BSDE driven by Dirichlet process and semi-linear parabolic PDE. Application to homogenization

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BSDE driven by Dirichlet Process and Semi-linear Parabolic PDE. Application to Homogenization

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Abstract: Backward Stochastic Differential Equations (BSDE) also gives the weak solution of a semi-linear system of parabolic PDEs with a second-order divergence-form partial differential operator and possibly discontinuous coefficients. This is proved here by approximation. After that, a homogenization result for such a system of semi-linear PDEs is proved using the weak convergence of the solution of the corresponding BSDEs in the S-topology.

Keywords: BSDE, divergence-form operator, homogenization, random media, periodic media

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1 Introduction

This article is devoted to the study of some Backward Stochastic Differential Equations (BSDE) for which the underlying process is associated to a divergence-form partial differential operator, and its connection with semi-linear parabolic PDE. An homogenization property is then proved.

The theory of BSDE is well known in the case of non-divergence form operators (see *e.g.*, Pardoux and Peng 1990, El Karoui 1997, Pardoux 1999a and references within). But it has been developed in the framework of Itô stochastic calculus and the classical or viscosity solutions of the corresponding PDE.

When the operator is of the form

$$L = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right) + b_i\frac{\partial}{\partial x_i},$$

for a bounded function b and a symmetric and bounded coefficient a satisfying the uniform ellipticity condition $\lambda|\xi|^2 \leqslant a(x)\xi\cdot\xi, \forall\xi \in \mathbb{R}^N$, $\forall x \in \mathbb{R}^N$ for some positive constant λ, the right notion of solution for the semi-linear parabolic PDE

$$\frac{\partial u(t,x)}{\partial t} + Lu(t,x) + h(t,x,u(t,x),\nabla u(t,x)) = 0 \text{ and } u(T,x) = g(x) \quad (1)$$

is the notion of weak — or generalized — solution. As it has been pointed first in Barles and Lesigne 1997, there exists also some connection between BSDE and weak solution. Our aim is to develop these links for divergence-form operators with discontinuous coefficients.

In the first step, results about the existence and uniqueness of the solutions for system of semi-linear parabolic PDEs are recalled. In particular, we state an approximation result which asserts the convergence of the solutions of a family of semi-linear system of parabolic PDEs to a solution of a semi-linear system of parabolic PDE when the coefficients of their operators and the non-linear term converges to that of the limiting non-linear equation.

If the coefficients in (1) are smooth, its generalized solution u is a classical solution. In this case, the couple

$$\mathsf{Y}_t = u(t,\mathsf{X}_t),\ \mathsf{Z}_t = \nabla u(t,\mathsf{X}_t) \quad (2)$$

is the solution to the BSDE

$$\mathsf{Y}_t = g(\mathsf{X}_T) + \int_t^T h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s - \int_t^T \mathsf{Z}_s\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}}, \quad (3)$$

where M^{X} is the martingale part of the process X whose generator is L.

Using the theory of semi-group and Dirichlet forms (Fukushima, Oshima and Takeda 1994), a strong Markov process X may be associated to a divergence-form operator. Furthermore, an approximation of the coefficients of L by smooth coefficients yields the convergence in distribution of the associated processes.

Combining this result with the convergence of the solution to some semi-linear parabolic PDE, we prove that the identification (2) of the solution of the BSDE (3) is also valid when the second-order differential operator is a divergence-form operator with possibly discontinuous coefficients.

We have to pass to the limit in the semi-martingale corresponding to the BSDE. But we have to face discontinuous functions, since the functional spaces that arises when studying divergence-form operators are Sobolev spaces. Our approach is inspired by that of Rozkosz and Słomiński 1991. As the process X is not in general solution to a BSDE, the use of the Krylov estimate has been replaced by the Aronson estimate on the transition density function.

Our proof does not require any knowledge about the stochastic calculus for processes associated to a Dirichlet Form. But results similar to ours may be proved directly without approximation: this is the approach chosen by V. Bally, É. Pardoux and L. Stoica (Bally, Pardoux and Stoica 2005).

The representation of the solution of semi-linear PDE given by (2) and (3) is extended in Section 5 to the system of semi-linear PDE

$$\frac{\partial u(t,x)}{\partial t}+Lu(t,x)+h(t,x,u(t,x),\nabla u(t,x))+\nabla u(t,x)\widehat{h}(t,x,u(t,x))=0. \quad (4)$$

Here, the first-order differential non-linear term $u(t,x)\widehat{h}(t,x,u(t,x))$ does not satisfy the same Lipschitz condition the term h satisfies. However, the system of semi-linear Parabolic PDE (4) has a unique solution, and a probabilistic representation for it using BSDE may be given too. With a Girsanov transform, this leads to construct the solution of the BSDE

$$\mathsf{Y}_t=g(\mathsf{X}_T)+\int_t^T h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s+\int_t^T \mathsf{Z}_s\widehat{h}(s,\mathsf{X}_s,\mathsf{Y}_s)\,\mathrm{d}s-\int_t^T \mathsf{Z}_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}}_s.$$

In Section 6, we consider the homogenization property for the family of semi-linear system of parabolic PDEs

$$\frac{\partial u^\varepsilon(t,x)}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}(x/\varepsilon)\frac{\partial u^\varepsilon(t,x)}{\partial x_j}\right) \\ +h(t,\tau_{x/\varepsilon}\omega,x,u^\varepsilon(t,x))+\nabla u^\varepsilon(t,x)\widehat{h}(t,\tau_{x/\varepsilon},x,u^\varepsilon(t,x))=0 \quad (5)$$

where the functions $a_{i,j}$, $\omega \mapsto h(\cdot, \omega, \cdot, \cdot, \cdot)$ and $\omega \mapsto \widehat{h}(\cdot, \omega, \cdot, \cdot, \cdot)$ are either periodic or stationary random fields in an ergodic random media $(\Omega, \mathcal{G}, \mu, (\tau_x)_{x\in\mathbb{R}^N})$.

The homogenization consists in proving the convergence of u^ε to the solution of a system of semi-linear parabolic PDE for which the coefficients of the linear operator are constant, and whose non-linear terms are given by some averaging of h and $\widehat{h}$.

This is a case of non trivial convergence of solutions of semi-linear parabolic PDEs. For that, a method introduced by É. Pardoux and A. Veretennikov (Pardoux and Veretennikov 1997) is used to prove the weak convergence of the solutions of the BSDEs.

This method relies on the use of a topology weaker than the Skorohod topology, but for which the tightness criterion is easy to prove. The homogenization property has yet been proved for semi-linear PDEs with a non divergence-form (Gaudron and Pardoux 2001, Pardoux 1999a, Pardoux 1999b), using the Meyer-Zheng topology (Meyer and Zheng 1984). In this article, we use the S-topology which has been recently introduced by A. Jakubowski (Jakubowski 1997).

However, the main drawback of this method is the fact that it does not allow to deal with non-linearity in ∇u, except for some special case of a quadratic term $|\nabla u|^2$ (see Gaudron and Pardoux 2001). Here, our assumptions are slightly different from that of Gaudron and Pardoux 2001, since we have to deal with a first-order differential term that is non-linear. In fact, we have first to prove that the BSDE also gives us a probabilistic representation of the solution of the semi-linear PDE (5).

An analytical proof of the homogenization property in periodic media, that gives only a convergence in $\mathrm{L}^2(\mathbb{R}^N)$ instead of a pointwise convergence, may be found in Section 16, p. 200 of Bensoussan, Lions and Papanicolaou 1978. A probabilistic study of the homogenization of linear-parabolic PDE with divergence-form operator having discontinuous coefficients may be found in Lejay 2001a for the periodic media and Lejay 2001b for the random media.

We have to note that another probabilistic technique has been developed using some stability Theorems for BSDE (Hu and Peng 1997) in order to prove some homogenization results in periodic media for non-linear PDE with non-divergence form operators : see Hu 1997, Buckdahn, Hu and Peng 1998. But while non-linear terms of the form $h(t, x, u(t, x), \nabla u(t, x))$ may be considered, more regularity of the coefficients is needed than in the previously cited works. Recently, F. Castell has adapted this method for random media (Castell 2001), but this is require more precise estimates than in the case of periodic media.

The paper is organized as follows: Section 2 contains some analytical facts about the solution of a system of semi-linear parabolic PDEs, mainly existence, regularity and approximation. Some facts about the process associated to the divergence-form operator L are recalled in Section 3, and a Martingale Representation Theorem with respect to the martingale part of such a process is proved. The link 2 between the solution of the semi-linear system of parabolic PDE and the BSDE is proved in Section 4. In section 5, the sequels of the addition of a non-linear first-order differential operator are studied. The homogenization property is considered in Section 6. A few remarks on some possible extension of the previous results are made in Section 7. This paper ends with an appendix containing some properties of the S-topology.

2 Semi-linear parabolic PDEs

We give in this Section the main results concerning semi-linear parabolic PDE, mainly existence, uniqueness and regularity of a solution, and a convergence results on the solutions.

We assume that $\mathcal{O}$ is a bounded, open, connected subset of $\mathbb{R}^N$, and that its boundary is smooth.

Let $a = (a_{i,j})_{i,j=1,\dots,N}$ be a measurable function on $\mathcal{O}$ with value in the space of symmetric matrices and satisfying the uniform ellipticity and boundedness condition:

$$\lambda|\xi|^2 \leqslant a(x)\xi\cdot\xi \leqslant \Lambda|\xi|^2,\ \forall\xi\in\mathbb{R}^N,\ \forall x\in\mathcal{O}, \tag{6}$$

for some positive constants λ and Λ.

The operator $(L, \mathrm{Dom}(L))$ is the self-adjoint operator defined by

$$L = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right),$$
$$\mathrm{Dom}(L) = \left\{\, f\in\mathrm{L}^2(\mathcal{O}) \,\middle|\, Lf\in\mathrm{L}^2(\mathcal{O}) \,\right\} \subset \mathrm{H}^1_0(\mathcal{O}).$$

The operator $(L, \mathrm{Dom}(L))$ is associated to $\mathcal{E}$ by the relation

$$\mathcal{E}(u,v) = -\langle Lu, v\rangle,\ \forall(u,v)\in\mathrm{Dom}(L)\times\mathrm{H}^1_0(\mathcal{O}),$$

where $\mathcal{E}$ is the bilinear form

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathcal{O}} a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,\mathrm{d}x \tag{7}$$

defined on $\mathrm{H}^1_0(\mathcal{O})\times\mathrm{H}^1_0(\mathcal{O})$.

Let h be a measurable function on $\mathbb{R}_+\times\mathcal{O}\times\mathbb{R}^m\times\mathbb{R}^{m\times N}$ satisfying

(h-i) $y \mapsto h_i(t,x,y,z)$ is continuous;

(h-ii) $|h(t,x,y,z) - h(t,x,y,z')| \leqslant K_0 \|z - z'\|$ with $\|z\| = \sqrt{\mathrm{Tr}(zz^{\mathrm{T}})}$;

(h-iii) $\langle h(t,x,y,z) - h(t,x,y',z), y - y' \rangle \leqslant K_1 |y - y'|^2$;

(h-iv) $|h(t,x,0,0)| \leqslant K_2$ and $|h(t,x,0,0)| \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O}))^m$;

(h-v) $|h(t,x,y,z)| \leqslant |h(t,x,0,0)| + K_3|y| + K_4|z|$

for some positive constants $K_1, \dots, K_4$.

Let T be a positive real. We consider the system of semi-linear parabolic PDEs

$$\begin{cases} \text{for } i = 1\dots,N,\ t \in [0,T] \text{ and } x \in \mathcal{O}, \\ \dfrac{\partial u^i(t,x)}{\partial t} + Lu^i(t,x) + h_i(t,x,u(t,x),\nabla u^1(t,x),\dots,\nabla u^m(t,x)) = 0, \\ u(T,x) = g(x), \\ u^i(t,\cdot) \in \mathrm{H}^1_0(\mathcal{O}),\ \forall t \in [0,T), \end{cases} \tag{8}$$

where the final condition g belongs to $\mathrm{L}^2(\mathcal{O})^m$

We have to note that the condition $u^i(t,\cdot) \in \mathrm{H}^1_0(\mathcal{O})$ means that the function u^i is equal to 0 on the set of points $\{ (t,x) \,|\, t \in [0,T),\ x \in \partial\mathcal{O} \}$. Hence, this is a *lateral boundary condition.*

Remark 1. We use here the final condition instead of an initial condition for reason of facility when dealing with BSDEs. The function $\widetilde{u}(t,x) = u(T-t,x)$ is solution to the system of semi-linear parabolic PDE

$$\frac{\partial \widetilde{u}(t,x)}{\partial t} = L\widetilde{u}(t,x) + h(T-t,x,\widetilde{u}(t,x),\nabla\widetilde{u}(t,x))$$

with the initial condition $\widetilde{u}(0,x) = g(x)$. If the non-linear term does not depend on the time t, then both systems are equivalent.

We say that u is a solution of (8) if it belongs to

$$\mathcal{W} = \left\{ f \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\mathcal{O}))^m \,\middle|\, \frac{\partial f}{\partial t} \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathcal{O}))^m \right\}$$

and satisfies for $i = 1,\dots,m$ — setting $u_t = u(t,\cdot)$ —,

$$\begin{aligned} &\int_0^T \left\langle u^i_t, v \right\rangle_{\mathrm{L}^2(\mathcal{O})} \varphi'(t)\,\mathrm{d}t + \langle g_i, v \rangle_{\mathrm{L}^2(\mathcal{O})} \varphi(T) - \left\langle u^i_0, v \right\rangle_{\mathrm{L}^2(\mathcal{O})} \varphi(0) \\ &\quad = -\int_0^T \mathcal{E}(u^i_t, v)\varphi(t)\,\mathrm{d}t + \int_0^T\!\!\int_{\mathcal{O}} h_i(t,x,u(t,x),\nabla u(t,x))v(x)\varphi(t)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{9}$$

for any v in $\mathrm{H}^1_0(\mathcal{O})$ and any smooth function φ in $\mathcal{C}^\infty([0,T];\mathbb{R})$. We have to note that there exists a version $\widehat{u}(t,x)$ of $u(t,x)$ such that $t\mapsto\widehat{u}(t,\cdot)$ is continuous from $[0,T]$ into $\mathrm{L}^2(\mathcal{O})^m$. We set

$$\mathcal{H}=\mathcal{C}(0,T;\mathrm{L}^2(\mathcal{O})^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^1_0(\mathcal{O}))^m,$$

so that $\widehat{u}$ belongs to $\mathcal{H}$. We systematically use the continuous version $\widehat{u}$ of $t\mapsto u(t,\cdot)$. The norm $|\cdot|_{\mathcal{H}}$ on $\mathcal{H}$ is defined by

$$|v|^2_{\mathcal{H}}=\sup_{0\leqslant t\leqslant T}\|v(t,\cdot)\|^2_{\mathrm{L}^2(\mathcal{O})}+\int_0^T\|\nabla v(t,\cdot)\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}t,$$

where we use the convention that for a function $v=(v_1,\dots,v_m)$ in $\mathrm{L}^2(\mathcal{O})^m$, $\|v\|^2_{\mathrm{L}^2(\mathcal{O})}=\sum_{i=1}^m\|v_i\|^2_{\mathrm{L}^2(\mathcal{O})}$. So $\|\nabla v\|_{\mathrm{L}^2(\mathcal{O})}$ is equal to $\sum_{\substack{i=1,\dots,N\\ j=1,\dots,m}}\|\partial v_j/\partial x_i\|^2_{\mathrm{L}^2(\mathcal{O})}$. We also suppress any further references in the name of functional spaces, *i.e.*, when there is no ambiguity, $\mathrm{L}^2(\mathcal{O})$ means $\mathrm{L}^2(\mathcal{O})^m$.

We say that a constant *depends only on the structure of* (8) if this constant depends only on λ, Λ, $K_0,\dots,K_4$, T and the dimension N.

Convention 1. We use the convention that in the proofs, the constants C_0, C_1, ... depend only on the structure of (8).

Theorem 1. i) *There exists a unique weak solution u to* (8). *Furthermore, this solution is bounded and Hölder continuous in any compact subset of* $(0,T)\times\mathcal{O}$.

ii) *If $\mathcal{O}$ is bounded, and if the final condition g is bounded, then u is bounded on* $[0,T]\times\mathcal{O}$.

iii) *If $\mathcal{O}$ is bounded and u is a solution to* (8)*, but with a boundary condition $u(t,x)=\varphi(t,x)$ on $[0,T]\times\partial\mathcal{O}\cup\{\,T\,\}\times\mathcal{O}$ such that φ is bounded and Hölder continuous, then u is bounded and Hölder continuous on* $[0,T]\times\mathcal{O}$.

The last point of this Theorem implies that if the final condition g is Hölder continuous and has compact support on $\mathcal{O}$, then u is Hölder continuous on $[0,T]\times\mathcal{O}$.

Proof. The existence and uniqueness of the solution is a well-known fact (see *e.g.*, Theorem 30.A, p. 771 in Zeidler 1990 for a proof with the Galerkin method, or by Theorem V.6.1, p. 466 in Ladyženskaja, Solonnikov and Ural'ceva 1968).

With our hypotheses on the coefficients of a and the non-linear term, we obtain the inequalities VII.(2.8) and VII.(3.3) p. 577 and p. 579 in Ladyženskaja, Solonnikov and Ural'ceva 1968, and Theorems VII.2.1, VII.2.2 and VII.3.1, p.578 and p. 582 in Ladyženskaja, Solonnikov and Ural'ceva 1968 may be applied. □

Remark 2. The condition (h-iv) we gave on h may not be the optimal condition, but we have to note that it is important that $\left(\int_0^T \left(\int_{\mathcal{O}} |h(t,x,0,0)|^p \,\mathrm{d}x\right)^{q/p} \mathrm{d}t\right)^{1/q}$ is satisfied for some couple (p,q) satisfying $N/2p + 1/q < 1$.

We recall here an approximation result for a system of non-linear parabolic PDEs, which yields that the convergence of the coefficients implies the convergence of the solutions.

Theorem 2. *Let* $(a^\varepsilon)_{\varepsilon\geqslant 0}$ *be a family of measurable functions satisfying*

$$a^\varepsilon \xrightarrow[\varepsilon\to 0]{} a^0 \text{ almost everywhere,} \tag{10a}$$

$$\text{and } \lambda|\xi|^2 \leqslant a^\varepsilon(x)\xi\cdot\xi \leqslant \Lambda|\xi|^2,\ \forall\xi\in\mathbb{R}^N,\quad \forall x\in\mathcal{O},\ \forall\varepsilon\geqslant 0. \tag{10b}$$

We also assume that there exists a family $(h^\varepsilon)_{\varepsilon\geqslant 0}$ *of functions satisfying* (h-i)-(h-v) *with constants independent of* ε, *and such that*

$$h^\varepsilon(t,x,y,z) \xrightarrow[\varepsilon\to 0]{} h^0(t,x,y,z) \text{ a.e. on } \mathbb{R}_+\times\mathcal{O}\times\mathbb{R}^m\times\mathbb{R}^{N\times m}. \tag{11}$$

Let L^ε *be the divergence-form operator* $L^\varepsilon = \frac{1}{2}\partial_{x_i}\left(a^\varepsilon_{i,j}\partial_{x_j}\right)$. *Then the solution* u^ε *to the system*

$$\begin{cases} \dfrac{\partial u^\varepsilon(t,x)}{\partial t} + L^\varepsilon u^\varepsilon(t,x) + h^\varepsilon(t,x,u^\varepsilon(t,x),\nabla u^\varepsilon(t,x)) = 0, \\ u^\varepsilon(T,x) = g(x)\in \mathrm{L}^2(\mathcal{O}) \text{ and } u^\varepsilon(t,x)\in \mathrm{H}^1_0(\mathcal{O})^m \text{ for } t\in[0,T), \end{cases} \tag{12}$$

converges for the norm $|\cdot|_{\mathcal{H}}$, *as* ε *decreases to* 0, *to the solution of the system* (12) *with* $\varepsilon = 0$.

