Polynomial Time Nondimensionalisation of Ordinary Differential Equations via their Lie Point Symmetries

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February 2006

Abstract

Lie group theory states that knowledge of a m-parameters solvable group of symmetries of a system of ordinary differential equations allows to reduce by m the number of equation.

We apply this principle by finding dilatations and translations that are Lie point symmetries of considered ordinary differential system. By rewriting original problem in an invariant coordinates set for these symmetries, one can reduce the involved number of parameters. This process is classically call nondimensionalisation.

We present an algorithm based on this standpoint and show that its arithmetic complexity is polynomial in input's size.

1 Introduction

This paper is devoted to the process of nondimensionalisation that it described in [10] as follow: *"Before analysing [a] model it is essential, or rather obligatory to express it in* nondimensional *terms. This has several advantages. For example, the units used in the analysis are then unimportant and the adjectives small and large have a definite relative meaning. It also always reduces the number of relevant parameters to* dimensionless groupings *which determine the dynamics".*

Example 1 — In order to illustrate this statement, let us consider the following Verhulst's logistic growth model with linear predation (see § 1.1 in [10]):

$$\mathrm{d}x/\mathrm{d}t = x(a - bx) - cx, \quad \dot{a} = \dot{b} = \dot{c} = 0, \tag{1}$$

for which all forthcoming computation could be easily performed by hand.
Step 1. One can remark that the following one parameter group of translation symmetries:

$$\mathcal{T}_\lambda : a \to a + \lambda, \quad c \to c + \lambda \tag{2}$$

and the following two parameters group of scale symmetries:

$$\mathcal{S}_{(\mu,\nu)} : \begin{array}{rcl} t & \to & t/\nu, \\ x & \to & \mu x, \end{array} \qquad \begin{array}{rcl} a & \to & \nu a, \\ b & \to & \nu b/\mu, \\ c & \to & \nu c, \end{array} \tag{3}$$

leave invariant solutions of system (1).
Step 2. Assuming that $a \neq c$, one can determine some rational invariants of these groups like:

$$\mathfrak{t} = (a-c)\,t, \qquad \mathfrak{x} = \frac{b}{a-c}\,x. \tag{4}$$

These invariants are the *dimensionless groupings* evoked in above citation.
Step 3. Using above rational invariants, system (1) can be rewritten in these new coordinates as follow:

$$\mathrm{d}\mathfrak{x}/\mathrm{d}\mathfrak{t} = \mathfrak{x}(1-\mathfrak{x}). \tag{5}$$

In fact, one can deduce from (1) and (4) a differential system that could be simplified in order to obtain (5). Thus, groups (2) and (3) allow to rewrite our original problem with a reduced number of parameters.

We have considered a variety of such dynamical systems that arise from biological, physical, etc. models. As often observed, those systems can be rewritten in a lower number of variables or parameters than initially appears.

There is no difficulty to perform manipulations done above by hand on simple systems. But while system's complexity increases, these manipulations require more avoidable work as illustrated by the following example.

Example 2 — Let us consider the following prey-predator model taken from page 88 in [10]:

$$\left\{\begin{array}{lcl} \frac{\mathrm{d}n}{\mathrm{d}t} & = & \left(\left(1-\frac{n}{k_1}\right)r - k_2\frac{p}{n+e}\right)n, \\ \frac{\mathrm{d}p}{\mathrm{d}t} & = & \left(1-h\frac{p}{n}\right)p\,s, \\ \dot{r} = \dot{s} = \dot{e} = \dot{h} = \dot{k}_1 = \dot{k}_2 = 0. & & \end{array}\right. \tag{6}$$

Step 1. One can determine that the following 3 parameters group of scale symmetries:

$$\mathcal{S}_{(\lambda,\mu,\nu)}: \begin{array}{ccccccccc} t & \rightarrow & t/\lambda, & r & \rightarrow & \lambda r, & h & \rightarrow & \nu h, \\ n & \rightarrow & \mu n, & s & \rightarrow & \lambda s, & k_1 & \rightarrow & \mu k_1, \\ p & \rightarrow & \mu p/\nu, & e & \rightarrow & \mu e, & k_2 & \rightarrow & \lambda\nu k_2 \end{array} \tag{7}$$

leaves invariant solutions of system (6). Computations show that this system does not have any translation symmetries.
Step 2 & 3. If group's parameters are specialized as follow:

$$\lambda = \frac{1}{r}, \quad \mu = \frac{1}{k_1}, \quad \nu = \frac{r}{k_2}, \tag{8}$$

resulting transformation $\mathcal{S}_{(1/r,1/k_1,r/k_2)}$ induces a change of coordinates that send coordinates r, k_1 and k_2 to 1 and other coordinates to rational invariants:

$$\mathfrak{t} = r\,t, \quad \mathfrak{n} = \frac{n}{k_1}, \quad \mathfrak{p} = \frac{k_2 p}{k_1 r}, \quad \mathfrak{s} = \frac{s}{r}, \quad \mathfrak{e} = \frac{e}{k_1}, \quad \mathfrak{h} = \frac{rh}{k_2}. \tag{9}$$

After applying transformation $\mathcal{S}_{(1/r,1/k_1,r/k_2)}$ on system (6), we obtain a new system expressed in above new coordinates:

$$\left\{\begin{array}{lcl} \frac{\mathrm{d}\mathfrak{n}}{\mathrm{d}\mathfrak{t}} & = & \left(1-\mathfrak{n}-\frac{\mathfrak{p}}{\mathfrak{n}+\mathfrak{e}}\right)\mathfrak{n}, \\ \frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}\mathfrak{t}} & = & \left(1-\mathfrak{h}\frac{\mathfrak{p}}{\mathfrak{n}}\right)\mathfrak{p}\,\mathfrak{s}, \\ \dot{\mathfrak{s}} = \dot{\mathfrak{e}} = \dot{\mathfrak{h}} = 0. & & \end{array}\right. \tag{10}$$

and as in example 1, we reduce by 3 the number or parameters. Thus, any further manipulations of system (1) (for example, its phase plane analysis or study of its bifurcation analysis) is simplified in the new set of coordinates (9).

Same process could be achieved in another set of invariants coordinates by choosing to send s, e and h to 1 using transformation $\mathcal{S}_{(1/s,1/e,1/h)}$.

Remark 1 — Example 2 shows how scale symmetries allow computation of a set of invariant coordinates and of resulting new system by an evaluation without any further algebraic manipulation (see Section 4.2).

1.1 Dimensional Analysis and some Lie point Symmetries

There is just seven primary units in the International Metric System and all physically meaningful equations could be written in these units. Such physical *primary dimension* (say time) can be expressed in different units (second, hour, etc.) and thus, these units can be scaled independently of each other (a hour is 3600 seconds, etc). This possibility induces also dilatations on *secondary dimension* like speed (m/s) or corporal mass index (kg/m^2), etc. Thus, there is likely some scale transformations group acting on functional relation among these kind of quantities.

Dimensional analysis is based on this remark. Bridgman explains in [3]: *"The principal use of dimensional analysis is to deduce from a study of the dimensions of the variables in any physical system certain limitations on the form of any possible relationship between those variables"*. Thus, dimensions analysis addresses very general problems and we consider in this paper just one of its consequences that is nondimensionalisation. This process is done because reducing the number of parameters, or possibly also the number of state variables, is an advantage for studying qualitative features of the model.

