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Rapports Techniques

1 9 9 2

ème
anniversaire

N° 135

Programme 6
*Calcul Scientifique, Modélisation et
Logiciel numérique par Ordinateur*

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Janvier 1992

RT_0135

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This work was completed while the autor was a visitor at:

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Abstract

In this work, at Chapiter I, a mathematical model for fully saturated elastic clay soil with a very low and settlement dependent permeability has been developed. The model was solved by the finite element method employing repeated fixed point techniques in order to obtain the results in displacement and pore water pressure. The model is consistent with the three-dimensional theory of Biot, where the applied loads can be dependent on time. The pore fluid may be selected as a compressible or incompressible one.

Later, at Chapter II, the user manual of solcxx is given. This program simulates the consolidation phenomenon of a soil, whose solid squeleton has an elastic behaviour, under different hypothesis.

Simulation du problème de consolidation d'un sol saturé

Résumé

Dans le Chapitre I de ce document on développe un modèle mathématique pour décrire les déplacements d'un sol élastique saturé, avec une perméabilité qui est fonction du degré de tassement. On a résolu le modèle numérique en employant une méthode de point fixe et en discrétisant avec des éléments finis conformes, pour obtenir les déplacements et la pression interstitielle. Le modèle concorde avec celui de Biot. Les forces appliquées peuvent être fonction du temps et l'eau interstitielle peut avoir un comportement incompressible ou compressible.

Dans le Chapitre II on donne un guide d'utilisation du module solcxx. Ce programme simule le phénomène de la consolidation d'un sol élastique sous différentes hypothèses.

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Chapter I

Consolidation theory summary

1.- Introduction

The first analysis of consolidation in porous media is atttributed to Karl Terzaghi[1]. His unidimensional consolidation theory relies on a set of suppositions, justified by practical results. In particular, Terzaghi suppose that there are only vertical strains which are directly related to the pore pressure dissipation.

In the one dimensional consolidation theory, pore pressure dissipation is given by the following law:

$$c_v \frac{\partial^2 p}{\partial z^2} = \frac{\partial p}{\partial t} \qquad (1)$$

where c_v is the consolidation coefficient and p the pore overpressure.

The edometric tests and the consolidation of a thin clay layer of negligible thickness as opposed to the dimensions of the load surface can be explained easily by means of the one dimensional theory, but these are practically the only ones. However, this theory has the advantage that it gives a very simple analytical solutions.

The Rendulic[2] theory, currently named the pseudo three dimensional consolidation theory, relies on the mistaken hypothesis that the octahedral mean stress σ_{oct} is constant in the domain and time independent; the equation 1 is changed to:

$$c_v \nabla^2 p = \frac{\partial p}{\partial t} \qquad (2)$$

In three dimensional analysis, taking into account the hypothesis of small strains, elastic squeleton, fully saturated soil with a incompressible water and fluid flow behaviour following Darcy's[3] Law, the consolidation is controlled by the static equilibrium equation together with the differential equation initially given by Biot[4]:

$$c_p \nabla^2 p = \frac{\partial p}{\partial t} - \frac{\partial \sigma_{oct}}{\partial t} \qquad (3)$$

where c_p is the consolidation coefficient. The three dimensional theory explains some consolidation attributes, such as the Mandel[5]-Cryer[6] effect, which is given by a pore pressure increment when the consolidation starts.

2.- Physical Model

For the governing equations of the porous medium, a continuum approach has been adopted[7], where each phase present in the system is assumed to fill up the entire porous medium domain, forming a *overlapping continuum*[8].

Let's consider a soil that occupies a volume Ω and whose boundary is divided into:

- A part in which the displacements are known and another over which the exterior loads (changing with time) are applied.

- Another two parts, equal or different, in which conditions are placed on the flow or on the existing interstitial pressure.

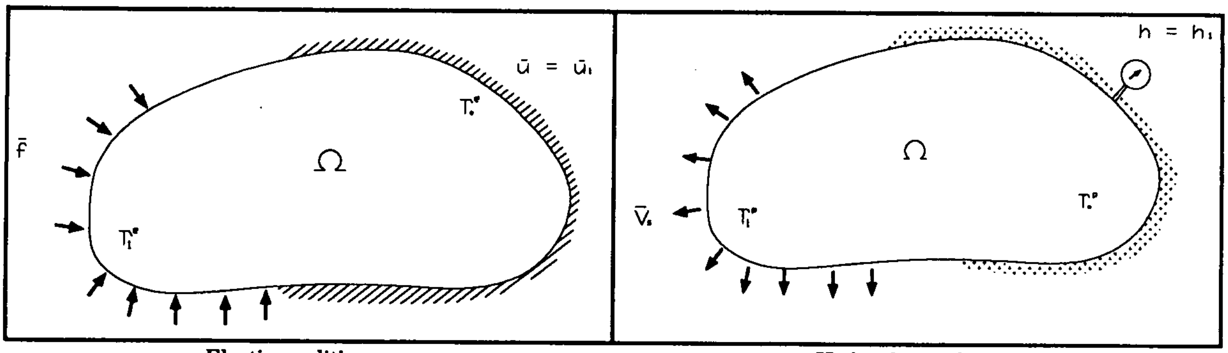

FIGURE 1 Divisions of the boundary.

In agreement with this proposal our problem will consist of *determining the evolution in pressure and displacement of the soil.*

In order to analyse the consolidation of a saturated elastic soil with a fluid, compressible or not, the following equations are employed:

- Equations relating to the soil squeleton
  - Elastic behaviour (effective stresses and strains relationship)
  - Small Strains
- Equations relating to the fluid
  - Fluid flow behaviour following Darcy's law
  - Mass flow for solid and fluid phases
  - Hydraulic boundary conditions (prescribed pressure and flows)
- Equations relating to the soil

- Dynamic equilibrium
- Principle of effective stresses
- Boundary conditions (prescribed loads and displacements)

All the above equations are explained below.

A.- Soil squeleton related equations

a) Elastic behaviour

In linear elasticity, the solid squeleton behaviour law is written as a linear relationship between the stress tension $\overleftrightarrow{\sigma}_e$ and the strain tensor $\overleftrightarrow{\varepsilon}$.

$$\overleftrightarrow{\sigma}_e = \overleftrightarrow{\mathbf{E}}\,\overleftrightarrow{\varepsilon} \quad \text{or} \quad \sigma_{e_{ij}} = \mathbf{E}_{ij}^{kl}\,\varepsilon_{kl} \tag{4}$$

using the Euler notation, where **E** is the squeleton elasticity tensor.

b) Small strains

This hypothesis allows the squeleton strain ε and the displacement vector $\vec{u}$ to be related by means of a linear relationship (first order approximation):

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{5}$$

B.- Pore fluid related equations

a) Darcy's law

This law, experimentally achieved, linearly relates the flow velocity across a porous medium (seepage velocity $\vec{v}_s$) and the hydraulic head h:

$$\vec{v}_s = -\overleftrightarrow{\mathbf{k}}\nabla h = -\mathbf{k}_{ij}\,h_{,j} \tag{6}$$

where $\overleftrightarrow{k}$ is the soil permeability tensor, and the hydraulic head is given as:

$$h = \frac{p}{\gamma_w} + Z_w \tag{7}$$

being p the pore fluid overpressure, Z_w the height above piezometric level of the soil element and γ_w the specific weight of the pore fluid.

b) Pore fluid continuity equation

The continuity equation is obtained from the mass conservation law, and declares that, for a control volume, the mass increment velocity in its interior is exactly equal to the net mass flow through this control volume. In the case of the flow across a porous medium, the continuity equation is written , given in macroscopic variable terms, as:

$$\frac{\partial(nS\rho)}{\partial t} + \nabla(\rho\vec{v}_s) = 0 \tag{8}$$

n being the soil porosity, S the saturation degree, ρ the fluid density, and $\vec{v}_s$ the seepage velocity. According to whether the fluid is incompressible or not, this equation changes to:

$$\nabla\vec{v}_s + \dot{\varepsilon}_V(\vec{u}) = 0 \qquad \text{or} \qquad \nabla\vec{v}_s + \dot{\varepsilon}_V(\vec{u}) + \frac{n}{K_w}\frac{\partial p}{\partial t} = 0 \tag{9}$$

c) Hydraulic boundary conditions

The hydraulic conditions imposed on the boundary domain Ω may have an effect on the hydraulic head value or on the flow velocity; i.e.:

$$\begin{cases} h = h_1 & \text{in } \Gamma_0^p \\ -\vec{v}_s\vec{n} + gh = q & \text{in } \Gamma_1^p \end{cases} \tag{10}$$

C.- Equations relating to the soil regarded as single phase

a) Dynamic equilibrium

The equilibrium equation, relating the total tension σ with the body forces $\vec{F}$, are declared as a displacement vector function:

$$\nabla \cdot \sigma + \vec{F} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{11}$$

being ρ the soil density and $\vec{u}$ its particle displacements.

b) Effective stress principle

It is assumed that a pure (external and internal) fluid pressure p causes only a uniform volumetric strain by compressing the grains of the soil and the major deformation of the porous skeleton is governed by the effective stress σ_e. This is defined as follows, with the sign convention that tension is positive,

$$\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_e - \overleftrightarrow{I}p \quad \text{or} \qquad \sigma_{ij} = \sigma_{e_{ij}} - \delta_{ij}p \tag{12}$$

where σ is the total stress and **I** is equal to unity for the normal stress components and zero for the shear stress components. Equation (12) can be obtained by pure statics which allows the total stress vector to be split into convenient superimposable parts. Equation (12) is also known as the effective stress principle first formulated by Terzaghi, and is considered as the basic equation of Soil Mechanics.

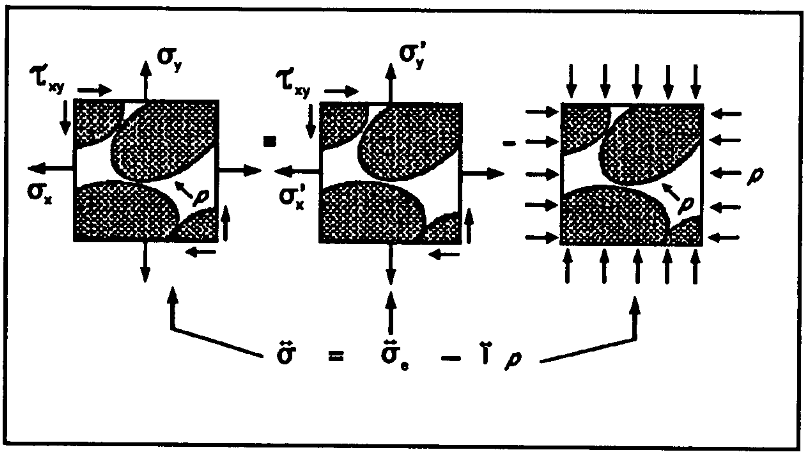

FIGURE 2
Total and effective stresses in a porous medium

c) Boundary conditions

The boundary conditions may be imposed, on the one hand, on the displacements and, on the other, on the stresses, making clear that both conditions can not be simultaneously imposed at the same point. Therefore, the boundary conditions are written as:

$$\begin{cases} u = u_1 & \text{in } \Gamma_0^* \\ \overleftrightarrow{\sigma} \cdot \vec{n} = \vec{f} & \text{in } \Gamma_1^* \end{cases} \tag{13}$$

3.- Differential Formulation

Let Ω be a bounded open subset of $\Re^m$, (m= 2, 3) simply connected and let $\{\Gamma_0^s, \Gamma_1^s\}$ and $\{\Gamma_0^p, \Gamma_1^p\}$ be two disjointed subsets of $\partial\Omega$, the consolidation problem is written as:

Find a pair $\vec{u}$ and p so that

$$\begin{cases} \rho\ddot{\vec{u}} - \nabla\cdot\overleftrightarrow{\sigma}_e = \vec{F} - \nabla\cdot(p\overleftrightarrow{\mathbf{I}}) & \text{in } \Omega \\ \varepsilon_{kl}(\vec{u}) = \frac{1}{2}(u_{k,l} + u_{l,k}) & \text{in } \Omega \\ \overleftrightarrow{\sigma}_e = \overleftrightarrow{\mathbf{E}}\,\overleftrightarrow{\varepsilon} & \text{in } \Omega \\ (\overleftrightarrow{\sigma}_e - p\overleftrightarrow{\mathbf{I}})\cdot\vec{n} = \vec{f} & \text{on } \Gamma_1^s \\ \vec{u} = \vec{u}_1 & \text{on } \Gamma_0^s \\ \vec{u}\,|_{t=t_0} = \vec{u}_0 & \text{in } \overline{\Omega} \\ \dot{\vec{u}}\,|_{t=t_0} = \vec{v}_0 & \text{in } \overline{\Omega} \end{cases} \qquad \begin{cases} \frac{\partial(nS\rho)}{\partial t} + \nabla\cdot(\rho\vec{v}_s) = 0 & \text{in } \Omega \\ \vec{v}_s = -\overleftrightarrow{\mathbf{k}}\cdot\nabla h & \text{in } \overline{\Omega} \\ h = Z_w + \frac{p}{\gamma_w} & \text{in } \overline{\Omega} \\ -\vec{v}_s\cdot\vec{n} + g h = q & \text{on } \Gamma_1^p \\ h = h_1 & \text{on } \Gamma_0^p \\ h\,|_{t=t_0} = h_0 & \text{in } \overline{\Omega} \end{cases}$$

4.- Variational problem

Given a soil occupying a region Ω, where Ω is a bounded open connected subset of $\Re^m$; m= 2,3, with a Lipschitz continuous boundary, such as that shown in figure 1 (see section 2.- Physical Model).

We define the space

$$W(\Omega) = \left\{ w; \quad w \in [H^1(\Omega)]^m; \quad w\,|_{\Gamma_0^s} = 0 \right\}$$

which represents the vector of displacement in each instant of each one of the points of the body and where Γ_0^s is a subset of the boundary of Ω with a strictly positive measurement. The space W is equipped with the product norm:

$$w = (w_1, \ldots, w_m) \to |\, w\,|_{1,\Omega} = \left[\sum_{i=1}^{m} \int_\Omega (w_i)^2 d\Omega \right]^{\frac{1}{2}} \qquad \forall w \in [H^1(\Omega)]^m$$

Let's define for each $w \in [H^1(\Omega)]^m$ the operators:

$$\sigma, \varepsilon \quad : [H^1(\Omega)]^m \to [L^2(\Omega)]^{m\times m}$$

as:

$$\sigma_{ij}(w) = \sigma_{ji}(w) = E_{ij}^{kl}\varepsilon_{kl}(w) \qquad 1 \le i,j,k,l \le m$$

$$\varepsilon_{ij}(w) = \frac{1}{2}(w_{i,j} + w_{j,i}) \qquad 1 \le i,j \le m$$

which play the role of stress tensor and strain tensor in each point of the soil and where the coefficients E_{ij}^{kl} are functions of a point in the domain Ω, that represent the properties of the soil, verifying[9]:

$$E_{ij}^{kl} = E_{ji}^{kl} = E_{ij}^{lk} \qquad E_{ij}^{kl} \in L^{\infty}(\Omega) \qquad \text{(symmetry)}$$

and

$$\exists \alpha \in \Re^{+};\ \forall X \in \Re^{m\times m} / X_{ij} = X_{ji} \Rightarrow\ {}^{t}X_{ij}E_{ij}^{kl}X_{kl} \ge \alpha \mid X \mid^{2}_{\Re^{m}\times\Re^{m}} \qquad \text{(ellipticity)}$$

We suppose that forces of volume dependent on time act on the soil:

$$F = [F_1, \dots F_m] \in H^1[0,T;(L^2(\Omega))^m]$$

and some surface forces, which are also dependent on time:

$$f = [f_1, \dots f_m] \in H^1\left[0,T;(H^{-1/2}(\Gamma_1^e))^m\right] \text{ where } \Gamma_1^e = \partial\Omega - \Gamma_0^e$$

Similarly for hydraulic behaviour of the soil we define

$$V(\Omega) = \left\{v;\quad v \in H^1(\Omega);\quad v\mid_{\Gamma_0^p} = 0\right\}$$

which represents the set of functions with pore hydraulic heads in each moment of each of the points of the body, being Γ_0^p a subset of $\partial\Omega$ with a strictly positive measurement. In this space we can define the norm:

$$\mid v \mid_{1,\Omega} = \left[\int_{\Omega} \nabla v \nabla v d\Omega\right]^{\frac{1}{2}}$$

Also we define the tensor which represents the relative permeability k_{ij} which is a function point in the domain Ω and verifies[10]

$$k_{ij} = k_{ji} \qquad k_{ij} \in L^{\infty}(\Omega) \qquad \text{(symmetry)}$$

and

$$\exists \alpha \in \Re^{+};\forall Y \in \Re^{m} \Rightarrow Y_i k_{ij} Y_j \ge \alpha \mid Y \mid^{2}_{\Re^{m}} \qquad \text{(ellipticity)}$$

We suppose that in the soil there exists a flow through the boundary Γ_1^p given by the function

$$q \in H^1[0,T;H^{-1/2}(\Gamma_1^p)]$$

The study of the phenomenon of consolidation can in reality begin from any state of initial equilibrium for which the deformation of the soil and the distribution of pore pressures (or hydraulic heads) are known. In the case of non-homogeneous conditions, the increase in the deformation and the excess of pore pressure in all the times is calculated from the initial state, which is taking as a reference point.