Proof. We remark that in the weak sense,

$$\begin{aligned} \frac{\partial u^\varepsilon(t,x) - u^0(t,x)}{\partial t} &+ L^\varepsilon(u^\varepsilon(t,x) - u^0(t,x)) \\ &+ h^\varepsilon(t,x,u^\varepsilon(t,x),\nabla u^\varepsilon(t,x)) - h^0(t,x,u^0(t,x),\nabla u^0(t,x)) \\ &\qquad + \frac{1}{2}\frac{\partial}{\partial x_i}\left((a^0_{i,j} - a^\varepsilon_{i,j})\frac{\partial u^0(t,x)}{\partial x_j}\right) = 0. \end{aligned}$$

For each $\varepsilon > 0$, we choose $\widehat{u}^\varepsilon(t,x) = u^\varepsilon(t,x) - u^0(t,x)$ as a test function. Hence,

$$\begin{aligned} \langle h^\varepsilon(t,x,u^\varepsilon(t,x),\nabla u^\varepsilon(t,x)) - h^\varepsilon(t,x,u^0(t,x),\nabla u^0(t,x)), \widehat{u}^\varepsilon(t,x)\rangle \\ \leqslant K_1|\widehat{u}^\varepsilon(t,x)|^2 + K_4\|\nabla\widehat{u}^\varepsilon(t,x)\|\,|\widehat{u}^\varepsilon(t,x)|, \end{aligned}$$

and that

$$f^\varepsilon(t,x) \stackrel{\text{def}}{=} h^\varepsilon(t,x,u^0(t,x),\nabla u^0(t,x)) \xrightarrow[\varepsilon\to 0]{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O}))} f^0(t,x) \stackrel{\text{def}}{=} h^0(t,x,u^0(t,x),\nabla u^0(t,x)),$$

because of the assumptions (h-i)-(h-v) on f^ε and the fact that h^ε is bounded almost everywhere by $K_2 + K_3|u^0| + K_4|\nabla u^0| \in \mathrm{L}^2(\mathcal{O})$.

Furthermore,

$$\sum_{i=1}^m \mathcal{E}^\varepsilon(\widehat{u}^{i,\varepsilon}(t,x),\widehat{u}^{i,\varepsilon}(t,x)) \geqslant \frac{\lambda}{2}\left\|\nabla\widehat{u}^\varepsilon(t,x)\right\|^2_{\mathrm{L}^2(\mathcal{O})}.$$

Using the inequality

$$2\alpha\beta \leqslant \delta\alpha^2 + \frac{1}{\delta}\beta^2,\ \forall\alpha,\beta\in\mathbb{R},\forall\delta>0, \tag{13}$$

in a judicious way, we obtain the existence of constants C_1, C_2 and C_3 such that, for any $\varepsilon > 0$,

$$\begin{aligned} C'\|\nabla\widehat{u}^\varepsilon(t,x)\|\,\|\widehat{u}^\varepsilon(t,x)\| &\leqslant \frac{\lambda}{12}\|\nabla\widehat{u}^\varepsilon(t,x)\|^2 + C_1\|\widehat{u}^\varepsilon(t,x)\|^2,\\ \langle(a^\varepsilon-a^0)\nabla u^0(t,x),\nabla\widehat{u}^\varepsilon(t,x)\rangle &\leqslant \frac{\lambda}{12}\|\nabla\widehat{u}^\varepsilon(t,x)\|^2 + C_2\|(a^\varepsilon-a^0)\widehat{u}^0(t,x)\|^2,\\ \langle f^\varepsilon(t,x)-f^0(t,x),\nabla\widehat{u}^\varepsilon(t,x)\rangle &\leqslant \frac{\lambda}{12}\|\widehat{u}^\varepsilon(t,x)\|^2 + C_3\|f^\varepsilon(t,x)-f^0(t,x)\|^2. \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} \frac{-1}{2}\frac{\partial}{\partial t}\|\widehat{u}^\varepsilon_t\|^2 + \frac{\lambda}{4}\|\nabla\widehat{u}^\varepsilon_t\|^2 \leqslant C_4\|\widehat{u}^\varepsilon_t\|^2 \\ + C_2\|(a^\varepsilon-a^0)\widehat{u}^0(t,\cdot)\|^2_{\mathrm{L}^2(\mathcal{O})} + \|f^\varepsilon(t,\cdot)-f^0(t,\cdot)\|^2_{\mathrm{L}^2(\mathcal{O})}, \end{aligned}$$

which yields, after an integration with respect to the time that

$$\begin{aligned} \frac{1}{2}\|\widehat{u}^\varepsilon_t\|^2_{\mathrm{L}^2(\mathcal{O})} + \frac{\lambda}{4}\int_t^T \|\nabla\widehat{u}^\varepsilon_\tau\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}\tau \leqslant C_4\int_t^T \|\widehat{u}^\varepsilon_\tau\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}\tau \\ + C_2\int_t^T \Big(\|(a^\varepsilon-a^0)\widehat{u}^0(\tau,\cdot)\|^2_{\mathrm{L}^2(\mathcal{O})} + \|f^\varepsilon(\tau,\cdot)-f^0(\tau,\cdot)\|^2_{\mathrm{L}^2(\mathcal{O})}\Big)\,\mathrm{d}\tau, \end{aligned} \tag{14}$$

since $\widehat{u}^\varepsilon_T = 0$. From the Lebesgue Dominated Convergence Theorem, the last two terms converge to 0 with ε. The Gronwall Lemma (see *e.g.*, Theorem 5.1, p. 498 in Ethier and Kurtz 1986) implies that $\sup_{0\leqslant t\leqslant T}\|\widehat{u}^\varepsilon_t\|_{\mathrm{L}^2(\mathcal{O})}$ converges to 0 as ε goes to 0. We easily deduce from (14) that $|\widehat{u}^\varepsilon|_{\mathcal{H}} \xrightarrow[\varepsilon\to 0]{} 0$. □

3 On the Markov process associated to some divergence-form operator

We quickly recall in this Section some results about stochastic processes associated to divergence-form operators. In Section 3.1, such a process is constructed using results on its transition functions. In Section 3.2, a martingale representation theorem with respect to the martingale part of processes generated by Divergence-form operators is proved. Section 3.3 is devoted to the relation between a process generated by a divergence-form defined on the whole space, and the process generated by the same operator defined only on some arbitrary domain.

3.1 Transition function and stochastic process

The quadratic bilinear form $\mathcal{E}$ defined in (7) is clearly a strong local, regular Dirichlet Form on $\mathrm{L}^2(\mathcal{O})$ (see Fukushima, Oshima and Takeda 1994, Ma and Röckner 1991). Let $(P_t)_{t>0}$ be the semi-group it generates.

Let us assume for a moment that $\mathcal{O} = \mathbb{R}^N$.

Let $p(t,x,y)$ be the fundamental solution — which existence is ensured by standard results in PDE theory (Friedman 1964) — of the equation (x is fixed)

$$\begin{cases} \dfrac{\partial p(t,x,y)}{\partial t} = Lp(t,x,y),\ \forall y \in \mathbb{R}^N,\ \forall t > 0, \\ p(0,x,y) = \delta_{x-y}. \end{cases}$$

This fundamental solution is continuous, symmetric in x and y, *i.e.*, $p(t,x,y) = p(t,y,x)$, and satisfies the Aronson estimate, *i.e.*, there exists a constant M depending only on Λ, λ and N such that

$$p(t,x,y) \leqslant \frac{M}{t^{N/2}} \exp\left(\frac{-|x-y|^2}{Mt}\right) \tag{15}$$

for any (t,x,y) in $\mathbb{R}_+^* \times \mathbb{R}^N \times \mathbb{R}^N$.

There exist also some constants $C = C(N,\lambda,\Lambda) > 0$ and $\alpha = \alpha(N,\lambda,\Lambda) \in (0,1)$ such that for every $\delta > 0$,

$$|p(t',x',y') - p(t,x,y)| \leqslant \frac{C}{\delta^N}\left(\frac{\sqrt{|t'-t|} \vee |x'-x| \vee |y'-y|}{\delta}\right)^\alpha \tag{16}$$

for all $(t',x',y'), (t',x',y') \in [\delta^2,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$ with $|y'-y| \vee |x'-x| \leqslant \delta$.

Proofs of these estimates may be found for example in Aronson 1968, Stroock 1988, or in the Appendix A, p. 536 of Jikov, Kozlov and Oleinik 1994 only for (15).

This fundamental solution p is the transition density function of the semi-group $(P_t)_{t>0}$ associated to $(L, \mathrm{Dom}(L))$, *i.e.*,

$$\text{for any } f \in \mathrm{L}^2(\mathbb{R}^N)\ , P_t f(x) = \int_{\mathbb{R}^N} p(t,x,y) f(y)\,\mathrm{d}y \text{ a.e.} \tag{17}$$

and this semi-group is a Feller semi-group, as it is proved for example in Theorem II.3.1, p. 341 in Stroock 1988.

With (15) and (16), for each $t > 0$, if we use the representation (17) of $P_t f$ for any $f \in \mathrm{L}^2(\mathbb{R}^N)$, P_t is continuous from $\mathrm{L}^2(\mathbb{R}^N)$ into the space $\mathcal{C}_0(\mathbb{R}^N;\mathbb{R})$ of continuous functions on $\mathbb{R}^N$ that vanish at infinity.

The resolvent $G_\alpha = (\alpha - L)^{-1}$ is linked to the semi-group by the relation

$$G_\alpha f(x) = \int_0^{+\infty} e^{-\alpha t} P_t f(x)\,\mathrm{d}t,\ \forall \alpha > 0 \tag{18}$$

and the domain $\mathrm{Dom}(L)$ of L is also equal to $G_\alpha(\mathrm{L}^2(\mathbb{R}^N))$.

The existence of a Hunt process $(\mathsf{X}_t, t \geqslant 0; \mathbb{P}_x, x \in \mathbb{R}^N)$ whose infinitesimal generator is $(L, \mathrm{Dom}(L))$ follows from the Feller property of the semi-group. Since $P_t 1 = 1$ for any $t > 0$, the life-time of this stochastic process is infinite (see Theorem 4.5.4, p. 165 in Fukushima, Oshima and Takeda 1994) The strong local property of its associated Dirichlet form $(\mathcal{E}, \mathrm{H}^1(\mathbb{R}^N))$ implies that X continuity. The filtration $(\mathcal{F}^{\mathsf{X}}_t)_{t\geqslant 0}$ of the process is the minimal complete admissible filtration (see Appendix A.2, p. 310 of Fukushima, Oshima and Takeda 1994 for definition) and is consequently right-continuous.

In fact, it may be proved that the process X is really a Dirichlet process, *i.e.*, the sum of a square-integrable martingale and a term locally of zero-quadratic variation (see Theorem 2.1, p. 19 in Rozkosz 1996a). Let us denote by M^{X} the martingale part of X.

3.2 A Martingale Representation Theorem

We give a Martingale Representation Theorem with respect to the martingale part M^{X} of the process X. This is required to prove the existence of the solution of some BSDE under the distribution $\mathbb{P}_x$ of X.

The space $\mathrm{Dom}_{\mathcal{C}}(L) = G_\alpha(\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^N;\mathbb{R}))$ is dense in $\mathrm{Dom}(L)$ and dense in the Banach space $\mathcal{C}_0(\mathbb{R}^N;\mathbb{R})$ consisting of those continuous functions which vanish at infinity (see *e.g.*, Proposition 5.3, p. 687 in Tomisaki 1980).

The Itô formula implies that for any $f \in \mathrm{Dom}_{\mathcal{C}}(L)$, $f(\mathsf{X}_t) - f(\mathsf{X}_0) - \int_0^t Lf(\mathsf{X}_s)\,\mathrm{d}s$ is a $(\mathcal{F}^{\mathsf{X}}, \mathbb{P}_x)$-martingale for any starting point x in $\mathbb{R}^N$ (this is true for any starting point x because f is continuous and Lf is bounded (See Chapter 5 in Fukushima, Oshima and Takeda 1994, and not only for quasi-every starting point).

Lemma 1. *For each x in $\mathbb{R}^N$, the distribution $\mathbb{P}_x$ is the unique solution to the martingale problem:*

$$f(\mathsf{X}_t) - f(x) - \int_0^t Lf(\mathsf{X}_s)\,\mathrm{d}s \text{ is a } (\mathcal{F}^{\mathsf{X}}, \mathbb{P}_x)\text{-martingale } \forall f \in \mathrm{Dom}_{\mathcal{C}}(L).$$

Proof. Let $(\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^N}$ be another family of solutions to the martingale problem. For a function f in $\mathrm{Dom}_{\mathcal{C}}(L)$, we set

$$\widetilde{G}_\alpha f(x) = \int_0^{+\infty} e^{-\alpha t}\widetilde{\mathbb{E}}_x\left[\, f(\mathsf{X}_t)\,\right]\,\mathrm{d}t.$$

It is then clear that $\|\widetilde{G}_\alpha f\|_\infty \leqslant \frac{1}{\alpha}\,\|f\|_\infty$. Then

$$\int_0^{+\infty} e^{-\alpha t}\widetilde{\mathbb{E}}_x\left[\, f(\mathsf{X}_t) - f(\mathsf{X}_0) - \int_0^t Lf(\mathsf{X}_s)\,\mathrm{d}s\right]\mathrm{d}t = 0$$

for any $\alpha > 0$ and any $x \in \mathbb{R}^N$. An integration by parts gives

$$\widetilde{G}_\alpha(\alpha - L)f(x) = f(x)$$

and $\widetilde{G}_\alpha = (\alpha - L)^{-1} = G_\alpha$ on $\mathrm{Dom}_{\mathcal{C}}(L)$. By density of this space in $\mathcal{C}_0(\mathbb{R}^N;\mathbb{R})$ equipped with the uniform norm and (18),

$$\mathbb{E}_x\left[\, f(\mathsf{X}_t)\,\right] = \widetilde{\mathbb{E}}_x\left[\, f(\mathsf{X}_t)\,\right],\ \forall f \in \mathcal{C}_0(\mathbb{R}^N;\mathbb{R}),\ \forall t \geqslant 0,\ \forall x \in \mathbb{R}^N.$$

According to Corollary 4.3, p. 186 in Ethier and Kurtz 1986, this is sufficient to prove that $\mathbb{P}_x = \widetilde{\mathbb{P}}_x$ for any $x \in \mathbb{R}^N$. □

Theorem 3 (A Martingale Representation Theorem). *Let x be a point of $\mathbb{R}^N$. Let M be a locally square-integrable $(\mathcal{F}_t^{\mathsf{X}}, \mathbb{P}_x)$-martingale. Then there exists some $\mathcal{F}_t^{\mathsf{X}}$-predictable process H such that*

$$\mathbb{E}_x\left[\sum_{i,j=1}^N \int_0^{+\infty} \mathsf{H}_s^i \mathsf{H}_s^j\,\mathrm{d}\langle \mathsf{M}^{\mathsf{X},i}, \mathsf{M}^{\mathsf{X},j}\rangle_s\right] < +\infty,$$

$$\text{and } \mathsf{M}_t = \mathsf{M}_0 + \sum_{i=1}^N \int_0^t \mathsf{H}_s^i\,\mathrm{d}\mathsf{M}_s^{\mathsf{X},i}.$$

Proof. Let σ be a matrix such that $\sigma\sigma^{\mathrm{T}} = a$. A Martingale Representation Theorem under $\mathbb{P}_x$ holds for the strongly orthogonal family of martingale $\left(\int_0^t \sigma_{1,j}^{-1}(\mathsf{X}_s)\,\mathrm{d}\mathsf{M}_s^{\mathsf{X},j}, \dots, \int_0^t \sigma_{N,j}^{-1}(\mathsf{X}_s)\,\mathrm{d}\mathsf{M}_s^{\mathsf{X},j}\right)$, if $\mathbb{P}_x$ is an extremal point of the set of probability measures $\widehat{\mathbb{P}}_x$ absolutely continuous with respect to $\mathbb{P}_x$ and such that the elements of this family are also square-integrable $\widehat{\mathbb{P}}_x$-martingales (see Theorem 39, p. 152 in Protter 1990).

Let f be a function in $\mathrm{Dom}_{\mathcal{C}}(L)$. For any $s > 0$, the square-integrable, continuous, local martingale $\mathsf{M}_t^{[f],s} = f(\mathsf{X}_t) - f(\mathsf{X}_s) - \int_s^t Lf(\mathsf{X}_r)\,\mathrm{d}r$ is equal to $\int_s^t \nabla f(\mathsf{X}_r)\,\mathrm{d}\mathsf{M}_r^{\mathsf{X}}$ $\mathbb{P}_x$-a.s. for a fixed starting point x. The result follows from Chapter 5 of Fukushima, Oshima and Takeda 1994. Generally, this is true for quasi-every point, but the existence of a transition density function, the Markov property and the fact that the previous expression of $f(\mathsf{X}_t)$ uses only the process between $s > 0$ and t allows to prove it for any starting point.

And this is true for any starting point because of the existence of a transition density function and the Markov property may be used.

Let x be a point in $\mathbb{R}^N$. Let $\widehat{\mathbb{P}}$ be a measure absolutely continuous with respect to $\mathbb{P}_x$ such that M^{X} is a square integrable martingale with respect to $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{P}}_{|\mathcal{F}_0^{\mathsf{X}}} = \mathbb{P}_{x|\mathcal{F}_0^{\mathsf{X}}}$. Hence $\mathsf{M}^{[f],s}$ is also a $(\mathcal{F}^{\mathsf{X}}, \widehat{\mathbb{P}})$-martingale for any $s > 0$.

As f and Lf are continuous and bounded, $\lim_{s\to 0} \mathsf{M}_t^{[f],s} = f(\mathsf{X}_t) - f(\mathsf{X}_0) - \int_0^t Lf(\mathsf{X}_s)\,\mathrm{d}s$ is a $(\mathcal{F}^{\mathsf{X}}, \widehat{\mathbb{P}})$-martingale. So, $\widehat{\mathbb{P}}$ is a solution of the martingale problem and is equal to $\mathbb{P}_x$. The result is proved. □

3.3 Killed process and generator on arbitrary domain

Now, if we work with an open, connected subset $\mathcal{O}$ of $\mathbb{R}^N$, the process associated to the Dirichlet form $\frac{1}{2}\int_{\mathcal{O}} a_{i,j}\partial_{x_i}u(x)\partial_{x_j}v(x)\,\mathrm{d}x$ on $\mathrm{H}_0^1(\mathcal{O}) \times \mathrm{H}_0^1(\mathcal{O})$ is the process X killed when exiting $\mathcal{O}$ (Theorem 4.4.2, p. 154 in Fukushima, Oshima and Takeda 1994).

This is done by adding to $\mathcal{O}$ an extra point Δ, and by defining the new process

$$\widehat{\mathsf{X}}_t = \begin{cases} \mathsf{X}_t & \text{if } t < \tau, \\ \Delta & \text{otherwise,} \end{cases}$$

where $\tau = \inf\{\, t \geqslant 0 \,|\, \mathsf{X} \notin \mathcal{O} \,\}$ is the exit time from $\mathcal{O}$. The infinitesimal generator of $\widehat{\mathsf{X}}$ is then the self-adjoint operator $L^{\mathcal{O}} = \frac{1}{2}\partial_{x_i}\left(a_{i,j}\partial_{x_j}\right)$ with domain $\left\{\, u \in \mathrm{H}_0^1(\mathcal{O}) \,\middle|\, L^{\mathcal{O}}u \in \mathrm{L}^2(\mathcal{O}) \,\right\}$.

The semi-group of this process $\widehat{\mathsf{X}}$ admits also a transition density function, which is symmetric on $\mathcal{O} \times \mathcal{O}$ and satisfies the upper bound of the Aronson estimate (15).

4 BSDEs driven by Dirichlet processes

This Section is devoted to prove that BSDE also gives in our cases the weak solution of semi-linear PDE.

4.1 On the BSDEs

Now, let $\mathcal{O}$ be a connected, open subset of $\mathbb{R}^N$ with a regular boundary.

Let us introduce a family $(a^\varepsilon)_{\varepsilon>0}$ of measurable functions with values in the space of symmetric matrices and a family of terms $(h^\varepsilon)_{\varepsilon>0}$ such that

(a^ε-i) The family $(a^\varepsilon)_{\varepsilon\geqslant 0}$ satisfies (10a)-(10b).