Previous related works. For large systems, nondimensionalisation could become a difficult process that motivate several implementations (see [8] and the references therein for more details). Up to our knowledge, there is no complexity result concerning these works that are related to Π theorem and to rules of thumbs based on the knowledge of units in which is expressed the problem.

As we notice at beginning of this section, there is often some scale symmetries group of ordinary differential system describing biological, physical, etc. phenomena. This fact could be considered in the framework of Lie symmetries group theory (see § 3.4 in [11] for such presentation of Π theorem). This standpoint allows to presents computation involved in nondimensionalisation in a very simple and efficient way that we did not found in literature.

The present study was motivated by this fact and by authors' inability to apply classical dimensional analysis' rules to systems composed of more then 7 equations and a dozen parameters.

Remark 2 — In this paper, we restrict ourself to translation and scale symmetries that occurs frequently in application. Same type of result could be obtain for other kind of Lie point symmetries (rotation, inversion, etc. See Example 3 in Section 5).

1.2 Main contribution

It is the aim of this paper to provide an algorithm to make the reduction evoked in previous section. In the example above and 90% that we tried, the system (11) is actually symmetric under a group of scalings and/or translations. New variables that we introduced in above examples and many others actually form a generating set of rational invariants for some group action.

Computing symmetry of a differential system has been an intensive field of application of computer algebra, especially to mathematical physics [1, 5, 11] and computing a generating set of (differential) invariants for general group action has received relatively recently firm foundations [7, 6, 11].

The viewpoint of this paper is not to be general nor theoretical but practical for a large class of problems. We shall apply known general theory to special cases. We obtain efficient algorithms for reducing the number of parameters in biological, chemical, etc. models basing ourselves on the observation of a general scenario: the invariance of the model under a group of scaling and translation. We provide efficient computer algebra algorithms for computing the scaling and translation symmetry of a differential system (11), compute their invariants and rewrite the system in terms of those. We thus obtain the reduced system. These result are summarized in the following statement:

Theorem 1 — Let Σ be a differential system bearing on n state variables and depending on ℓ parameters that is coded by a straight-line program of size L (see Section 2). There exists a probabilistic algorithm that determines if a Lie point symmetries group of Σ composed of dilatation and translation exists; in that case, a rational set of invariant coordinates is computed and Σ is rewrite in this set with a reduced number of parameters.

The arithmetic complexity of this algorithm is bounded by

$$\mathcal{O}\Big((n+\ell+1)\big(L+(n+\ell+1)(2n+\ell+1)\big)\Big).$$

Outline of the paper. In the next section, we recall some basic definitions of differential algebra and we present in this framework the relationship between dilatation/translation transformations and induced derivations that are theirs infinitesimal generators. Then, we recall that such infinitesimal generators are defined by a partial differential equations system i.e. some infinitesimal conditions which are presented and used in the sequel.

In the last part of this paper, we show that infinitesimal conditions allow by a Gaussian elimination performed on a constant field to determine Lie groups of scale/translation symmetries. In fact, for these transformations, infinitesimal conditions split into a linear system of algebraic equations. We point out that computation of sets of rational invariant coordinates and rewriting of original system in these sets is reduced to linear algebra and we estimate the related arithmetic complexity.

2 Mathematical framework

Hereafter, we consider an algebraic ordinary differential system bearing on n state variables $X := (x_1, \ldots, x_n)$ and depending on ℓ parameters $\Theta := (\theta_1, \ldots, \theta_\ell)$:

$$\Sigma \qquad \begin{cases} \dot{t} = 1, \quad \dot{\Theta} = 0, \\ \dot{X} = F(t, X, \Theta). \end{cases} \tag{11}$$

The letter $\dot{X}$ stands for first order derivatives of state variables $(\dot{x}_1, \ldots, \dot{x}_n)$ w.r.t. time t; we use the standard notation $X^{(i)} = \big({x_1}^{(i)}, \ldots, {x_n}^{(i)}\big)$ for higher derivatives of order i. We assume that $F := (f_1, \ldots, f_n)$ consist of rational functions over a subfield $\mathbb{K}$ ($\mathbb{Q}$ for example) of $\mathbb{C}$ i.e. F is a finite subset of $\mathbb{K}(t, X, \Theta)$.

Complexity model. We evaluate the complexity of our algorithms within the model of *straight-line program* (see § 4 in [4]). For instance the expression $e := (x+1)^5$ is represented by the following kind of instructions sequence:

$$e_1 := x+1, \quad e_2 := {e_1}^2, \quad e_3 := {e_2}^2, \quad e := e_3 e_1. \tag{12}$$

The complexity is measured in terms of the following date of the input: $n+\ell$ and the number L of arithmetic operations needed to compute the numerators and denominators of F.

2.1 Differential Algebraic Setting

We use differential algebra, founded by J.F. Ritt, in order to introduce forthcoming definitions (see [13] for a complete description).

The differential algebra $\mathbb{K}\{t, X, \Theta\}$ is the $\mathbb{K}$-algebra of multivariate polynomials defined by the infinite set of indeterminates $\{t, \Theta, X^{(j)} | \forall j \in \mathbb{N}\}$ and equipped with time derivation $\mathrm{d}/\mathrm{d}t$ denoted by $\mathcal{L}$. Thus, for any y in $\mathbb{K}\{t, X, \Theta\}$, relations $\mathcal{L}y^{(i)} = y^{(i+1)}$ hold.

System (11) defines a prime differential ideal I of the algebra $\mathbb{K}\{t, X, \Theta\}$. This ideal encodes all relations between coefficients of power series solutions of Σ. In the sequel, we are going to focus our attention on the quotient differential fraction field $\mathbb{K}\{t, X, \Theta\}/I$ —denoted by $\mathcal{K}$—associated to I.

Derivations vector space. Let us recall that derivations acting on $\mathbb{K}(t, X, \Theta)$ form a vector space over $\mathbb{K}(t, X, \Theta)$ denoted by $\mathrm{Der}(\mathbb{K}(t, X, \Theta)/\mathbb{K})$ and equipped with a canonical base given by elementary derivations:

$$\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \theta_l} \;\middle|\; 1 \le i \le n,\ 1 \le l \le \ell \right\}. \tag{13}$$

This vector space equipped with canonical Lie bracket forms a solvable Lie algebra (see [12, 9] and references therein for some algorithmic tools used in study of Lie algebra in this context). In the sequel, we are going to consider other such algebras (see remark 13).

Canonical field isomorphism. As our input system defines explicitly a vector field, any high order derivatives could be rewritten to 0th order ones using relations (11). Thus, our differential field $\mathcal{K}$ is isomorphic to the differential field $\mathbb{K}(t, X, \Theta)$ equipped with the following formal Lie derivation in $\mathrm{Der}(\mathbb{K}(t, X, \Theta)/\mathbb{K})$:

$$\mathcal{D} := \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}. \tag{14}$$

We are going to use directly the canonical isomorphism between differential fields $(\mathcal{K}, \mathcal{L})$ and $(\mathbb{K}(t, X, \Theta), \mathcal{D})$. All forthcoming developments are based on derivation $\mathcal{D}$. Thus, hereafter, we denote by $\mathcal{K}$ the field $\mathbb{K}(t, X, \Theta)$, the set $(\mathcal{D}f_1, \ldots, \mathcal{D}f_n)$ by $\mathcal{D}F$ and the composition $\underbrace{\mathcal{D} \circ \cdots \circ \mathcal{D}}_{j \text{ times}}$ by $\mathcal{D}^j$.