Next we carried out the homogenization of the essential boundary conditions introducing a new change of variables. In this way, considering as unknown the increase in value of the displacements and the hydraulic heads in respect of the initial values[11][12], the problem is expressed as[13]

Find a pair (u,h) so that, $(u,h) \in \Sigma(\Omega)$ and $\ddot{u} \in L^2[0,T;W'(\Omega)]$ verifying

$$\int_\Omega w_i \rho_T \ddot{u}_i \, d\Omega + \int_\Omega \varepsilon_{ij}(w) \cdot E_{ij}^{kl} \cdot \varepsilon_{kl}(u)\, d\Omega + \left[\int_\Omega \varepsilon_{ij}(w) \cdot \sigma_{0_{ij}}\, d\Omega\right] - \int_\Omega \varepsilon_{ii}(w) p \, d\Omega =$$

$$= \int_\Omega w_i F_i \, d\Omega + \int_{\Gamma_0^s} w_i f_i \, d\Gamma + \int_\Omega w_i \rho_T \ddot{u}_{1_i} \, d\Omega +$$

$$+ \int_\Omega \varepsilon_{ij}(w) \cdot E_{ij}^{kl} \cdot \varepsilon_{kl}(u_1)\, d\Omega - \int_\Omega \varepsilon_{ii}(w) \gamma_w h_1 \, d\Omega \quad \forall w \in W \; c.s.t \in (0,T) \tag{14}$$

$$\int_\Omega v \frac{n}{K_w} \frac{\partial p}{\partial t} d\Omega + \int_\Omega \nabla v \cdot k \cdot \nabla\left(\frac{p}{\gamma_w}\right) d\Omega + \int_{\Gamma_0^p} v g \left(\frac{p}{\gamma_w}\right) d\Gamma = \int_{\Gamma_0^p} v q \, d\Omega - \int_\Omega v \varepsilon_{ii}(\dot{u})\, d\Omega -$$

$$+ \left[\int_\Omega \nabla v \cdot k \cdot \nabla h_1 \, d\Omega + \int_{\Gamma_0^p} v g h_1 \, d\Gamma + \int_\Omega v \varepsilon_{ii}(\dot{u}_1)\, d\Omega\right] \qquad \forall w \in W \; c.s.t \in (0,T)$$

$$u(0,x) = 0 \; ; \; \dot{u}(0,x) = v_0 \; ; \; p(0,x) = 0 \qquad \forall x \in \Omega$$

where:

$$\Sigma(\Omega) = [H^1[0,T;(L^2(\Omega))^m] \cap L^2[0,T;W(\Omega)]] \times [H^1[0,T;L^2(\Omega)] \cap L^2[0,T;V(\Omega)]]$$

$$\rho(x), \gamma_w(x) \in L^\infty(\Omega); \quad \rho(x), \gamma_w(x) > 0 \quad \forall x \in \Omega \qquad g(x) \in L^\infty(\Gamma_1^p); \quad g(x) \geq 0 \quad \forall x \in \Gamma_1^p$$

and σ_0, u_1, h_1 represent the initial stresses of the soil and the known boundary values of the displacements and the hydraulic heads in excess of the initial values.

5.- The Computer Model

In this section, the stages of implementation of a consolidation problem are outlined. We shall begin with a complete description of the process for the elastic case and later deal with the hydraulic case.

After the rewriting of the equilibrium equation, we will obtain the matrices of stiffness and second elemental members, as well as the coupling matrices in both cases. Next, the assembly process will be developed, and finally the problem of consolidation will be strictly implemented.

The exposition is carried out without the loss of generality with *m=3*.

A.- Rewriting in the matrix form

We represent in the three-dimensional case

i) The strain tensor in the form[14]:

$$\varepsilon(w)=\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\end{pmatrix}(w)=[\Lambda]\,\{Dw\}\quad \text{with}\quad [\Lambda]=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&1&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&1&0&0&0&1&0&0\end{pmatrix}$$

and

$$\{Dw\}'=\begin{pmatrix}\dfrac{\partial w_1}{\partial x_1}&\dfrac{\partial w_1}{\partial x_2}&\dfrac{\partial w_1}{\partial x_3}&\dfrac{\partial w_2}{\partial x_1}&\dfrac{\partial w_2}{\partial x_2}&\dfrac{\partial w_2}{\partial x_3}&\dfrac{\partial w_3}{\partial x_1}&\dfrac{\partial w_3}{\partial x_2}&\dfrac{\partial w_3}{\partial x_3}\end{pmatrix}$$

ii) The stress tensor, through the elastic behaviour law

$$\sigma(w)=E\varepsilon(w)$$

being (in the isotropic case)

$$\sigma=\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{pmatrix}=[E]\,[\Lambda]\,\{Dw\}\quad\text{with}\quad E=\begin{pmatrix}\lambda+2\mu&\lambda&\lambda&0&0&0\\ \lambda&\lambda+2\mu&\lambda&0&0&0\\ \lambda&\lambda&\lambda+2\mu&0&0&0\\ 0&0&0&\mu&0&0\\ 0&0&0&0&\mu&0\\ 0&0&0&0&0&\mu\end{pmatrix}$$

where λ, μ are Lamé's parameters[9].

iii) The initial stress tensor σ_0

$$\sigma_0^t = (\sigma_{0_{11}} \quad \sigma_{0_{22}} \quad \sigma_{0_{33}} \quad \sigma_{0_{12}} \quad \sigma_{0_{23}} \quad \sigma_{0_{31}}) = [S]^t$$

iv) The volumetric strain

$$\varepsilon_V(w) = \varepsilon_{11}(w) + \varepsilon_{22}(w) + \varepsilon_{33}(w) = (1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0)\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{pmatrix}(w) = [D]\,[\Lambda]\,\{Dw\}$$

v) The flow velocity

$$k\nabla h = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix}\begin{pmatrix} \dfrac{\partial h}{\partial x_1} \\ \dfrac{\partial h}{\partial x_2} \\ \dfrac{\partial h}{\partial x_3} \end{pmatrix} = [k]\,\{Dh\}$$

In agreement with the above, we can write:

$$\int_{\Omega_d} \{w\}\rho\{\ddot{u}\}d\Omega_d + \int_{\Omega_d} {}^t\{Dw\}^t[\Lambda]\,[E]\,[\Lambda]\,\{Du\}d\Omega_d = \int_{\Omega_d} {}^t\{Dw\}^t[\Lambda]\,[S]d\Omega_d +$$

$$+ \int_{\Omega_d} {}^t\{w\}\,\{F\}d\Omega_d + \int_{\Gamma_0^s} {}^t\{w\}\,\{f\}d\Gamma_d + \int_{\Omega_d} {}^t\{Dw\}^t[\Lambda]^t[D]\,\{\gamma_w h\}d\Omega_d$$

$$\forall w \in W_d^1 \; c.s.t \in (0,T) \qquad (15)$$

$$\int_{\Omega_d} {}^t\{v\}\frac{n\gamma_w}{K_w}\{h\}d\Omega_d + \int_{\Omega_d} {}^t\{Dv\}\,[k]\,\{Dh\}d\Omega_d + \int_{\Gamma_0^p} \{v\}g\{h\}d\Gamma_d = \int_{\Gamma_0^p} \{v\}q\,d\Gamma_d -$$

$$- \int_{\Omega_d} \{v\}\,[D]\,[\Lambda]\,[E]\,\{D\dot{u}\}d\Omega_d \qquad \forall v \in V_d^1 \; c.s.t \in (0,T)$$

a) Interpolation

We suppose that each element T contains n_T nodes and we carry out a *Lagrange* interpolation of degree m, so that the function of displacement in the element T can be expressed as:

$$u_{j|T}(x)=\sum_{i=1}^{n_T} p_i^m(x)u_j(a_i^T)=\begin{pmatrix}p_1^m(x) & p_2^m(x) & \dots & p_{n_T}^m(x)\end{pmatrix}\begin{pmatrix}u_j(a_1^T)\\ u_j(a_2^T)\\ \dots \\ u_j(a_{n_T}^T)\end{pmatrix} \qquad j=1,2,3$$

where

i) a_i^T is the vector position of the node i of the element T.

ii) $u_{j|T}(x)$ represents the jth component of the solution of the approximate displacement problem, restricted to the element T in the point x.

iii) $p_i^m(x)$ represent the polynomial of local interpolation of degree m in the node i of the element T.

That is, representing in a matrix form each one of the strain components

$$u_{j|T}(x)=[P^{mT}(x)]\,\{u_j^T\}$$

and the strain vector

$$\{u\}\,(x)=\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}(x)=\begin{pmatrix}P^{mT}(x) & 0 & 0\\ 0 & P^{mT}(x) & 0\\ 0 & 0 & P^{mT}(x)\end{pmatrix}\begin{pmatrix}u_1^T\\ u_2^T\\ u_3^T\end{pmatrix}=[\vec{P}^{mT}]\,\{u^T\}$$

Similarly, we can express the partial differentials of the components of the strain function

$$\begin{pmatrix}\dfrac{\partial u_j}{\partial x_1}\\ \dfrac{\partial u_j}{\partial x_2}\\ \dfrac{\partial u_j}{\partial x_3}\end{pmatrix}_{|T}(x)=\begin{pmatrix}\dfrac{\partial p_1^m}{\partial x_1} & \dfrac{\partial p_2^m}{\partial x_1} & \dots & \dfrac{\partial p_{n_T}^m}{\partial x_1}\\ \dfrac{\partial p_1^m}{\partial x_2} & \dfrac{\partial p_2^m}{\partial x_2} & \dots & \dfrac{\partial p_{n_T}^m}{\partial x_2}\\ \dfrac{\partial p_1^m}{\partial x_3} & \dfrac{\partial p_2^m}{\partial x_3} & \dots & \dfrac{\partial p_{n_T}^m}{\partial x_3}\end{pmatrix}\{u_j^T\}=[DP^{mT}(x)]\,\{u_j^T\} \qquad j=1,2,3$$

then

$$\{Du\}=\begin{pmatrix}\dfrac{\partial u_1}{\partial x_1}\\ \dfrac{\partial u_1}{\partial x_2}\\ \dots\\ \dfrac{\partial u_3}{\partial x_3}\end{pmatrix}_{|T}(x)=\begin{pmatrix}[DP^{mT}(x)] & [0] & [0]\\ [0] & [DP^{mT}(x)] & [0]\\ [0] & [0] & [DP^{mT}(x)]\end{pmatrix}\begin{pmatrix}u_1^T\\ u_2^T\\ u_3^T\end{pmatrix}=[\vec{DP}^{mT}]\,\{u^T\}$$

Proceeding in a similar way for the hydraulic head function, we get

$$\{h\}(x) = [P^{mT}]\,\{h^T\}$$

$$\begin{pmatrix} \dfrac{\partial h}{\partial x_1} \\ \dfrac{\partial h}{\partial x_2} \\ \dfrac{\partial h}{\partial x_3} \end{pmatrix}_{|T} (x) = \begin{pmatrix} \dfrac{\partial p_1^m}{\partial x_1} & \dfrac{\partial p_2^m}{\partial x_1} & \cdots & \dfrac{\partial p_{n_T}^m}{\partial x_1} \\ \dfrac{\partial p_1^m}{\partial x_2} & \dfrac{\partial p_2^m}{\partial x_2} & \cdots & \dfrac{\partial p_{n_T}^m}{\partial x_2} \\ \dfrac{\partial p_1^m}{\partial x_3} & \dfrac{\partial p_2^m}{\partial x_3} & \cdots & \dfrac{\partial p_{n_T}^m}{\partial x_3} \end{pmatrix} \{h^T\} = [DP^{mT}(x)]\,\{h^T\}$$

In this way, the initial equations (15) are rewritten

$$\sum_{T\in\Im_d}\left(\int_T {}^t\{w^T\}\,{}^t[\overrightarrow{P}^{mT}]\,\rho\,[\overrightarrow{P}^{mT}]\,\{\ddot{u}^T\}\,dx + \int_T {}^t\{w^T\}\,{}^t[\overrightarrow{DP}^{mT}]\,{}^t[\Lambda]\,[E]\,[\Lambda]\,[\overrightarrow{DP}^{mT}]\,\{u^T\}\,dx\right) =$$

$$= \sum_{T\in\Im_d}\left(\int_T {}^t\{w^T\}\,{}^t[\overrightarrow{P}^{mT}]\,\{F\}\,dx + \int_{\partial T\cap\Gamma_0^{\sigma}} {}^t\{w^T\}\,{}^t[\overrightarrow{P}^{mT}]\,\{f\}\,d\Gamma\right) +$$

$$+ \sum_{T\in\Im_d}\left(\int_T {}^t\{w^T\}\,{}^t[\overrightarrow{DP}^{mT}]\,{}^t[\Lambda]\,[S]\,dx + \int_T {}^t\{w^T\}\,{}^t[\overrightarrow{DP}^{mT}]\,{}^t[\Lambda]\,{}^t[D]\,[P^{mT}]\,{}^t\{\gamma_w h^T\}\,dx\right)$$

$$\forall w \in (\Re^3)^{n_T} \qquad (16)$$

$$\sum_{T\in\Im_d}\left(\int_T {}^t\{v^T\}\,{}^t[P^{mT}]\,\frac{n\gamma_w}{K_w}\,[P^{mT}]\,\{\dot{h}^T\}\,dx + \int_T {}^t\{v^T\}\,{}^t[DP^{mT}]\,[k]\,[DP^{mT}]\,\{h^T\}\,dx\right) +$$

$$+ \sum_{T\in\Im_d}\left(\int_{\partial T\cap\Gamma_0^{p}} {}^t\{v^T\}\,{}^t[P^{mT}]\,\{g\}\,[P^{mT}]\,\{h^T\}\,d\Gamma\right) =$$

$$= \sum_{T\in\Im_d}\left(\int_{\partial T\cap\Gamma_0^{p}} {}^t\{v^T\}\,{}^t\{P^{mT}\}\,\{q\}\,d\Gamma - \int_T {}^t\{v^T\}\,{}^t\{P^{mT}\}\,[D]\,[\Lambda]\,[\overrightarrow{DP}^{mT}]\,\{\dot{u}\}\,dx\right)$$

$$\forall v \in (\Re)^{n_T}$$

where ∂T is the edge or face of the element T.

If we now denote

$$[M_e^T] = \int_T {}^t[\overrightarrow{P}^{mT}]\rho[\overrightarrow{P}^{mT}]dx$$

$$[K_e^T] = \int_T {}^t[\overline{DP}^{mT}]{}^t[\Lambda]\,[E]\,[\Lambda]\,[\overline{DP}^{mT}]dx$$

$$[b_e^T] = \int_T {}^t[\overrightarrow{P}^{mT}]\,\{F\}dx + \int_{\partial T\cap\Gamma_0^e} {}^t[\overrightarrow{P}^{mT}]\,\{f\}d\Gamma + \int_T {}^t[\overline{DP}^{mT}]{}^t[\Lambda]\,[S]dx$$

$$[N^T] = \int_T {}^t[\overline{DP}^{mT}]{}^t[\Lambda]{}^t[D]\,[P^{mT}]dx$$

$$[M_p^T] = \int_T {}^t[P^{mT}]\left[\frac{n\gamma_w}{K_w}\right][P^{mT}]dx$$

$$[K_p^T] = \int_T {}^t[DP^{mT}]\,[k]\,[DP^{mT}]dx + \int_{\partial T\cap\Gamma_0^p} {}^t[P^{mT}]\,\{g\}\,[P^{mT}]d\Gamma$$

$$[b_p^T] = \int_{\partial T\cap\Gamma_0^p} {}^t[P^{mT}]\,\{q\}d\Gamma$$

We can write the expression (16) in a compact form:

$$\sum_{T\in\Im_d}({}^t\{w^T\}\,[M_e^T]\,\{\ddot{u}^T\} + {}^t\{w^T\}\,[K_e^T]\,\{u^T\}) = \sum_{T\in\Im_d}({}^t\{w^T\}\,[b_e^T] + {}^t\{w^T\}\,[N^T]\,\{\gamma_w h^T\}) \quad \forall w \in (\Re^3)^{n_T}$$

$$\sum_{T\in\Im_d}({}^t\{v^T\}\,[M_p^T]\,\{\dot{h}^T\} + {}^t\{v^T\}\,[K_p^T]\,\{h^T\}) = \sum_{T\in\Im_d}({}^t\{v^T\}\,[b_p^T] - {}^t\{v^T\}\,[N^T]\,\{\dot{u}^T\}) \qquad \forall v \in (\Re)^{n_T}$$

b) Assembly

Let $[B_e^T]$ and $[B_p^T]$ be boolean matrices that relate local degrees of freedom in the element T and the global ones:

$$\{u^T\} = [B_e^T]\,\{u^g\} \qquad [B_e^T] \in \Re^{(3\cdot n_T)\times(3\cdot n_g)} \qquad \{u^g\} \in \Re^{3\cdot n_g}$$

$$\{h^T\} = [B_p^T]\,\{h^g\} \qquad [B_p^T] \in \Re^{n_T\times n_g} \qquad \{h^g\} \in \Re^{n_g}$$

being n_T the number of nodes per element and n_g the total number of nodes; and where

$$\{u^g\} = {}^t\{u_1(a_1), u_2(a_1), u_3(a_1), \ldots, u_1(a_{n_g}), u_2(a_{n_g}), u_3(a_{n_g})\}$$

$$\{h^g\} = {}^t\{h(a_1), h(a_2), \ldots, h(a_{n_g})\}$$

According to this, we can rewrite (16)

$$
\begin{aligned}
&{}^t\{w^g\}\left(\sum_{T\in\Im_d}{}^t[B_e^T]\,[M_e^T]\,[B_e^T]\right)\{\ddot{u}^g\}+{}^t\{w^g\}\left(\sum_{T\in\Im_d}{}^t[B_e^T]\,[K_e^T]\,[B_e^T]\right)\{u^g\}=\\
&\quad={}^t\{w^g\}\left(\sum_{T\in\Im_d}{}^t[B_e^T]\,[b_e^T]\right)+{}^t\{w^g\}\left(\sum_{T\in\Im_d}{}^t[B_e^T]\,[N^T]\,[B_p^T]\right)\{\gamma_w h^g\}\qquad\forall w^g\in\Re^{3\cdot n_e}\\
&{}^t\{v^g\}\left(\sum_{T\in\Im_d}{}^t[B_p^T]\,[M_p^T]\,[B_p^T]\right)\{\dot{h}^g\}+{}^t\{v^g\}\left(\sum_{T\in\Im_d}{}^t[B_p^T]\,[K_p^T]\,[B_p^T]\right)\{h^g\}=\\
&\quad={}^t\{v^g\}\left(\sum_{T\in\Im_d}{}^t[B_p^T]\,[b_p^T]\right)+{}^t\{v^g\}\left(\sum_{T\in\Im_d}{}^t[B_p^T]\,{}^t[N^T]\,[B_e^T]\right)\{\dot{u}^g\}\qquad\forall v^g\in\Re^{n_e}
\end{aligned}
\tag{17}
$$