(a^ε-ii) For each $\varepsilon > 0$, a^ε is smooth, *i.e.*, of class $\mathcal{C}^\infty$.

(h^ε-i) For each $\varepsilon \geqslant 0$, h^ε satisfies (h-i)-(h-v) and $h^\varepsilon(t,x,y,z) \xrightarrow[\varepsilon\to 0]{} h^0(t,x,y,z)$ a.e. on $\mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}^m \times \mathbb{R}^{N\times m}$.

(h^ε-ii) For each $\varepsilon > 0$, h^ε is smooth.

The hypothesis of Theorem 2 are satisfied with these conditions.

For technical reason, we assume that $a^\varepsilon(x)$ is extended to the Identity matrix when $x \notin \mathcal{O}$, and that every other function defined on $\mathcal{O}$ is extended to 0 outside $\mathcal{O}$.

We denote by X^ε the process associated to the self-adjoint divergence-form operator $L^\varepsilon = \frac{1}{2}\partial_{x_i}\left(a^\varepsilon_{i,j}\partial_{x_j}\right)$ on $\mathbb{R}^N$. For each $\varepsilon \geqslant 0$, we denote by τ^ε the exit time from $\mathcal{O}$ of the process X^ε.

For any $\varepsilon > 0$, the coefficients a_ε are some smooth approximation of a_0, and then

$$\forall \varepsilon > 0,\ L^\varepsilon = \frac{1}{2}a^\varepsilon \frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{2}\frac{\partial a^\varepsilon_{i,j}}{\partial x_j}\frac{\partial}{\partial x_j}.$$

Of course, if $\varepsilon > 0$, then X^ε is solution to the SDE

$$\mathrm{d}\mathsf{X}^\varepsilon_t = \sigma^\varepsilon_{i,j}(\mathsf{X}^\varepsilon_t)\,\mathrm{d}\mathsf{B}^\varepsilon_t + \frac{1}{2}\frac{\partial a^\varepsilon_{i,j}}{\partial x_j}(\mathsf{X}^\varepsilon_t)\,\mathrm{d}t$$

for some Brownian Motion B^ε, and where σ^ε is a matrix such that $\sigma^\varepsilon \cdot (\sigma^\varepsilon)^{\mathrm{T}} = a^\varepsilon$. But this in not the case for $\varepsilon = 0$.

We assume that the processes X^ε are defined on the same probability space. Since a Martingale Representation Theorem holds with respect to the martingale part $\mathsf{M}^{\mathsf{X}^\varepsilon}$ of X^ε (see Theorem 3), for any $T > 0$, any $x \in \mathbb{R}^N$ and each $\varepsilon \geqslant 0$, there exists a unique pair $(\mathsf{Y}^\varepsilon_t, \mathsf{Z}^\varepsilon_t)_{t\in[0,T]}$ solution to the problem

$$\mathsf{Y}^\varepsilon_t = g(\mathsf{X}^\varepsilon_T)\mathbf{1}_{\{T<\tau^\varepsilon\}} + \int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} h^\varepsilon(s, \mathsf{X}^\varepsilon_s, \mathsf{Y}^\varepsilon_s, \mathsf{Z}^\varepsilon_s)\,\mathrm{d}s - \int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} \mathsf{Z}^\varepsilon_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s,\ \forall t \in [0,T],\ \mathbb{P}_x\text{-a.s.}, \tag{19a}$$

$$\mathsf{Y}^\varepsilon \text{ and } \mathsf{Z}^\varepsilon \text{ are } \mathcal{F}^{\mathsf{X}^\varepsilon}\text{-progressively measurable}, \tag{19b}$$

$$\mathsf{Y}^\varepsilon_t = 0 \text{ and } \mathsf{Z}^\varepsilon_t = 0 \text{ when } t \in [\tau^\varepsilon, T], \tag{19c}$$

$$\mathbb{E}_x\left[\sup_{0\leqslant t\leqslant T} |\mathsf{Y}^\varepsilon_t|^2 + \int_0^T \|\mathsf{Z}^\varepsilon_s\|^2\,\mathrm{d}s\right] < +\infty, \tag{19d}$$

for a non-linear term h^ε satisfying (h-i)-(h-v) (see *e.g.*, Theorem 4.1 and Remark 3.5, p. 511 and p. 523 in Pardoux 1999a).

Under conditions (a^ε-ii) and (h^ε-ii), with the additional assumption that

(g-i) The function g is smooth with compact support on $\mathcal{O}$,

the weak solution u^ε to (12) is a classical solution, *i.e.*, a function in $\mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathcal{O})$. Using the Itô formula, it immediately follows that The Itô formula yields that for any $\varepsilon > 0$,

$$u^\varepsilon(t\wedge\tau^\varepsilon, \mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon}) = u^\varepsilon(T\wedge\tau^\varepsilon, \mathsf{X}^\varepsilon_{T\wedge\tau^\varepsilon}) - \int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} \nabla u^\varepsilon(s, \mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s + \int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} h^\varepsilon(s, \mathsf{X}^\varepsilon_s, u^\varepsilon(s, \mathsf{X}^\varepsilon_s), \nabla u^\varepsilon(s, \mathsf{X}^\varepsilon_s))\,\mathrm{d}s,\ \forall t \in [0,T] \tag{20}$$

almost surely.

The following Lemma allows to identify the solution to (19a)-(19d).

Lemma 2. *Let ε be a fixed non-negative real. We assume that a^ε satisfies* (6)*, g belongs to* $\mathrm{L}^2(\mathcal{O})$*, h^ε satisfies* (h-i)-(h-v)*, and the solution $u^\varepsilon \in \mathcal{H}$ to* (12) *satisfies* (20)*. Then for almost every starting point x,*

$$\mathbb{E}_x\left[\sup_{0\leqslant t\leqslant T\wedge\tau^\varepsilon} |u^\varepsilon(t, \mathsf{X}^\varepsilon_t)|^2\right] < +\infty \text{ and } \mathbb{E}_x\left[\int_0^{T\wedge\tau^\varepsilon} \|\nabla u^\varepsilon(t, \mathsf{X}^\varepsilon_t)\|^2\right] < +\infty.$$

Proof. Let $(P^\varepsilon_t)_{t>0}$ be the semi-group of X^ε with density transition function p^ε. Let φ be a non-negative bounded function in $\mathrm{L}^1(\mathcal{O})$, and f be a function in $\mathrm{L}^1([0,T]\times\mathcal{O})$. Then, with the Cauchy-Schwarz inequality,

$$\int_0^T \int_{\mathcal{O}} |\varphi(x)P^\varepsilon_s f(s,x)|\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_0^T \int_{\mathcal{O}}\int_{\mathcal{O}} \|\varphi\|_\infty\, p^\varepsilon(s,x,y)|f(s,y)|\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \leqslant \|\varphi\|_\infty \|f\|_{\mathrm{L}^1([0,T]\times\mathcal{O})},$$

since by symmetry of $p^\varepsilon(t,\cdot,\cdot)$, $\int_{\mathcal{O}} p^\varepsilon(s,x,y)\,\mathrm{d}x \leqslant 1$. As $\|\nabla u^\varepsilon\|^2$ belongs to $\mathrm{L}^1([0,T]\times\mathcal{O})$, the local martingale $\int_0^{t\wedge\tau^\varepsilon} \nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s$ is then a square-integrable martingale under $\int_{\mathcal{O}} \mathrm{d}x\varphi(x)\mathbb{P}_x$ for any initial distribution with a bounded density φ. Hence, this is true under $\mathbb{P}_x$ for almost every $x\in\mathcal{O}$, since $x\mapsto \mathbb{E}_x\left[\int_0^{t\wedge\tau^\varepsilon} \|\nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\|^2\,\mathrm{d}s\right]$ is universally measurable.

With (20), it is clear that $\mathbb{E}_x\left[\sup_{0\leqslant t\leqslant T}|u^\varepsilon(t\wedge\tau^0,\mathsf{X}^\varepsilon_{t\wedge\tau^0})|^2\right]$ may be compared to $\mathbb{E}_x\left[\int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} |h^\varepsilon(s,\mathsf{X}^\varepsilon_s,u^\varepsilon(s,\mathsf{X}^\varepsilon_s),\nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s))|^2\,\mathrm{d}s\right]$, to $\mathbb{E}_x\left[\,|g(\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon})|^2\,\right]\leqslant C\,\|g\|^2_{\mathrm{L}^2(\mathcal{O})}$, and to $\mathbb{E}_x\left[\int_0^{t\wedge\tau^\varepsilon}\|\nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\|^2\,\mathrm{d}s\right]$. These quantities are finite under $\mathbb{P}_x$ for almost every point x. □

Hence for any $\varepsilon>0$, the unique solution to (19a)-(19d) satisfies

$$u^\varepsilon(t\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon})=\mathsf{Y}^\varepsilon_t \text{ and } \nabla u^\varepsilon(t\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon})=\mathsf{Z}^\varepsilon_t. \tag{21}$$

The goal of this Section is to prove that the previous identifications (20) and (21) are also true for $\varepsilon=0$, when u^ε is not necessarily differentiable and for which the Itô formula does not work.

Theorem 4. *If a^0 satisfies (6), g belongs to $\mathrm{L}^2(\mathcal{O})$ and h^0 satisfies (h-i)-(h-v), then, for any $T>0$, the solution $(\mathsf{Y}^0_t,\mathsf{Z}^0_t)_{t\in[0,T]}$ of (19a)-(19d) with $\varepsilon=0$ is equal to*

$$\begin{aligned}\mathsf{Y}^0_t&=u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0}),\ \forall t\in[0,T],\mathbb{P}_x\text{-a.s.}\\ \text{and } \mathsf{Z}^0_t&=\nabla u^0(t\wedge\tau^0,\mathsf{X}^0_t\wedge\tau^0),\ t\text{-a.e.}\in[0,T],\ \mathbb{P}_x\text{-a.s.},\end{aligned}$$

for almost every x, where u^0 is the continuous version of the solution of (8).

This Theorem is proved in Section 4.2.

Remark 3. Strictly speaking, this Theorem identifies Y_0 and $u(0,x)$ when $\mathsf{X}_0=x$, but translating everything in time allows to identify $u(s,x)$ with the initial value Y_0 of some BSDE for any $s\in[0,T]$.

To conclude this Section, we state a Lemma which will be used intensively.

Lemma 3. *Let $(\mathsf{Y}^\varepsilon,\mathsf{Z}^\varepsilon)$ be the solution of (19a)-(19d). The local martingale $\left(\int_0^{t\wedge\tau^\varepsilon}\mathsf{Y}^\varepsilon_s\cdot\mathsf{Z}^\varepsilon_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}}_s\right)_{t\in[0,T]}$ is a martingale, and there exists some constants C that depend only on λ, Λ such that*

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^{t\wedge\tau^\varepsilon}\mathsf{Y}^\varepsilon_s\cdot\mathsf{Z}^\varepsilon_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s\right|\right]\\ &\qquad\leqslant C\,\mathbb{E}\left[\delta\sup_{0\leqslant t\leqslant T}|\mathsf{Y}^\varepsilon_t|^2+\frac{1}{\delta}\int_0^{T\wedge\tau^\varepsilon}\|\mathsf{Z}^\varepsilon_s\|^2\,\mathrm{d}s\right]<+\infty\end{aligned}\tag{22}$$

for any $\delta>0$.

Proof. With the Burkholder-Davis-Gundy Inequality,

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^{t\wedge\tau^\varepsilon} \mathsf{Y}_s^\varepsilon\cdot\mathsf{Z}_s^\varepsilon\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}^\varepsilon}\right|\right]\leqslant C\mathbb{E}\left[\left(\int_0^{T\wedge\tau^\varepsilon}|\mathsf{Y}_s^\varepsilon|^2\,\|\mathsf{Z}_s^\varepsilon\|^2\,\mathrm{d}s\right)^{1/2}\right]$$
$$\leqslant C\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}_s^\varepsilon|\left(\int_0^{T\wedge\tau^\varepsilon}\|\mathsf{Z}_s^\varepsilon\|^2\,\mathrm{d}s\right)^{1/2}\right],$$

that yields (22) with (13), and consequently, $\int_0^{\cdot\wedge\tau^\varepsilon}\mathsf{Y}_s^\varepsilon\cdot\mathsf{Z}_s^\varepsilon\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}^\varepsilon}$ is a martingale. □

4.2 Identification of the solutions of the BSDE

This Section is devoted to the proof of Theorem 4.

For a given non-linear term h^0 satisfying (h-i)-(h-v), we set

$$h^\varepsilon(t,x,y,z)=\rho_\varepsilon(x,y,z)\star h^0(t,x,y,z), \tag{23}$$

where $(\rho_\varepsilon)_{\varepsilon>0}$ is a sequence of mollifiers, and $\star$ denotes the convolution operation. It is clear that h^ε also satisfies (h-i)-(h-v) with the same constants as h^0. Hence (h^ε-i) and (h^ε-ii) are also satisfied.

For any $x\in\mathbb{R}^N$, we have

$$\mathsf{X}_t^\varepsilon=\mathsf{M}_t^{\mathsf{X}^\varepsilon}+\mathsf{N}_t^\varepsilon,\ \forall t\geqslant 0,\ \mathbb{P}_x\text{-a.s.},$$

where $\mathsf{M}_t^{\mathsf{X}^\varepsilon}$ is a martingale with cross-variations

$$\langle\mathsf{M}^{\mathsf{X}^\varepsilon,i},\mathsf{M}^{\mathsf{X}^\varepsilon,j}\rangle_t=\int_0^t a_{i,j}(\mathsf{X}_s^\varepsilon)\,\mathrm{d}s,\ \text{for } i,j=1,\dots,N,$$

and N^ε is a process locally of zero-quadratic variations (see Theorem 2.1, p. 19 in Rozkosz 1996a). Furthermore, from the results in Rozkosz 1996a,

$$\mathcal{L}(\mathsf{X}^\varepsilon,\mathsf{M}^{\mathsf{X}^\varepsilon},\mathsf{N}^\varepsilon\mid\mathbb{P}_x)\xrightarrow[\varepsilon\to 0]{}\mathcal{L}(\mathsf{X}^0,\mathsf{M}^{\mathsf{X}^0},\mathsf{N}^0\mid\mathbb{P}_x). \tag{24}$$

We may assume without loss of generality that $\mathcal{O}$ is bounded, so that the exit time τ^ε from $\mathcal{O}$ is almost surely finite under each $\mathbb{P}_x$, for any x in $\mathcal{O}$. A localization argument will be used at the end of this proof.

Let us denote by ν the Lebesgue measure on $\mathcal{O}$ and we set $\mathbb{P}_\nu[\,\cdot\,]=\int_{\mathcal{O}}\mathbb{P}_x[\,\cdot\,]\,\mathrm{d}x$.

The function which associate to a continuous path its first exit time from an open domain $\mathcal{O}$ is discontinuous if the paths hits the boundary of $\mathcal{O}$ and remains for some positive time in the closure of $\mathcal{O}$. But, the irregular points of a domain for a process $(\mathsf{X}, \mathbb{P})$ associated to the divergence-form operator $(L, \mathrm{Dom}(L))$ are the same of those of the Brownian motion (Littman, Stampacchia and Weinberger 1963). Hence, the set of discontinuities of the function $\omega \in \mathcal{C}([0,T];\mathcal{O}) \mapsto \tau(\omega)$, if $\tau(\omega) = \inf\{\, t \geqslant 0 \,|\, \omega_t \notin \mathcal{O}\,\}$ is of null measure for $\mathbb{P}$ if the boundary of $\mathcal{O}$ is smooth enough.

Under the assumption that the boundary of $\mathcal{O}$ is regular, it follows from (24) that

$$\mathcal{L}(\mathsf{X}^\varepsilon, \mathsf{M}^{\mathsf{X}^\varepsilon}, \mathsf{N}^\varepsilon, \tau^\varepsilon \mid \mathbb{P}_\nu) \xrightarrow[\varepsilon\to 0]{} \mathcal{L}(\mathsf{X}^0, \mathsf{M}^{\mathsf{X}^0}, \mathsf{N}^0, \tau^0 \mid \mathbb{P}_\nu).$$

The following Lemma replaces the Krylov estimate (Krylov 1980) for the Dirichlet process X^ε which in general is not the solution of a Stochastic Differential Equation.

We define by $\mathrm{L}^1([0,T]\times\mathcal{O})$ the space of measurable functions f on $[0,T]\times\mathcal{O}$ such that

$$\int_0^T \int_{\mathcal{O}} |f(t,x)|\,\mathrm{d}x\,\mathrm{d}t < +\infty.$$

Lemma 4. *Let f be a function in $\mathrm{L}^1([0,T]\times\mathcal{O})$ and let $(f^n)_{n\in\mathbb{N}}$ be a sequence of functions converging to f in $\mathrm{L}^1([0,T]\times\mathcal{O})$. Then*

$$\mathbb{E}_\nu\left[\int_0^{T\wedge\tau^\varepsilon} |f^n(s,\mathsf{X}^\varepsilon_s) - f(s,\mathsf{X}^\varepsilon_s)|\,\mathrm{d}s\right] \xrightarrow[n\to+\infty]{\text{uniformly in } \varepsilon} 0.$$

Proof. As ν is the Lebesgue measure on $\mathcal{O}$,

$$\begin{aligned} &\mathbb{E}_\nu\left[\int_0^{T\wedge\tau^\varepsilon} |f^n(s,\mathsf{X}^\varepsilon_s) - f(s,\mathsf{X}^\varepsilon_s)|\,\mathrm{d}s\right] \qquad (25)\\ &\quad\leqslant \int_0^T \mathbb{E}_\nu\left[\int_0^T \mathbf{1}_{\mathcal{O}}(\mathsf{X}^\varepsilon_s)|f^n(s,\mathsf{X}^\varepsilon_s) - f(s,\mathsf{X}^\varepsilon_s)|\,\mathrm{d}s\right]\\ &\quad\leqslant \int_0^T \int_{\mathcal{O}\times\mathcal{O}} p^\varepsilon(s,x,y)|f^n(s,y) - f(s,y)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

where p^ε is the transition density of X^ε. By the symmetry of p^ε, $\int_{\mathcal{O}} p^\varepsilon(s,x,y)\,\mathrm{d}x \leqslant 1$. So, (25) is smaller than $\int_{[0,T]\times\mathcal{O}} |f^n(s,x) - f(s,x)|\,\mathrm{d}s\,\mathrm{d}x$ which converges to 0 as $n \to +\infty$, and the Lemma is proved. □

Two Lemmas may be deduced from this one.

Since we need some joint convergence, we denote by $(\mathsf{U}^\varepsilon)_{\varepsilon>0}$ a family of continuous stochastic processes, which will be set to $\int_0^{\cdot\wedge\tau^\varepsilon} f(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}s$ and to $\int_0^{\cdot\wedge\tau^\varepsilon} \nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s$.

Lemma 5. *Let f be a function in* $\mathrm{L}^1([0,T]\times\mathcal{O})$ *and* $(\mathsf{U}^\varepsilon)_{\varepsilon\geqslant 0}$ *be a family of continuous stochastic processes such that* $\mathcal{L}(\mathsf{X}^\varepsilon,\mathsf{U}^\varepsilon\,|\,\mathbb{P}_\nu)$ *converges weakly to* $\mathcal{L}(\mathsf{X}^0,\mathsf{U}^0\,|\,\mathbb{P}_\nu)$ *for the topology of the uniform norm. Then*

$$\mathcal{L}\left(\mathsf{X}^\varepsilon,\mathsf{U}^\varepsilon,\int_0^{\cdot\wedge\tau^\varepsilon} f(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}s\,\middle|\,\mathbb{P}_\nu\right) \xrightarrow[\varepsilon\to 0]{\text{dist.}} \mathcal{L}\left(\mathsf{X}^0,\mathsf{U}^0,\int_0^{\cdot\wedge\tau^0} f(s,\mathsf{X}^0_s)\,\mathrm{d}s\,\middle|\,\mathbb{P}_\nu\right) \quad (26)$$

for the topology of the uniform norm.