Formal power series. Let us denote by $\Xi(t, X, \Theta)$ formal power series with coefficients in $\mathcal{K}$ that are solutions of $\dot{\Xi} = F(t, \Xi, \Theta)$ with initial condition $\Xi(0, X, \Theta) := X$.

These power series could be define using derivation $\mathcal{D}$ by the following formal relations:

$$\Xi(t, X, \Theta) = \sum_{j \in \mathbb{N}} \mathcal{D}^j X \, \frac{t^j}{j!}. \tag{15}$$

Remark 3 — Higher order derivatives could be considered via differential field $\mathbb{K}\langle t, X, \Theta \rangle$ that is the field $\mathbb{K}(t, \Theta, X^{(i)}, i \in \mathbb{N})$ and the formal derivation:

$$\mathcal{D}_\infty = \mathcal{D} + \sum_{j \in \mathbb{N} \setminus \{0,1\}} \sum_{i=1}^{n} \mathcal{D}^j f_i \frac{\partial}{\partial x_i^{(j)}}, \tag{16}$$

as $\mathcal{D}_\infty X^j \subset \mathcal{K}$, that allows to compute higher order derivatives of X.

In the sequel, we are going to exploit the fact that the differential field $(\mathcal{K}, \mathcal{D})$ encodes all formal informations associated to formal power series solution of Σ.

3 Infinitesimal Generators of Scale Symmetries acting on $\mathcal{K}$

We are looking for a m-parameters group $\sigma_{(\lambda_1, \ldots, \lambda_m)}$ acting on $\mathcal{K}$ that is a symmetries group leaving the solution of Σ invariant. First, we are going to recall how a derivation in $\mathrm{Der}(\mathcal{K}/\mathbb{K})$ could be associated to a scale (resp. translation) transformation. Then, we explicit that such a derivation defines a scale (resp. translation) symmetries of Σ.

3.1 Scale/translation transformation groups and associated derivations vector spaces.

In order to explicit classical basis of forthcoming computations, let us consider the following one-parameter group of scale transformations:

$$\sigma_\lambda : \begin{array}{ccc} t & \rightarrow & \lambda^{\alpha_t}\, t, \\ x_1 & \rightarrow & \lambda^{\alpha_{x_1}}\, x_1, \\ & \vdots & \\ x_n & \rightarrow & \lambda^{\alpha_{x_n}}\, x_n, \end{array} \quad \begin{array}{ccc} \theta_1 & \rightarrow & \lambda^{\alpha_{\theta_1}}\theta_1, \\ & \vdots & \\ \theta_\ell & \rightarrow & \lambda^{\alpha_{\theta_\ell}}\theta_\ell, \end{array} \tag{17}$$

the group parameter is denoted by λ and taken in a field $\mathbb{F}$; exponents α_i are in a constant field ($\mathbb{Q}$ for example) and they define considered group.

Remark 4 — For translation, we consider analogous expressions $y \rightarrow y + \alpha_y \lambda$ with y in (t, X, Θ). In Example 1, we already considered the one-parameter translation given by the exponents set defined by relations $\alpha_a = \alpha_c = 1$ and $\alpha_t = \alpha_x = \alpha_b = 0$ that leads to transformations

$$\sigma_\lambda : t \rightarrow t,\ x \rightarrow x,\ a \rightarrow a + \lambda,\ b \rightarrow b,\ c \rightarrow c + \lambda.$$

Actions of these groups could be defined on power series Ξ solution of Σ (i.e. they remain invariant under group action $\Xi(\sigma_\lambda(t, X, \Theta)) = \Xi(t, X, \Theta)$) and on $\mathbb{K}(t, X, \Theta)$. We consider this last case in the sequel.

Infinitesimal generators vector space. When a transformation group is connected—as it is always the case for transformations considered in this paper—each symmetries of group (17) is associated to an infinitesimal generator (for geometric description, see § 2 in [11]). It is a derivation on $\mathcal{K}$ in the vector space $\mathbf{S}$ generated in $\mathrm{Der}(\mathcal{K}/\mathbb{K})$ by the derivation:

$$\mathcal{S} = \sum_{y \in (t, X, \Theta)} \alpha_y y \frac{\partial}{\partial y}, \tag{18}$$

with the α_y are constant exponents taken from (17). In fact, given any element $\mathcal{S}_\rho$ in $\mathbf{S}$ equal to $\rho\mathcal{S}$ with ρ an element of a constant field, one can construct a scale transformation using exponential map as follow:

$$\sigma_\lambda : \begin{array}{ccc} \mathcal{K} & \rightarrow & \mathcal{K}(\lambda) \\ y & \rightarrow & \sum_{i \in \mathbb{N}} \mathcal{S}_\rho^{\,i}(y)/i! \end{array} \tag{19}$$

if exponential $\lambda := \exp(\rho)$ is defined. At opposite, one can determine a derivation associated to the application σ_λ but we are not going to use this fact in the sequel.

Remark 5 — In the case of a m-parameters group, the vector space $\mathbf{S}$ is generated by m derivations $\mathcal{S}_1, \ldots, \mathcal{S}_m$ of the same type then (18). More precisely $\mathbf{S}$ is a Lie algebra s.t. $[\mathcal{S}_i, \mathcal{S}_j] = 0$.

Remark 6 — Same considerations are true for translation but in this case infinitesimal generators are:

$$\mathcal{T} = \sum_{y \in (t, X, \Theta)} \alpha_y \frac{\partial}{\partial y}. \tag{20}$$

The vector space of infinitesimal generators associated to a translation symmetries group is denoted by $\mathbf{T}$ in the sequel.

3.2 Constraint on infinitesimal generators of scale and translation symmetries — Lie symmetry determining equation

In order to compute derivations (18) and thus, the associated scale symmetries group, we use the classical property that a derivation $\mathcal{C}$ associated to a symmetry of a differential system commutes with the induced derivation $\mathcal{D}$ (in that case, the derivation $\mathcal{C}$ is called a symmetry of $\mathcal{D}$ by extension). This fact can be state using Lie bracket as follows:

$$[\mathcal{C}, \mathcal{D}] := \mathcal{C} \circ \mathcal{D} - \mathcal{D} \circ \mathcal{C} = \lambda \mathcal{D}, \tag{21}$$

the parameters λ is a constant; it is different from 0 if considered symmetry does act on times and is 0 otherwise.

Infinitesimal conditions defining symmetries groups. For reader's convenience, we derive from (21) infinitesimal conditions that a derivation is an infinitesimal generators of a scale (resp. translation) symmetry (see § 2.5 in [11] for a presentation of these infinitesimal conditions based on jet space and prolongation and [5] for resolution algorithms based on them).