If we denote by

$[M_e^g]=\sum_{T\in\Im_d}{}^t[B_e^T]\,[M_e^T]\,[B_e^T]$ elastic mass global matrix

$[K_e^g]=\sum_{T\in\Im_d}{}^t[B_e^T]\,[K_e^T]\,[B_e^T]$ elastic stiffness global matrix

$[b_e^g]=\sum_{T\in\Im_d}{}^t[B_e^T]\,[b_e^T]$ elastic second member global matrix

$[N^g]=\sum_{T\in\Im_d}{}^t[B_e^T]\,[M^T]\,[B_p^T]$ coupling global matrix

$[M_p^T]=\sum_{T\in\Im_d}{}^t[B_e^T]\,[M_p^T]\,[B_e^T]$ hydraulic mass global matrix

$[K_p^T]=\sum_{T\in\Im_d}{}^t[B_e^T]\,[K_p^T]\,[B_e^T]$ hydraulic stiffness global matrix

$[b_p^T]=\sum_{T\in\Im_d}{}^t[B_e^T]\,[b_e^T]\,[B_e^T]$ hydraulic second member global matrix

we can write the consolidation problem in a discrete form:

$$
\begin{aligned}
&{}^t\{w^g\}\,[M_e^g]\,\{\ddot{u}^g\}+{}^t\{w^g\}\,[K_e^g]\,\{u^g\}={}^t\{w^g\}\,[b_e^g]+{}^t\{w^g\}\,[N^g]\,\{\gamma_w h^g\}\qquad\forall w^g\in\Re^{3\cdot n_e}\\
&{}^t\{v^g\}\,[M_p^g]\,\{\dot{h}^g\}+{}^t\{v^g\}\,[K_p^g]\,\{h^g\}={}^t\{v^g\}\,[b_p^T]+{}^t\{v^g\}\,{}^t[N^g]\,\{\dot{u}^g\}\qquad\forall v^g\in\Re^{n_e}\\
&[M_e^g]\in\Re^{3\cdot n_v\times 3\cdot n_v};\qquad [K_e^g]\in\Re^{3\cdot n_v\times 3\cdot n_v};\qquad [b_e^g]\in\Re^{3\cdot n_v};\\
&[M^g]\in\Re^{n_v\times n_v};\qquad [K_p^g]\in\Re^{n_v\times n_v};\qquad [b_p^g]\in\Re^{n_v};\\
&[N^g]\in\Re^{3\cdot n_v\times n_v};
\end{aligned}
\tag{18}
$$

Since (18) must be satisfied for every w^g and v^g, the problem is reduced to solve the system of coupled differential equations

$$\left|\begin{array}{l}[M_e^g]\,\{\ddot{u}^g\} + [K_e^g]\,\{u^g\} = [b_e^g] + [N^g]\,\{\gamma_w h^g\} \\ [M_p^g]\,\{\dot{h}^g\} + [K_p^g]\,\{h^g\} = [b_p^T] + {}^t[N^g]\,\{\dot{u}^g\}\end{array}\right. \qquad (19)$$

which together with the boundary conditions in $\partial\Omega$ will provide the solution to the problem.

Once the displacement vectors and the global hydraulic heads are known, it is possible to calculate the stress and elemental flows as

$$\sigma(u) = [E]\,[\Lambda]\,\{Du\} = [E]\,[\Lambda]\,[\overrightarrow{DP}^{mT}]\,\{u^T\} = [E]\,[\Lambda]\,[\overrightarrow{DP}^{mT}]\,[B_e^T]\,\{u^g\} = [\sigma^T]\,[B_e^T]\,\{u^g\}$$

$$\phi(h) = [k]\,\{Dh\} = [k]\,[DP^{mT}]\,\{h^T\} = [k]\,[DP^{mT}]\,[B_p^T]\,\{h^g\} = [\Phi^T]\,[B_p^T]\,\{h^g\}$$

being

$[\sigma^T] = [E]\,[\Lambda]\,[\overrightarrow{DP}^{mT}]$	Stress elemental matrix
$[\Phi^T] = [k]\,[DP^{mT}]$	Flow elemental matrix

B.- Temporal discretization

There exist many ways of solving the coupling equations system (19) which give rise to the model. We are going to resolve the problem after a discretization in time through a multi-step method, that is

$$\sum_{i=0}^{nq_e} \alpha_e^i \mathbf{M}_e \mathbf{u}^{n-i} = \sum_{i=0}^{nq_e} \beta_e^i \delta^2 (\mathbf{b}_e^{n-i} + \gamma_w \mathbf{N} \mathbf{h}^{n-i} - \mathbf{K}_e \mathbf{u}^{n-i})$$

$$\sum_{i=0}^{nq_p} \alpha_p^i \mathbf{M}_p \mathbf{h}^{n-i} = \sum_{i=0}^{nq_p} \beta_p^i \delta \left(\mathbf{b}_p^{n-i} + \mathbf{N} \frac{\mathbf{u}^{n-i} - \mathbf{u}^{n-i-1}}{\delta} - \mathbf{K}_p \mathbf{h}^{n-i} \right)$$

using fixed point techniques, such as

$$
\begin{aligned}
(\alpha_p^0 \mathbf{M}_p + \beta_e^0 \delta \mathbf{K}_p)\mathbf{h}_{k+1}^n &= \beta_p^0 \delta\left(\mathbf{b}_e^n + \mathbf{N}\frac{\mathbf{u}_{k+1}^n - \mathbf{u}^{n-1}}{\delta}\right) + \\
&+ \sum_{i=1}^{nq_p}\left\{\beta_p^i \delta\left(\mathbf{b}_p^{n-i} + \mathbf{N}\frac{\mathbf{u}^{n-i} - \mathbf{u}^{n-i-1}}{\delta} - \mathbf{K}_p \mathbf{h}^{n-i}\right) - \alpha_p^i \mathbf{M}_p \mathbf{h}^{n-i}\right\} \\
(\alpha^0 \mathbf{M}_e + \beta^0 \delta^2 \mathbf{K}_e)\mathbf{u}_{k+1}^n &= \beta^0 \delta^2(\mathbf{b}_e^n + \gamma_w \mathbf{N}\mathbf{h}_{k+1}^n) + \\
&+ \sum_{i=1}^{nq_e}\{\beta^i \delta^2(\mathbf{b}_e^{n-i} + \gamma_w \mathbf{N}\mathbf{h}^{n-i} - \mathbf{K}_e \mathbf{u}_d^{n+i}) - \alpha^i \mathbf{M}_e \mathbf{u}^{n-i}\}
\end{aligned}
\tag{20}
$$

where now the subscript k indicates the iteration in the nth step of time, and the solution (u_∞^n, h_∞^n) is considered to have been reached when

$$\| u_{k-1}^n - u_k^n \|_\infty \le o(u) \quad , \quad \| h^n - h_k^n \|_\infty \le o(h)$$

In this way, the generalization of the model is made easier in cases in which there exist no linearities . However, owing to the sensibility of h to variations of u, it is convenient to smoothen the solution in hydraulic heads by using a parameter θ. Such a parameter depends on the size of the increments of time, and also on the permeability and the geometry of the mesh. After this, the discrete model is expressed as:

$$
\begin{aligned}
(\alpha_p^0 \mathbf{M}_p + \beta_e^0 \delta \mathbf{K}_p)\mathbf{h}^n &= \beta_p^0 \delta\left(\mathbf{b}_e^n + \mathbf{N}\frac{\mathbf{u}_{k+1}^n - \mathbf{u}^{n-1}}{\delta}\right) + \\
&+ \sum_{i=1}^{nq_p}\left\{\beta_p^i \delta\left(\mathbf{b}_p^{n-i} + \mathbf{N}\frac{\mathbf{u}^{n-i} - \mathbf{u}^{n-i-1}}{\delta} - \mathbf{K}_p \mathbf{h}^{n-i}\right) - \alpha_p^i \mathbf{M}_p \mathbf{h}^{n-i}\right\} \\
\mathbf{h}_{k+1}^n &= \theta \mathbf{h}^n + (1-\theta)\mathbf{h}_k^n \\
(\alpha^0 \mathbf{M}_e + \beta^0 \delta^2 \mathbf{K}_e)\mathbf{u}_{k+1}^n &= \beta^0 \delta^2(\mathbf{b}_e^n + \gamma_w \mathbf{N}\mathbf{h}_{k+1}^n) + \\
&+ \sum_{i=1}^{nq_e}\{\beta^i \delta^2(\mathbf{b}_e^{n-i} + \gamma_w \mathbf{N}\mathbf{h}^{n-i} - \mathbf{K}_e \mathbf{u}_d^{n+i}) - \alpha^i \mathbf{M}_e \mathbf{u}^{n-i}\}
\end{aligned}
\tag{21}
$$

The proof of convergence and the error estimation can be seen in former jobs[13].

C.- Implementation

On the one hand, the following date are taken for granted:

- M_e and M_p: mass matrices of the elastic and the hydraulic problem, stored in profile (skyline) form.

- K_e and K_p : stiffness matrices of the elastic and hydraulic problem respectively, also stored in the profile form.

- Two vectors of initial conditions of displacements and pressures.

- Two second member vectors, one arising from the elastic problem, and the other from the hydraulic problem.

On the other hand, once the time increment δ has been chosen, the method coefficients α^i y β^i are provided.

In order to obtain the solution of the problem in the time T it is necessary to solve the equation (19) in each time step from the first instant. It is therefore interesting to change this into a more accessible method from the point of view of effective computing.

Initially h^0, u^0 and u^1 are known, which allows us to determine h^1. Once the problem of calculating the values of supplementary steps (which are necessary to start the method) is solved, the process will be defined. If we denote by:

$$\beta'_i = \delta^2 \beta^i_e \qquad i = 1, 2, \ldots, nq_e$$

$$\mathbf{X}^i = \mathbf{b}^i_e + \gamma_w \mathbf{N}\mathbf{h}^i\ ,\ \ \mathbf{Y}^i = \mathbf{M}_e \mathbf{u}^i\ ,\ \ \mathbf{Z}^i = \mathbf{K}_e \mathbf{u}^i \qquad i = 1, 2, \ldots$$

the system (19) is written as

$$\begin{aligned}
(\alpha^0_p \mathbf{M}_p + \beta^0_e \delta \mathbf{K}_p)\mathbf{h}^n &= \beta^0_p \delta \left(\mathbf{b}^n_e + \mathbf{N}\frac{\mathbf{u}^n_{k+1} - \mathbf{u}^{n-1}}{\delta} \right) + \\
&\quad + \sum_{i=1}^{nq_p} \left\{ \beta^i_p \delta \left(\mathbf{b}^{n-i}_p + \mathbf{N}\frac{\mathbf{u}^{n-i} - \mathbf{u}^{n-i-1}}{\delta} - \mathbf{K}_p \mathbf{h}^{n-i} \right) - \alpha^i_p \mathbf{M}_p \mathbf{h}^{n-i} \right\} \\
\mathbf{h}^n_{k+1} &= \theta \mathbf{h}^n + (1-\theta)\mathbf{h}^n_k \\
(\alpha^0 \mathbf{M}_e + \beta'_0 \mathbf{K}_e)\mathbf{u}^n_{k+1} &= \beta'_0 \mathbf{H}^n + \sum_{i=1}^{nq_e} \{ \beta'_i (\mathbf{X}^{n-i} - \mathbf{Z}^{n+i}) - \alpha^i_e \mathbf{Y}^{n-i} \}
\end{aligned}$$

As M_e and K_e are constants, we can write $\mathbf{A} = \alpha^0_e \mathbf{M}_e + \beta'^0 \mathbf{K}_e$ which is equally a constant (given that δ is fixed), and the system takes the form:

$$\mathbf{A}\mathbf{u}^n = \mathbf{J}^n \quad \text{where} \quad \mathbf{J}^n = \sum_{i=0}^{nq_e} \beta'_i \mathbf{X}^{n-i} - \sum_{i=1}^{nq_e} (\alpha^i_e \mathbf{Z}^{n-i} + \beta'_i \mathbf{Y}^{n-i})$$

To determine u^n it is necessary to know J^n and the factorised matrix of Cholesky $A={}^tLL$, in order to apply forward elimination and back substitution to the system. As A is a constant, it is only necessary to factorize once. At the same time, the determining of J^n requires knowing X^i for i from 0 up to nq, and Y^i and Z^i for i from 1 up to nq. However, the previous time step is known X^i for i from 1 up to $nq+1$, and also Y^i and Z^i for i from 2 up to $nq+1$, and it is therefore only necessary to calculate X^0, Y^1 and Z^1. Given that u^{n-1} is known from the previous step, Y^1 and Z^1 are calculated in the following way:

$$\mathbf{Z}^1 = \mathbf{M}_e \mathbf{u}^{n-1} \text{ y } \mathbf{Y}^1 = \frac{\mathbf{J}^1 - \alpha_e^0 \mathbf{Z}^1}{\beta'_0}$$

In this way it isn't necessary to store the mass matrix M_e for the calculation of Y^1, avoiding also the multiplication of a matrix for a vector in each step of time.

a) Flow Chart

It is necessary to point out that the scheme adopted for the numerical resolution, based on convergent iterative techniques, is especially adequate for its implementation in Modulef. Thus, the implemented modules are written according to the standards of the Modulef Club[15], i.e., they are divided in two parts:

- A first part where the initiation and addressing of the arrays are performed.
- The second part, used by the former, consists of the main algorithm by using the subroutines and libraries of the Modulef Club.

Figure 3 shows the flowchart for the algorithm, when a compressible fluid or a variable permeability are taken. The calculation of the mass or stiffnes matrices for the hydraulic case and their mixed factorisation may be made in each time step.

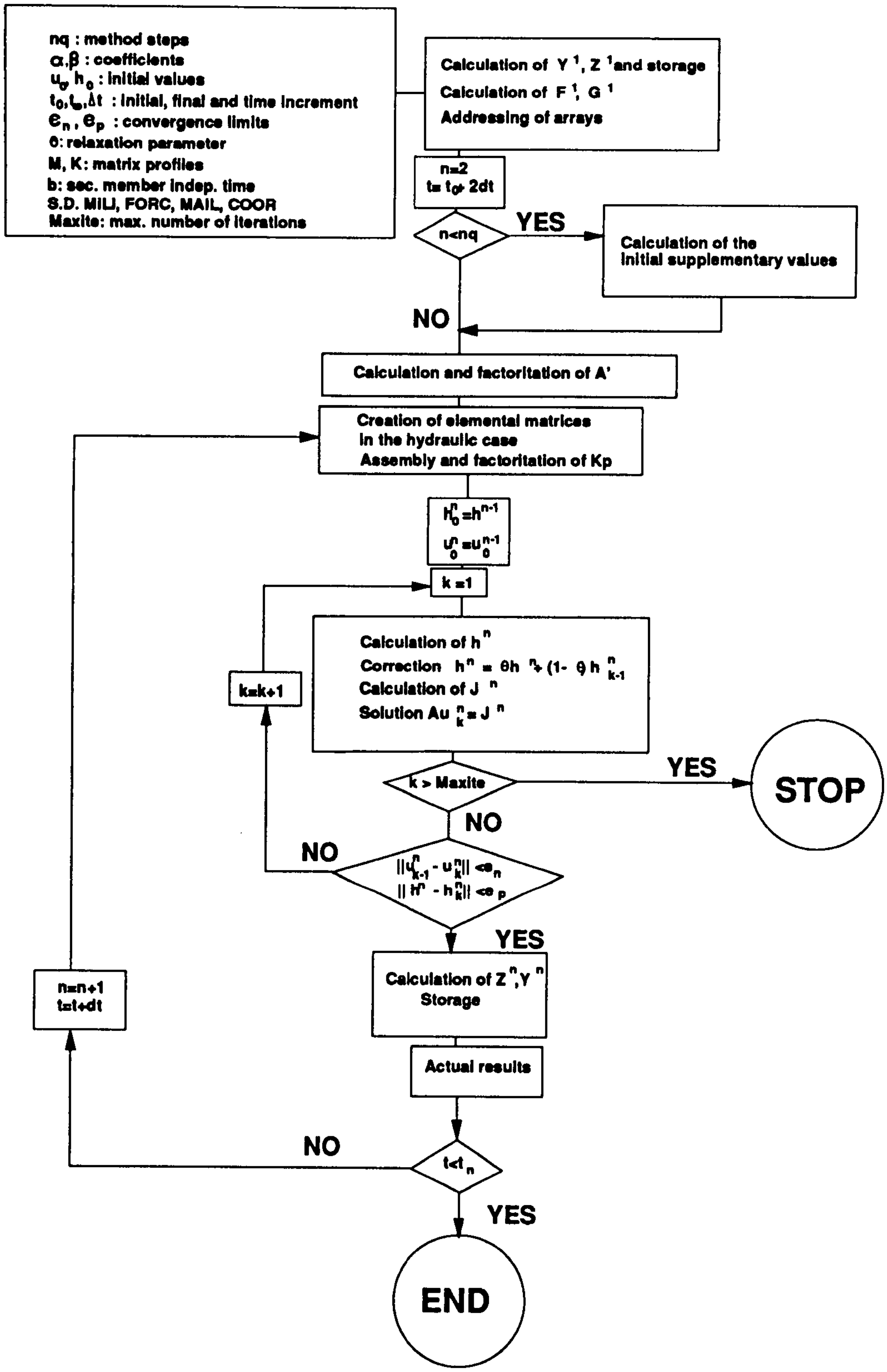
nq : method steps
α,β : coefficients
u0, h0 : initial values
t0,t∞,Δt : initial, final and time increment
εn , εp : convergence limits
θ: relaxation parameter
M, K: matrix profiles
b: sec. member indep. time
S.D. MILI, FORC, MAIL, COOR
Maxite: max. number of iterations
Calculation of Y 1, Z 1 and storage
Calculation of F 1, G 1
Addressing of arrays
n=2
t= t0+ 2dt
n<nq
YES
Calculation of the initial supplementary values
NO
Calculation and factoritation of A'
Creation of elemental matrices in the hydraulic case
Assembly and factoritation of Kp
k =1
Calculation of h n
Calculation of J n
k=k+1
k > Maxite
YES
STOP
NO
NO
YES
Calculation of Z n, Y n
Storage
n=n+1
t=t+dt
Actual results
NO
YES
END

FIGURE 3

Chapter II

Solcxx user manual

1.- Introduction

The *solcxx* program simulates the consolidation phenomenon of a soil, whose solid squeleton has an elastic behaviour, under differents types of flow and saturation. Such module has been programmed under the Club Modulef norms, for which a knowledge of its use, at least superficial, is required.