Proof. Let $(f^n)_{n\in\mathbb{N}}$ be a family of continuous functions on $[0,T]\times\mathcal{O}$ such that f^n converges to f in $\mathrm{L}^1([0,T]\times\mathcal{O})$. It is clear that $\int_0^{\cdot\wedge\tau^\varepsilon} f^n(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}s$ converges in distribution to $\int_0^{\cdot\wedge\tau^\varepsilon} f^n(s,\mathsf{X}^0_s)\,\mathrm{d}s$, and (26) follows from the use of Lemma 4 and Theorem 4.2, p. 25 in Billingsley 1968. □

Lemma 6. *Let* $(\mathsf{U}^\varepsilon)_{\varepsilon\geqslant 0}$ *be a family of continuous stochastic processes such that* $\mathcal{L}(\mathsf{X}^\varepsilon,\mathsf{U}^\varepsilon\,|\,\mathbb{P}_\nu)$ *converges weakly to* $\mathcal{L}(\mathsf{X}^0,\mathsf{U}^0\,|\,\mathbb{P}_\nu)$. *Then*

$$\mathcal{L}\left(\mathsf{X}^\varepsilon,\mathsf{U}^\varepsilon,\int_0^{\cdot\wedge\tau^\varepsilon} \nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s\,\middle|\,\mathbb{P}_\nu\right) \xrightarrow[\varepsilon\to 0]{\text{dist.}} \mathcal{L}\left(\mathsf{X}^0,\mathsf{U}^0,\int_0^{\cdot\wedge\tau^0} \nabla u^0(s,\mathsf{X}^0_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^0}_s\,\middle|\,\mathbb{P}_\nu\right) \quad (27)$$

for the topology of the uniform norm.

Proof. We remark that if $(f^n)_{n\in\mathbb{N}}$ is a sequence of functions converging in $\mathrm{L}^1(0,T;\mathrm{L}^2(\mathcal{O}))$ to some function f, then

$$\mathbb{E}_\nu\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^{t\wedge\tau^\varepsilon}(f-f^n)(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s\right|^2\right] \leqslant \mathbb{E}_\nu\left[\int_0^{T\wedge\tau^\varepsilon}\Lambda|f-f^n|^2(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}s\right] \xrightarrow[n\to+\infty]{\text{uniformly in }\varepsilon} 0.$$

The study of the limit of $\int_0^{\cdot\wedge\tau^\varepsilon}\nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s$ is then reduced to that of $\int_0^{\cdot\wedge\tau^\varepsilon}\nabla u^0(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s$, and u^0 may be approximated by some continuous functions.

We remark that

$$\sup_{\varepsilon\geqslant 0}\mathbb{E}_\nu\left[\langle\mathsf{M}^{\mathsf{X}^\varepsilon,i},\mathsf{M}^{\mathsf{X}^\varepsilon,j}\rangle_T\mathbf{1}_{\{T<\tau^\varepsilon\}}\right]\leqslant\sup_{\varepsilon\geqslant 0}\Lambda T\nu(\mathcal{O})$$

Hence Theorem 7.10 in Kurtz and Protter 1995 (see also Jakubowski, M'emin and Pag'es 1989) implies the convergence (27) in the Skorohod topology.

On the other side, we remark that if $u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)$, then for $i, j = 1, \dots, m$,

$$\left(\left\langle \int_0^{\cdot\wedge\tau^\varepsilon} \nabla u_i^\varepsilon(s, \mathsf{X}_s^\varepsilon)\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}^\varepsilon}, \int_0^{\cdot\wedge\tau^\varepsilon} \nabla u_j^\varepsilon(s, \mathsf{X}_s^\varepsilon)\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}^\varepsilon} \right\rangle_t\right)_{t\in[0,T]} \\ = \left(\int_0^{t\wedge\tau^\varepsilon} \langle a^\varepsilon \nabla u_i^\varepsilon, \nabla u_j^\varepsilon\rangle(s, \mathsf{X}_s^\varepsilon)\,\mathrm{d}s\right)_{t\in[0,T]}$$

converges in distribution in the space of continuous functions to the cross-variations of $\int_0^{\cdot\wedge\tau^\varepsilon} \nabla u^\varepsilon(s, \mathsf{X}_s^\varepsilon)\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}^\varepsilon}$. Hence, the sequence of quadratic variations is tight in the space of continuous function, which also proves the tightness of $(\int_0^{\cdot\wedge\tau^\varepsilon} \nabla u^\varepsilon(s, \mathsf{X}_s^\varepsilon)\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}^\varepsilon})_{\varepsilon>0}$ (see Theorem VI.4.13, p. 322 in Jacod and Shiryaev 1987). The Skorohod topology is weaker than the uniform topology, so that any limit in distribution in the space of continuous function of the last sequence is equal to $\int_0^{\cdot\wedge\tau^0} \nabla u^0(s, \mathsf{X}_s^0)\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}^0}$, which is consequently continuous. The convergence in (27) holds in fact in the uniform topology (see Proposition VI.1.17, p. 292 in Jacod and Shiryaev 1987). □

4.2.1 Case of a non-linear term with a Lipschitz growth

Theorem 4 will be proved in three steps. In the first two steps, we assume that the final condition g satisfies condition (g-i), *i.e.*, g is a smooth function with compact support on $\mathcal{O}$.

We assume in a first step that for all $\varepsilon \geqslant 0$, the functions h^ε are Lipschitz in y with the same constant, *i.e.*,

(h-iii') $|h^\varepsilon(t, x, y, z) - h^\varepsilon(t, x, y', z)| \leqslant C|y - y'|$.

In fact, if this condition is true for h^0, (h-iii') is satisfied for h^ε defined by (23).

In this first Section, under the additional hypotheses (g-i) and (h-iii') we prove that both sides of (20) will converges in distribution as ε goes to 0 to the same equation (20) with $\varepsilon = 0$. Hence, it will be possible to identify the solution of the BSDE (19a)-(19d) with $\varepsilon = 0$ as $(u^0(t\wedge\tau^0, \mathsf{X}_{t\wedge\tau^0}^0), \nabla u^0(t\wedge\tau^0, \mathsf{X}_{t\wedge\tau^0}^0))$.

We set $f^\varepsilon(s, x) = h^\varepsilon(s, x, u^\varepsilon(s, x), \nabla u^\varepsilon(s, x))$ for any $\varepsilon \geqslant 0$. We remark that

$$|f^\varepsilon(s,x) - f^0(s,x)| \leqslant C_0\|\nabla u^\varepsilon(s,x) - \nabla u^0(s,x)\| + C_1|u^\varepsilon(s,x) - u^0(s,x)| \\ + C_2|h^\varepsilon(s,x,u^0(s,x),\nabla u^0(s,x)) - h^0(s,x,u^0(s,x),\nabla u^0(s,x))|.$$

The hypotheses on h^ε and h^0 imply that f^ε converges to f^0 in $\mathrm{L}^1([0,T]\times\mathcal{O})$, because from Theorem 2, u^ε and ∇u^ε converge respectively to u^0 and ∇u^0 in $\mathrm{L}^1([0,T]\times\mathcal{O})$.

From Lemmas 4, 5 and 6,

$$\mathcal{L}\left(\mathsf{X}^\varepsilon, \int_0^{\cdot\wedge\tau^\varepsilon} u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}s, \int_0^{\cdot\wedge\tau^\varepsilon} \nabla u^\varepsilon(s,\mathsf{X}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s \,\middle|\, \mathbb{P}_\nu\right) \xrightarrow[\varepsilon\to 0]{\text{dist.}} \mathcal{L}\left(\mathsf{X}^0, \int_0^{\cdot\wedge\tau^0} u^0(s,\mathsf{X}^0_s)\,\mathrm{d}s, \int_0^{\cdot\wedge\tau^0} \nabla u^0(s,\mathsf{X}^0_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^0}_s \,\middle|\, \mathbb{P}_\nu\right).$$

Lemma 7. *For any $t\in[0,T]$,*

$$u^\varepsilon(t\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon}) \xrightarrow[\varepsilon\to 0]{\text{dist.}} u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0}). \tag{28}$$

Proof. We have

$$\begin{aligned}\mathbb{E}_\nu&\left[\,|u^\varepsilon(t\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon})-u^0(t\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon})|^2\,\right]\\ &\leqslant \int_{\mathcal{O}}\int_{\mathcal{O}} p^\varepsilon(t,x,y)|u^\varepsilon(t,y)-u^0(t,y)|^2\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \int_{\mathcal{O}} |u^\varepsilon(t,y)-u^0(t,y)|^2\,\mathrm{d}y \xrightarrow[\varepsilon\to 0]{} 0,\end{aligned}\tag{29}$$

so we have to study the limit of $u^0(t\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon})$. With Theorem 1, u^0 is continuous on $[0,T]\times\mathcal{O}$, so that $u^0(t\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{t\wedge\tau^\varepsilon})$ converges in distribution to $u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$. □

The continuity of u^0 on $[0,T]\times\mathcal{O}$, equations (20) and (28) lead to

$$u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0}) = u^0(T\wedge\tau^0,\mathsf{X}^0_{T\wedge\tau^0}) + \int_{t\wedge\tau^0}^{T\wedge\tau^0} h^0(s,\mathsf{X}^0_s,u^0(s,\mathsf{X}^0_s),\nabla u^0(s,\mathsf{X}^0_s))\,\mathrm{d}s - \int_{t\wedge\tau^0}^{T\wedge\tau^0} \nabla u^0(s,\mathsf{X}^0_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}^0}_s,\ \forall t\in[0,T],\ \mathbb{P}_\nu\text{-a.s.} \tag{30}$$

Because of the lateral condition, $u^\varepsilon(T\wedge\tau^\varepsilon,\mathsf{X}^\varepsilon_{T\wedge\tau^\varepsilon})$ is equal to $g(\mathsf{X}^\varepsilon_T)\mathbf{1}_{\{T<\tau^\varepsilon\}}$. If $\mathcal{O}$ is not bounded, we have only to use a localization argument to obtain the same result. For any event K in $\mathcal{F}_T$, the function $x\mapsto\mathbb{P}_x[\,K\,]$ is universally measurable. Hence, (30) also holds under $\mathbb{P}_x$ for almost every x,

We have then proved that u^0 is solution of (20) even if $\varepsilon=0$. But we have not proved yet that $u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$ and $\nabla u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$ are in the good spaces. If is clear that these functions are $\mathcal{F}^{\mathsf{X}^0}$-progressively measurable.

With Lemma 2, our candidate to be the solution of the BSDE satisfies (19d), and $(\mathsf{Y}^0_t,\mathsf{Z}^0_t)=(u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0}),\nabla u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0}))$ under $\mathbb{P}_x$, but only for almost every x.

Hence Theorem 4 is proved under the conditions (h-iii') and (g-i).

4.2.2 Case of monotone non-linear terms

The condition (h-iii') may be relaxed, but the strategy is changed. For that, we work now with the process X^0 associated to $L^0 = \frac{1}{2}\partial_{x_i}(a^0_{i,j}\frac{\partial}{\partial x_j})$. Now, we do not assume that h^0 satisfies (h-iii'). But the functions h^ε defined by (23) satisfy this condition with some constant C_ε depending on ε. The following proof is inspired by the results in Hu 1997.

We still assume condition (g-i), and that $\mathcal{O}$ is bounded. For any $\varepsilon \geqslant 0$, let u^ε be the solution to the system of semi-linear parabolic PDE

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} + L^0 u^\varepsilon(t,x) + h^\varepsilon(t,x,u^\varepsilon(t,x),\nabla u^\varepsilon(t,x)) = 0$$

with the final condition $u^\varepsilon(T,x) = g(x)$ and the lateral boundary condition $u^\varepsilon(t,\cdot) \in \mathrm{H}^1_0(\mathcal{O})$, $\forall t < T$.

For any $\varepsilon \geqslant 0$, let $(\mathsf{Y}^\varepsilon_t, \mathsf{Z}^\varepsilon_t)_{t\in[0,T]}$ be the unique $\mathcal{F}^{\mathsf{X}^0}$-progressively measurable solution to

$$\mathsf{Y}^\varepsilon_t = u^\varepsilon(T\wedge\tau^0, \mathsf{X}^0_{T\wedge\tau^0}) + \int_{t\wedge\tau^0}^{T\wedge\tau^0} h^\varepsilon(s,\mathsf{X}^0_s,\mathsf{Y}^\varepsilon_s,\mathsf{Z}^\varepsilon_s)\,\mathrm{d}s - \int_{t\wedge\tau^0}^{T\wedge\tau^0} \mathsf{Z}^\varepsilon_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}^0}_s,$$

such that $(\mathsf{Y}^\varepsilon_t, \mathsf{Z}^\varepsilon_t) = 0$ on $(\tau^0, T]$ and

$$\mathbb{E}_\nu\left[\sup_{0\leqslant t\leqslant T} |\mathsf{Y}^\varepsilon_t|^2 + \int_0^{T\wedge\tau^0} \|\mathsf{Z}^\varepsilon_s\|^2\,\mathrm{d}s\right] < +\infty.$$

We remark that

$$\begin{aligned}\langle h^\varepsilon(t,\mathsf{X}^0_t,\mathsf{Y}^\varepsilon_t,\mathsf{Z}^\varepsilon_t) &- h^0(t,\mathsf{X}^0_t,\mathsf{Y}^0_t,\mathsf{Z}^0_t), \mathsf{Y}^\varepsilon_t - \mathsf{Y}^0_t\rangle \\ &\leqslant \langle h^\varepsilon(t,\mathsf{X}^0_t,\mathsf{Y}^\varepsilon_t,\mathsf{Z}^\varepsilon_t) - h^\varepsilon(t,\mathsf{X}^0_t,\mathsf{Y}^0_t,\mathsf{Z}^0_t), \mathsf{Y}^\varepsilon_t - \mathsf{Y}^0_t\rangle \\ &\quad + \langle h^\varepsilon(t,\mathsf{X}^0_t,\mathsf{Y}^0_t,\mathsf{Z}^0_t) - h^0(t,\mathsf{X}^0_t,\mathsf{Y}^0_t,\mathsf{Z}^0_t), \mathsf{Y}^\varepsilon_t - \mathsf{Y}^0_t\rangle\end{aligned}$$

and that

$$\begin{aligned}f^\varepsilon(t,x) &\stackrel{\text{def}}{=} h^\varepsilon(t,x,u^\varepsilon(t,x),\nabla u^\varepsilon(t,x)) \\ &\xrightarrow[\varepsilon\to 0]{\mathrm{L}^2([0,T]\times\mathcal{O})} f^0(t,x) \stackrel{\text{def}}{=} h^0(t,x,u^0(t,x),\nabla u^0(t,x)).\end{aligned}$$

With Lemma 3, and the Gronwall Lemma, there exist some constants C_1 and C_2 such that

$$\begin{aligned}\mathbb{E}_\nu\left[|\mathsf{Y}^\varepsilon_t - \mathsf{Y}^0_t|^2\right] &+ C_0\mathbb{E}_\nu\left[\int_t^{T\wedge\tau^0} \|\mathsf{Z}^\varepsilon_s - \mathsf{Z}^0_s\|^2\,\mathrm{d}s\right] \\ &\leqslant \mathbb{E}_\nu\left[|u^\varepsilon(T\wedge\tau^0,\mathsf{X}^0_{T\wedge\tau^0}) - u^0(T\wedge\tau^0,\mathsf{X}^0_{T\wedge\tau^0})|^2\right] \\ + C_1\mathbb{E}_\nu&\left[\int_{t\wedge\tau^0}^{T\wedge\tau^0} |\mathsf{Y}^\varepsilon_s - \mathsf{Y}^0_s|^2\,\mathrm{d}s\right] + C_2\mathbb{E}_\nu\left[\int_{t\wedge\tau^0}^{T\wedge\tau^0} (f^\varepsilon - f^0)^2(s,\mathsf{X}^0_s)\,\mathrm{d}s\right].\end{aligned}$$

Again with the Gronwall Lemma, Y^ε_t converges to Y^0_t in $\mathrm{L}^2(\mathbb{P}_\nu)$, because

$$\mathbb{E}_\nu\left[\int_0^{T\wedge\tau^0}(f^\varepsilon-f^0)^2(s,\mathsf{X}^0_s)\,\mathrm{d}s\right]\xrightarrow[\varepsilon\to 0]{}0,$$

and (29) holds.

As $\mathsf{Y}^\varepsilon_t=u^\varepsilon(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$ and $\mathsf{Z}^\varepsilon_t=\nabla u^\varepsilon(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$ for any $\varepsilon>0$, the identification of the limit leads to $\mathsf{Y}^0_t=u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$ and $\mathsf{Z}^0_t=\nabla u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$. With a localization argument, this is true even if $\mathcal{O}$ is not bounded. Again with Lemma 2, we may identify Y^0 and Z^0 with $u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$ and $\nabla u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})$ even if condition (h-iii') is not satisfied.

4.2.3 Final condition

Now, let g^0 be a function in $\mathrm{L}^2(\mathcal{O})$. We know that there exists a family $(g^\varepsilon)_{\varepsilon\geqslant 0}$ of smooth functions with compact support on $\mathcal{O}$ such that g^ε converges to g^0 in $\mathrm{L}^2(\mathcal{O})$. For all $\varepsilon\geqslant 0$, let $(\mathsf{Y}^\varepsilon,\mathsf{Z}^\varepsilon)$ be the unique solution to the BSDE

$$\mathsf{Y}^\varepsilon_t=g^\varepsilon(\mathsf{X}^0_T)\mathbf{1}_{\{T<\tau^0\}}+\int_{t\wedge\tau^0}^{T\wedge\tau^0}h^0(s,\mathsf{X}^0_s,\mathsf{Y}^\varepsilon_s,\mathsf{Z}^\varepsilon_s)\,\mathrm{d}s-\int_{t\wedge\tau^0}^{T\wedge\tau^0}\mathsf{Z}^\varepsilon_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}^0}_s,\ \mathbb{P}_\nu\text{-a.s.}$$

satisfying (19b)-(19d). Hence, the Gronwall Lemma yields

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\mathbb{E}_\nu\left[|\mathsf{Y}^\varepsilon_t-\mathsf{Y}^0_t|^2\right]&+\mathbb{E}_\nu\left[\int_0^{T\wedge\tau^\varepsilon}\|\mathsf{Z}^\varepsilon_s-\mathsf{Z}^0_s\|^2\,\mathrm{d}s\right]\\&\leqslant C_0e^{C_1T}\mathbb{E}_\nu\left[|g(\mathsf{X}^0_T)-g^\varepsilon(\mathsf{X}^0_T)|^2\right]\leqslant C_0e^{C_1T}\|g-g^\varepsilon\|_{\mathrm{L}^2(\mathcal{O})}\xrightarrow[\varepsilon\to 0]{}0.\end{aligned}$$

But, with Theorem 2,

$$\mathbb{E}_\nu\left[|u^\varepsilon(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})-u^0(t\wedge\tau^0,\mathsf{X}^0_{t\wedge\tau^0})|^2\right]\xrightarrow[\varepsilon\to 0]{}0$$
$$\text{and }\mathbb{E}_\nu\left[\int_0^T\|\nabla u^\varepsilon(s,\mathsf{X}^0_s)-\nabla u^0(s,\mathsf{X}^0_s)\|^2\,\mathrm{d}s\right]\xrightarrow[\varepsilon\to 0]{}0,$$

and, by concluding as previously, Theorem 4 is proved for any final condition in $\mathrm{L}^2(\mathcal{O})$.

4.3 Two estimates on the solution of the BSDE

In this Section, we prove some estimates on Y and Z better than (19d), but under the assumption that

(g-ii) The function g is bounded on $\mathcal{O}$.

These estimates will be used to deal with a non-linear first order differential term in the homogenization result.

We drop the superscript 0.

A probabilistic proof of the boundedness of the process Y is given here.

Boundedness of Y. We prove that under our conditions, the process Y is bounded.