Lemma 1 — *Infinitesimal conditions of order* 0 *that an infinitesimal generator given by* $\sum_{y \in (t,X,\Theta)} \alpha_y y \partial/\partial y$ *defines a scale symmetry of ordinary differential system* Σ *are:*

$$\sum_{y \in (t,X,\Theta)} \alpha_y y \frac{\partial f_i}{\partial y} + (\alpha_t - \alpha_{x_i}) f_i = 0, \text{ for } i = 1, \ldots, n. \tag{22}$$

The infinitesimal conditions of order 0 *that an infinitesimal generator* $\sum_{y \in (t,X,\Theta)} \alpha_y \partial/\partial y$ *defines a translation symmetry of ordinary differential system* Σ *are:*

$$\sum_{y \in (t,X,\Theta)} \alpha_y \frac{\partial f_i}{\partial y} = 0, \text{ for } i = 1, \ldots, n. \tag{23}$$

Sketch of proof. Recall that we work in vector space $\mathrm{Der}(\mathcal{K}/\mathbb{K})$, thus one could consider each component of relation (21) on element of canonical base (13). Thus, coefficient of element $\partial/\partial t$ is $\mathcal{SD}t - \mathcal{DS}t = \lambda \mathcal{D}t$ and we deduce that the equality $\lambda = \alpha_t$ holds. Then, remark that the n other element's coefficient $[\mathcal{S}, \mathcal{D}]x_i = \lambda \mathcal{D} x_i$ of canonical base could be expanded as:

$$\alpha_{x_i} f_i - \sum_{y \in (t,X,\Theta)} \alpha_y y \frac{\partial f_i}{\partial y} = \lambda f_i, \text{ for } i = 1, \ldots, n. \tag{24}$$

Infinitesimal conditions of order 0 are obtained by replacing λ by α_t in these relations. The proof of the second assertion is similar. □

Example 2 (revisited) — These infinitesimal conditions associated to example 2 are given by the matricial relation $MA = 0$ with matrix M and vector A defined in figure 1.

$$A := \begin{pmatrix} \alpha_t & \alpha_n & \alpha_p & \alpha_r & \alpha_{k_1} & \alpha_{k_2} & \alpha_h & \alpha_s & \alpha_e \end{pmatrix}$$
$$M := \begin{pmatrix} 0 & \left(\frac{k_2 p}{(n+e)^2} - \frac{r}{k_1}\right) rn & -\frac{k_2 p}{n+e} & r - \frac{rn}{k_1} & \frac{rn}{k_1} & -\frac{pk_2}{n+e} & 0 & 0 & \frac{k_2 pe}{(n+e)^2} \\ 0 & \frac{hp}{n} & -\frac{hp}{n} & 0 & 0 & 0 & -\frac{hp}{n} & 1 - \frac{hp}{n} & 0 \end{pmatrix} \tag{25}$$

Figure 1: Matrix defining infinitesimal conditions associated to example 2

Exponents vector space. Considered as relations with coefficients in $\mathbb{K}(t, X, \Theta)$, the above 0th order infinitesimal conditions are not apparently sufficient to define completely the vector space S (resp. T) of exponents associated to derivation vector space **S** (resp. **T**) presented in Section 3.1 and thus to determine searched symmetries groups.

In fact, there is $(n + \ell + 1)$ unknowns and n relations; hence there is apparently no enough relations to define a basis of S.

Remark 7 — Nevertheless, this vector space is well defined because derivations $\mathcal{D}$ and $\mathcal{S}$ could be prolongated by the derivation $\mathcal{D}_\infty$ defined by formula (16) (see remark 3) and by $\mathcal{S}_\infty$ defined as follow:

$$\mathcal{S}_\infty = \mathcal{S} + \sum_{j\in\mathbb{N}^\star} \sum_{y\in X} (\alpha_y - j\alpha_t) y^{(j)} \frac{\partial}{\partial y^{(j)}}. \tag{26}$$

These derivations act on higher order derivatives of initial states variables i.e. on $\mathbb{K}\langle t, X, \Theta\rangle$; The first part of $\mathcal{S}_\infty$ is related to relationship between classical twisted Euler derivations (18) and scaling; the second one is related to the fact that

$$\sigma_\lambda\left(\frac{\mathrm{d}^j x_i}{\mathrm{d}t^j}\right) = \lambda^{(\alpha_{x_i} - j\alpha_t)} \frac{\mathrm{d}^j x_i}{\mathrm{d}t^j},\ j \in \mathbb{N},\ i = 1, \ldots, n. \tag{27}$$

Applying manipulations of lemma 1 on prolongated derivations, we obtain enough relations to define desired vector space S by considering prolongated base elements $\partial^j/\partial x_i{}^j$. To be more precise, the following higher order infinitesimal conditions hold:

$$\sum_{y\in(t,X,\Theta)} \alpha_y y \frac{\partial \mathcal{D}^j x_i}{\partial y} - (\alpha_{x_i} - j\alpha_t)\mathcal{D}^j x_i = 0, \tag{28}$$

for i in $(1, \ldots, n)$ and j in $\mathbb{N}$. From this infinite set of relations, one can deduce bases of S when this vector space is not reduced to 0 (this is generically the case, but as shown in Section 1.1, we considered here non generic differential systems coming from biology, physics, etc).

The dimension m of S gives the number of parameters of our scales symmetries group; once a basis of S is chosen, each of its vector is associated to a one parameter group of scale symmetries and vector's coefficients allows to determine exponents—the αs in (17)—of this group.

One can choose m of the components of the α_i quite arbitrarily. We shall actually want this arbitrariness to bear on the components corresponding to Θ (see Section 4.2).

Remark 8 — All forthcoming computations are based and devoted to this vector space but we are going to see that there is no need to derive supplementary infinitesimal conditions from prolongated derivations (16) and (26).

4 Algorithm

4.1 Infinitesimal generators computation

Let us recall that infinitesimal conditions presented in lemma 1, show that computation of considered symmetries groups is associated to computation of a kernel of a matrix of size $n \times (n + \ell + 1)$ with coefficients in the field $\mathbb{K}(t, X, \Theta)$ (this kernel defines vector space S presented at end of Section 3.2).

Remark 9 — Fortunately, the vector space S is defined over a constant field and thus its computation does not require computations in $\mathbb{K}(t, X, \Theta)$. Coordinates t, X and Θ can be specialized to some generic values of a constant field in the considered matrix and so, computations could be performed numerically with high probability of success (i.e. computation failed when specialization are done in a Zariski closed set).

Multiple specializations vs. higher order computations. In order to compute a basis of vector space S as sketched in remark 7, one could specialize coordinates t, X, Θ and their higher order derivations in formula (28) i.e. compute numerical a single power series solution of linear variational system derived from Σ (see appendix A.1 for more details and [14] for another application of that principle and more details on this computational strategy). This process could be done several time at different order.

But instead, one could also perform $\lceil (n+\ell+1)/n \rceil$ specializations of coordinates t, X, Θ in 0th order infinitesimal conditions presented in lemma (1) and thus, obtain a square system that allows to compute a base of S (see appendix A.2, for an example). Thus, we consider $\lceil (n+\ell+1)/n \rceil$ series computations at order 0 in the sequel.