The manual is divided into four differents parts. In the first part, after an introduction, it is showed an tree diagram of modules joined in the analytical work. Next, it is presented the modules dependent of the nature of the simulation, and which, on changing the case, could become necessary to modify. Some of them come from the Modulef library, but it has been necessary to add some more, for the control of the time outputs and the variation of density with the pressure.

In the second one, the input data file is defined for the *solcxx* module. The character in brackets after each instruction refer to the type of data and/or their length, since they are read with a free format. At the same time, the data appear separated in four blocks. The first block are given by the general parameters (titles, common data structures, etc). Nexts ones gives the necessary parameters in the hydraulic and elastic cases respectively. Finally, there gives the control parameters, which define the interval of study and the control parameters of the algorithm convergence.

In the third part, it is defined the input data file for the *dacoxx* module, which carries out the formats exchange. This permits the graphic representation of the time evolution of the differents nodes.

In the last part, it is showed some numerical test.

A.- The algorithms

The algorithm adopted for the numerical resolution is based on a convergent iterative techniques. The coefficients of the time discretization are given by means of the *evcoef* module (See F.- Subroutines and functions required).

B.- Limits and Failures

- Two dimensional domains are considered
- Currently, only elements TRIA 2P1D and QUAD 2Q1D are implemented.
- In the event that the grid elements of hydraulic and elastic formulation are different, it is necessary that the numeration of the nodes of both meshes coincide.

The possible failures of the algorithm are the following:

Non convergence, that is, after a MAXITE number of iterations a stable solution is not reached yet. In such case, the program stops and the user has to change the data and try again. The algorithm convergence can be achieved of many ways:

- Refining the mesh
- Decreasing the relaxation parameter
- Increasing the time step
- Increasing the MAXITE value

C.- Libraries Modulef to be declared

The next libraries are needed:

util	utsd	cosd	ela2	elas
evol	ther	nop2	nopo	poba
ppal	resb	resd	resr	trac

D.- Auxiliary Files

The file of number NFCOOR, containing the input S.D. COOR.

Two files of number NFMAIL?, containing the input S.D. MAIL for each problem (hydraulic or elastic ones).

One or two files of number NFMUAM?, containing the input S.D. MUA of mass, it is necessary, for each problem (hydraulic or elastic ones).

Two files of number NFMUAK?, containing the input S.D. MUA of stiffness for each problem.

Two files of number NFBDCL?, containing the input S.D. BDCL for each problem.

One or two files of number NFBCTE?, containing the input S.D. B of second member constants, it is necessary, for each problem.

Two files of number NFBS?, containing the output S.D. B for each problem.

E.- Output of the program

A new arrangement is associated with the S.D. B (tableau associé), described by:

Name= *time*, Type= *Real*, Length= *ltime* (time numbers in which a graphics solution is desired).

The vector B4 of the each S.D. B contain the solution, stored a arrangement with the nodal pressure or nodal displacements of the mesh nodes at the selected times.

F.- Subroutines and functions required

Various defined subroutines exist for the user depending on the problem to be dealt with. They affect the processes such as the construction of elemental matrices, the temporal behaviour of the loads and the number of times in which it is desired to obtain a solution.

FORCE y MILIEU

Modules of creation of elemental arrangements associated with the elements employed when the permeability matrix is settlement depending[1].

COTTIM and TPLOTT

Modules of creation of the time arrangement. It generates a time arrangement which stores the times in which it is desirable to obtain results. The selection of such times is carried out by using the module *tplott*.

1) See Modulef Manual N°14

```
      REAL FUNCTION tplott(tinic,dt,t)
c     Calculate de next time in which a solution is desired
c
      tinic        : initial time
c     dt           : time increment
c     t            : current time
      t0 = t
      IF (t .LT. (tinic+1.8*dt)) THEN
           hinc = dt
      ELSE
           hinc = hinc + dt
      END IF
      IF (hinc .LT. dt) hinc = dt
      tplott = t0 + hinc
      RETURN
      END
```

In the above example, as time passes, the graphic results become more distant.

TIMEDE and TIMECH

In all cases it is supposed that the second members given from the discretization by the finite element method have separable components in t and x. This is applicable in most of the cases and permit dealing with cyclical or Heaviside loads. The functions *timede* and *timech* provide the time dependent loads for the elastic and hydraulic problem respectively. These names are not fixed, since they pass as external functions from the main program *solcxx*. Below an example is given in which, in order to simulate more realistic conditions, the load will be applied linearly to a value of time, and then it will be assumed to be constant.

```
      REAL FUNCTION timede (t,dt)
c     Calculate the time dependent load factor.
c     Simulate a linear applied load between two times
c     t:           current time
c     dt:          time increment
      t0 = 0.0000E+00
      t1 = 0.1000E+05
      IF (t .LT. t0) THEN
           timede = 0.0
      ELSE IF (t .LT. t1) THEN
           timede = (t - t0) /(t1-t0)
      ELSE
           timede = 1.0
      END IF
      IF (hinc .LT. dt) hinc = dt
      tplott = t0 + hinc
      RETURN
      END
```

EVCOEF

This subroutine determines the coefficients that define the employed multistep method. It serves simultaneously to give the coefficients of the hydraulic problem (MF=0 or MF=1) and the elastic problem (MF>1)[2].

PEAGUA

A function that, in the case of compressible fluid, relates the hydraulic head to the fluid density. Its only parameter is the increment of hydraulic head from one taken as reference. The default relation chosen is given by De Wiest[16].

$$\gamma_w = \gamma_0 e^{\left(\frac{\gamma_0 \Delta h}{K_w}\right)}$$

```
      REAL FUNCTION peagua (h)
c     Calculate the specific weight of the fluid
      REAL gamma0, Kw
      gamma0 = 0.1000E+05
      Kw = 0.2000E+10
      peagua = gamma0 * exp (h*gamma0/Kw)
      RETURN
      END
```

G.- Modules structural relationship

Here we present a structural relation between the differents modules employed in a form which expedite future enlargements.

The tree diagram in figure four shows the subprograms employed in the case of incompressible pore fluid and constant permeability

2) See Modulef Manual Nº 63

DIAGRAM TREE

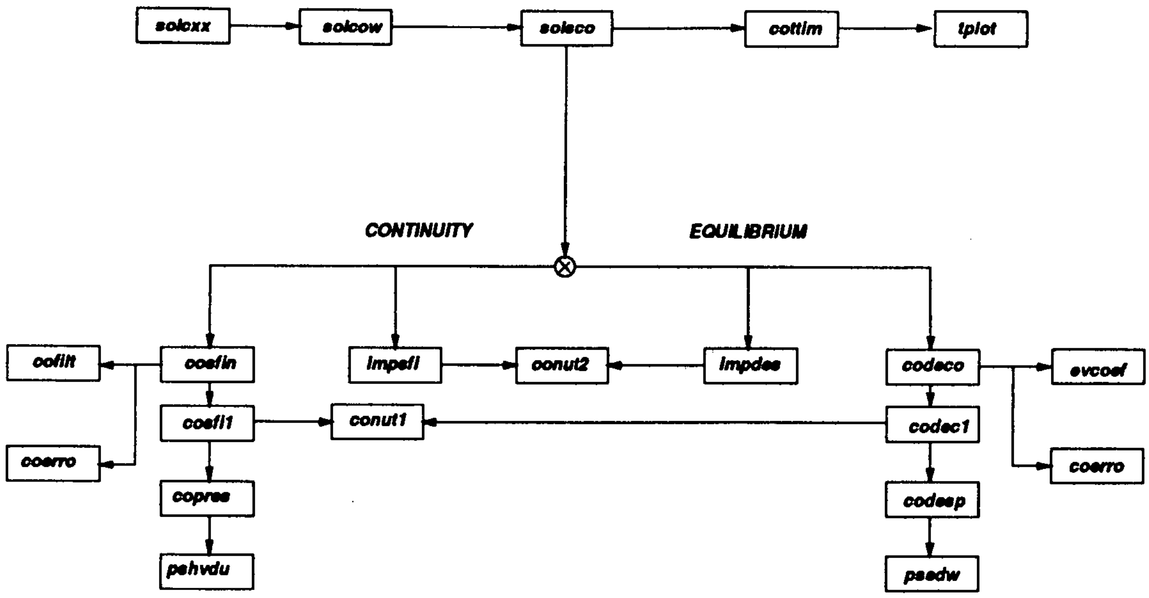

FIGURE 4

On the left we can see the subroutines employed in the resolution of the continuity equation, whose job is:

cosfin Initiation and addressing of arrangements, solution relaxation and iterations control

coerro Local error control between two iterations.

cofilt Solution filter in order to avoid the passing of non-valid values.

cosfi1 Resolution of the system in each iteration and, where necessary, generation of the starting solutions in order to begin the multistep method.

conut1 Creation of the second member of the system from the forces and solutions in previous times.

copres Global arrangement generation for the coupling term, and assembling the elemental arrangement.

pshvdu Elemental coupling matrix for the differents elements.

impsfi Back-up disc for the results and user information.

conut2 Arrangement displacement in order to prepare the next time step.

On the right of the tree diagram we find the subroutines employed in the equilibrium equation resolution, whose job is:

codeco Initiation and addressing of arrangements, combination of the mass and stiffness matrices, factorisation and iteration control

evcoef Definition of the multistep method coefficients.

codecl Resolution of the system in each iteration and, or where necessary, generation of the starting solutions in order to begin the multistep method.

codesp Global arrangement generation for the coupling term, and assembling the elemental arrangement.

psepdw Elemental coupling matrix for the differents elements.

impdes Back-up disc for the results and user information.

On employing the compressible fluid model, but maintaining the constant permeability, a time dependent term in the continuity equation appears. The tree diagram in this case is shown in figure 5. The equilibrium equation doesn't undergo any modifications

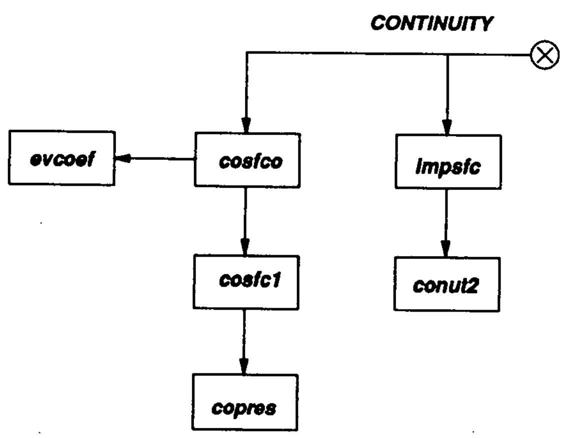

FIGURE 5

The job of each subroutine is now:

cosfco Initiation and addressing of arrangements, combination of the mass and stiffness matrices, factorisation, solution relaxation and iteration control

cosfcl Resolution of the system in each iteration and, or where necessary, generation of the starting solutions in order to begin the multistep method.

impsfc Back-up disc for the results and user information.

On employing the model of settlement dependent permeability, it is necessary to generate the elemental matrices in each time step and assembly them. Depending on whether the fluid is incompressible or not, we will obtain the tree diagrams of figures 6 or 7 respectively.

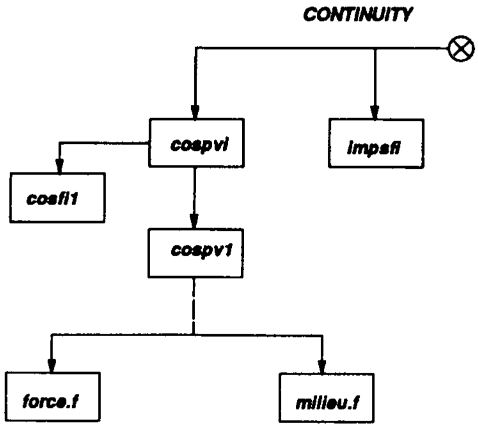

FIGURE 6

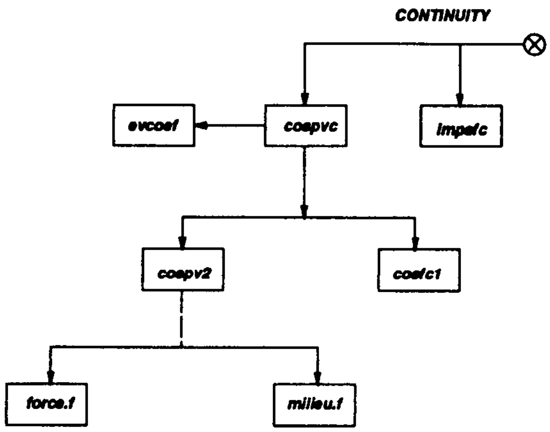

FIGURE 7

where:

cospv1 and cospv2 — carries the calling up to the library module (thelas) for the generation of the elemental matrices after the preparation and assembly, for the cases of incompressible and compressible fluid respectively

2.- Input File Instructions

A.- General Parameters

Instruction 1.1 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	TITLE	Title of the problem and comments

Notes

(1) The title of the problem is written between quotation marks in the first line, and is employed in all data structure (S.D.) or data arrangement generated by the programme. In the absence of a title, a line with at least one blank space must be left.

Instruction 1.2 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFCOOR	File that contains the S.D. COOR.

Notes

(1) The data structure COOR[3] contains the coordinates of the points and of the nodes of a meshing. This S.D. is the only one for both problems. Therefore, in the event that the grid elements of hydraulic and elastic formulation are different, it is necessary that the numeration of the nodes of both meshes coincide.

B.- Parameters of the Hydraulic Problem

Instruction 2.1 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFMAIL	File which contains the S.D. MAIL.

Notes

(1) The data structure MAIL describes the topology of the elements of a mesh that cover a mono, Bi or three dimensional domain. It equally contains the numbers associated with subdomains and with borders. Such numbers allow us to characterise the physical and/or geometric properties of the different materials.

Instruction 2.2 [I10]

Notes	Columns	Variable	Description
(1)	1-10	FLOW	Type of the hydraulic problem

Notes

(1) Integer number between 0 and 5.

It defines the problem of flow across a porous medium associated with the phenomenon of consolidation.

FLOW=0 Fully saturated soil with incompressible fluid.

FLOW=1 Fully saturated soil with compressible fluid.

3) See Modulef Manual N°2

FLOW=2 Fully saturated soil with incompressible fluid (and having a variable permeability). The permeability of the soil depends on the settlement produced, being less the filtration according to the decrease in the void ratio of the porous medium.

FLOW=3 Fully saturated soil with compressible fluid and variable permeability. The permeability of the soil depends on the settlement produced too.

FLOW=4 Unsaturated soil with incompressible fluid.

FLOW=5 Unsaturated soil with compressible fluid.

Instruction 2.3 [I10]

Notes	Columns	Variable	Description
(1)	1-10	IEPOBA	Existence of the base polynomials

Notes

(1) Positive integer or zero.

The direct access file (POBA) is utilized in certain finite elements in order to store the values of the base polynomials (shape functions) and of their derivatives in the points of numerical integration. The existence or not of base polynomials is given, for each element, in its technique record card[4].

IEPOBA=0, Base polynomials aren't necessary.

IEPOBA≠0, Base polynomials are necessary.

If generation of elemental matrices don't exist (FLOW=0 or FLOW=1) the POBA file isn't necessary, and therefore we can proceed to instruction 2.5.

Instruction 2.4 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	POBA	Name of the direct access file which contains the base polynomials.

Notes

(1) The file *poba* is found in the *symd* library. The path to find it is given by the function *kinfo*. In its current installation it is found in */usr/modulef_sta/s/symd/poba*.

If the fluid is compressible (FLOW=1,3 or 5) the instruction 2.6 is carried out.

4) See Modulef Manuals Nº 100 and 101.

Instruction 2.5 [E12.5]

Only in the case that the fluid is incompressible (FLOW=0,2 or 4).

Notes	Columns	Variable	Description
(1)	1-12	PESPWA	Specific weight of the water

Notes

(1) Positive real.

Relates the variation of pore pressure that appears in the equilibrium equation of the elastic problem with the variation of hydraulic head that solves the problem of flow. When the fluid is incompressible then $p = \gamma_w h$.

When the fluid is compressible, a function is employed, called *peagua*(SEE F.- Subroutines and functions required) and passed as an external function by the main program *solcxx*, which determines the specific weight as a function of the hydraulic head in the fluid. The default De Wiest[16] relation has been considered.

$p = \gamma_w(h)\, h$ with $\gamma_w = \gamma_0 e^{\left(\frac{\gamma_0 h}{K_w}\right)}$ where K_w is the water bulk modulus

Instruction 2.6 [1A72]

Only in the case of compressible fluid (FLOW=1,3 or 5).

Notes	Columns	Variable	Description
(1)	1-72	NFMUAM	File that contain the S.D. MUA

Notes

(1) The data structure MUA stores the coefficients coming from the discretization of the problem by the finite element method in profile (skyline) form. In this case is the mass matrix, obtained to discretize the term with the pressure derivative.

If the permeability isn't variable (FLOW=1 or 2), skip to instruction 2.10.

Instruction 2.7 [1A72]

Only in the case of variable permeability (FLOW>1).

Notes	Columns	Variable	Description
(1)	1-72	NFFORC	File which contains the S.D. FORC

Notes

(1) The data structure FORC describes as the form in which the forces or flows, necessary for the calculation of the second member elementals of an elastic and heat problem, are given. In this case it describes the flow of water through the boundaries and also, in case of the existence, the flow sources or sink.

