product $(\mathbb{R} \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0})$, where M_0 is a complete Riemannian spin manifold with a non-trivial parallel spinor field. Moreover, $q_\varphi = 0$ if and only if there exists a unit vector field ξ on N such that $\gamma(\xi)\varphi = i\varphi$. In fact, we can easily prove that the vector field ξ is the normal field of $\{t\} \times M_0$ for all $t \in \mathbb{R}$. So consider the domain given by the warped product $\mathcal{O} := ([a, b] \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0})$, where M_0 is a compact spin manifold carrying a non-trivial parallel spinor field and with $-\infty < a < b < +\infty$. The domain \mathcal{O} carries an imaginary Killing spinor φ of type I, so there exists ξ normal to $\{t\} \times M_0$ for all $t \in [a, b]$

such that $\gamma(\xi)\varphi = i\varphi$. The boundary of \mathcal{O} has two connected components which are slices $\{a\} \times M_0$ and $\{b\} \times M_0$ of \mathcal{O} and with mean curvature $H_a = H_b = 2\alpha$, where H_t is the mean curvature of a slice $\{t\} \times M_0$. The spinor field φ clearly satisfies the boundary conditions (17).

REFERENCES

- [Bär98] C. Bär, *Extrinsic bounds of the Dirac operator*, Ann. Glob. Anal. Geom. **16** (1998), 573–596.
- [Bau89a] H. Baum, *Complete Riemannian manifolds with imaginary Killing spinors*, Ann. Glob. Anal. Geom. **7** (1989), 205–226.
- [Bau89b] ———, *Odd-dimensional Riemannian manifolds admitting imaginary Killing spinors*, Ann. Glob. Anal. Geom. **7** (1989), 141–153.
- [BBW93] B. Booß-Bavnbek and K.P. Wojciechowski, *Elliptic boundary problems for the Dirac operator*, Birkhäuser, Basel, 1993.
- [BFGK90] H. Baum, T. Friedrich, R. Grunewald, and I. Kath, *Twistor and Killing spinors on Riemannian manifolds*, vol. 108, Seminarbericht, 1990, Humboldt-Universität zu Berlin.
- [BHMM] J.P. Bourguignon, O. Hijazi, J.L. Milhorat, and A. Moroianu, *A spinorial approach to Riemannian and conformal geometry*, Monograph (In Preparation).
- [CJJ⁺74] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, and V.F. Weisskopf, *New extended model of hadrons*, Phys. Rev. D **9** (1974), 3471–3495.
- [CJJT74] A. Chodos, R.L. Jaffe, K. Johnson, and C.B. Thorn, *Baryon structure in the bag theory*, Phys. Rev. D **10** (1974), 2599–2604.
- [Fri80] T. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nicht negativer Skalarkrümmung*, Math. Nach. **97** (1980), 117–146.
- [Fri00] ———, *Dirac operators in Riemannian geometry*, vol. 25, Amer. Math. Soc. Graduate Studies in Math., 2000.
- [HMR02] O. Hijazi, S. Montiel, and S. Roldán, *Eigenvalue boundary problems for the Dirac operator*, Commun. Math. Phys. **231** (2002), 375–390.
- [HMR03] ———, *Dirac operators on hypersurfaces of manifolds with negative scalar curvature*, Ann. Global Anal. Geom. **23** (2003), 247–264.
- [HMZ02] O. Hijazi, S. Montiel, and X. Zhang, *Conformal lower bounds for the Dirac operator on embedded hypersurfaces*, Asian J. Math. **6** (2002), 23–36.
- [Joh75] K. Johnson, *The M.I.T bag model*, Acta Phys. Pol. **B6** (1975), 865–892.
- [LM89] H.B. Lawson and M.L. Michelsohn, *Spin Geometry*, Princeton University Press ed., vol. 38, Princeton Math. Series, 1989.
- [Lop53] Ya.B. Lopatinskiĭ, *On a method for reducing boundary problems for a system of differential equations of elliptic type to regular integral equations*, Ukrain. Math. Ž. **5** (1953), 123–151, (Russian).
- [Mon99] S. Montiel, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), 711–748.

Author address:

Simon Raulot,
 Institut Élie Cartan BP 239
 Université de Nancy 1
 54506 Vandœuvre-lès -Nancy Cedex
 France

E-Mail: raulot@iecn.u-nancy.fr