Example 2 (continued) — After 6 specialization of matrix M defined in figure 1, one can compute numerically its kernel and construct the following matrix:

$$K := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \tag{29}$$

such that $MK = 0$. Rows of this matrix define the vector space S in $\mathbb{K}^9$ if this vector space is given by coordinates $(\alpha_t, \alpha_n, \alpha_p, \alpha_r, \alpha_{k_1}, \alpha_{k_2}, \alpha_h, \alpha_s, \alpha_e)$. By Section 3, this is sufficient to retrieve the searched 3 parameters group (7) of scale symmetries. In fact, the last row of matrix (29) is associated to derivation:

$$\mathcal{S}_\nu = h\frac{\partial}{\partial h} + k_2\frac{\partial}{\partial k_2} - p\frac{\partial}{\partial p}, \tag{30}$$

that defines the one-parameter group of scale symmetries: $p \to p/\nu,\ h \to \nu h,\ k_2 \to \nu k_2$.

Computation of Gradients. Infinitesimal conditions are based on gradient computation of functions F defining our input differential system; we show in this section that these computations could be efficiently performed in our model of complexity which is described below.

Definition 1 — *Let $\mathcal{A}$ be a finite set of variables. A* straight-line program *over $k[\mathcal{A}]$ is a finite sequence of assignments $b_i \leftarrow b' \circ_i b''$ s.t. $\circ_i$ is in $\{+, -, \times, \div\}$ and $\{b', b''\}$ is in $\bigcup_{j=1}^{i-1}\{b_j\} \cup \mathcal{A} \cup k$. Its complexity of evaluation is measured by its length L, which is the number of its arithmetic operations. Hereafter, we use the abbreviation* SLP *for straight-line program.*

A SLP representing a rational expression f is a program that computes the value of f from any values of the ground field such that every division of the program is possible. The following constructive results taken form [2] allows us to determine a SLP representing the gradient of f.

Theorem 2 — Let us consider a straight-line program computing the value of a rational expression f in a point of the ground field and let us denote by L_f its complexity of evaluation. One can construct a SLP of length $5L_f$ that computes the value of f and of its gradient $\mathrm{grad}(f)$.

Complexity result. We gather all elements presented above and used in our computation of symmetries in the following proposition:

Proposition 1 — Let Σ be a differential system as described in Section 2. There exists an algorithm that determines a m-parameters group of scale (resp. translation) symmetries of Σ. The arithmetic complexity of this algorithm is bounded by

$$\mathcal{O}\Big((n+\ell+1)\big(L + (n+\ell+1)(2n+\ell+1)\big)\Big). \tag{31}$$

Sketch of proof. Using theorem 2, the evaluation complexity of linear system derived from infinitesimal conditions (22) is bounded by $\mathcal{O}((L + (n+\ell+1)n)n)$. This system should be evaluated $\lceil (n+\ell+1)/n \rceil$

times on some generic specializations elements in a constant field in order to obtain a square system. Using Gaussian elimination, a base of the resulting system kernel could be computed with $\mathcal{O}\big((n+\ell+1)^3\big)$ arithmetic operation. Thus, we obtain our complexity (31). □

Remark 10 — The algorithm presented in this section is not probabilistic even if it is based on specializations that could lead to computation of some spurious symmetries. In fact, it could occurs—with a very small probability—that coordinates t, X, Θ are specialized on several points in the orbit of a symmetries group of Σ; in this case, some spurious symmetries are obtain.

Fortunately, a simple evaluation of computed symmetries on our original system allows to show if it is a computational artifact or not. The complexity associated to these tests is bounded by $\mathcal{O}((n + \ell + 1)L)$.

Remark 11 — There is an infinite way to choose a basis of S. But, one can use Lenstra, Lenstra and Lovász' basis reduction algorithm in order to obtain a reduced basis in the sense that exponent αs are smaller then what could be obtained using classical Gram-Schmidt orthogonalization.

4.2 Computation of some rational invariants and original system rewritting

Generally speaking, when a differential system admits a group action as symmetry, we can rewrite it in terms of the invariants of the group action. Fels and Olver revised the moving frame the construction for (differential) invariants [11]. The constructed invariants allow for a trivial rewriting. We do not recall the general theory but rather show how it works on the group actions of interest in this paper.

While computation of *non-specific* rational invariants i.e. a total description of invariant field for any algebraic group action could be done using reduced Gröbner basis computation (see [7]), we do not need such a general tool because we restrict ourself to scale and translation transformations. In fact for scale/translation symmetries, we could restrict ourself to compute $\ell - m$ time independent rational invariants in the multiplicative group in $\mathbb{K}(\Theta)$ generated by the set:

$$\mathbb{M} := \big\{\theta^\beta \;\big|\; (\theta, \beta) \subset \Theta \times \mathbb{Q},\ \forall \mathcal{S} \in \mathbf{S},\ \mathcal{S}\theta \neq 0\big\}. \tag{32}$$

A set of generators of this multiplicative group is denoted by $\pi_1, \ldots, \pi_{\ell-m}$. Furthermore, we restrict ourself to looking for following kind of time dependent rational invariants:

$$\pi_t = p_t t, \quad \pi_{x_i} = p_i x_i,\ 1 \leq i \leq n, \tag{33}$$

where the p_i are in $\mathbb{M}$. This arbitrary choice simplify invariants' computation and is well suited with our purpose; we are trying to use computed group of symmetries to reduce the number of parameters and not to reduce the number of equations.

Group action. Given $\sigma_{(\lambda_1,\ldots,\lambda_m)}$ a m-parameters symmetries group, let us consider its action ψ on field $\mathbb{K}(t, X, \Theta)$:

$$\psi : \begin{array}{ccccc} \mathbb{F}^m & \times & \mathbb{K}(t, X, \Theta) & \rightarrow & \mathbb{K}(t, X, \Theta), \\ (\lambda_1, \ldots, \lambda_m) & \times & y & \rightarrow & \sigma_{(\lambda_1,\ldots,\lambda_m)}(y). \end{array} \tag{34}$$

with $\mathbb{F}$ a field where parameters could be specified. For scale symmetries, we could wrote ψ as the substitution given by:

$$\forall y \in (t, X, \Theta), \quad \psi(y) = y\, \Pi_{i=1}^{n} \lambda_i^{a_{y,i}}, \tag{35}$$

where exponents are given by the basis $\big\{(a_{t,i}, a_{x_1,i}, \ldots, a_{x_1,i}, a_{\theta_1,i}, \ldots, a_{\theta_\ell,i}),\ i = 1, \ldots, m\big\}$ of $\mathbf{S}$.

Rational invariant computation. By classical canonical homomorphism, the multiplicative set $\mathbb{F}^m \times \mathbb{M}$ could be considered as a $\mathbb{Z}$ module of dimension 2ℓ. Thus, the subset of relations (35) involving only parameters could be represented by the following $\ell \times 2\ell$ matrix:

$$\begin{array}{c} \quad\lambda_1 \;\dots\; \lambda_m \;\; \theta_1 \;\dots\; \theta_\ell \\ \begin{pmatrix} a_{\theta_1,1} & \dots & a_{\theta_1,m} & 1 & 0 & \dots & 0 \\ a_{\theta_2,1} & \dots & a_{\theta_2,m} & 0 & 1 & & \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ a_{\theta_{\ell-1},1} & \dots & a_{\theta_{\ell-1},m} & & & 1 & 0 \\ a_{\theta_\ell,1} & \dots & a_{\theta_\ell,m} & 0 & \dots & 0 & 1 \end{pmatrix} \end{array}. \tag{36}$$

One can consider the matrix obtained after a permutation of (36) lines :

$$\begin{array}{c} \quad\lambda_1 \;\dots\; \lambda_m \;\; \hat\theta_1 \;\dots\; \hat\theta_\ell \\ \begin{pmatrix} a_{\hat\theta_1,1} & \dots & a_{\hat\theta_1,m} & 1 & 0 & \dots & 0 \\ a_{\hat\theta_2,1} & \dots & a_{\hat\theta_2,m} & 0 & 1 & & \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ a_{\hat\theta_{\ell-1},1} & \dots & a_{\hat\theta_{\ell-1},m} & & & 1 & 0 \\ a_{\hat\theta_\ell,1} & \dots & a_{\hat\theta_\ell,m} & 0 & \dots & 0 & 1 \end{pmatrix} \end{array}. \tag{37}$$

such in order to ensure that the determinant of the submatrix $(a_{\hat\theta_i,j})_{i=1,\dots m}^{j=1,\dots,m}$ is not 0.