The default option is to calculate the data with the FORCE module, in which flow sources are not defined and flow does not exist through the references.

Instruction 2.8 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFMILI	File that contains the S.D. MILI

Notes

(1) The data structure MILI describes as the form in which the properties of the materials, necessary for the calculation of the elemental matrices for a thermal or elastic problem, are given.

In the default case, we consider that there exists a bilinear relation between the void ratio e and the logarithm of the permeability k which represents fairly well the behaviour of the soil. Such a relation, calculated experimentally by Monte and Kritzen[17], is assumed considering that the vertical is the direction of maximum stress. If k_c and e_c are the critical values of the permeability and of the void ratio in which a change of slope is produced (see figure 8) and if e_0 is the initial void ratio, then the permeability k for a void ratio e is given by

$$\ln\left(\frac{k}{k_c}\right)=\frac{e-e_c}{S} \quad \text{where} \quad \begin{cases} S=S_1, & e<e_c \\ S=S_2, & e>e_c \end{cases}$$

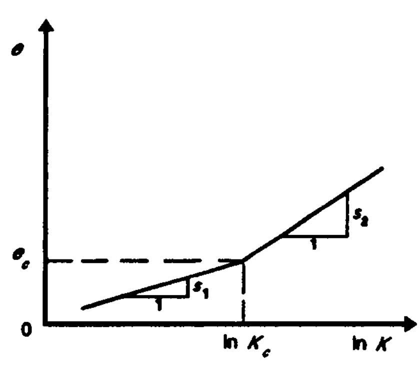

FIGURE 8
Void ratio versus permeability logarithm

Instruction 2.9 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFTAE	File that contains the S.D. TAE

Notes

(1) The data structure TAE describes the mass, stiffness and second members elemental matrices. Given the volume of required information, it must be stored in secondary memory, for which, even when it is a structure of intermediate data in the problem, it is necessary to assign it a file. After its assembly, we obtain S.D. MUA of mass and of stiffness, and a S.D. B of seconds member constants.

Skip to instruction 2.12.

Instruction 2.10 [1A72]

Only in case of constant permeability (FLOW=0 or 1).

Notes	Columns	Variable	Description
(1)	1-72	NFMUAK	File that contains the S.D. MUA

Notes

(1) The data structure MUA stores the coefficients coming from the discretization of the problem by the finite element method in the profile form. In this case it is the matrix of stiffness, obtained to discretize the term $\nabla v \cdot \overleftrightarrow{k} \cdot \nabla v$.

If the permeability is variable (FLOW=3 or 4) we skip over the next line.

Instruction 2.11 [1A72]

Only in the case of constant permeability (FLOW=0 or 1).

Notes	Columns	Variable	Description
(1)	1-72	NFBCTE	File that contains the S.D. B

Notes

(1) The data structure B stores the coefficients coming from the discretization of the problem by the finite element method second members independent of time[5]. This is

5) See Modulef Manual N°63.

due to the fact that the second member is considered as that can be separated from function space and time, i.e., $F(x,t) = f(x)\, g(t)$. In this case it is the vector that stores the terms due to sink or sources, and also the flow through the boundary.

Instruction 2.12 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFBS	File that contains the S.D. BS

Notes

(1) The data structure BS stores the solution in hydraulic heads for different times.

Instruction 2.13 [I10]

Notes	Columns	Variable	Description
(1)	1-10	IEBDCL	Existence of boundary conditions

Notes

(1) Positive or nil integer.

IEBDCL=0, boundary conditions don't exist.

IEBDCL≠0, boundary conditions do exist.

If boundary conditions aren't present, skip over the next line

Instruction 2.14 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFBDCL	File that contains the S.D. BDCL

Notes

(1) The data structure BDCL describes the Dirichlet boundary conditions:

$$h_i = \pi_i$$

i.e., of the known values of pressure.

Instruction 2.15 [1F12.5]

Notes	Columns	Variable	Description
(1)	1-12	HMIN	Minimum value of the hydraulic head

Notes

(1) Real.

Describes the minimum value of the hydraulic head, and is used to prevent the existence of inferior depressions to the vacuum, for which it is necessary that the initial hydraulic head together with the pore overpressure aren't lower than the nil pressure.

If FLOW=0, skip to instruction 3.1.

Instruction 2.16 [1A72]

Notes	*Columns*	*Variable*	*Description*
(1)	1-72	NFB0	File that contains the S.D. B0

Notes

(1) The data structure B0 describes the initial conditions of hydraulic heads.

If FLOW=2 (incompressible fluid), skip to the instruction 3.1.

Instruction 2.17 [2I10]

Notes	*Columns*	*Variable*	*Description*
(1)	1-10	MF	Problem type
(2)	11-20	NQ	Method steps

Notes

(1) Positive integer

MF=0 The time dependent problem is parabolic, and the data for starting up the process are given by multistep methods of order 1.

MF=1 The time depending problem is parabolic, and the data for starting up are calculated by the Runge-Kutta method (to three steps).

(2) Positive integer.

NQ determines the number of method steps. Depending on its value, it will be necessary to calculate previous values or not, using the option given in MF.

Instruction 2.18 [2I10]

This line is kept eventually for tests with the different methods.

Notes	*Columns*	*Variable*	*Description*
(1)	1-10	NMET	Method type employed
(2)	11-20	NMET1	Method type, in case it is necessary to calculate previous solutions.

C.- Parameters of the Elastic Problem

Instruction 3.1 [1A72]

Notes	*Columns*	*Variable*	*Description*
(1)	1-72	NFMAIL	File which contains the S.D. MAIL.

Notes

(1) The data structure MAIL describes the topology of the elements of a mesh that cover a mono, Bi or three dimensional domain. It equally contains the numbers associated with subdomains and with borders. Such numbers allow us to characterise the physical and/or geometric properties of the different materials.

Instruction 3.2 [1A72]

Notes	*Columns*	*Variable*	*Description*
(1)	1-72	NFMUAM	File that contain the S.D. MUA

Notes

(1) The data structure MUA stores the coefficients coming from the discretization of the problem by the finite element method in profile (skyline) form. In this case the mass matrix obtained to discretize the term $\rho\ddot{w}$.

Instruction 3.3 [1A72]

Notes	*Columns*	*Variable*	*Description*
(1)	1-72	NFMUAK	File that contains the S.D. MUA

Notes

(1) The data structure MUA stores the coefficients coming from the discretization of the problem by the finite element method in the profile form. In this case it is the matrix of stiffness, obtained to discretize the term $\varepsilon_{ij}(w) \cdot E_{ij}^{kl} \cdot \varepsilon_{kl}(w)$

Instruction 3.4 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFBCTE	File that contains the S.D. BS

Notes

(1) The S.D. B store the coefficients coming from the discretization of the problem through the finite element method for second members. For this case, it is the vector that stores the part of the time independent forces *f*, since the second member can be put as $F(x,t) = f(x)\ g(t)$. These forces can be volumetric (self weight, etc), superficial (applied forces, etc) or punctual ones (punctual heads due to cables, etc).

Instruction 3.5 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFBS	File that contains the S.D. B

Notes

(1) The S.D. BS stores the solution of the displacements for different times.

Instruction 3.6 [I10]

Notes	Columns	Variable	Description
(1)	1-10	IEBDCL	Existence of boundary conditions.

Notes

(1) Whole positive number or zero.

IEBDCL=0, Boundary conditions don't exist.

IEBDCL≠0, Boundary conditions do exist.

In the absence of boundary conditions, we skip over the next line.

Instruction 3.7 [1A72]

Only in the case where boundary conditions exist (IEBDCL≠0).

Notes	Columns	Variable	Description
(1)	1-72	NFBDCL	File that contains the S.D. BDCL

Notes

(1) The structure of data BDCL describes the boundary conditions and it can be of two types:

Dirichlet: $u_i = \upsilon_i$

Lineal combinations: $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$

In the latter case, the definition can be necessary through the module *VALCLR*, of the lineal relations that join the conditions in the boundary nodes[6].

Instruction 3.8 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFB0	File that contains the S.D. B0

Notes

(1) The structure of data B0 describes the initial conditions in displacement. It need two initial values, which depending on the type of problem will be the displacements in two consecutive instants or the displacement and the velocity in an instant.

Instruction 3.9 [I10]

Notes	Columns	Variable	Description
(1)	1-10	IESIGMA	Existence of the initial stress.

Notes

(1) Positive integer or zero.

IESIGMA=0, initial stresses don't exist.

IESIGMA≠0, initial stresses do exist.

6) See Modulef Manual Nº 57.

If initial stresses don't exist, skip over the next line.

Instruction 3.10 [1A72]

Notes	*Columns*	*Variable*	*Description*
(1)	1-72	NFSIGMA	File that contains the S.D. BSIGMA.

Notes

(1) The data structure BSIGMA describe the initial stresses of the soil. Such S.D. is obtained by running the module *thelxx*[7] or *thecxx* with initial stresses and without loads.

Instruction 3.11 [2I10]

Notes	*Columns*	*Variable*	*Description*
(1)	1-10	MF	Problem type.
(2)	11-20	NQ	Method steps.

Notes

(1) Positive integer greater than one.

MF=2 The time dependent problem is hyperbolic, and the data to start up are displacements in consecutive times.

MF=3 The time dependent problem is hyperbolic, and the data to start up are displacements and velocity in a time.

MF=4 The time dependent problem is hyperbolic, and the data to start up are displacements in consecutive times. Viscous damping matrix are substituted by a combination of the mass and stiffness matrices.

MF=5 The time dependent problem is hyperbolic, and the data to start up are displacements and velocity in a time. Viscous damping also exist.

(2) A positive whole number greater than one and lesser than five.

NQ determines the number of method steps. Such number indicates the number of coefficients needed in the subprogram *evcoef.*

If viscous damping doesn't exist, go to the next time.

7) See manual Nº14 Club Modulef.

Instruction 3.12 [2F12.5]

Notes	Columns	Variable	Description
(1)	1-10	CM	Viscous damping coefficient for mass
(1)	11-20	CK	Viscous damping coefficient for stiffness

Notes

(1) A particularly convenient form of B is the Rayleigh damping matrix $B = c_m A + c_k C$, where c_m and c_k are coefficients. The two constituent of Rayleigh damping are seen to be mass and stiffness proportional. Thus the relation

$$\mathrm{A}\frac{\partial^2 u}{\partial t^2} + \mathrm{B}\frac{\partial u}{\partial t} + \mathrm{C}u \quad \text{is changed to} \quad \mathrm{A}\frac{\partial^2 u}{\partial t^2} + (c_m \mathrm{A} + c_k \mathrm{C})\frac{\partial u}{\partial t} + \mathrm{C}u$$

Instruction 3.13 [2I10]

This line is kept eventually for tests with the different methods.

Notes	Columns	Variable	Description
(1)	1-10	NMET	Method type employed
(2)	11-20	NMET1	Method type, in case it is necessary to calculate previous solutions.

D.- Control Parameters

Instruction 4.1 [3F12.5]

Notes	Columns	Variable	Description
(1)	1-10	TINI	Starting time
(1)	11-20	TFIN	Final time
(1)	21-30	DELTAT	Time increment

Notes

Positive reals

The time increment value must be chosen in a way which satisfies the conditions of stability and convergence. Given that a multistep method is employed, the solutions average the time dependent values of the hydraulic head and the displacements. The smaller time increment the greater will be the accuracy obtained, and also a greater number of steps will be necessary in order to obtain a static equilibrium. Likewise, a greater number of iterations will be necessary in each time step in order to reach the convergence.

Instruction 4.2 [2I10]

Notes	*Columns*	*Variable*	*Description*
(1)	1-10	CONTRO	Program control
(2)	11-20	MAXITE	Maximum number of iterations in each time step

Notes

(1) Positive or nil integer.

CONTRO=0 In each time step the program carries out the iterations given by maxite

CONTRO=1 In each time step the program carries out the necessary iterations up to a maximum of MAXITE, in order to reach an estimated error between two consecutive iterations.

(2) Positive integer.

When the number of iterations exceed this value, if CONTRO=1, it produces the stopping of the program and a warning indicating that the desired convergence has not been obtained.

Instruction 4.3 [2F12.5]

Notes	*Columns*	*Variable*	*Description*
(1)	1-10	RELAX	Relaxation factor of the solution in hydraulic head
(2)	11-20	ESCALA	Scale factor

Notes

(1) Real number between zero and one.

The value of RELAX controls the convergence of the algorithm. RELAX small values help its convergence, but by means of carrying out a high number of iterations in each time step. On the contrary, the nearest values to one permit a quick adjustment of the system requirements, but they can make it divergent.

(2) Positive real

It permit the definition of a scale value, correction of the hydraulic matrix, in order to correct the rounding errors. It must be of inverse order to the lowest permeability.

Instruction 4.4 [2F12.5]

Notes	Columns	Variable	Description
(1)	11-20	ERRH	Estimated error in hydraulic head
(1)	21-30	ERRD	Estimated error in displacements

Notes

(1) Positive reals.

ERRH and ERRD permit the definition of, in the case of CONTRO=1, the admissible errors, in absolute value, of the hydraulic head and the displacements between two successive iterations.

3.- Instructions for the Graphic Output

A simple program *dacoxx* has been added in order to obtain a graphic output for the unknown variable time distribution. The point number in which it is plotted is given by the number of time steps in which they have obtained the results

Instruction 1.1 [1A72]

Notes	Columns	Variable	Description
(1)	1-72	NFBS	File contains the solutions S.D. B

Notes

(1) File that contains the S.D. B with hydraulic heads or displacement solutions from *solcxx*.

Instruction 1.2 [1I10]

Notes	Columns	Variable	Description
(1)	1-10	INTERF	Interface drawing type

Notes

(1) Zero or positive integer less than 3

INTERF=1 Create a legible file for the programme of the Modulef library *tracxx* (curve plotting).

INTERF=2 Create a legible file for the program *excel.*

INTERF=0 Create a legible file for the program *tracxx* and another for *excel.*

Instruction 1.3 [3E12.5]

Notes	Columns	Variable	Description
(1)	1-12	ESCALX	X axis scale
(1)	13-24	ESCALY	Y axis scale
(2)	25-36	DESPX	Displacement in the X axis.

Notes

(1) Positive reals

Determine the solution representation scale; this permits using in the graph measures which are not of the International System, but simplify the interpretation of the results.

(2) Real.

Determine the displacement of the X axis origin. Allow graphic comparison of the results with different starting times on displacing the measure origin. If it is positive, the graph has a displacement on the X axis indicated by the magnitude of DESPX; else, the absolute value will be taken by the graph as origin.

Instruction 1.4 [3I10]

Notes	Columns	Variable	Description
(1)	1-10	ND	Freedom degree
(2)	11-20	NPTIME	Time steps
(3)	21-30	NOE	Number of curves

Notes

(1) Integer greater than zero.

In the hydraulic case ND=1, there doesn't exist more than one degree of freedom for each node.

In the elastic case, ND is a positive integer taken between 1 and the dimension of the space.

(2) Positive integer or zero.
It indicates the number of time steps, counting from the beginning, which we have to take. If its value is nil, take all the existing ones in the solution file.

(3) Positive integer less than ten
NOE indicates the number of nodes whose behaviour we want to represent in a time, therefore each node having a different curve.

Instruction 1.5 [10I5]

Notes	Columns	Variable	Description
(1)	1-5	NP1	First node number
(1)	6-10	NP2	Second node number
...	...	...	...
(1)	45-50	NP10	Tenth node number

Notes

(1) All of these must be positive integers, not repeated and taken between one and the total node number of the mesh.

Instruction 1.6 [1A72]

This instruction is necessary when INTERF=0 or INTERF=1

Notes	Columns	Variable	Description
(1)	1-72	NFTRAC	Input file for the program tracxx.

Instruction 1.7 [1A72]

This instruction is necessary when INTERF=0 or INTERF=2

Notes	Columns	Variable	Description
(1)	1-72	NFEXCE	Input file for the program excel.

4.- Numerical Test

Initially we present a quasi-unidimensional model that allows comparison with Terzaghi's solution, thus noting the effects that the variation of the different parameters have on it.

Then, we can study a more realistic example, in which there exist different materials for studying the settlement of a slab.

A.- Unidimensional Case

The problem of unidimensional consolidation of a elastic soil is analysed under the following conditions:

a.- We consider a horizontal layer of infinite extension and thickness H, placed over an impermeable rock layer.

b.- Water flow is considered unidirectional, draining through a free surface.

c.- We apply a instant load on the surface of the soil in the initial instant, and maintain it constant during the drainage.

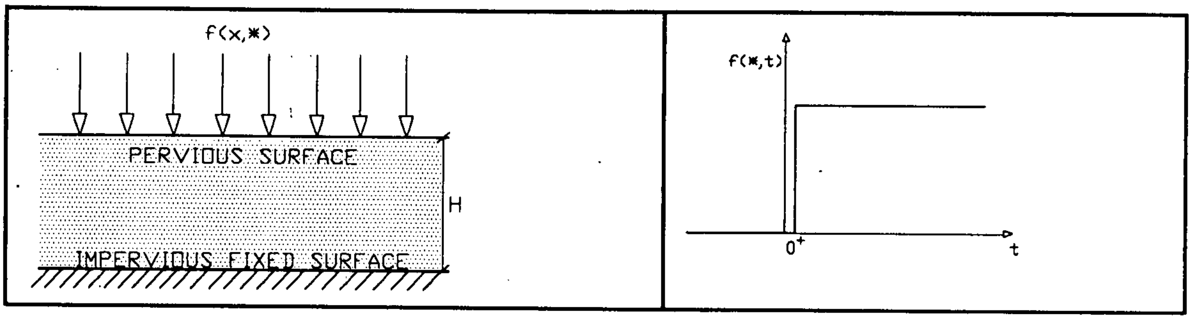

FIGURE 9 Unidimensional consolidation

According to this hypothesis (figure 9), the soil behaves according to Terzaghi's model, and the solution (vertical displacements and pore pressure) depends on the following parameters: *k*, f/E_m, ν and *H*, where *k* is the permeability of the soil, *f* the applied load, E_m the edometric module of the soil and ν is the Poisson's ratio.