Let α be a positive real. The Itô formula applied to $e^{\alpha t}|\mathsf{Y}_t|^2$ yields

$$\begin{aligned} e^{\alpha t}|\mathsf{Y}_t|^2 = e^{\alpha T}|g(\mathsf{X}_T)|^2 \mathbf{1}_{\{T<\tau\}} + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}\mathsf{Y}_s \cdot h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s \\ - 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}\mathsf{Y}_s\cdot\mathsf{Z}_s\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}} - \sum_{i=1}^m \int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}\left\langle a(\mathsf{X}_s)\mathsf{Z}_s^i, \mathsf{Z}_s^i\right\rangle \mathrm{d}s \\ - \alpha\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|\mathsf{Y}_s|^2\,\mathrm{d}s. \end{aligned}$$

But, with (h-ii)-(h-iv),

$$\begin{aligned} \mathsf{Y}_s\cdot h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s) &\leqslant K_1|\mathsf{Y}_s|^2 + K_4|\mathsf{Y}_s|\,\|\mathsf{Z}_s\| + K_2|\mathsf{Y}_s| \\ &\leqslant \left(K_1 + \frac{K_4}{2\delta} + 1\right)|\mathsf{Y}_s|^2 + K_2^2 + \frac{\delta}{2}\|\mathsf{Z}_s\|^2. \end{aligned}$$

Hence, for δ small enough,

$$\begin{aligned} e^{\alpha t}|\mathsf{Y}_t|^2 + C_0\int_{t\wedge\tau}^{T\wedge\tau}\|\mathsf{Z}_s\|^2\,\mathrm{d}s \leqslant C_1 + \left(K_1+\frac{K_4}{2\delta}+1-\alpha\right)\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}|\mathsf{Y}_s|^2\,\mathrm{d}s \\ - 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}\mathsf{Y}_s\cdot\mathsf{Z}_s\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}} \quad (31) \end{aligned}$$

for some constant C_0 and C_1 that depend on the bound of g and λ, Λ, α and T. If α is large enough, we obtain that

$$e^{\alpha t}|\mathsf{Y}_t|^2 \leqslant C_1 - 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\alpha s}\mathsf{Y}_s\cdot\mathsf{Z}_s\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}}.$$

But, as in Lemma 3, $\int_0^\cdot e^{\alpha s}\mathsf{Y}_s\cdot\mathsf{Z}_s\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}}$ is a martingale, and applying the conditional expectation $\mathbb{E}\left[\,\cdot\,\middle|\,\mathcal{F}_t^{\mathsf{X}}\right]$ to the last expression, we obtain that Y_t is bounded by some constant that depends only on λ, Λ, $K_1,\dots,K_4$, $\|g\|_\infty$ and T.

An estimate on $\int_0^\cdot \mathsf{Z}_s\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}}$. If we set α to 0 in (31), it is clear now that

$$\int_0^{T\wedge\tau}\|\mathsf{Z}_s\|^2\,\mathrm{d}s \leqslant C_2 + C_3\int_0^{T\wedge\tau}\mathsf{Y}_s\cdot\mathsf{Z}_s\,\mathrm{d}\mathsf{M}_s^{\mathsf{X}}.$$

Taking the expectation of the square in each side of the inequality leads to

$$\mathbb{E}\left[\left(\int_0^{T\wedge\tau}\|\mathsf{Z}_s\|^2\,\mathrm{d}s\right)^2\right] \leqslant C_4. \quad (32)$$

5 Case of a non-linear first-order differential term

5.1 Solution of semi-linear PDE and BSDE

We are now interested by the system of parabolic PDE

$$\begin{cases}\text{for } i=1,\dots,m,\\ \dfrac{\partial u^i}{\partial t}+\dfrac{1}{2}\dfrac{\partial}{\partial x_k}\left(a_{k,j}\dfrac{\partial u^i}{\partial x_j}\right)+\widehat{h}(u^1,\dots,u^m)\nabla u^i\\ \qquad\qquad +h_i(u^1,\dots,u^m,\nabla u^1,\dots,\nabla u^m)=0,\\ u(T,x)=g(x) \text{ and } u^i(t,\cdot)\in \mathrm{H}^1_0(\mathcal{O})\ \forall t\in[0,T),\end{cases} \tag{33}$$

where $\widehat{h}$ is a measurable bounded function defined on $[0,T]\times\mathcal{O}\times\mathbb{R}^m$ with values in $\mathbb{R}^m$ and satisfying

($\widehat{h}$-i) $|\widehat{h}(s,x,y)|\leqslant\Lambda$ for any $(s,x,y)\in[0,T]\times\mathcal{O}\times\mathbb{R}^m$;

($\widehat{h}$-ii) there exists some constant K such that $|\widehat{h}(s,x,y)-\widehat{h}(s,x,y)|\leqslant K|y-y'|$ for any (s,x,y,y') in $[0,T]\times\mathcal{O}\times\mathbb{R}^m\times\mathbb{R}^m$.

We insist on the fact that the non-linear first-order terms is the same for any component on the system. This system may not be reduced to (8) since for any (x,y,y',z),

$$|\widehat{h}(x,y)z-\widehat{h}(x,y')z|\leqslant K|y-y'|\cdot|z|$$

and thus $\widehat{h}$ does not satisfies (h-iii).

Theorem 5. *There exists a unique solution in $\mathcal{H}$ to the system* (33).

Proof. We remark first that since $|\widehat{h}(x,u(t,x))\nabla u^i(t,x)|\leqslant\Lambda|u^i(t,x)|$, using classically the Gronwall inequality, for any $\tau\in[0,T]$,

$$\sup_{\tau\leqslant t\leqslant T}\|u_t\|^2_{\mathrm{L}^2(\mathcal{O})}+\int_\tau^T\|\nabla u_t\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}t\leqslant\alpha\,\|g\|_{\mathrm{L}^2(\mathcal{O})}\,e^{\beta(T-\tau)}, \tag{34}$$

where α and β depend only on the structure of (33).

For the existence, we have only to consider the following scheme: Let $u^{(0)}$ be an arbitrary function in the Banach space $\mathcal{H}$, and let $u^{(n)}=(u^{1,(n)},\dots,u^{m,(n)})$ be the recursively defined solution in $\mathcal{H}$ to

$$\begin{cases}\text{for } i=1,\dots,m,\\ \dfrac{\partial u^{i,(n)}}{\partial t}+\dfrac{1}{2}\dfrac{\partial}{\partial x_k}\left(a_{k,j}\dfrac{\partial u^{i,(n)}}{\partial x_j}\right)+\widehat{h}(u^{1,(n-1)},\dots,u^{m,(n-1)})\nabla u^{i,(n)}\\ \qquad\qquad +h_i(u^{1,(n)},\dots,u^{m,(n)},\nabla u^{1,(n)},\dots,\nabla u^{m,(n)})=0,\\ u^{(n)}(T,x)=g(x) \text{ and } u^{i,(n)}(t,\cdot)\in\mathrm{H}^1_0(\mathcal{O})\ \forall t\in[0,T).\end{cases}$$

The existence of $u^{(n)}$ when $u^{(n-1)}$ is given is ensured by Theorem 1, since for this equation, the term $\widehat{h}(x, u^{(n-1)}(x))z$ satisfies (h-iii). Some computations similar to that of the proof of Theorem 2 imply that

$$\|u_t^{(n+1)} - u_t^{(n)}\|^2_{\mathrm{L}^2(\mathcal{O})} + \int_t^T \|\nabla u_s^{(n+1)} - \nabla u_s^{(n)}\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}s \\ \leqslant \int_t^T (C_1 + C_2\|\nabla u_s^{(n)}\|^2_{\mathrm{L}^2(\mathcal{O})})\|u_s^{(n+1)} - u_s^{(n)}\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}s \\ + \int_t^T \|u_s^{(n)} - u_s^{(n-1)}\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}s$$

As $\int_0^T \|\nabla u_s^{(n)}\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}s$ is bounded by some constant depending only on the structure of (8) and which is so independent from n. Hence, by the Gronwall Lemma again, for any $0 \leqslant \tau \leqslant T$,

$$\sup_{\tau\leqslant t\leqslant T} \|u_t^{(n+1)} - u_t^{(n)}\|^2_{\mathrm{L}^2(\mathcal{O})} + \int_\tau^T \|\nabla u_s^{(n+1)} - \nabla u_s^{(n)}\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}s \\ \leqslant \left(\int_\tau^T e^{s\alpha\|g\|_{\mathrm{L}^2(\mathcal{O})}e^{\beta T}}\,\mathrm{d}s\right) \\ \times \left(\sup_{\tau\leqslant t\leqslant T} \|u_t^{(n)} - u_t^{(n-1)}\|^2_{\mathrm{L}^2(\mathcal{O})} + \int_\tau^T \|\nabla u_t^{(n)} - \nabla u_t^{(n-1)}\|^2_{\mathrm{L}^2(\mathcal{O})}\,\mathrm{d}t\right).$$

We set

$$\tau = T - \frac{1}{2}\exp(-T\alpha\,\|g\|_{\mathrm{L}^2(\mathcal{O})}\,e^{\beta T}),$$

so that $\int_\tau^T e^{s\alpha\|g\|_{\mathrm{L}^2(\mathcal{O})}e^{\beta T}}\,\mathrm{d}s \leqslant 1/2$. It follows from the Fixed Point Theorem that $u^{(n)}$ converges to the solution u of (33) on the time interval $[\tau, T]$ and the limit is unique. Setting $\tau_0 = T$ and $\tau_1 = T$, we may iteratively construct a sequence of decreasing times in order to solve (33) on $[\tau_k, \tau_{k-1}]$ with $u_{\tau_{k-1}}$ as final condition. However, the norm of $u_{\tau_{k-1}}$ may increase, but using (34), we may set

$$\tau_k = \tau_{k-1} - \frac{1}{2}\exp\left(-T\alpha\,\|g\|_{\mathrm{L}^2(\mathcal{O})}\,e^{\beta T}e^{\beta(T-\tau_k)}\right),$$

so that the distance between τ_k and τ_{k-1} decreases. However, if $(\tau_k)_{k\in\mathbb{N}}$ is bounded by below, then $\tau_k - \tau_{k-1}$ does not converges to 0, and then τ_k could not be bounded by below. Although $\tau_k - \tau_{k-1}$ decreases to 0, it is possible to solve recursively (33) on the whole interval $[0, T]$. □

Now, once the solution u to (33) has been found, one may consider the process X generated by the differential operator

$$L = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right) + \widehat{h}_i(x, u(t,x))\frac{\partial}{\partial x_i}.$$

The distribution $\widehat{\mathbb{P}}_{s,x}$ of this inhomogeneous in time stochastic process X satisfying $\widehat{\mathbb{P}}_{s,x}[\mathsf{X}_s = x] = 1$ is also given by the Girsanov transform

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}_{s,x}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \exp\left(\int_s^t a^{-1}\widehat{h}(r,\mathsf{X}_r,u(r,\mathsf{X}_r))\,\mathrm{d}\mathsf{M}^{\mathsf{X}}_r - \frac{1}{2}\int_s^t a\widehat{h}\cdot\widehat{h}(r,\mathsf{X}_r,u(r,\mathsf{X}_r))\,\mathrm{d}r\right),$$

where $\mathbb{P}_x$ is the distribution of the process generated by $L = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right)$ starting at x, and M^{X} is the martingale part of this process. The absolute continuity of the measure $\widehat{\mathbb{P}}_{s,x}$ with respect to the measure $\mathbb{P}_x$ is easily derived from the results in Lyons and Zhang 1996 and in Chen and Zhao 1995.

Remark 4. If the coefficient a is smooth, then X is the solution of the Stochastic Differential Equation

$$\mathsf{X}_t = \mathsf{X}_0 + \int_0^t \sigma(\mathsf{X}_s)\,\mathrm{d}\mathsf{B}_s + \int_0^t \widehat{h}(s,\mathsf{X}_s,\mathsf{Y}_s)\,\mathrm{d}s$$

where σ is such that $\sigma\sigma^{\mathrm{T}} = a$ and B is a Brownian Motion. We face here a Forward-Backward Stochastic Differential Equation. The idea of the previous representation coupling a BSDE and a non-linear operator seen as a linear operator by "freezing" the solution appearing in the coefficient is close in its spirit from the Four Step Scheme introduce by Ma, Protter and Yong (Ma, Protter and Yong 1994). It may be used for quasi-linear PDE with a divergence-form operator (Lejay 2004).

Theorem 6. *The conclusion of Theorem 4 are valid under* $\widehat{\mathbb{P}}_{s,x}$.

Proof. Although Theorem 4 has been proved for time-homogeneous stochastic process whose infinitesimal generator is self-adjoint, it is still valid under distribution $\widehat{\mathbb{P}}_{s,x}$. In fact Aronson estimate (15) and estimate (16) have a version in for non-homogeneous probability transition function. Furthermore, we have used the fact that $\int_{\mathbb{R}^N} p(t,x,y)\,\mathrm{d}x = 1$ for any y, and this comes from the symmetry of the operator. But if the operator has a bounded first-order differential operator, with the version of the non-homogeneous in time Aronson estimate, it is easily proved that there exists some constant C depending only on λ, Λ and n such that $\int_{\mathbb{R}^N} p(s,t,x,y)\,\mathrm{d}x \leqslant C$. □

Theorem 4 has to be re-proved from scratch. One may think to use the Girsanov theorem on the conclusion of 4, but in this case, the BSDE becomes under the new distribution

$$\mathsf{Y}_t = g(\mathsf{X}_T)\mathbf{1}_{\{T<\tau\}} + \int_{t\wedge\tau}^{T\wedge\tau} h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s + \int_{t\wedge\tau}^{T\wedge\tau} \mathsf{Z}_s\widehat{h}(s,\mathsf{X}_s,\mathsf{Y}_s)\,\mathrm{d}s - \int_{t\wedge\tau}^{T\wedge\tau} \mathsf{Z}_s\mathsf{M}^{\mathsf{X}}_s,$$

which is different from (19a). But, as we will see in the next Section, this allow to prove a new existence result.

5.2 Girsanov Theorem and an existence result for BSDE

We will see how Theorem 6 may be used together with the Girsanov theorem to prove some new existence result of BSDE.

Proposition 1. *Let $\widehat{h}$ be satisfying ($\widehat{h}$-i)-($\widehat{h}$-ii). We assume (g-ii), i.e., that g is bounded. Let $(\mathsf{X}, \widehat{\mathbb{P}}_x)$ be the process generated by the self-adjoint operator $L = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right)$. The martingale part of this process is denoted by $\widehat{\mathsf{M}}^{\mathsf{X}}$. Then there exists a unique solution to the BSDE*

$$\begin{cases} \mathsf{Y}_t = g(\mathsf{X}_T)\mathbf{1}_{\{T<\tau\}} + \int_{t\wedge\tau}^{T\wedge\tau} h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s \\ \qquad + \int_{t\wedge\tau}^{T\wedge\tau} \mathsf{Z}_s\widehat{h}(s,\mathsf{X}_s,\mathsf{Y}_s)\,\mathrm{d}s - \int_{t\wedge\tau}^{T\wedge\tau} \mathsf{Z}_s\,\mathrm{d}\widehat{\mathsf{M}}^{\mathsf{X}}_s,\ \widehat{\mathbb{P}}_{0,x}\text{-a.s.}, \\ \widehat{\mathbb{E}}_x\left[\sup_{0\leqslant t\leqslant T\wedge\tau}|\mathsf{Y}_t|^2 + \int_0^{T\wedge\tau}\|\mathsf{Z}_t\|^2\,\mathrm{d}t\right] < +\infty, \\ \mathsf{Y} \text{ and } \mathsf{Z} \text{ are } \mathcal{F}^{\mathsf{X}}\text{-progressively measurable} \end{cases} \tag{35}$$

for almost every x in $\mathcal{O}$. Moreover, this solution is given by $\mathsf{Y}_t = u(t,\mathsf{X}_t)$ and $\mathsf{Z}_t = \nabla u(t,\mathsf{X}_t)$ on $[0,\tau]$, where u is the solution of (33).

Proof. The idea is to consider first the BSDE $\mathsf{Y}_t = g(\mathsf{X}_T)+\int_t^T h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s - \int_t^T \mathsf{Z}_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}}_s$, where the driving process X is generated by the operator $L - \widehat{h}(t,x,u(t,x))\nabla$. Here, u is the solution to (33) with $-\widehat{h}$ instead of $\widehat{h}$ as a first-order differential term. The process X denotes the martingale part of the process X. Let us denote by $\mathbb{P}_x$ the distribution of this process starting at point x. By Theorem 6, $\mathsf{Y}_t = u(t,\mathsf{X}_t)$ and $\mathsf{Z}_t = u(t,\mathsf{Z}_t)$.

Let L be the martingale

$$\mathsf{L}_t = \int_0^{t\wedge\tau} a^{-1}(\mathsf{X}_s)\widehat{h}(s,\mathsf{X}_s,\mathsf{Y}_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}}_s.$$

The boundedness of a^{-1} and $\widehat{h}$ implies that $\exp\left(\mathsf{L}_t - \frac{1}{2}\langle\mathsf{L}\rangle_t\right)$ is an exponential martingale. Let $\widehat{\mathbb{P}}_{0,x}$ be the distribution defined by

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}_{0,x}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}^{\mathsf{X}}_t} = \exp\left(\mathsf{L}_t - \frac{1}{2}\langle\mathsf{L}\rangle_t\right)$$

for any $t \in [0,T]$. The solution of the BSDE

$$\mathsf{Y}_t = g(\mathsf{X}_T)\mathbf{1}_{\{T<\tau\}} + \int_{t\wedge\tau}^{T\wedge\tau} h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s - \int_{t\wedge\tau}^{T\wedge\tau} \mathsf{Z}_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}}_s,\ \forall t\in[0,T],$$

becomes

$$\mathsf{Y}_t = g(\mathsf{X}_T)\mathbf{1}_{\{T<\tau\}} + \int_{t\wedge\tau}^{T\wedge\tau} h(s,\mathsf{X}_s,\mathsf{Y}_s,\mathsf{Z}_s)\,\mathrm{d}s \\ + \int_{t\wedge\tau}^{T\wedge\tau} \mathsf{Z}_s\cdot\widehat{h}(s,\mathsf{X}_s,\mathsf{Y}_s)\,\mathrm{d}s - \int_{t\wedge\tau}^{T\wedge\tau} \mathsf{Z}_s\,\mathrm{d}\widehat{\mathsf{M}}^{\mathsf{X}}_s,\ \forall t\in[0,T],\ \widehat{\mathbb{P}}_{0,x}\text{-a.s.}$$

under $\widehat{\mathbb{P}}_{0,x}$, where

$$\widehat{\mathsf{M}}^{\mathsf{X}}_t = \mathsf{M}^{\mathsf{X}}_t - \int_0^{t\wedge\tau} \widehat{h}(s,\mathsf{X}_s,\mathsf{Y}_s)\,\mathrm{d}s$$

is a $(\mathcal{F}^{\mathsf{X}},\widehat{\mathbb{P}}_{0,x})$-martingale. The boundedness of Y and (32) (see also (46) below) implies that

$$\widehat{\mathbb{E}}_{0,x}\left[\sup_{0\leqslant t\leqslant T\wedge}|\mathsf{Y}_s|^2 + \int_0^{T\wedge\tau}\|\mathsf{Z}_s\|^2\,\mathrm{d}s\right] \\ \leqslant C_0\mathbb{E}_x\left[\sup_{0\leqslant t\leqslant T\wedge}|\mathsf{Y}_s|^4 + \left(\int_0^{T\wedge\tau}\|\mathsf{Z}_s\|^2\,\mathrm{d}s\right)^2\right] < +\infty,$$

so that (Y,Z) is also the solution of the system of BSDE (35) under the distribution $\widehat{\mathbb{P}}_{0,x}$.

Under the new distribution $\widehat{\mathbb{P}}_{0,x}$, the infinitesimal generator of X is the divergence-form operator $L=\frac{1}{2}\partial_{x_i}\left(a_{i,j}\partial_{x_j}\right)$. □

6 Homogenization

We prove now a homogenization result for a family of semi-linear parabolic PDEs. More precisely, a convergence result of the solution of semi-linear parabolic PDE is proved when a parameter that represents the scale of the heterogeneities decreases to 0.

Two cases of homogenization are considered: the periodic media and the random media.

We assume that the coefficients of the linear operators and the non-linear terms are defined on $\mathbb{R}^N$, and not only on $\mathcal{O}$.