A Gaussian elimination performed on this matrix and terminated at $m+1$ column position leads to the matrix :

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \gamma_{1,\hat\theta_1} & \dots & \gamma_{1,\hat\theta_\ell} \\ 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \gamma_{m,\hat\theta_1} & \dots & \gamma_{m,\hat\theta_\ell} \\ 0 & & \dots & 0 & \beta_{m+1,\hat\theta_1} & \dots & \beta_{m+1,\hat\theta_\ell} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & & \dots & 0 & \beta_{\ell,\hat\theta_1} & \dots & \beta_{\ell,\hat\theta_\ell} \end{pmatrix}. \tag{38}$$

This computation is sufficient to determine the following generators of the multiplicative set (32) of rational invariants:

$$\sigma_{(\lambda_1,\dots,\lambda_m)}\left(\prod_{j=1}^{\ell}\hat\theta_j^{\beta_{m+1,\hat\theta_j}}\right) = \prod_{j=1}^{\ell}\hat\theta_j^{\beta_{m+1,\hat\theta_j}}, \dots, \sigma_{(\lambda_1,\dots,\lambda_m)}\left(\prod_{j=1}^{\ell}\hat\theta_j^{\beta_{\ell,\hat\theta_j}}\right) = \prod_{j=1}^{\ell}\hat\theta_j^{\beta_{\ell,\hat\theta_j}}. \tag{39}$$

Thus, this elimination construct elements π_i of $\mathcal{K}$ that are invariant under the action ψ; remark that considering the whole action of ψ i.e. a $(n+\ell+1)\times 2(n+\ell+1)$ matrix, the same process allows to construct time dependent invariants of type (33) but we are not going to do so.

Example 2 (continued) — In Section 4.1, we determine by numerical computation, a basis of vector space S presented in matrix (29); taking the 6 last columns of this matrix, one obtains the 3 first columns of the following matrix:

$$\begin{array}{c} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \mu \quad \lambda \quad \nu \quad r \quad k_1 \quad k_2 \quad h \quad s \quad e \end{array}. \tag{40}$$

After a Gaussian elimination on the first 3 columns of above matrix, we obtain:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{41}$$

The first line of above matrix defines the rational invariants $\mathfrak{h} = rh/k_2$ and the last line defines $\mathfrak{e} = e/k_1$, etc.

Algebraic counterpart of Frobenius theorem in considered cases. First, let us remark that the action ψ is surjective. In fact, one can consider the group's parameters $\lambda_1 = \ldots = \lambda_m = \exp(0) = 1$ that correspond to identity map. This map is associated to derivation 0 in $\mathbf{S}$ by formula (19).

The arbitrary choice of parameters made at beginning of Section 4.2 is motivated by the possibility to rewrite our original dynamic (11) in order to reduce the number of parameters; we want to determine an expression of the dynamic $\mathcal{D}$ on an intermediate invariant field $\mathcal{F} = \mathcal{K}^{\sigma}$ i.e.

$$\mathbb{K} \hookrightarrow \mathcal{F} = \mathbb{K}(\pi_t, \pi_{x_1}, \ldots, \pi_{x_n}, \pi_1, \ldots, \pi_{\ell-m}) \hookrightarrow \mathcal{K},$$

which transcendence dimension w.r.t. $\mathbb{K}$ is smaller then the original one but where the number of variables depending on times does not change.

In the following proposition, we gives an elementary algebraic proof of a general results for reader's convenience (see [7] for definition, computation and use of replacement invariants).

Proposition 2 — There exists $\hat{\lambda}_1, \ldots, \hat{\lambda}_m$ in $\mathbb{M}$ such that the following equality holds :

$$\psi^{-1}\left(\mathcal{K}^{\sigma(\lambda_1,\ldots,\lambda_m)}\right) = \left(\hat{\lambda}_1, \ldots, \hat{\lambda}_m\right) \times \mathcal{K}. \tag{42}$$

Sketch of proof. This proposition is a reformulation of matrix (38)'s structure inherited from matrix (36). In fact, the submatrix $(\gamma_{i,\hat{\theta}_j})_{i=1,\ldots,m}^{j=1,\ldots,\ell}$ keep a track of the performed gaussian elimination i.e. the following equality holds:

$$\begin{pmatrix} a_{\hat{\theta}_1,1} & \cdots & a_{\hat{\theta}_1,m} \\ \vdots & & \vdots \\ a_{\hat{\theta}_m,1} & \cdots & a_{\hat{\theta}_m,m} \end{pmatrix} \begin{pmatrix} \gamma_{1,\hat{\theta}_1} & \cdots & \gamma_{1,\hat{\theta}_\ell} \\ \vdots & & \vdots \\ \gamma_{m,\hat{\theta}_1} & \cdots & \gamma_{m,\hat{\theta}_\ell} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}. \tag{43}$$

If, for $i = 1, \ldots, m$, we define $\hat{\lambda}_i := \prod_{j=1}^{\ell} \hat{\theta}_j^{-\gamma_{i,\hat{\theta}_j}}$ and notice that the above equality shows that the following relations hold:

$$\sigma_{(\hat{\lambda}_1,\ldots,\hat{\lambda}_m)}(\hat{\theta}_i) = \hat{\theta}_i \prod_{h=1}^{m} \left(\prod_{j=1}^{\ell} \hat{\theta}_j^{-\gamma_{h,\hat{\theta}_j}} \right)^{a_{\hat{\theta}_i,h}} = \hat{\theta}_i \prod_{j=1}^{\ell} \hat{\theta}_j^{-\sum_{h=1}^{m} a_{\hat{\theta}_i,h}\gamma_{h,\hat{\theta}_j}} = \hat{\theta}_i \hat{\theta}_i^{-1} = 1. \tag{44}$$