Given the symmetry of the layer, it is enough for this study to consider a surface with a fixed area, imposing on its lateral faces neither flow nor horizontal displacement conditions. This case has been widely studied to test different models [18][19][20][21][22][23], since its analytical solution is known.

The data employed is as follows

Layer thickness (*H*)	*7 m*
Edometric module (E_m)	*5 MPa*
Poisson's ratio (ν)	*0.2*
Surface load (*f*)	*500* kN/m^2
Permeability (*k*)	$10^{-8}\ m/s$

The mesh employed is shown in graph 1. The influence of the time increment on the results can be seen in graphs 2 and 3. We used the central node of the surface (node 62) for the study of the displacement, and the next inferior one (node 58) for the study of pore water pressure. It was observed that with the reduction in the time increment, the settlement curves became smoother, arriving at the analytical solution in the initial moments. An increase in the peak of the pore pressure was observed with the reduction of the time increment, which is reasonable considering that, in the following instant to the application of the load, it is supported by the water. When taking a smaller increment, the pore pressure decreases, dissipated by the flow of water through the surface.

In graphs 4 and 5 the result obtained with a 500s time increment are compared with Terzaghi's model for distinct depths (nodes). In such graphs, we can observe the coincidence of both models for the expected settlement once they pass the starting time. In the same way, we can see the similarity of the results in the nodes further from the drainage surface, and also the ones nearer the above surface once the consolidation is in process.

It is unusual to find works in the numerical examples whose permeability is less than $10^{-6}\ m/s$. In graphs 6 and 7, we show, for the nodes 62 and 58, the settlement and the pore pressure curves in a permeability range decreasing from 10^{-4} to 10^{-10} m/s.

B.- Consolidation under a slab

Finally, we shall study a practical case where the load is time-depending and the soil is composed of two layer, one below consisting of sand, and the other above consisting of clay (figure 10).

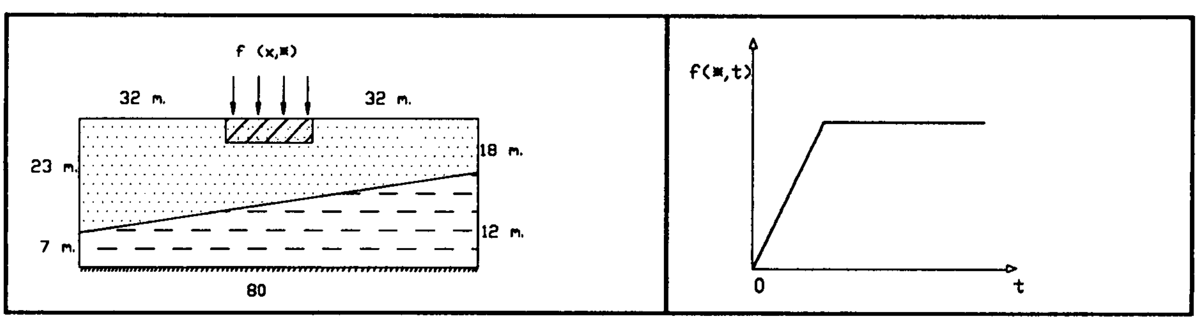

FIGURE 10 Consolidation under a slab

a.- Let's consider a horizontal layer 80 m long and 30 m thick, placed over an impermeable rocky layer. Such a layer is composed of a sandy layer below, 23 m in depth at one extreme, and 18 m at the other, and a clay layer above, in which a slab 18 m long and 1 m thick is placed.

b.- Water flow takes place through the surface of the terrain and lateral sides of the sandy layer.

c.- A load f is applied increasing lineally until it reaches $300\ kN/cm^2$ for a time of $2.5x10^5 s$, remaining constant for that value during the rest of the process.

d.- The properties of the material are

	Clay	Concrete	Sand
Edometric Module (E_m)	*5 MPa*	*20000 MPa*	*5 MPa*
Poisson's Ratio (v)	*0,2*	*0,3*	*0,45*
Permeability (k)	$10^{-8} m/s$	$10^{-9} m/s$	$10^{-3} m/s$

The mesh employed is shown in graph 8, and an enlargement of the slab area is given in graph 9. Next, the settlement and the pore pressure curves are given (graphs 10 and 11). It can be observed that the settlement curve has two clearly different parts. A first part, where the settlements are proportional to the increase in the load and a second part, with no change in load, in which settlement still takes place in a slower way. The first part is explained as an elastic deformation caused in the sand layer by the load (permeability being high, the response to the load is immediate) while the second part is due to the deformation of the clay layer, which starts with the draining of the pore fluid. Similarity, it can be seen that at the beginning, the pressure in the central inferior part of the slab is lower than that in the extremes, a fact already experimentally proved[24].

Finally, a study has been carried out, for different times in the evolution of pore pressure expressed in water column meters (graphs 12-15). It can be pointed out that peaks of the pressure appear at the extremes of the slab, a fact referred to, which later develop forming a pressure bulb[25].

TIME	0.5000E+05	0.1600E+06	0.5500E+06	0.9250E+06
GRAPH	12	13	14	15

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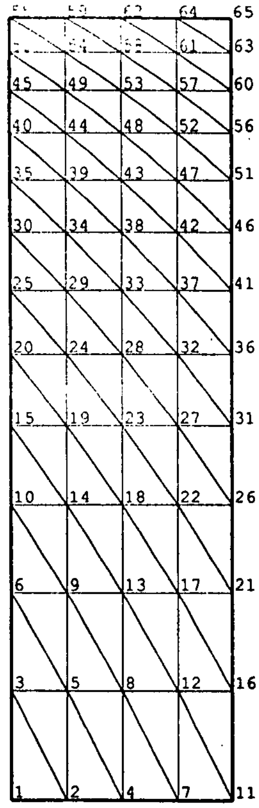
Cross References

GRAPH 1



MODULEF : cesarm

ONE-DIMENSIONAl CONSOLIDATION

08/05/91

DT/NOPO

65 POINTS

65 NOEUDS

96 ELEMENTS

96 TRIANGLES

0 TROU(S)

COIN BAS GAUCHE :

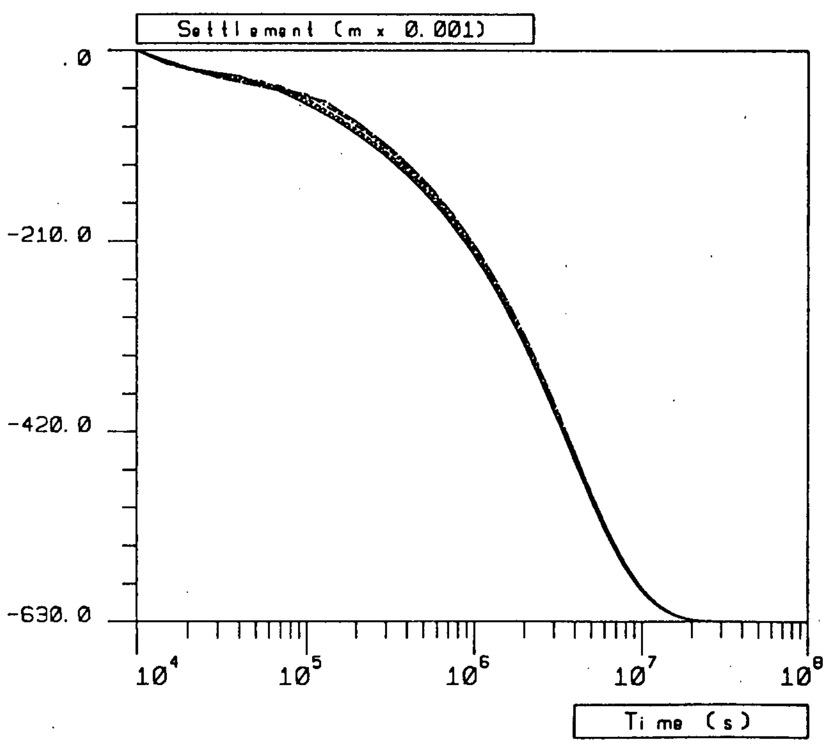
-3.6 -7.4

COIN HAUT DROIT :

5.6 .35

NUMERO NOEUD

GRAPH 2



MODULEF : cesarm

VARIATION OF DT

08/05/91

DT/tracd

NOMBRE DE COURBES : 5

EXTREMA EN X :

.25E+04 .50E+08

EXTREMA EN Y :

-.63E+03 .00E+00

: t1 = 0.5000E+04

: t2 = 0.7500E+04

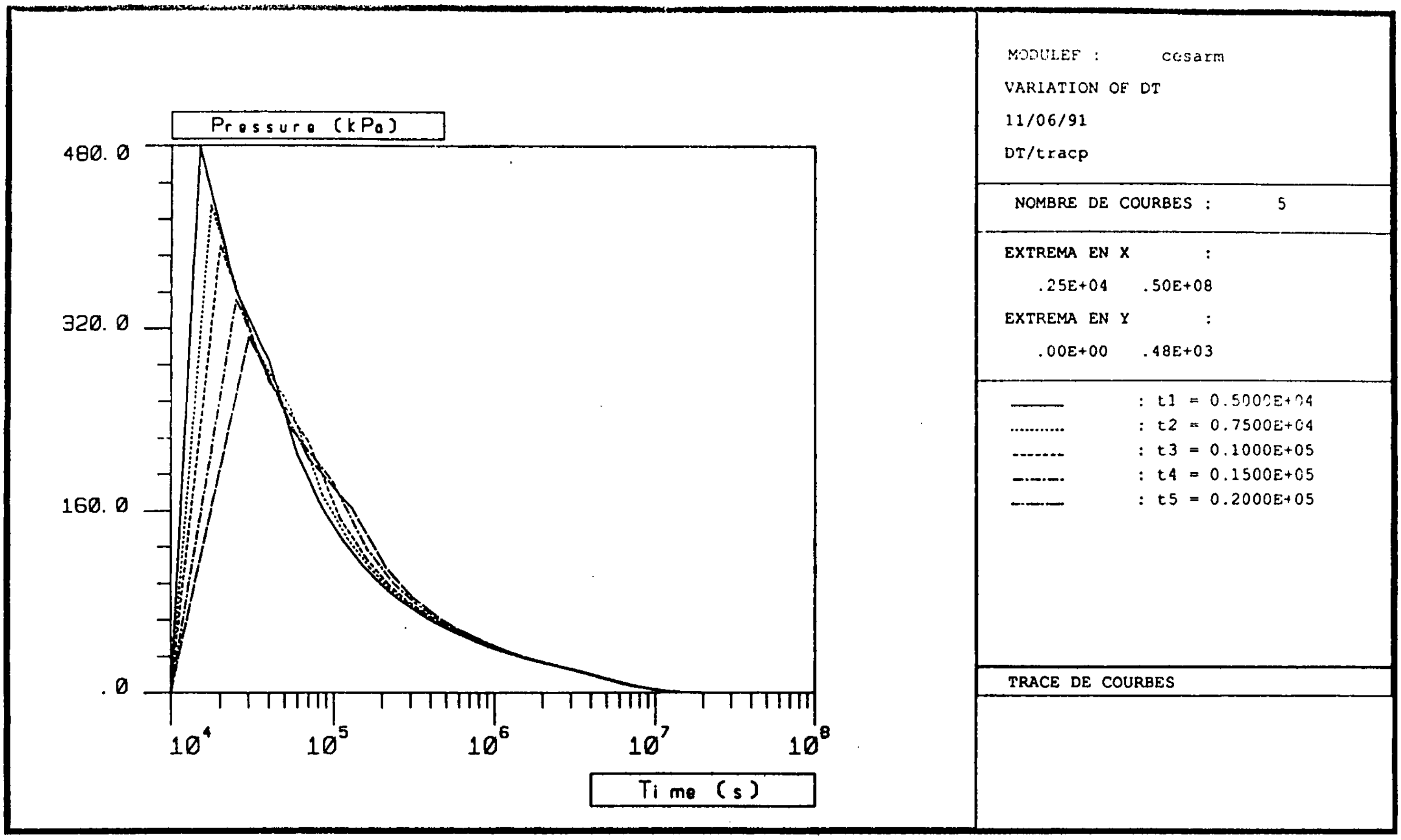
: t3 = 0.1000E+05

: t4 = 0.1500E+05

: t5 = 0.2000E+05

TRACE DE COURBES

GRAPH 3



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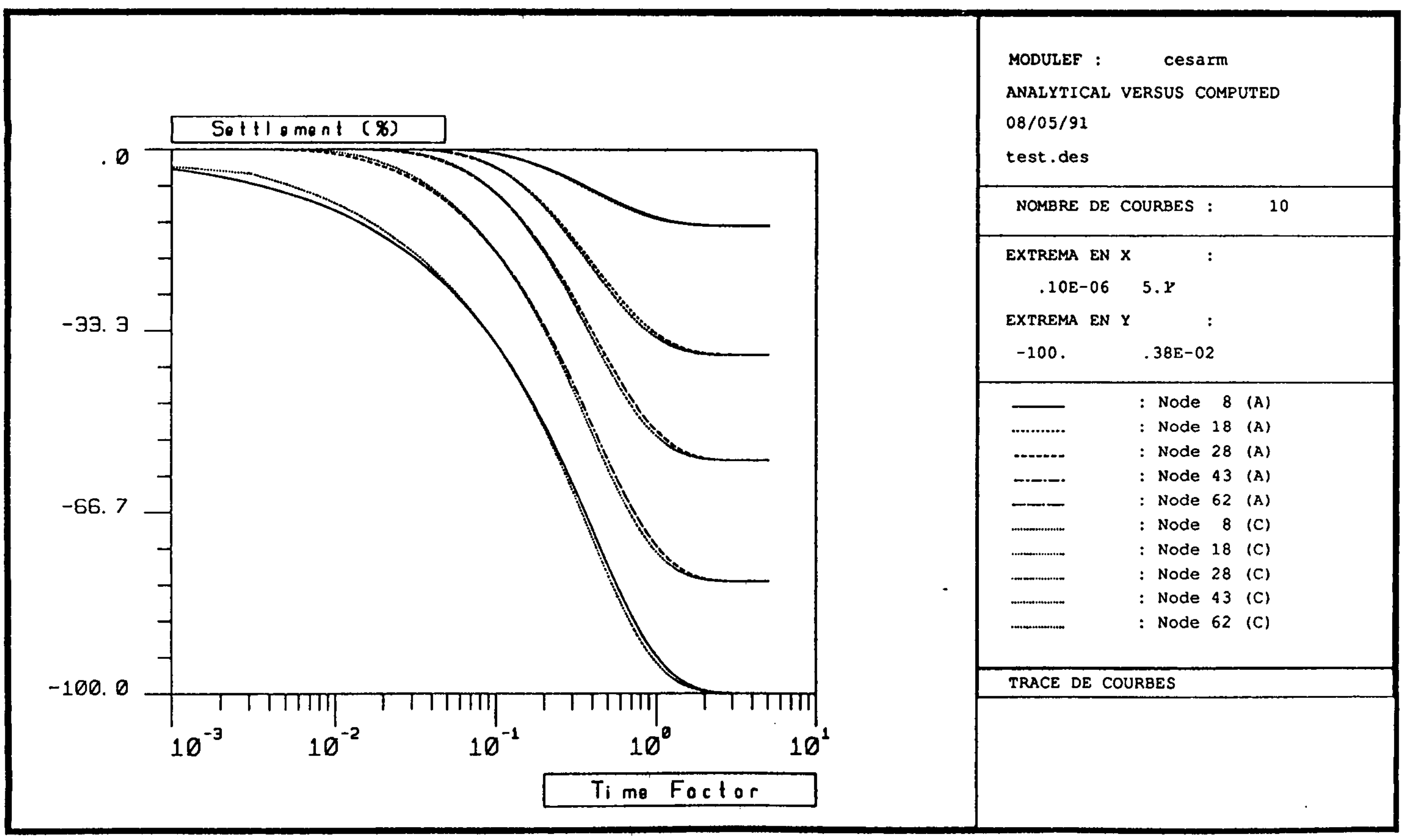