We start by recalling quickly some homogenization results on the linear part of the PDE.

6.1 Homogenization in periodic media of a divergence-form operator

We give here briefly the result about the homogenization of the family of operators L^ε defined by

$$L^\varepsilon = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}(\cdot/\varepsilon)\frac{\partial}{\partial x_j}\right) + \frac{1}{\varepsilon}b_i(\cdot/\varepsilon)\frac{\partial}{\partial x_i}, \tag{36}$$

assuming that a, b are measurable, 1-periodic, a satisfies hypotheses 6 as above, and $b = (b_i)_{i=1}^N$ is measurable, bounded by Λ.

A probabilistic approach of the the convergence of X^ε to a Gaussian process $\overline{\mathsf{X}}$ for divergence-form operator is considered in Lejay 2001a, Lejay 2000. The method used here adapts to our particular class of processes the general procedure: see Bensoussan, Lions and Papanicolaou 1978, Olla 1994, Freidlin 1996 for example.

This convergence of X^ε to $\overline{\mathsf{X}}$ may also be obtained by some analytical way, because the convergence of the solutions of the parabolic equations is equivalent to the convergence in distribution of the associated processes, as it is proved in Rozkosz 1996b. The analytical ways of proving a homogenization result are well-known (Jikov, Kozlov and Oleinik 1994, Bensoussan, Lions and Papanicolaou 1978).

In fact, the process X^ε starting from x is equal in distribution to the process $(\varepsilon \mathsf{X}_{t/\varepsilon^2})_{t\geqslant 0}$, where $\mathsf{X} \stackrel{\text{def}}{=} \mathsf{X}^1$ starting from x/ε.

There exists a unique 1-periodic solution π locally in $\mathrm{H}^1(\mathbb{R}^N)$ to the problem

$$\frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial \pi}{\partial x_j}\right) - \frac{\partial}{\partial x_i}(b_i\pi) = 0,\ \int_{[0,1]^N}\pi(x)\,\mathrm{d}x = 1.$$

This function π is the density of the invariant measure of the projection on the torus $\mathbb{R}^N/\mathbb{Z}^N$ of the Markov process associated to $\frac{1}{2}\partial_{x_i}\left(a_{i,j}\partial_{x_j}\right) + b_i\partial_{x_i}$.

With some spectral gap technique (see *e.g.*, Corollary 4, p. 21 in Lejay 2001a or Chapter 1 in Olla 1994), if f is locally integrable and 1-periodic, then for any $p \geqslant 1$,

$$\forall t \geqslant 0,\ \sup_{x\in\mathbb{R}^N}\mathbb{E}_x\left[\left|\int_0^t f(\mathsf{X}_s^\varepsilon/\varepsilon)\,\mathrm{d}s - t\int_{[0,1]^N} f(x)\pi(x)\,\mathrm{d}x\right|^p\right] \xrightarrow[\varepsilon\to 0]{} 0. \tag{37}$$

For $k = 1, \dots, N$, there exists a unique 1-periodic solution v_k locally in $\mathrm{H}^1(\mathbb{R}^N)$ of the problem

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial}{\partial x_i}\left(a_{i,j}\dfrac{\partial v_k}{\partial x_j}\right) + b_i\dfrac{\partial v_k}{\partial x_i} = \left(\dfrac{-1}{2}\dfrac{\partial a_{k,j}}{\partial x_j} - b_j\right) \text{ in the weak sense,} \\ \displaystyle\int_{[0,1]^N} v_k(x)\pi(x)\,\mathrm{d}x = 0, \end{cases}$$

considering that

$$\frac{1}{2}\int_{[0,1]^N} a_{i,j}(x)\frac{\partial \pi(x)}{\partial x_j}\,\mathrm{d}x = \int_{[0,1]^N} b_i(x)\pi(x)\,\mathrm{d}x. \qquad (38)$$

If $x = (x_1, \dots, x_N)$, we set $w_k(x) = x_k + v_k(x)$, $w_k^\varepsilon(x) = x_k + \varepsilon v_k(x/\varepsilon)$ and $w^\varepsilon(x) = (w_1^\varepsilon(x), \dots, w_N^\varepsilon(x))$. Then w_k is a harmonic function for $\frac{1}{2}\partial_{x_i}\left(a_{i,j}\partial_{x_j}\right) + b_i\partial_{x_i}$ and

$$w^\varepsilon(\mathsf{X}_t^\varepsilon) = w^\varepsilon(\mathsf{X}_0^\varepsilon) + \widetilde{\mathsf{M}}_t^\varepsilon, \qquad (39)$$

where $\widetilde{\mathsf{M}}^\varepsilon$ is a local martingale with cross-variations

$$\langle \widetilde{\mathsf{M}}^{\varepsilon,i}, \widetilde{\mathsf{M}}^{\varepsilon,j}\rangle_t = \int_0^t a_{i,j}^\varepsilon \frac{\partial w^\varepsilon}{\partial x_i}\frac{\partial w^\varepsilon}{\partial x_j}(\mathsf{X}_s^\varepsilon)\,\mathrm{d}s.$$

Furthermore, v_k is 1-periodic and is then bounded. We assume that for any $\varepsilon > 0$, $\mathsf{X}_0^\varepsilon = x$. It follows from (37) and a Central Limit Theorem for the martingales (see *e.g.*, Theorem 7.1.4, p. 339 in Ethier and Kurtz 1986, in which the convergence in probability of the cross-variations implies the converge in distribution) that X^ε converges in distribution to

$$\overline{\mathsf{X}}_t = x + \mathsf{M}_t^{\overline{\mathsf{X}}}$$

where $\mathsf{M}^{\overline{\mathsf{X}}}$ is a martingale with cross-variations

$$\langle \mathsf{M}^{\overline{\mathsf{X}},i}, \mathsf{M}^{\overline{\mathsf{X}},j}\rangle = t\overline{a}_{i,j} = t\int_{[0,1]^N} a_{p,q}(x)\frac{\partial u_i(x)}{\partial x_p}\frac{\partial u_j(x)}{\partial x_q}\pi(x)\,\mathrm{d}x. \qquad (40)$$

With (37) and the Burkholder-Davis-Gundy Inequality that

$$\sup_{\varepsilon>0}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|\widetilde{\mathsf{M}}_t^\varepsilon|^2\right] < +\infty, \qquad (41)$$

since $a_{i,j}\partial_{x_i}w\partial_{x_j}w$ is 1-periodic and belongs locally to $\mathrm{L}^1(\mathbb{R}^N)$.

Theorem 4.4, p. 27 in Billingsley 1968 asserts that $(\mathsf{X}^\varepsilon, \widetilde{\mathsf{M}}^\varepsilon)_{\varepsilon>0}$ converges in distribution to $(\overline{\mathsf{X}}, \mathsf{M}^{\overline{\mathsf{X}}})$.

6.2 Homogenization of a divergence-form operator in random media

Let $(\Omega_\mathcal{M}, \mathcal{G}, \mu)$ be a probability space. We consider now a group $(\tau_x)_{x\in\mathbb{R}^N}$ of transformation on $(\Omega_\mathcal{M}, \mathcal{G}, \mu)$ that preserves the measure μ and such that it is ergodic. Furthermore, we assume that $(\tau_x)_{x\in\mathbb{R}^N}$ is stochastically continuous.

A function f measurable on $\Omega_{\mathcal{M}} \times \mathbb{R}^N$ such that there exists a random function $\mathbf{f}$ on Ω for which

$$f(x,\omega) = \mathbf{f}(\tau_x\omega),\ \forall x \in \mathbb{R}^N,\ \forall \omega \in \Omega_{\mathcal{M}},$$

is called a *stationary random field.*

Let us consider the family of operators L^ε defined by

$$L^{\varepsilon,\omega} = \frac{1}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}(x/\varepsilon,\omega)\frac{\partial}{\partial x_j}\right) \tag{42}$$

where the coefficient a is the stationary random field associated to $\mathbf{a}$ such that μ-almost surely,

$$\lambda|\xi|^2 \leqslant \mathbf{a}\xi\cdot\xi,\ \forall \xi \in \mathbb{R}^N,\ \text{and } \mathbf{a} \text{ is bounded by } \Lambda.$$

Let $\mathsf{X}^{\varepsilon,\omega}$ be the Dirichlet process whose infinitesimal generator is $L^{\varepsilon,\omega}$. As in the case of periodic media, there exists an operator $\overline{L} = \frac{1}{2}\overline{a}_{i,j}\partial^2_{x_i,x_j}$ with constant coefficients such that $\mathsf{X}^{\varepsilon,\omega}$ converges in distribution to the process $\overline{\mathsf{X}}$ associated to $\overline{L}$.

Let us introduce the space

$$\mathcal{V}^2_{\mathrm{pot}} = \left\{ (\mathbf{f}_1,\dots,\mathbf{f}_N) \in \mathrm{L}^2(\Omega_{\mathcal{M}})^N \,\middle|\, \begin{array}{l} \int_{\Omega_{\mathcal{M}}} \mathbf{f}_i \,\mathrm{d}\mu = 0,\ \text{for } i = 1,\dots,N, \\ \mathbf{rot}\,(f_1(\cdot,\omega),\dots,f_N(\cdot,\omega)) = 0 \end{array} \right\},$$

where f_i is the stationary random field associated to $\mathbf{f}_i$.

It is possible to find for $k = 1,\dots,N$ a unique element $(\mathbf{f}^k_1,\dots,\mathbf{f}^k_N)$ in $\mathcal{V}^2_{\mathrm{pot}}$ such that

$$\int_{\Omega_{\mathcal{M}}} \mathbf{a}_{i,j}(\delta_{i,k} + \mathbf{f}^k_i)\mathbf{g}_j \,\mathrm{d}\mu = 0,\ \forall (\mathbf{g}_1,\dots,\mathbf{g}_N) \in \mathcal{V}^2_{\mathrm{pot}}.$$

Hence, there exists a unique family of (continuous) functions $w_i(x,\omega)$ such that

$$\frac{\partial w_i(x,\omega)}{\partial x_j} = \delta_{i,j} + \mathbf{f}^i_j(\tau_x\omega)\ \forall i,j = 1,\dots,N \text{ and } w_i(0,\omega) = 0.$$

Each of the functions $w_i(x,\omega)$ is not a stationary random field. But there are harmonic for $L^{1,\omega}$, *i.e.*, $L^{1,\omega}w_i(\cdot,\omega) = 0$ in the weak sense. Furthermore, for almost every realization ω, $\varepsilon w_i(x/\varepsilon,\omega)$ converges uniformly on each compact of $\mathbb{R}^N$ to the i^{th} projection $x \mapsto x_i$. Using the Aronson estimate (15), it is easily proved that for any $T \geqslant 0$,

$$\sup_{0\leqslant t\leqslant T} |\mathsf{X}^{\varepsilon,\omega}_t - \varepsilon w(\mathsf{X}^{\varepsilon,\omega}_t/\varepsilon,\omega)| \xrightarrow[\varepsilon\to 0]{\text{proba.}} 0.$$

Now, we set

$$\begin{aligned}\text{and } \check{\mathbf{a}}_{i,j}(\tau_x\omega) &= \mathbf{a}_{p,q}(\delta_{i,p}+\mathbf{f}^i_p)(\delta_{j,q}+\mathbf{f}^j_q)(\tau_x\omega)\\ &= a_{p,q}(x,\omega)\frac{\partial w_i(x,\omega)}{\partial x_p}\frac{\partial w_j(x,\omega)}{\partial x_q}(x,\omega).\end{aligned} \tag{43}$$

For almost every realization ω, there exists a local martingale $\widetilde{\mathsf{M}}^{\varepsilon,\omega}$ such that

$$\widetilde{\mathsf{M}}^{\varepsilon,\omega,i}_t = \varepsilon w_i(\mathsf{X}^{\varepsilon,\omega}_t/\varepsilon,\omega) - \mathsf{X}^{\varepsilon,\omega,i}_t \text{ and } \widetilde{\mathsf{M}}^{\varepsilon,\omega} \xrightarrow[\varepsilon\to 0]{\text{dist.}} \mathsf{M}^{\overline{\mathsf{X}}} = \overline{\mathsf{X}},$$

where $\mathsf{M}^{\overline{\mathsf{X}}}$ is a martingale with quadratic variations $\langle \mathsf{M}^{\overline{\mathsf{X}},i}, \mathsf{M}^{\overline{\mathsf{X}},j}\rangle_t = t\int_{\Omega_{\mathcal{M}}} \check{\mathbf{a}}_{i,j}(\omega)\,\mathrm{d}\mu(\omega)$. The convergence holds for almost every starting point x and almost every realization ω.

Details may be found in Lejay 2001b and Lejay 2000. Concerning the probabilistic approach, the reader is also referred to Olla 1994 and Osada 1998.

6.3 Homogenization of semi-linear PDEs

In this section, the process X^ε is the process associated to the operator L^ε, *i.e.*,

$$\begin{cases} L^\varepsilon = \dfrac{1}{2}\dfrac{\partial}{\partial x_i}\left(a_{i,j}(\cdot/\varepsilon)\dfrac{\partial}{\partial x_j}\right) + \dfrac{1}{\varepsilon}b_i(\cdot/\varepsilon)\dfrac{\partial}{\partial x_i} & \text{in periodic media,}\\ L^\varepsilon = \dfrac{1}{2}\dfrac{\partial}{\partial x_i}\left(a_{i,j}(\tau_{\cdot/\varepsilon}\omega)\dfrac{\partial}{\partial x_j}\right) & \text{in a random media.}\end{cases}$$

We assume that these processes are conservative, *i.e.*, the operators L^ε are defined on $\mathbb{R}^N$. We also assume that they are defined on the canonical probability space of continuous functions.

The distribution $\mathbb{P}$ is either $\mathbb{P}_x$ for a fixed x in the case of periodic media, or $\mathbb{P}_{x,\omega}$ which is the distribution of the process starting at x when the random environment is fixed to be ω.

In fact, the periodic media is a particular part of the homogenization in random media. Indeed, in the periodic case, we let the space $\Omega_{\mathcal{M}}$ be equal to the unit torus $\mathbb{R}^N/\mathbb{Z}^N$, and $\tau_x y = y + x \mod 1$ for any x, y in $\mathbb{R}^N$. The measure μ is the Lebesgue measure.

So, we use only the notation for a general random media. If we are in the case of periodic media, we have to replace the measure μ by the measure $\pi(x)\,\mathrm{d}x$ when averaging the non-linear terms.

We shall precise our assumptions on the non-linear terms. We assume that h is a measurable function on $[0,T]\times\Omega_{\mathcal{M}}\times\mathbb{R}^N\times\mathbb{R}^m$ satisfying:

(h-vi) The function $(x,y) \mapsto h(s,\omega,x,y)$ is equicontinuous uniformly with respect to the two first variables.

(h-vii) The family of functions $s \in [0,T] \mapsto h(s,\omega,x,y)$ is equicontinuous uniformly with respect to ω, x, z.

(h-viii) The function $\omega \mapsto h(s,\omega,x,y)$ is a function in $\mathrm{L}^1(\Omega_{\mathcal{M}},\mu)^m$ for any $(s,x,y) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^m$.

Let $\mathcal{O}$ be a open domain of $\mathbb{R}^N$ with regular boundary. We prove the following Theorem.

Theorem 7 (Homogenization of semi-linear PDE). *Let L^ε be defined by* (36) *or* (42)*. Let h and $\widehat{h}$ be some functions on $\mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R}^m$ such that*

(i) *$(t,x,y) \mapsto h(t,\omega,x,y)$ and $(t,x,y) \mapsto \widehat{h}(t,\omega,x,y)$ satisfy* (h-i)-(h-v) *and the constants $K_1,\dots,K_4$ do not depend on ω;*

(ii) *The function h satisfies* (h-vi)-(h-vii)*;*

(iii) *The function $\widehat{h}$ satisfies* ($\widehat{h}$-i)-($\widehat{h}$-ii)*;*

Let $u^\varepsilon = (u^\varepsilon_1,\dots,u^\varepsilon_m)$ be the solution of the system

$$\begin{cases} \dfrac{\partial u^\varepsilon_i(t,x,\omega)}{\partial t} + L^\varepsilon u^\varepsilon_i(t,x,\omega) + h_i(t,\tau_{x/\varepsilon}\omega,x,u^\varepsilon(t,x,\omega)) \\ \qquad + \dfrac{\partial u^\varepsilon_i(t,x))}{\partial x_j}\widehat{h}_j(t,\tau_{x/\varepsilon}\omega,x,u^\varepsilon(t,x,\omega)) = 0,\ x \in \mathcal{O},\ t \in [0,T) \\ u^\varepsilon(T,x,\omega) = g(x) \in \mathrm{L}^2(\mathcal{O})^m, \\ u^\varepsilon(T,\cdot,\omega) \in \mathrm{H}^1_0(\mathcal{O})^m,\ \forall t \in [0,T). \end{cases} \tag{44}$$

Then, for almost every realization ω, almost every $x \in \mathcal{O}$ and any $t \geqslant 0$, the solution $u^\varepsilon(t,x,\omega)$ converges almost surely to the solution of the system

$$\begin{cases} \dfrac{\partial u_i(t,x)}{\partial t} + \dfrac{1}{2}\overline{a}_{k,j}\dfrac{\partial^2 u_i(t,x)}{\partial x_k \partial x_j} + \overline{h}_i(t,x,u(t,x)) \\ \qquad + \dfrac{\partial u_i(t,x)}{\partial x_j}\overline{\overline{h}}_j(t,x,u(t,x)) = 0,\ x \in \mathcal{O},\ t \in [0,T), \\ u(T,x) = g(x), \\ u(t,\cdot) \in \mathrm{H}^1_0(\mathcal{O})^m,\ \forall t \in [0,T), \end{cases} \tag{45}$$

where $\overline{a}$ is defined by (40)*,*

$$\overline{h}_i(t,x,y) = \int_{\Omega_{\mathcal{M}}} h(t,\omega,x,y)\,\mathrm{d}\mu(\omega)$$
$$\text{and } \overline{\overline{h}}_i(t,x,y) = \int_{\Omega_{\mathcal{M}}} (\delta_{i,j} + \mathbf{f}^i_j)(\omega)\widehat{h}_j(t,\omega,x,y)\,\mathrm{d}\mu(\omega).$$

The choice of the hypotheses (h-vi)-(h-viii) will be justified by Proposition 2 below.

The method used here does not allow to deal with a general non-linearity in ∇u, because of the lack on control on the term Z^ε (see the comments p. 535 in Pardoux 1999a).

The remainder of this section is devoted to prove Theorem 7. We assume that we have fixed an arbitrary realization ω of the media, with a starting point x.

The distribution $\widehat{\mathbb{P}}^\varepsilon$ is the distribution of the process starting at point x and time 0 whose infinitesimal generator is

$$L^\varepsilon + \widehat{h}(s, \tau_x\omega, x, u^\varepsilon(s,x)),$$

where u is the solution of (44). The martingale part of X under the distribution $\widehat{\mathbb{P}}^\varepsilon$ is denoted by $\widehat{\mathsf{M}}^{\mathsf{X}^\varepsilon}$. We denote by τ^ε the first exit time of X of the open, connected subset $\mathcal{O}$ of $\mathbb{R}^N$.

Let us denote by $\mathsf{Y}^\varepsilon, \mathsf{Z}^\varepsilon$ the unique $\mathcal{F}^{\mathsf{X}^\varepsilon}$-progressively measurable solution to the BSDE

$$\begin{cases} \mathsf{Y}^\varepsilon_t = g(\mathsf{X}^\varepsilon_T)\mathbf{1}_{\{T<\tau^\varepsilon\}} + \int_t^T h(s, \tau_{\mathsf{X}^\varepsilon_s}\omega, \mathsf{X}^\varepsilon_s, \mathsf{Y}^\varepsilon_s)\,\mathrm{d}s - \int_t^T \mathsf{Z}^\varepsilon_s\,\mathrm{d}\widehat{\mathsf{M}}^{\mathsf{X}^\varepsilon}_s, \\ \widehat{\mathbb{E}}\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}^\varepsilon_t|^2 + \int_0^T \|\mathsf{Z}^\varepsilon_t\|^2\,\mathrm{d}t\right]. \end{cases}$$

We use the convention that Y^ε_t and Z^ε_t are equal to zero when $t \geqslant \tau^\varepsilon$, and that $x \mapsto h(s,x,y)$ and $x \mapsto \widehat{h}(s,x,y)$ are extended to be equal to 0 in the cemetery point Δ. We set $\mathsf{M}^\varepsilon_t = \int_0^t \mathsf{Z}^\varepsilon_s\,\mathrm{d}\widehat{\mathsf{M}}^{\mathsf{X}^\varepsilon}_s$.