Therefore, there exists a subset $(\hat{\theta}_1, \ldots, \hat{\theta}_m)$ of parameter's set and a subset $(\hat{\lambda}_1, \ldots, \hat{\lambda}_m)$ of $\mathcal{K}$ such that the relations $\sigma_{(\hat{\lambda}_1,\ldots,\hat{\lambda}_m)}(\hat{\theta}_i) = 1$ hold for $i = 1, \ldots, m$. The same type of result is valid for the submatrix $(\beta_{i,j})_{i=m+1,\ldots,\ell}^{j=1,\ldots,\ell}$ of (38):

$$\begin{pmatrix} a_{\hat{\theta}_{m+1},1} & \cdots & a_{\hat{\theta}_{m+1},m} \\ \vdots & & \vdots \\ a_{\hat{\theta}_\ell,1} & \cdots & a_{\hat{\theta}_\ell,m} \end{pmatrix} \begin{pmatrix} \gamma_{1,\hat{\theta}_1} & \cdots & \gamma_{1,\hat{\theta}_\ell} \\ \vdots & & \vdots \\ \gamma_{m,\hat{\theta}_1} & \cdots & \gamma_{m,\hat{\theta}_\ell} \end{pmatrix} = \begin{pmatrix} -\beta_{m+1,\hat{\theta}_1} & \cdots & 1-\beta_{m+1,\hat{\theta}_m+1} & \cdots & -\beta_{m+1,\hat{\theta}_\ell} \\ \vdots & & \vdots & \ddots & \vdots \\ -\beta_{\ell,\hat{\theta}_1} & \cdots & -\beta_{m+1,\hat{\theta}_m+1} & \cdots & 1-\beta_{\ell,\hat{\theta}_\ell} \end{pmatrix}. \tag{45}$$

This matricial relation prove that, for $i = m+1, \ldots, \ell$ the following equalities hold:

$$\sigma_{(\hat{\lambda}_1,\ldots,\hat{\lambda}_m)}(\hat{\theta}_i) = \hat{\theta}_i \prod_{h=1}^{m} \left(\prod_{j=1}^{\ell} \hat{\theta}_j^{-\gamma_{h,\hat{\theta}_j}} \right)^{a_{\hat{\theta}_i,h}} = \hat{\theta}_i \prod_{j=1}^{\ell} \hat{\theta}_j^{-\sum_{h=1}^{m} a_{\hat{\theta}_i,h}\gamma_{h,\hat{\theta}_j}} = \prod_{j=1}^{\ell} \hat{\theta}_j^{\beta_{m+1,\hat{\theta}_j}} \tag{46}$$

To conclude, remark that the same properties hold for time dependent variables. □

Thus, after Gaussian elimination performed on (36), we obtain a description of $\mathcal{K}^\sigma$ and an application $\sigma_{(\hat{\lambda}_1,\ldots,\hat{\lambda}_m)}$ that maps $\mathcal{K}$ to $\mathcal{K}^\sigma$. In fact, This action allows to determine rational invariants and rewrite original system in an set of invariant coordinates with a reduce number of parameters.

Example 2 (continued) — The second, third and fourth lines of matrix (41) give the specialization (8) of parameters that allows to determine time dependent invariants and system rewriting. We summarized computation done in this section by the following proposition:

Proposition 3 — Computation of rational invariants defined in (32) and by (33) could be performed by Gaussian elimination and thus its complexity is bounded by $\mathcal{O}(\ell^3)$.

Remark 12 — The computation of rational invariants presented in this section suppose that the considered symmetries groups is composed of dilatation. The same consideration holds for translation. In fact, the group action is in this case

$$\forall y \in (t, X, \Theta), \quad \psi(y) = y + \sum_{i=1}^{n} a_{y,i}\lambda_i, \tag{47}$$

and is thus linear; that allows exactly the same type of computation.

Remark 13 — If $\mathcal{S}$ and $\mathcal{T}$ are two symmetries of $\mathcal{D}$, the Lie bracket $[\mathcal{S}, \mathcal{T}]$ is also a symmetry of $\mathcal{D}$ by Jacobi identity. Thus, symmetries form a Lie algebra and if $\mathcal{S}_i$ is in $\mathbf{S}$ and $\mathcal{T}_i$ is in $\mathbf{T}$, we have the classical facts that $[\mathcal{S}_i, \mathcal{S}_j] = 0$ and $[\mathcal{T}_i, \mathcal{T}_j]$ and $[\mathcal{S}_i, \mathcal{T}_j]$ are in $\mathbf{T}$ i.e. induced derivation is of the same type then (20). Typically, scaling are symmetries of translation's invariant while the opposite is not true.

Thus, $\mathcal{S}$ and $\mathcal{T}$ generate a solvable Lie algebra and above commutation relations show that we have to use translation symmetries first in our algorithm to reduce parameter's number and then use scale symmetries (see § 2.5 in [11]).

Remark 14 — As mentioned in remark 11 there is some freedom in choosing the components of a basis of S (resp. T), the set of exponents that define a scaling (resp. translation) symmetry of a differential system. We shall always try to have the freedom on the components of α that correspond to Θ. This is achieved by placing correctly the unknown when solving the system by Gaussian elimination; this fact allows user to choose parameters to eliminate. When this is not achievable, the general Lie method's could be use to solve—partially—the system by quadrature.

5 Conclusion and future work

In this paper, we consider the computation of scale and/or transformation group that are symmetries of an ordinary differential system and the determination of some of their invariants. We use these groups and associated invariants in order to rewrite the ordinary differential system in an set of invariant coordinates with less parameters. The complexity of this process is polynomial in input's size.

Remark 15 — For the sake of simplicity, we do not include control time dependent variables U in our input system (11); if such variables occur, the ground field is in practice the differential fraction field $\mathbb{K}\langle U \rangle$. Computations are performed after specialization of variables U on power series with random integer coefficients and which are truncated at order $n + \ell + 1$.

Same type of result could be proved for more general symmetries that allow to reduce further the number of significant parameters as shown below.

Example 3 — Let us consider a FitzHugh Nagumo model (see § 7 in [10]) defined as follow:

$$\dot{a} = \dot{b} = \dot{c} = \dot{d} = 0, \quad \mathrm{d}x/\mathrm{d}t = c(x - x^3/3 - y + d), \quad \mathrm{d}y/\mathrm{d}t = (x + a - by)/c. \tag{48}$$

This system does not have scale or translation symmetries that are considered in this paper but one can determine that the derivation:

$$\frac{\partial}{\partial y} + b\frac{\partial}{\partial a} + \frac{\partial}{\partial d} \tag{49}$$

is an infinitesimal generators of the one-parameter symmetries group (which is not of type (20)):

$$y \to y + \lambda, \quad a \to a + b\lambda, \quad d \to d + \lambda. \tag{50}$$

Up to our knowledge, there is likely no polynomial time algorithm that compute infinitesimal generators (49). In fact, this type of symmetries can be found by supposing that all coefficients of seeked infinitesimal generators of symetries are rational function of parameters; in that case the solution of determining system of PDE (21) is reduced to the computation of a polynomial matrix kernel. But as done in this paper, one can use the following invariant coordinates $\mathfrak{y} = y - d$ and $\mathfrak{a} = a + bd$ to rewrite system (48) as follow:

$$\mathrm{d}x/\mathrm{d}t = c(x - x^3/3 - \mathfrak{y}), \quad \mathrm{d}\mathfrak{y}/\mathrm{d}t = (x + \mathfrak{a} - b\mathfrak{y})/c \tag{51}$$

Acknowledgments. The second author is grateful to F. Lemaire and M. Safey El Din for many useful suggestions that helped to considerably improve correctness and presentation of this paper.

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A Two computation methods

Considering a classical example taken from biology, we are going to explicit some computation evoked in Section 4.1. First, we determine a symmetries group using power series approach and then, we retrieve the same result using just specialization of 0th order infinitesimal conditions.