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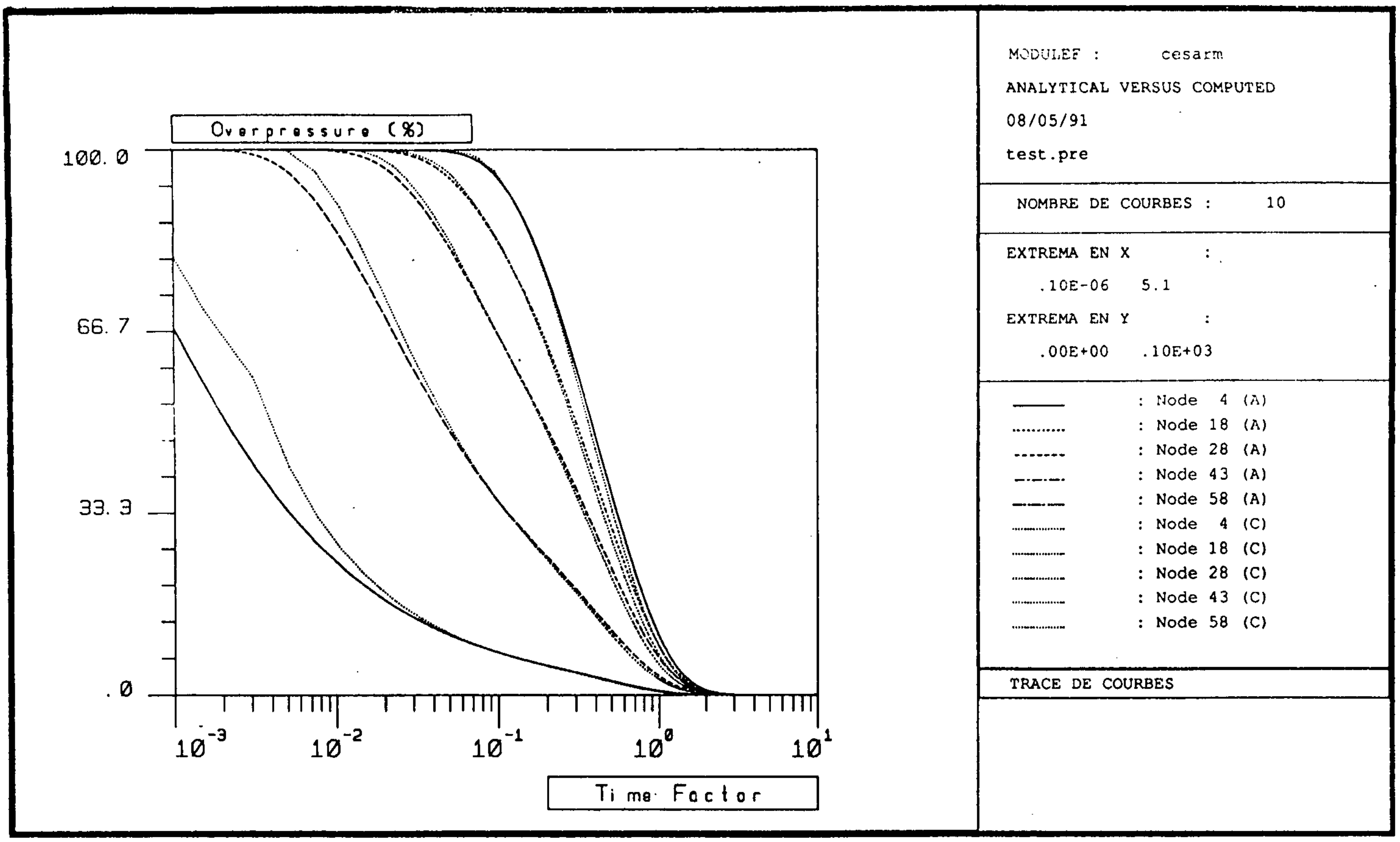

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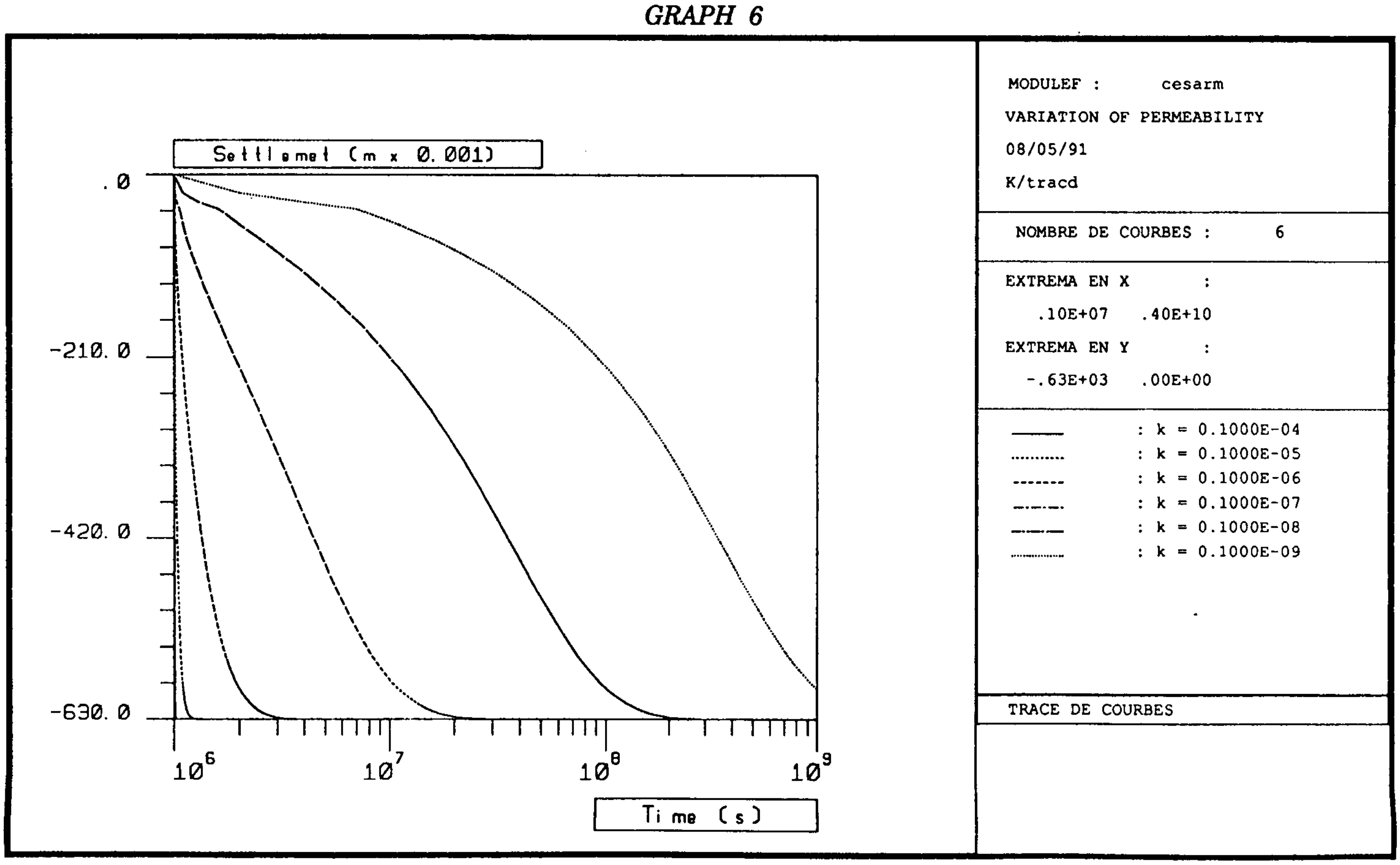

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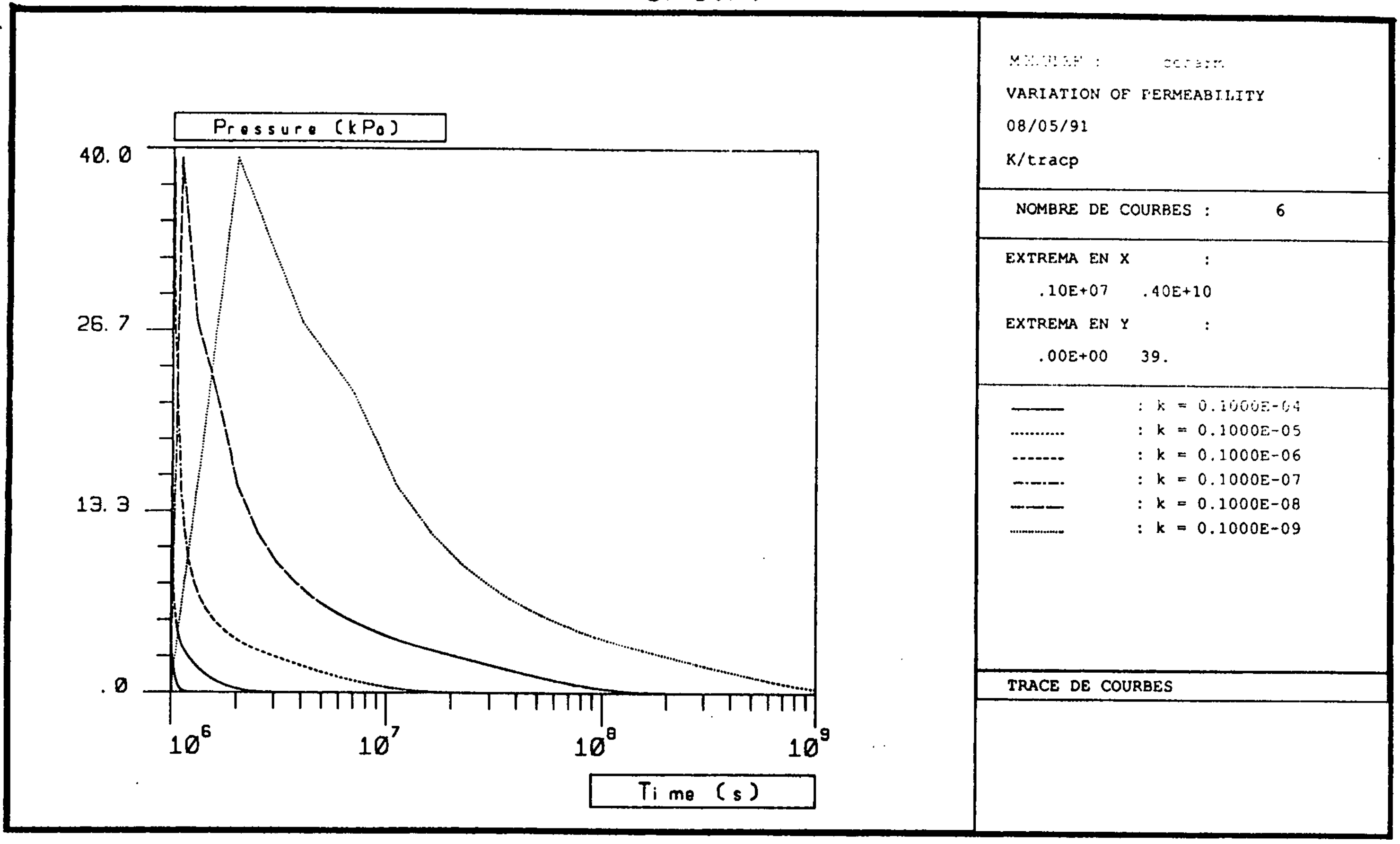
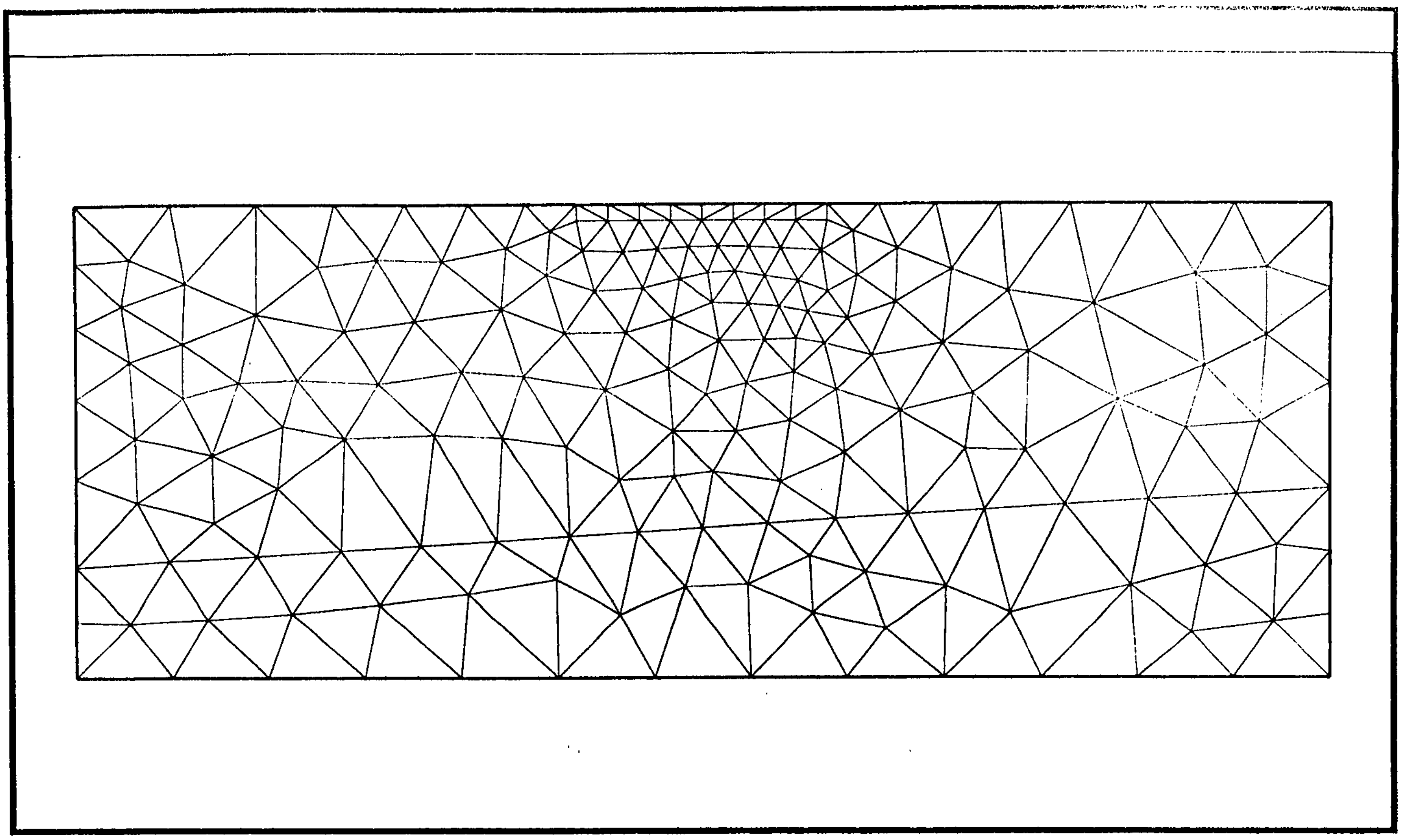

Pressure (kPa)
40.0
26.7
13.3
.0
10^6
10^7
10^8
10^9
Time (s)
VARIATION OF PERMEABILITY
08/05/91
K/tracp
NOMBRE DE COURBES : 6
EXTREMA EN X :
.10E+07 .40E+10
EXTREMA EN Y :
.00E+00 39.
: k = 0.1000E-04
: k = 0.1000E-05
: k = 0.1000E-06
: k = 0.1000E-07
: k = 0.1000E-08
: k = 0.1000E-09
TRACE DE COURBES

GRAPH 8



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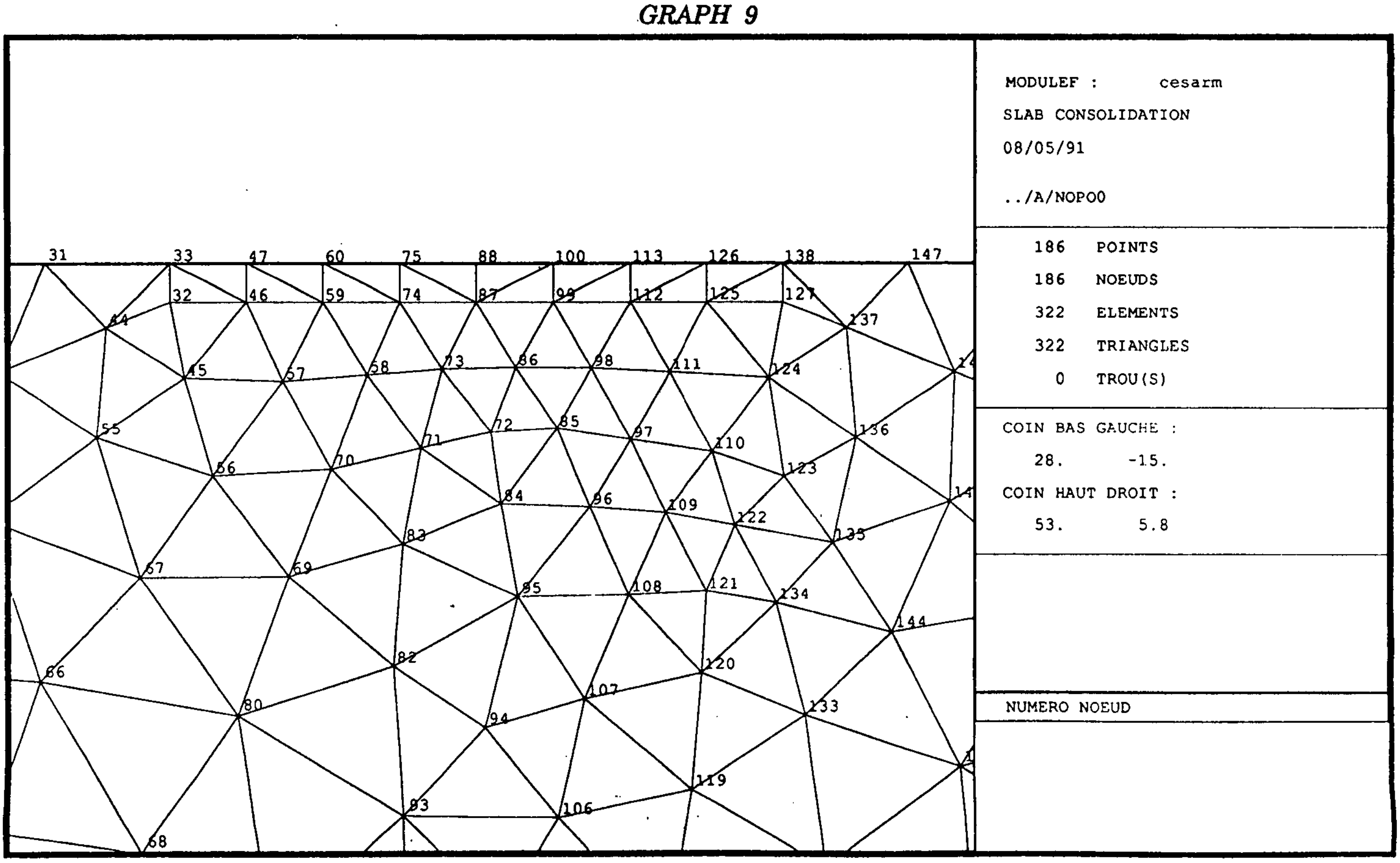
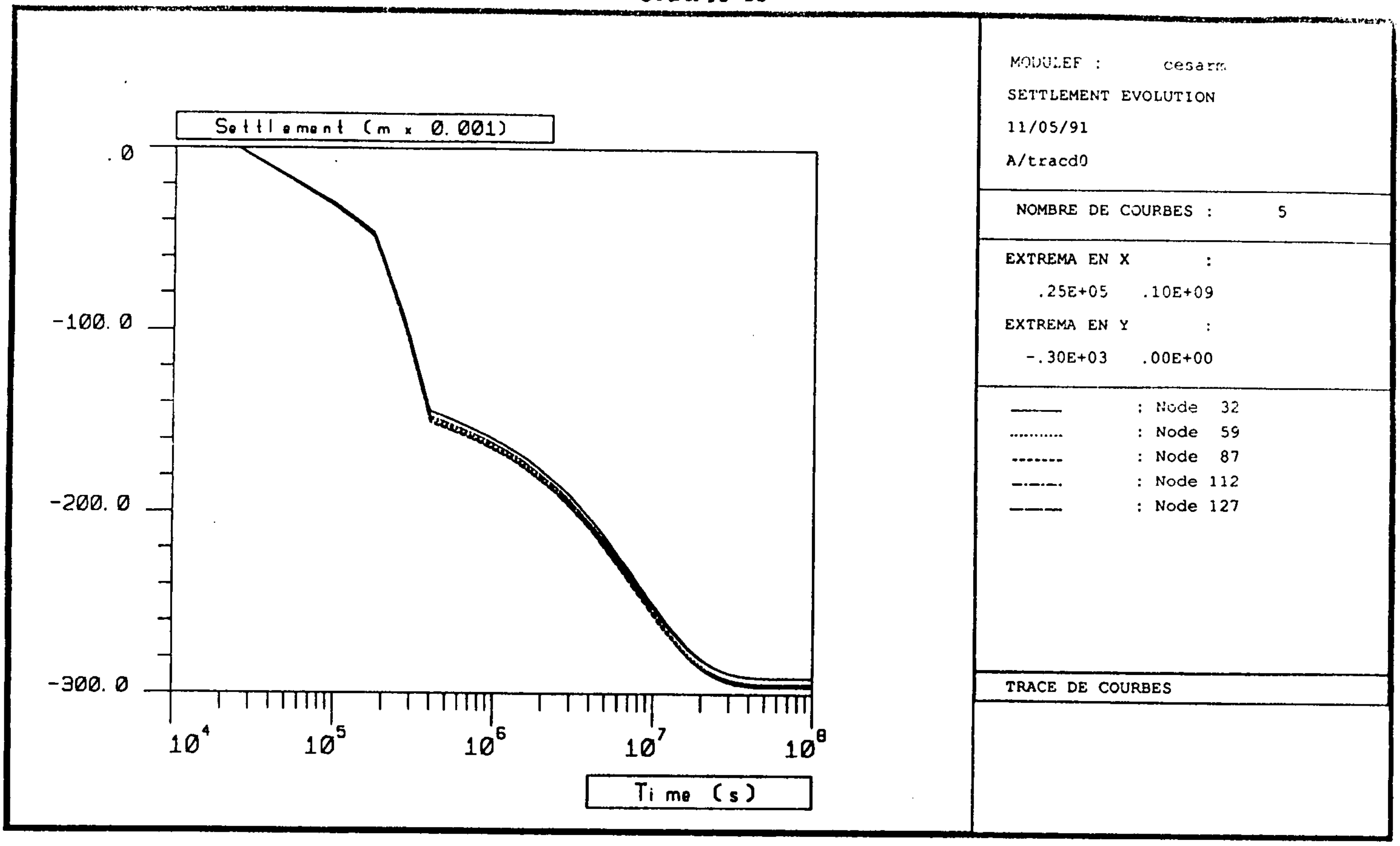