The distribution $\mathbb{P}^\varepsilon$ is that of the process whose generator is L^ε and starting from some point x.

The canonical stochastic process will be denoted by X^ε while working under the distributions $\mathbb{P}^\varepsilon$ or $\widehat{\mathbb{P}}^\varepsilon$.

We know from the results in Section 6.1 for periodic media and 6.2 for random media that $\mathcal{L}(\mathsf{X}^\varepsilon \mid \mathbb{P}^\varepsilon)$ converges to the distribution of some non-standard Brownian Motion.

We assume in a first time that the final condition g is a smooth function with compact support on $\mathcal{O}$. We will see in Step 10 how to drop this condition.

Step 1: Use of the Girsanov transform. As we have seen it in Section 5.1, the distribution $\widehat{\mathbb{P}}^\varepsilon$ is the probability measure defined on the space of continuous functions by the relation,

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}^\varepsilon}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}^{\mathsf{X}}_t} = \exp\left(\mathsf{L}^\varepsilon_t - \frac{1}{2}\langle\mathsf{L}^\varepsilon\rangle_t\right),\ \forall t \in [0,T],$$

where $\mathsf{L}^\varepsilon_t = \int_0^{t\wedge\tau^\varepsilon} a^{-1}(\mathsf{X}^\varepsilon_s/\varepsilon)\widehat{h}(s,\tau_{\mathsf{X}^\varepsilon_s}\omega,\mathsf{X}^\varepsilon_s,\mathsf{Y}^\varepsilon_s)\,\mathrm{d}\mathsf{M}^{\mathsf{X}}$. We remark that for any $\gamma>1$, for any $\varepsilon>0$, any $t\in[0,T]$ and for any function Φ $\mathcal{F}^{\mathsf{X}^\varepsilon}_T$-measurable,

$$\begin{aligned}\widehat{\mathbb{E}}^\varepsilon[\,\Phi\,] &\leqslant \mathbb{E}^\varepsilon\left[\exp\left(\gamma'\mathsf{L}^\varepsilon_T+\frac{\gamma'}{2}\langle\mathsf{L}^\varepsilon\rangle_T\right)\right]^{1/\gamma'}\mathbb{E}^\varepsilon\left[\exp\left(\frac{\gamma'}{2}\langle\mathsf{L}^\varepsilon\rangle_T\right)\Phi^\gamma\right]^{1/\gamma}\\ &\leqslant \exp\left(\frac{\gamma'\lambda\widehat{K}^2}{2}T\right)\mathbb{E}^\varepsilon[\,\Phi^\gamma\,]^{1/\gamma},\end{aligned}\tag{46}$$

where $\mathbb{P}^\varepsilon$ is the distribution of X^ε and $1/\gamma+1/\gamma'=1$. Hence, it follows that the convergence in probability under $\mathbb{P}^\varepsilon$ implies the convergence in probability under $\widehat{\mathbb{P}}^\varepsilon$.

Step 2: An estimate on Y^ε and M^ε.

We have already seen that u^ε is bounded. But for the periodic media, we work with the operator $L^\varepsilon+\frac{1}{\varepsilon}b$, and in this case, it has not been proved that u^ε is bounded by a constant that does not depend on ε, since the bounds of the coefficients vary with ε.

But as in Section 4.3, Y^ε is bounded by a constant that does not depend on ε, because we need only to know the bounds of the coefficient a. In fact, we only require

$$\widehat{\mathbb{E}}^\varepsilon\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}^\varepsilon_t|^2\right]\leqslant C_0$$

which may be directly proved setting $\alpha=0$ in the proof of Section 4.3.

Since

$$\widehat{\mathbb{E}}^\varepsilon[\,\langle\mathsf{M}^\varepsilon\rangle_T\,]\leqslant\Lambda\widehat{\mathbb{E}}^\varepsilon\left[\int_0^{T\wedge\tau^\varepsilon}\|\mathsf{Z}^\varepsilon_s\|^2\,\mathrm{d}s\right],$$

it follows that

$$\sup_{\varepsilon>0}\widehat{\mathbb{E}}^\varepsilon\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}^\varepsilon_t|^2+\sup_{0\leqslant t\leqslant T}|\mathsf{M}^\varepsilon_t|^2\right]<+\infty.\tag{47}$$

We also remark that from (h-v),

$$\mathrm{CV}(\mathsf{Y}^\varepsilon)\leqslant\widehat{\mathbb{E}}^\varepsilon\left[\int_0^{T\wedge\tau^\varepsilon}|h^\varepsilon(s,\mathsf{X}^\varepsilon_s,\mathsf{Y}^\varepsilon_s)|\,\mathrm{d}s\right]\leqslant TK'\widehat{\mathbb{E}}^\varepsilon\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}^\varepsilon_s|^2\right]+TK'.$$

With (A1), this is sufficient to assert that $(\mathsf{Y}^\varepsilon,\mathsf{M}^\varepsilon)$ is S-tight.

Step 3: Convergence of $(\mathsf{X}^\varepsilon,\mathsf{Y}^\varepsilon,\mathsf{M}^\varepsilon)$. From (47), the sequence $\mathcal{L}(\mathsf{Y}^\varepsilon,\mathsf{M}^\varepsilon\mid\widehat{\mathbb{P}}^\varepsilon)$ is S-tight (see (A1)). The sequence $\mathcal{L}(\mathsf{X}^\varepsilon\mid\mathbb{P}^\varepsilon)$ is tight for the topology U of the uniform norm as we have seen it in Section 6.1 and 6.2. With the result of Step 1, it is clear that $\mathcal{L}(\mathsf{X}^\varepsilon\mid\widehat{\mathbb{P}}^\varepsilon)$ is tight.

As a result, there exist some processes $\overline{\mathsf{X}}$ in $\mathcal{C}([0,T];\mathbb{R}^N)$, $\overline{\mathsf{Y}}$ and $\overline{\mathsf{M}}$ in $\mathcal{D}([0,T];\mathbb{R}^m)$ such that $\mathcal{L}(\mathsf{X}^\varepsilon,\mathsf{Y}^\varepsilon,\mathsf{M}^\varepsilon \mid \widehat{\mathbb{P}}^\varepsilon) \xrightarrow[\varepsilon\to 0]{U\times S\times S\text{-topology}} \mathcal{L}\left(\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}\middle|\overline{\mathbb{P}}\right)$ along a subsequence.

Step 4: Convergence of X^ε. Under $\widehat{\mathbb{P}}^\varepsilon$, for $i=1,\dots,N$,

$$\begin{aligned}\mathsf{X}^{\varepsilon,i}_t+\varepsilon v_i(\mathsf{X}^\varepsilon_t/\varepsilon)&=\mathsf{X}^{\varepsilon,i}_{\varepsilon^2}+\varepsilon v_i(\mathsf{X}^\varepsilon_{\varepsilon^2}/\varepsilon)\\&\quad+\widehat{\mathsf{M}}^{\varepsilon,i}_t+\int_{\varepsilon^2}^t\left(\delta_{i,j}+\frac{\partial v_i}{\partial x_j}(\mathsf{X}^\varepsilon_s,\omega)\right)\widehat{h}_j(s,\tau_{\mathsf{X}^\varepsilon_s}\omega,\mathsf{X}^\varepsilon_s,\mathsf{Y}^\varepsilon_s)\,\mathrm{d}s,\end{aligned}\tag{48}$$

where $\widehat{\mathsf{M}}^\varepsilon$ is a local martingale with cross-variations

$$\langle\widehat{\mathsf{M}}^i,\widehat{\mathsf{M}}^j\rangle_t=\int_{\varepsilon^2}^t a_{p,q}\left(\delta_{i,p}+\frac{\partial v_i}{\partial x_p}\right)\left(\delta_{j,q}+\frac{\partial v_j}{\partial x_q}\right)(\mathsf{X}^\varepsilon_s,\omega)\,\mathrm{d}s.$$

From (46), the Central Limit Theorem for martingales (See Theorem 7.1.4, p. 339 in Ethier and Kurtz 1986 and Proposition 2, there exists a distribution $\overline{\mathbb{P}}$ on the space of continuous function such that $\mathcal{L}(\widehat{\mathsf{M}}^{\mathsf{X}^\varepsilon}\mid\widehat{\mathbb{P}}^\varepsilon)$ converges weakly to $\mathcal{L}(\mathsf{M}^{\overline{\mathsf{X}}}\mid\overline{\mathbb{P}})$.

The following Proposition is used to prove the convergence of the last term in the right-hand side of (48).

Proposition 2. *Let $f(t,\omega,x,y)$ a function satisfying* (h-vi)-(h-viii). *Then for almost every realization of the media, the following convergence holds:*

$$\sup_{0\leqslant t\leqslant T}\left|\int_0^{t\wedge\tau^\varepsilon}f^\varepsilon(s,\tau_{\mathsf{X}^\varepsilon_s}\omega,\mathsf{X}^\varepsilon_s,\mathsf{Y}^\varepsilon_s)\,\mathrm{d}s-\int_0^{t\wedge\tau^\varepsilon}\overline{f}(s,\mathsf{X}^\varepsilon_s,\mathsf{Y}^\varepsilon_s)\,\mathrm{d}s\right|\xrightarrow[\varepsilon\to0]{\text{proba.}}0,$$

where $\overline{f}(s,x,y)=\int_{\Omega_{\mathcal{M}}}f(s,\omega,x,y)\,\mathrm{d}\mu(\omega)$.

Proof. The argument used in the proof of Lemma 4.2 in Pardoux 1999b are easily adapted to our case. □

We denote by $\overline{\tau}$ the first exit time from $\mathcal{O}$ for the process $\overline{\mathsf{X}}$. From results on the S-topology (see Corollary 2.11, p.10 in Jakubowski 1997 and Proposition 2, it is clear that $\int_0^{t\wedge\tau^\varepsilon}\overline{\widehat{h}}(s,\mathsf{X}^\varepsilon_s,\mathsf{Y}^\varepsilon_s)\,\mathrm{d}s$ converges in $\mathcal{C}([0,T];\mathbb{R}^m)$ to $\int_0^{t\wedge\overline{\tau}}\overline{\widehat{h}}(s,\overline{\mathsf{X}}_s,\overline{\mathsf{Y}}_s)\,\mathrm{d}s$. Passing to the limit,

$$\overline{\mathsf{X}}_t=\mathsf{M}^{\overline{\mathsf{X}}}_t+\int_0^{t\wedge\overline{\tau}}\overline{\widehat{h}}(s,\overline{\mathsf{X}}_s,\overline{\mathsf{Y}}_s)\,\mathrm{d}s,\ \forall t\in[0,T],\ \overline{\mathbb{P}}\text{-a.s.}\tag{49}$$

and $\mathsf{M}^{\overline{\mathsf{X}}}$ is a $(\mathcal{F}^{\overline{\mathsf{X}}},\overline{\mathbb{P}})$-martingale with cross-variations $t\overline{a}$.

Step 5: Convergence of Y^ε at the terminal time. As u^ε is the solution to (44), if $\tau^\varepsilon \leqslant T$, then $\mathsf{X}^\varepsilon_{\tau^\varepsilon}$ belongs to $\partial\mathcal{O}$, and then $u^\varepsilon(T\wedge\tau^\varepsilon, \mathsf{X}^\varepsilon_{T\wedge\tau^\varepsilon}) = g(\mathsf{X}^\varepsilon_T)\mathbf{1}_{\{T<\tau^\varepsilon\}}$. Furthermore, since g is a continuous function with compact support in $\mathcal{O}$, it is also clear that $g(\mathsf{X}^\varepsilon_T)\mathbf{1}_{\{T<\tau^\varepsilon\}} = g(\mathsf{X}^\varepsilon_{T\wedge\tau^\varepsilon})$ and it converges to $g(\overline{\mathsf{X}}_T)\mathbf{1}_{\{T<\overline{\tau}\}}$, where $\overline{\tau}$ is the first time the process $\overline{\mathsf{X}}$ exits from $\mathcal{O}$.

Step 6: Passage to the limit for the BSDEs. With Theorem A.1 and Proposition A.1, there exists a countable set D in $[0,T)$ such that for any $t\in[0,T]\setminus D$,

$$\mathcal{L}\left(\tau^\varepsilon, \mathsf{X}^\varepsilon_t, \mathsf{Y}^\varepsilon_t, \mathsf{M}^\varepsilon_t, \int_t^T \overline{h}(s,\mathsf{X}^\varepsilon_s,\mathsf{Y}^\varepsilon_s)\,\mathrm{d}s, \mathsf{X}^\varepsilon_T, \mathsf{M}^\varepsilon_T \,\middle|\, \widehat{\mathbb{P}}^\varepsilon\right)$$
$$\xrightarrow[\varepsilon\to 0]{} \mathcal{L}\left(\overline{\tau}, \overline{\mathsf{X}}_t, \overline{\mathsf{Y}}_t, \overline{\mathsf{M}}_t, \int_t^T \overline{h}(s,\overline{\mathsf{X}}_s,\overline{\mathsf{Y}}_s)\,\mathrm{d}s, \overline{\mathsf{X}}_T, \overline{\mathsf{M}}_T \,\middle|\, \overline{\mathbb{P}}\right), \quad (50)$$

along a subsequence.

Using the right-continuity of $\overline{\mathsf{Y}}$ and $\overline{\mathsf{M}}$, $\overline{\mathbb{P}}$-a.s.,

$$\overline{\mathsf{Y}}_t = g(\overline{\mathsf{X}}_T)\mathbf{1}_{\{T<\overline{\tau}\}} + \int_{t\wedge\overline{\tau}}^{T\wedge\overline{\tau}} \overline{h}(s,\overline{\mathsf{X}}_s,\overline{\mathsf{Y}}_s)\,\mathrm{d}s + \overline{\mathsf{M}}_{T\wedge\overline{\tau}} - \overline{\mathsf{M}}_{t\wedge\overline{\tau}},\ \forall t\in[0,T], \quad (51)$$

Step 7: The martingales $\mathsf{M}^{\overline{\mathsf{X}}}$ and $\overline{\mathsf{M}}$ are $(\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}, \overline{\mathbb{P}})$-martingales. The processes Y^ε and M^ε are $\mathcal{F}^{\mathsf{X}^\varepsilon}$-adapted. We denote by $\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}$ the minimal admissible and complete filtration generated by $(\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}})$.

We remark from (51) that $\overline{\mathsf{M}}_t$ is a function of $\overline{\mathsf{Y}}_s$, $\overline{\mathsf{X}}_s$ for $s\leqslant t$, and $\overline{\mathsf{M}}_0$. Hence, $\overline{\mathsf{M}}$ is $\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}$-adapted.

With the Burkholder-Davis-Gundy Inequality and (47), we may apply Lemma A.1 to $(\mathsf{X}^\varepsilon,\mathsf{Y}^\varepsilon,\mathsf{M}^\varepsilon)$, that asserts that $\overline{\mathsf{M}}$ is a $(\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}, \overline{\mathbb{P}})$-martingale.

Since $\mathsf{M}^{\overline{\mathsf{X}}}_t$ is $\mathcal{F}^{\overline{\mathsf{X}}}_t$-adapted, it is also $\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}_t$-adapted.

In the case of a periodic media, we have only to use (37) with (46) and Lemma A.1 to prove $\mathsf{M}^{\overline{\mathsf{X}}}$ is a $\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}$-martingale.

The function $\breve{\mathbf{a}}$ defined by (43) belongs in fact to $\mathrm{L}^{1+\gamma}(\Omega_{\mathcal{M}},\mu)$ for some $\gamma>0$ that depends only on λ and Λ. For that, we have to use the Meyers' estimate on the gradient — which is a stationary random field — of the harmonic function $w_i(x,\omega)$ (for $i=1,\dots,N$) (see Meyers 1963). Hence, with (46),

$$\widehat{\mathbb{E}}_\mu\left[\sup_{0\leqslant t\leqslant T}|\widehat{\mathsf{M}}^\varepsilon_t|^2\right] \leqslant C_0\widehat{\mathbb{E}}_\mu\left[\int_0^T |\breve{\mathbf{a}}|(\tau_{\mathsf{X}^\varepsilon_s/\varepsilon}\omega)\,\mathrm{d}s\right]$$
$$\leqslant C_1\mathbb{E}_\mu\left[\varepsilon^2\int_0^{T/\varepsilon^2} |\breve{\mathbf{a}}|^{1+\gamma}(\tau_{\mathsf{X}_s}\omega)\,\mathrm{d}s\right] = C_1\int_\Omega |\breve{\mathbf{a}}|^{1+\gamma}(\omega)\,\mathrm{d}\mu(\omega)$$

for some constants C_0 and C_1, where C_1 depends on γ. Hence Lemma A.1 is valid for $(\mathsf{X}^\varepsilon, \mathsf{Y}^\varepsilon, \mathsf{M}^\varepsilon, \widehat{\mathsf{M}}^\varepsilon)$ if we work under $\widehat{\mathbb{P}}_\mu$. Since $\overline{\mathbb{P}}$ does not depend on a realization of the media, $\mathsf{M}^{\overline{\mathsf{X}}}$ is a $(\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}, \overline{\mathbb{P}})$-martingales.

Step 8: Identification of the limit. Let $(\mathsf{Y}_t, \mathsf{Z}_t)_{t\in[0,T]}$ be the unique $\mathcal{F}^{\overline{\mathsf{X}}}_t$-progressively measurable solution of the BSDE

$$\mathsf{Y}_t = g(\overline{\mathsf{X}}_T)\mathbf{1}_{\{T<\overline{\tau}\}} + \int_{t\wedge\overline{\tau}}^{T\wedge\overline{\tau}} \overline{h}(s, \overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}s - \int_{t\wedge\overline{\tau}}^{T\wedge\overline{\tau}} \mathsf{Z}_s\,\mathrm{d}\mathsf{M}^{\overline{\mathsf{X}}}_s \tag{52}$$

with $\overline{\mathbb{E}}\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}_t|^2 + \int_0^{T\wedge\tau^\varepsilon}\|\mathsf{Z}_s\|^2\,\mathrm{d}s\right] < +\infty$ and $(\mathsf{Y}_t, \mathsf{Z}_t) = 0$ on $[\overline{\tau}, T]$. The process Y is $\mathcal{F}^{\overline{\mathsf{X}}}_t$-adapted. Hence it is $\mathcal{F}^{\overline{\mathsf{X}},\overline{\mathsf{Y}},\overline{\mathsf{M}}}$-adapted. We set $\mathsf{M}_t = \int_0^{t\wedge\overline{\tau}} \mathsf{Z}_s\,\mathrm{d}\mathsf{M}^{\overline{\mathsf{X}}}_s$.

With Lemma A.2, $\overline{\mathsf{Y}}_t$ and $\overline{\mathsf{M}}_t$ are square-integrable for each $t \geqslant 0$, and from Lemma 3, $\int_0^{T\wedge\overline{\tau}} \overline{\mathsf{Y}}_s\,\mathrm{d}\overline{\mathsf{M}}_s$ is a martingale.