A.1 Series based computations

Let us consider the linear variational system:

$$\nabla \quad \left\{ \begin{array}{rcl} \dot{\Xi} & = & F(t, \Xi, \Theta), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \Xi}{\partial X} & = & \dfrac{\partial F}{\partial X}(t, \Xi, \Theta)\dfrac{\partial \Xi}{\partial X}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \Xi}{\partial \Theta} & = & \dfrac{\partial F}{\partial X}(t, \Xi, \Theta)\dfrac{\partial \Xi}{\partial \Theta} + \dfrac{\partial F}{\partial \Theta}(t, \Xi, \Theta), \end{array} \right. \tag{52}$$

with the initial conditions $\partial \Xi / \partial X = \mathrm{Id}_{n \times n}$ and $\partial \Xi / \partial \Theta = 0$ when $t = 0$. Power series solutions of this system are (15) and:

$$\frac{\partial \Xi}{\partial X} = \sum_{j \in \mathbb{N}} \frac{\partial \mathcal{D}^j X}{\partial X} \frac{t^j}{j!}, \qquad \frac{\partial \Xi}{\partial \Theta} = \sum_{j \in \mathbb{N}} \frac{\partial \mathcal{D}^j X}{\partial \Theta} \frac{t^j}{j!}. \tag{53}$$

Coefficients of these series are used in generalized infinitesimal conditions (28) that allows symmetries computations.

Hence, one can construct a system of ordinary differential equations that allows to compute directly a specialization of $\mathcal{D}^j x_i$ and $\partial \mathcal{D}^j x_i / \partial y$ with y in (X, Θ). In fact, on can compute power series solutions of this system and, the wanted quantities are coefficients of these power series.

Example 4 — We perform above computations on the following Michaelis Menten's equation (see § 6.3 in [10]):

$$\dot{\xi} = \frac{k_1 \xi}{k_2 + \xi} = f(k_1, k_2, \xi). \tag{54}$$

As $\partial f/\partial t \equiv 0$, the associated linear variational system is:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi}{\partial x} &= \frac{k_1 k_2}{(k_2+\xi)^2}\frac{\partial \xi}{\partial x}, \quad \dot{k}_1 = \dot{k}_2 = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi}{\partial k_1} &= \frac{k_1 k_2}{(k_2+\xi)^2}\frac{\partial \xi}{\partial k_1} + \frac{\xi}{(k_2+\xi)}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi}{\partial k_2} &= \frac{k_1 k_2}{(k_2+\xi)^2}\frac{\partial \xi}{\partial k_2} - \frac{k_1 \xi}{(k_2+\xi)^2}. \end{cases} \tag{55}$$

Using specializations defined by $k_1 = 7, k_2 = 2$ and $\xi(0) = 3$, at order 5 power series solutions $\bar{\Xi}(t, 3, 7, 2)$ of above equations are:

$$\begin{array}{rl} \xi &= 3 + \frac{21}{5}t + \frac{147}{125}t^2 - \frac{1372}{3125}t^3 + \frac{2401}{31250}t^4 + \mathcal{O}(t^5), \\ \frac{\partial \xi}{\partial k_1} &= \frac{3}{5}t + \frac{42}{125}t^2 - \frac{588}{3125}t^3 + \frac{686}{15625}t^4 + \mathcal{O}(t^5), \\ \frac{\partial \xi}{\partial k_2} &= -\frac{21}{25}t - \frac{147}{1250}t^2 + \frac{1029}{3125}t^3 - \frac{69629}{312500}t^4 + \mathcal{O}(t^5), \\ \frac{\partial \xi}{\partial x} &= 1 + \frac{14}{25}t - \frac{196}{625}t^2 + \frac{686}{9375}t^3 + \frac{16807}{234375}t^4 + \mathcal{O}(t^5). \end{array} \tag{56}$$

Using these values, we construct linear system associated to commutation condition (21):

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{63}{25} & \frac{21}{5} & -\frac{42}{25} \\ \frac{294}{125} & -\frac{2646}{625} & \frac{588}{125} & -\frac{294}{625} \\ -\frac{8232}{3125} & \frac{4116}{3125} & -\frac{16464}{3125} & \frac{12348}{3125} \end{pmatrix} \begin{pmatrix} \alpha_t \\ \alpha_x \\ \alpha_{k_1} \\ \alpha_{k_2} \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \frac{21}{5} \\ \frac{294}{125} \\ -\frac{8232}{3125} \end{pmatrix} \tag{57}$$

and by a Gaussian elimination, we determine the basis of S defined by the relations $\alpha_t = -\lambda, \alpha_{k_1} = \lambda + \mu$ and $\alpha_{k_2} = \mu, \alpha_x = \mu$. Thus, the dimension of vector space **S** is 2 and the following derivations:

$$\mathcal{S}_\lambda = t\frac{\partial}{\partial t} - k_1\frac{\partial}{\partial k_1}, \quad \mathcal{S}_\mu = x\frac{\partial}{\partial x} + k_1\frac{\partial}{\partial k_1} + k_2\frac{\partial}{\partial k_2} \tag{58}$$

form one of its bases (for the sake of simplicity, higher order derivatives are omitted in these derivations). We deduce from (19) that the following 2 parameters groups:

$$\begin{array}{rclcrcl} t & \rightarrow & \lambda t, & & k_1 & \rightarrow & \mu k_1/\lambda, \\ x & \rightarrow & \mu x, & & k_2 & \rightarrow & \mu k_2 \end{array} \tag{59}$$

acts on (54) and that $\mathfrak{t} = k_1 t/k_2$ and $\mathfrak{x} = x/k_2$ are a convenient set of new coordinates in which system (54) could be rewritten as $\mathrm{d}\mathfrak{x}/\mathrm{d}\mathfrak{t} = \mathfrak{x}/(1+\mathfrak{x})$.

A.2 Multiple specialization

Example 4 (continued) — For this example, 0th order infinitesimal condition (22) leads to consider the following vector:

$$\left(\frac{k_1\, x}{k_2 + x}, -\frac{k_1\, x^2}{(k_2 + x)^2}, \frac{k_1\, x}{k_2 + x}, -\frac{x k_1\, k_2}{(k_2 + x)^2} \right), \tag{60}$$

Using specialization:

$$\begin{array}{cc} x = -2, k_1 = 10, k_2 = -2, & x = -4, k_1 = -7, k_2 = 1, \\ x = 2, k_1 = 8, k_2 = -1, & x = 4, k_1 = -2, k_2 = 1 \end{array}$$

one can obtain the following matrix associated to infinitesimal condition (22):

$$\begin{pmatrix} 5 & -5/2 & 5 & -5/2 \\ -28/3 & 112/9 & -28/3 & -28/9 \\ 16 & -32 & 16 & 16 \\ -8/5 & 32/25 & -8/5 & 8/25 \end{pmatrix} \tag{61}$$

whose kernel is defined by vectors $(1, 1, 0, 1)$ and $(-1, 0, 1, 0)$ that generates the vector space S already determined above (applying LLL reduction on these vectors, we retrieve exactly the previous basis).

Computation of numerical power series solutions of (54) and (55) performed above could be done using specialization (61). Instead of considered series solutions associated to a single specialization, one can consider two or more such series. The series' order needed by our computation decreases with the number of used specialization.

So, there is no need to compute power series as described above even if theoretical structures are clearly defined using this approach (see remark 7).