GRAPH 10



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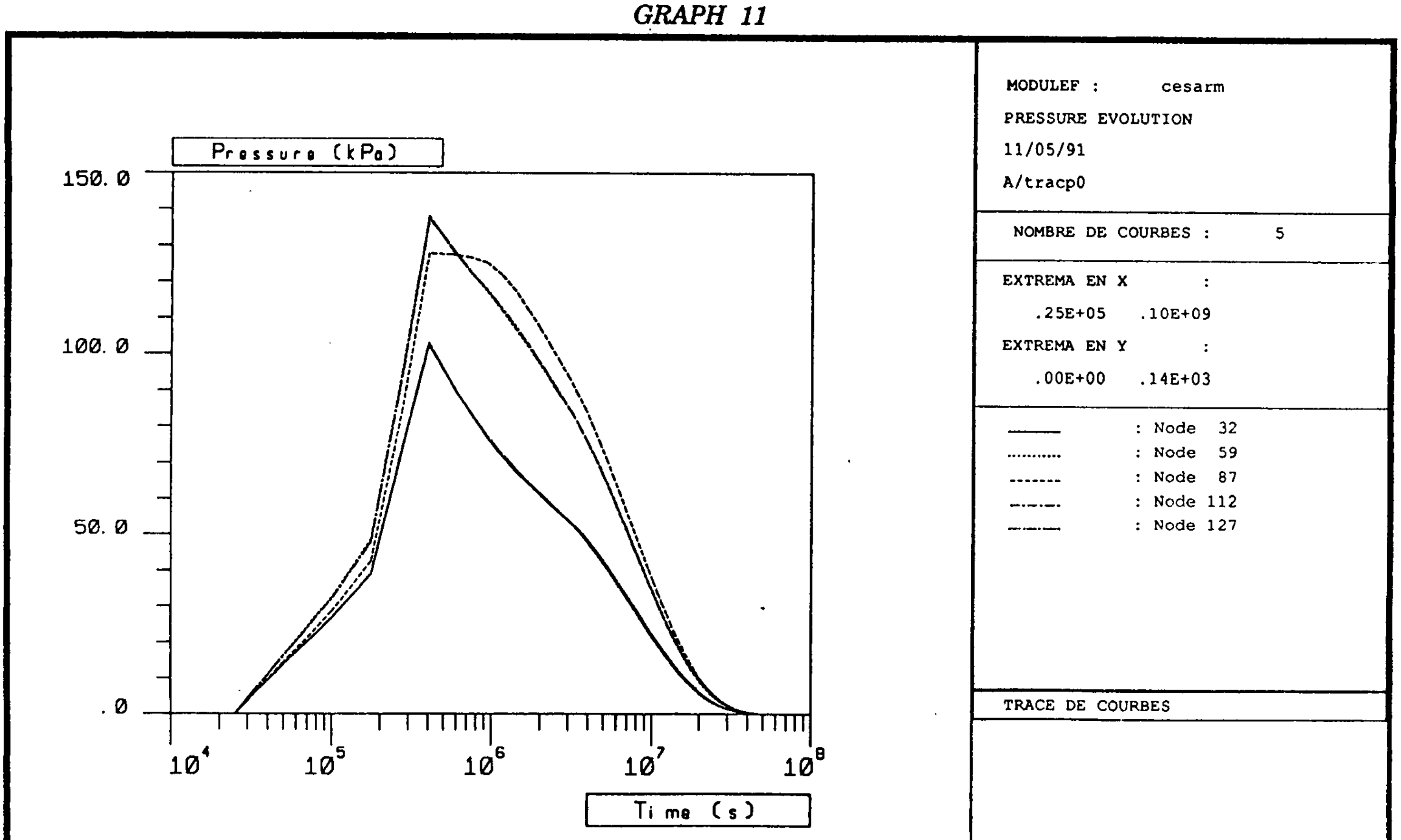
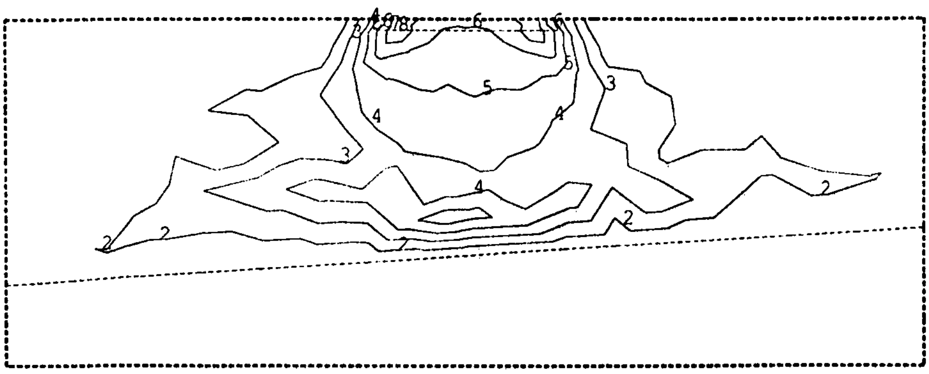


GRAPH 12



MODULEF : cesarm

11/05/91
A/MAILPO
A/COOR
A/BSPO

186 POINTS
186 NOEUDS
322 ELEMENTS
322 TRIANGLES

INCONNUE : 1 MNEMO :VN

8 ____ 2.049

7 ____ 1.756

6 ____ 1.463

5 ____ 1.171

4 ____ .8781

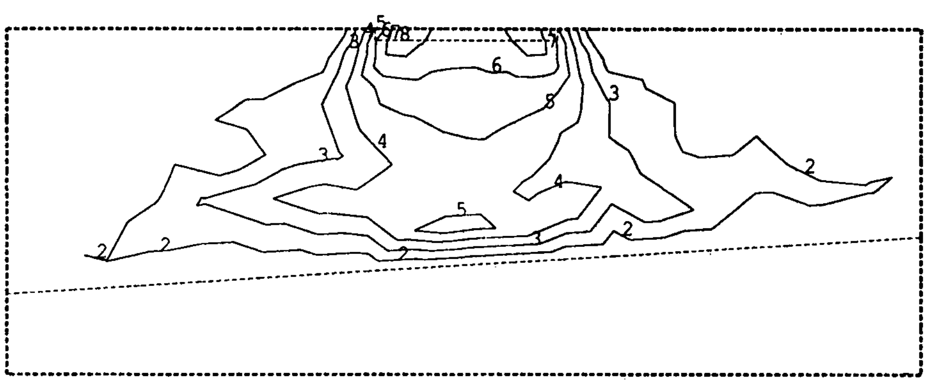
3 ____ .5854

2 ____ .2927

1 ____ .0000E+00

8 ISOVALEURS

GRAPH 13



MODULEF : cesarm

11/05/91
A/MAILPO
A/COOR
A/BSPO

186 POINTS
186 NOEUDS
322 ELEMENTS
322 TRIANGLES

INCONNUE : 1 MNEMO :VN

8 ____ 5.558

7 ____ 4.764

6 ____ 3.970

5 ____ 3.176

4 ____ 2.382

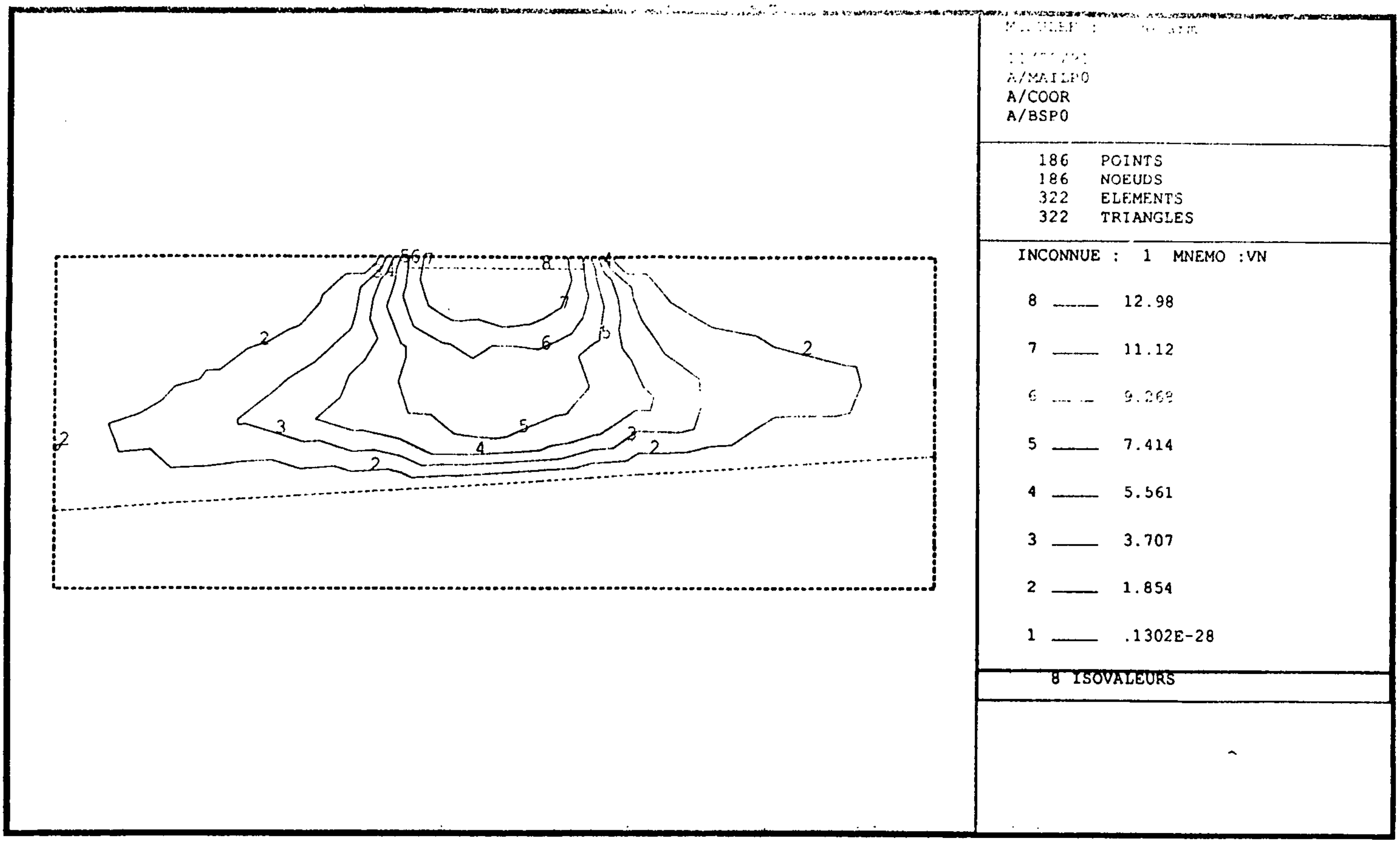
3 ____ 1.588

2 ____ .7940

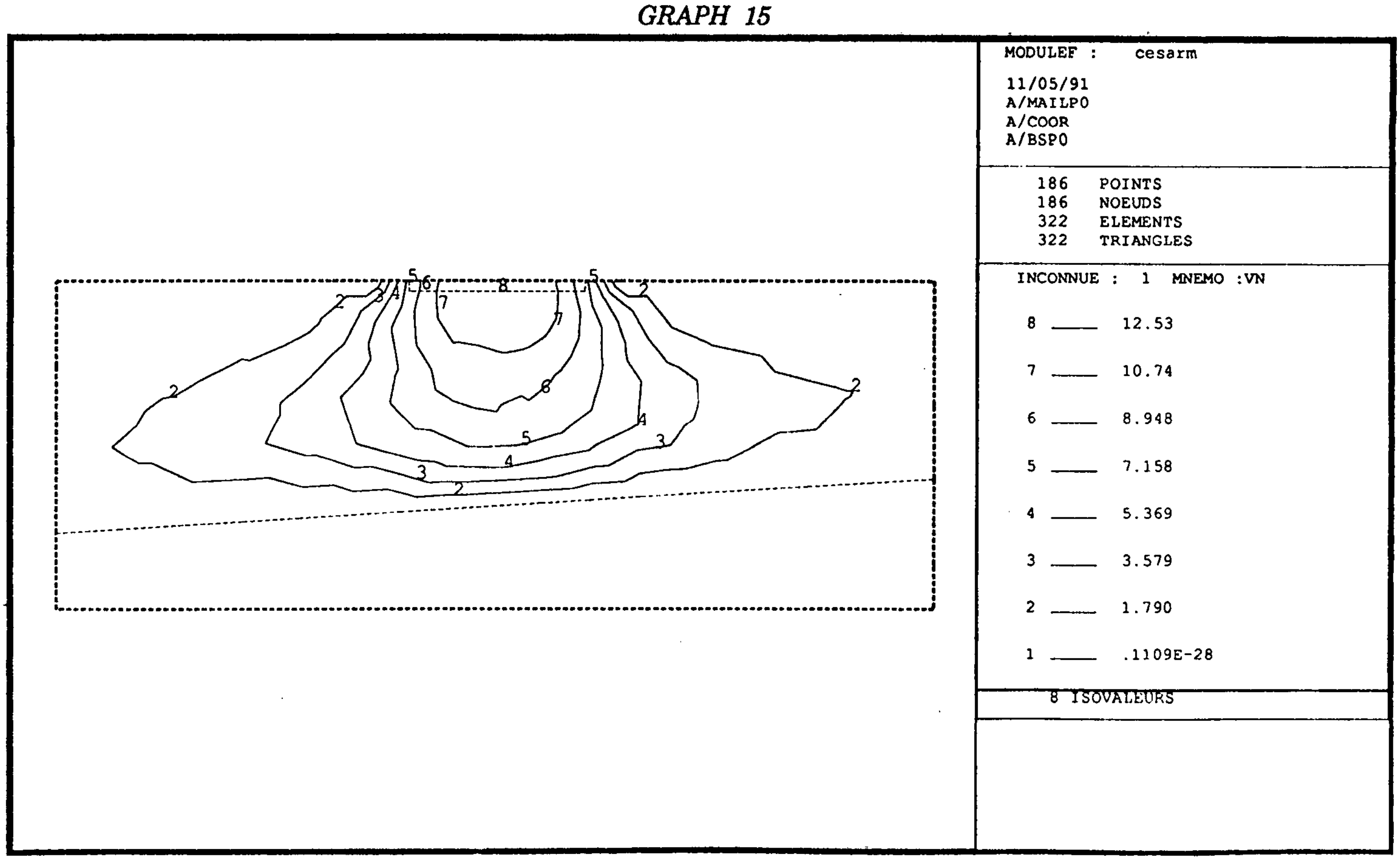
1 ____ .3946E-29

8 ISOVALEURS

GRAPH 14



GRAPH 15



Imprimé en France
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l'Institut National de Recherche en Informatique et en Automatique