Applying the Itô formula for possibly discontinuous martingale $\overline{\mathsf{M}}$ and using (47),

$$\begin{aligned}
\overline{\mathbb{E}}\left[|\mathsf{Y}_t - \overline{\mathsf{Y}}_t|^2\right] &+ \sum_{i=1}^m 2\overline{\mathbb{E}}\left[[\mathsf{M}^i - \overline{\mathsf{M}}^i]_T - [\mathsf{M}^i - \overline{\mathsf{M}}^i]_t\right] \\
&= 2\overline{\mathbb{E}}\left[\int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} (\overline{h}(s, \overline{\mathsf{X}}_s, \mathsf{Y}_s) - \overline{h}(s, \overline{\mathsf{X}}_s, \overline{\mathsf{Y}}_s))\cdot(\mathsf{Y}_s - \overline{\mathsf{Y}}_s)\,\mathrm{d}s\right] \\
&\qquad\qquad\qquad\qquad \leqslant 2K\int_t^T |\mathsf{Y}_s - \overline{\mathsf{Y}}_s|^2\,\mathrm{d}s.
\end{aligned}$$

Using the Gronwall Lemma $\mathsf{Y} = \overline{\mathsf{Y}}$ and $\mathsf{M} = \overline{\mathsf{M}}$. This proves that $(\mathsf{Y}^\varepsilon, \mathsf{M}^\varepsilon)$ converge to (Y, M) is the S-topology.

Step 9: Convergence of Y_0^ε to Y_0. The convergence in the S-topology does not ensure us that Y_0^ε converges to Y_0. But the value of Y^ε at time 0 is given by

$$\mathsf{Y}_0^\varepsilon = g(\mathsf{X}_T^\varepsilon)\mathbf{1}_{\{T<\tau^\varepsilon\}} + \int_0^{T\wedge\overline{\tau}} h^\varepsilon(s, \mathsf{X}_s^\varepsilon, \mathsf{Y}_s^\varepsilon)\,\mathrm{d}s - \mathsf{M}_T^\varepsilon.$$

It follows from the continuity of the projection $\pi_T : y \in (\mathcal{D}([0,T];\mathbb{R}), S) \mapsto y(T)$ (*cf.* Remark A.2) that M_T^ε converges in distribution to M_T. Then Y_0^ε converges in distribution to Y_0. Since Y_0^ε and Y_0 are deterministic,

$$\begin{aligned}
\mathsf{Y}_0^\varepsilon &= g(\mathsf{X}_T^\varepsilon)\mathbf{1}_{\{T<\tau^\varepsilon\}} + \int_0^{T\wedge\overline{\tau}} h^\varepsilon(s, \mathsf{X}_s^\varepsilon, \mathsf{Y}_s^\varepsilon)\,\mathrm{d}s + \mathsf{M}_T^\varepsilon \\
&\xrightarrow[\varepsilon\to 0]{} \mathsf{Y}_0 = g(\mathsf{X}_T)\mathbf{1}_{\{T<\overline{\tau}\}} + \int_0^{T\wedge\overline{\tau}} \overline{h}(s, \overline{\mathsf{X}}_s, \mathsf{Y}_s)\,\mathrm{d}s + \mathsf{M}_T.
\end{aligned}$$

Step 10: Extension to any initial condition in $\mathrm{L}^2(\mathcal{O})$. If the final condition g belongs only to $\mathrm{L}^2(\mathcal{O})$, there exists a sequence of continuous functions g_η such that g_η converges to g in $\mathrm{L}^2(\mathcal{O})$. We use again a stability result and Theorem 4.2, p. 25 in Billingsley 1968.

Let $(\mathsf{Y}^{\varepsilon,\eta},\mathsf{Z}^{\varepsilon,\eta})$ be the solution to

$$\mathsf{Y}^{\varepsilon,\eta}_t = g_\eta(\mathsf{X}^\varepsilon_T)\mathbf{1}_{\{T<\tau^\varepsilon\}} + \int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} h(s,\tau_{\mathsf{X}^\varepsilon_s/\varepsilon},\mathsf{X}^\varepsilon_s,\mathsf{Y}^{\varepsilon,\eta}_s)\,\mathrm{d}s - \int_{t\wedge\tau^\varepsilon}^{T\wedge\tau^\varepsilon} \mathsf{Z}^{\varepsilon,\eta}_s\,\mathrm{d}\mathsf{M}^{\mathsf{X}^\varepsilon}_s$$

satisfying (19b)-(19d), and $(\mathsf{Y}^\eta,\mathsf{Z}^\eta)$ the solution to

$$\mathsf{Y}^\eta_t = g_\eta(\overline{\mathsf{X}}_T)\mathbf{1}_{\{T<\overline{\tau}\}} + \int_{t\wedge\overline{\tau}}^{T\wedge\overline{\tau}} \overline{h}(s,\overline{\mathsf{X}}_s,\mathsf{Y}^{\varepsilon,\eta}_s)\,\mathrm{d}s - \int_{t\wedge\overline{\tau}}^{T\wedge\overline{\tau}} \mathsf{Z}^{\varepsilon,\eta}_s\,\mathrm{d}\mathsf{M}^{\overline{\mathsf{X}}}_s.$$

Using the results of Section 4.2.3, we deduce that

$$\widehat{\mathbb{E}}\left[\,|\mathsf{Y}^\varepsilon_0-\mathsf{Y}^{\varepsilon,\eta}_0|^2\,\right] \leqslant C\widehat{\mathbb{E}}\left[\,|g(\mathsf{X}^\varepsilon_T)-g_\eta(\mathsf{X}^\varepsilon_T)|^2; T<\tau^\varepsilon\,\right].$$

If there is no highly-oscillatory first order term in the operator L^ε (this is the case for the random media), the Aronson estimate (15) implies that there exists some constant C depending only on λ, Λ, the dimension N and T such that

$$\widehat{\mathbb{E}}\left[\,|g(\mathsf{X}^\varepsilon_T)-g_\eta(\mathsf{X}^\varepsilon_T)|^2; T<\tau^\varepsilon\,\right] \leqslant C\,\|g-g_\eta\|^2_{\mathrm{L}^2(\mathcal{O})} \xrightarrow[\eta\to 0]{\text{uniformly in }\varepsilon} 0.$$

It is now clear from Theorem 4.2, p. 25 in Billingsley 1968 that $g(\mathsf{X}^\varepsilon_T)\mathbf{1}_{\{T<\tau^\varepsilon\}}$ converges in distribution to $g(\overline{\mathsf{X}}_T)\mathbf{1}_{\{T<\overline{\tau}\}}$.

For the homogenization in periodic media with the presence of a highly-oscillatory first order term, we have to work a bit more.

Lemma 8. *We assume that the hypothesis on the coefficients a and b defined in Section 6.1 are satisfied. Let f be a function in $\mathrm{L}^1(\mathcal{O})$. Then there exists some constant C depending only on λ, Λ such that*

$$\sup_{x\in\mathcal{O}} \mathbb{E}_x\left[\,|f(\mathsf{X}^\varepsilon_T)|\,\right] \leqslant C\,\|f\|_{\mathrm{L}^1(\mathcal{O})}.$$

Proof. Using a partition of the unity, we may assume that the support of f is contained in the cube $\xi+[0,1]^N$, and it may be extended to a periodic function $\widetilde{f}$. With Proposition 1 in Lejay 2001a, we know that

$$\begin{aligned}\sup_{x\in\mathcal{O}} \mathbb{E}_x\left[\,|f(\mathsf{X}^\varepsilon_T)|\,\right] &\leqslant \sup_{x\in\mathbb{R}^N} \mathbb{E}_x\left[\,|\widetilde{f}(\varepsilon^{-1}\mathsf{X}^\varepsilon_{T/\varepsilon^2})|\,\right] \\ &\leqslant C\int_{[0,1]^N} |\widetilde{f}(x/\varepsilon)|\,\mathrm{d}x \leqslant (C+1)\int_{[0,1]^N} |\widetilde{f}(x)|\,\mathrm{d}x\end{aligned}$$

for some constant C that depends only on λ and Λ. □

We have only to conclude as previously, by using again a Girsanov transform to work under the distribution $\widehat{\mathbb{P}}$.

Conclusion The proof of Theorem 7 is then complete, and the expressions (49) and (52), together the identification of Step 8 and Theorem 6 allows to express $\overline{\mathsf{Y}}_0$ as $u(0,x)$, where u is the solution of (45). With Step 9, we have then proved that $u^\varepsilon(0,x,\omega)$ converges to $u(0,x)$. There is no difficulty to prove using the same method that $u^\varepsilon(t,x,\omega)$ converges to $u(t,x)$ for any $t\in[0,T]$.

7 Concluding remarks

1. Our results may be extended to the case the self-adjoint operator L takes the form

$$L=\frac{e^{2V}}{2}\frac{\partial}{\partial x_i}\left(a_{i,j}e^{-2V}\frac{\partial}{\partial x_j}\right)$$

where V is a measurable bounded function. In this case, it is natural to work with the space $\mathrm{L}^2(\mathcal{O};e^{2V(x)}\,\mathrm{d}x)$ instead of $\mathrm{L}^2(\mathcal{O})$. For the homogenization in random media, the invariant measure is $e^{-2\mathbf{V}(\omega)}\,\mathrm{d}\mu(\omega)$ if $\mathbf{V}$ is the random variable on $(\Omega_{\mathcal{M}},\mathcal{G},\mu)$ corresponding to the stationary random field V and $\int_{\Omega_{\mathcal{M}}}e^{-2\mathbf{V}(\omega)}\,\mathrm{d}\mu(\omega)=1$. For the homogenization in periodic media, the density π of the invariant measure is given by $L^*\pi=0$, where L^* is the adjoint of L seen as an operator on the space $\mathrm{L}^2_{\mathrm{per}}(e^{-2V(x)}\,\mathrm{d}x)$ of periodic functions.

2. There should be no difficulty to extend the results of Section 4 and 5 when the coefficients of the linear operator depend on time, and are bounded and uniformly elliptic independently of the time variable.

3. All the previous proofs may be adapted for studying systems of semi-linear elliptic PDEs of the form

$$\begin{cases}Lu^i(x)+h_i(x,u^i(x),\nabla u^i(x))=f_i(x),\\ u^i\in\mathrm{H}^1_0(\mathcal{O}),\ i=1,\dots,n.\end{cases}$$

A The S-topology

The S-topology has been introduced by A. Jakubowski in Jakubowski 1997. It is a topology defined on the Skorohod space $\mathcal{D}([0,T];\mathbb{R})$ of càdlàg (right continuous with left limit at each point) functions that is weaker than the Skorohod topology. But tightness criterions are easier to establish with this topology using the same tightness criteria as the one introduced by P.A. Meyer and W.A. Zheng in Meyer and Zheng 1984.

Let z be a function in $\mathcal{D}([0,T];\mathbb{R})$. Let $N^{a,b}(z)$ the number of *up-crossing* given levels $a < b$, *i.e.*, $N^{a,b}(z) \geqslant k$ if there exist numbers $0 \leqslant t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} \leqslant T$ such that $z(t_{2k-1}) < a$ and $z(t_{2k}) > b$.

We recall here some propositions about the S-topology.

Proposition A.1 (A criteria for S-tightness). *A sequence $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is S-tight if and only if it is relatively compact with respect to the S-topology.*

Let $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ be a family of stochastic processes in $\mathcal{D}([0,T];\mathbb{R})$. Then this family is tight for the S-topology if and only if $(\|\mathsf{Y}^\varepsilon\|_\infty)_{\varepsilon>0}$ and $(N^{a,b}(\mathsf{Y}^\varepsilon))_{\varepsilon>0}$ are tight for each $a < b$.

Remark A.1. The S-topology is defined on $\mathcal{D}([0,T];\mathbb{R})$, but may be easily extended to the N-dimensional case. We remark that from the very definition, $(\mathsf{Y}^{1,\varepsilon},\dots,\mathsf{Y}^{N,\varepsilon})_{\varepsilon>0}$ is S-tight if and only if $(\mathsf{Y}^{i,\varepsilon})_{\varepsilon>0}$ is S-tight for $i = 1,\dots,N$.

If $(\mathsf{Y},\mathbb{P},(\mathcal{F}_t)_{t\geqslant 0})$ is a process in $\mathcal{D}([0,T];\mathbb{R})$ such that Y_t is integrable for any t, the *conditional variation* of Y is defined by

$$\mathrm{CV}(\mathsf{Y}) = \sup_{0\leqslant t_1<\cdots<t_n=T \text{ partition of } [0,T]} \sum_{i=1}^{n-1} \mathbb{E}\left[\left|\mathbb{E}\left[\mathsf{Y}_{t_{i+1}} - \mathsf{Y}_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right]\right|\right].$$

The process Y is called a *quasimartingale* if $\mathrm{CV}(\mathsf{Y}) < +\infty$.

When Y is a $\mathcal{F}_t$-martingale, then $\mathrm{CV}(\mathsf{Y}) = 0$.

A variation of the Doob inequality (*cf.* Lemma 3, p. 359 in Meyer and Zheng 1984, where it is assumed that $\mathsf{Y}_T = 0$) implies that

$$\mathbb{P}\left[\sup_{t\in[0,T]} |\mathsf{Y}_t| \geqslant \kappa\right] \leqslant \frac{2}{\kappa}\left(\mathrm{CV}(\mathsf{Y}) + \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\mathsf{Y}_t|\right]\right),$$
$$\mathbb{E}\left[N^{\alpha,\beta}(\mathsf{Y})\right] \leqslant \frac{1}{\beta-\alpha}\left(|\alpha| + \mathrm{CV}(\mathsf{Y}) + \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\mathsf{Y}_t|\right]\right).$$

It follows that a sequence $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ is S-tight if

$$\sup_{\varepsilon>0}\left(\mathrm{CV}(\mathsf{Y}^\varepsilon) + \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\mathsf{Y}^\varepsilon_t|\right]\right) < +\infty. \tag{A1}$$

Theorem A.1. *Let $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ be a S-tight family of stochastic processes in $\mathcal{D}([0,T];\mathbb{R})$. Then there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing to 0, some process Y in $\mathcal{D}([0,T];\mathbb{R})$ and a countable subset $D \in [0,T)$ such that for any n and any $(t_1,\dots,t_n) \subset [0,T] \setminus D$,*

$$(\mathsf{Y}^{\varepsilon_k}_{t_1},\dots,\mathsf{Y}^{\varepsilon_k}_{t_n}) \xrightarrow[k\to\infty]{\text{dist.}} (\mathsf{Y}_{t_1},\dots,\mathsf{Y}_{t_n}).$$

Remark A.2. We have to note that the projection $\pi_T : y \in (\mathcal{D}([0,T];\mathbb{R}), S) \mapsto y(T)$ is continuous (see Remark 2.4, p. 8 in Jakubowski 1997), but $y \mapsto y(t)$ is not continuous for each $0 \leqslant t < T$.

We can now state two Lemmas concerning the convergence of some sequence of random variable in the S-topology.

Lemma A.1. *Let* $(\mathsf{X}^\varepsilon, \mathsf{M}^\varepsilon)$ *be a multidimensional process in* $\mathcal{D}([0,T];\mathbb{R}^p)$ *(*$p \in \mathbb{N}^*$*) converging to* (X, M) *in the* S*-topology. Let* $(\mathcal{F}_t^{\mathsf{X}^\varepsilon})_{t\geqslant 0}$ *(resp.* $(\mathcal{F}_t^{\mathsf{X}})_{t\geqslant 0}$*) be the minimal complete admissible filtration for* X^ε *(resp.* X*). We assume that*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\mathsf{M}_t^\varepsilon|^2\right] < C_T, \quad \forall T > 0, \tag{A2a}$$

$$\mathsf{M}^\varepsilon \text{ is a } \mathcal{F}^{\mathsf{X}^\varepsilon}\text{−martingale}, \tag{A2b}$$

$$\text{and } \mathsf{M} \text{ is } \mathcal{F}^{\mathsf{X}}\text{-adapted.} \tag{A2c}$$

Then M *is a* $\mathcal{F}^{\mathsf{X}}$*-martingale.*

Proof. We may assume without loss of generality that X and M are one-dimensional processes in $\mathcal{D}([0,T];\mathbb{R})$. Let $t_0 \in [0,t]$.

Let D be a countable dense subset of $[0,T]$ such that X converges in finite-dimensional distribution on $[0,T] \setminus D$ (see Theorem A.1). Let Φ_{t_0} be a function on the space $\mathcal{D}([0,T];\mathbb{R}^p)$ of the form $\Phi_{t_0}(\mathsf{X}_t) = \sum_{i=1}^k f_i(\mathsf{X}_{t_i})$, where k is an arbitrary integer, $(f_i)_{i=1,\dots,k}$ a family of continuous bounded functions and $t_1, \dots, t_k$ belongs to $[0,t_0] \setminus D$.

We set

$$S_{t,\delta}(\mathsf{V}) = \frac{1}{\delta}\int_t^{t+\delta} \mathsf{V}_s \,\mathrm{d}s,$$

which is a continuous function from $(\mathcal{D}([0,T];\mathbb{R}), S)$ to $(\mathcal{C}([0,T];\mathbb{R}), U)$. Furthermore,

$$\mathbb{E}\left[\, S_{t,\delta}(\mathsf{V})^2 \,\right] \leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\mathsf{V}_t|^2\right].$$

Consequently, $(S_{t,\delta}(\mathsf{M}^\varepsilon))_{\varepsilon>0}$ is uniformly integrable.

So, with (A2a) and (A2b), for any $r \geqslant 0$,

$$0 = \mathbb{E}\left[\, \Phi_{t_0}(\mathsf{X}^\varepsilon)(S_{t_0+r,\delta}(\mathsf{M}^\varepsilon) - S_{t_0,\delta}(\mathsf{M}^\varepsilon)) \,\right] \xrightarrow[\varepsilon\to 0]{} \mathbb{E}\left[\, \Phi_{t_0}(\mathsf{X})(S_{t_0+r,\delta}(\mathsf{M}) - S_{t_0,\delta}(\mathsf{M})) \,\right].$$

As M is right-continuous and $\mathcal{F}^{\mathsf{X}}$-adapted, if δ decreases to 0,

$$\mathbb{E}\left[\, \Phi_{t_0}(\mathsf{X})\mathsf{M}_{t_0+r} \,\right] = \mathbb{E}[\, \Phi_{t_0}(\mathsf{X})\mathbb{E}[\, \mathsf{M}_{t_0+r} \mid \mathcal{F}_{t_0}^{\mathsf{X}} \,]\,] = \mathbb{E}\left[\, \Phi_{t_0}(\mathsf{X})\mathsf{M}_{t_0} \,\right],$$

and from the freedom of choice of Φ_{t_0} and a monotone class Theorem, M is a $\mathcal{F}^{\mathsf{X}}$-martingale. □

Remark A.3. The condition (A2a) may be replaced by the weaker condition that $(\sup_{0\leqslant t\leqslant T}|\mathsf{M}^\varepsilon_t|)_{\varepsilon>0}$ is uniformly integrable. But under the condition (A2a), M is a square-integrable martingale.

Remark A.4. We have assumed that $(\sup_{0\leqslant t\leqslant T}|\mathsf{M}^\varepsilon_t|)_{\varepsilon>0}$ is uniformly integrable to deal with some càdlàg processes. But if M^ε converges to M in the topology of the uniform convergence, the previous Lemma is true with (A2a) replaced by the weaker condition that $(\mathsf{M}^\varepsilon_t)_{\varepsilon>0,t\in[0,T]}$ is uniformly integrable (see Proposition IX.1.12, p. 484 in Jacod and Shiryaev 1987).

Lemma A.2. *Let* $(\mathsf{Y}^\varepsilon)_{\varepsilon>0}$ *be a sequence of processes converging weakly in* $(\mathcal{D}([0,T];\mathbb{R}^p),S)$ *to* Y. *We assume that* $\sup_{\varepsilon>0}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}^\varepsilon_t|^2\right] < +\infty$. *Hence, for any* $t\geqslant 0$, $\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}_t|^2\right] < +\infty$.

Proof. With Definition 3.3 and Corollary 2.9, p. 13 and p. 10 of Jakubowski 1997, there exists a countable set $D\subset[0,T)$ such that, with the help of the Fatou Lemma,

$$\mathbb{E}\left[\sup_{t\in D^c}|\mathsf{Y}_t|^2\right]\leqslant\mathbb{E}\left[\liminf_{\varepsilon\to 0}\sup_{t\in D^c}|\mathsf{Y}^\varepsilon_t|^2\right]\leqslant\sup_{\varepsilon>0}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|\mathsf{Y}^\varepsilon_t|^2\right]<+\infty.$$

The Lemma is proved using the density of D^c in $[0,T]$ and the right-continuity of Y. □

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