Choosability of the square of planar subcubic graphs with large girth

Frédéric Havet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Choosability of the square of planar subcubic graphs with large girth

Frédéric Havet

N° 5800

Janvier 2006

Thème COM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5800--FR+ENG

Choosability of the square of planar subcubic graphs with large girth

Frédéric Havet*

Thème COM — Systèmes communicants
Projet Mascotte

Rapport de recherche n° 5800 — Janvier 2006 — 17 pages

Abstract: We first show that the choose number of the square of a subcubic graph with maximum average degree less than 18/7 is at most 6. As a corollary, we get that the choose number of the square of a planar graph with girth at least 9 is at most 6. We then show that the choose number of the square of a subcubic planar graph with girth at least 13 is at most 5.

Key-words: colouring, list colouring, planar graph, girth

* Partially supported by the Integrated Project IST-15964 *AEOLUS*

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Choisissabilité du carré des graphes planaires subcubiques de grande maille

Résumé : Nous montrons que le nombre de choix du carré d'un graphe subcubique de degré moyen maximum inférieur à 18/7 est au plus 6. Ceci implique que le carré d'un graphe planaire subcubique de maille au moins 9 est 6-choisissable. Ensuite, nous montrons que le nombre de choix du carré d'un graphe planaire subcubique de maille au moins 13 est au plus 5.

Mots-clés : coloration, coloration par listes, graphe planaire, maille

1 Introduction

Let G be a (simple) graph.

Let v be a vertex of G. Its *neighbourhood*, denoted $N_G(v)$ is the set of its *neighbours*, i.e. is vertices y such that xy is an edge. The *degree* of a vertex v in G, denoted $d_G(v)$, is its number of neighbours. Often, when the graph G is clearly understood from the context, we omit the subscript G.

A graph is *subcubic* if every vertex has degree at most 3.

Let $p : V(G) \rightarrow \mathbb{N}$. A *p-list-assignment* is a list-assignment L such that $L(v) = p(v)$ for any $v \in V(G)$. G is *p-choosable* if it is L-colourable for any p-list-assignment. By extension , if k is an integer, we say that G is *k-choosable* if it is p-choosable when p is the constant function with value k (i. e. $p(v) = k$ for all $v \in V$). The *choose number* of G is the smallest integer k such that G is k-choosable. Clearly the choose number of G is at least as large as $\chi(G)$ the *chromatic number* of G.

The *square* of G, is the graph G^2 with vertex set $V(G)$ such that two vertices are linked by an edge of G^2 if and only if x and y are at distance at most 2 in G (either xy is an edge or x and y share a neighbour). Formally, $E(G^2) = \{xy, (xy \in E(G) \text{ or } \exists z \in N(x) \cap N(y))\}$.

Wegner [4] proved that the square of a subcubic planar is 8-colourable. He also conjectured it is 7-colourable.

Conjecture 1 (Wegner [4]) *Let G be a subcubic planar graph. Then $\chi(G^2) \leq 7$.*

The *average degree* of G, denoted $Ad(G)$ is $\dfrac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \dfrac{2|E(G)|}{|V(G)|}$.

The *maximum average degree* of G, denoted $Mad(G)$, is $\max\{Ad(H), H \text{ subgraph of } G\}$.

In [1], Dvořák, Škrekovski and Tancer proved that the choose number of the square of a subcubic graph G is at most 4 if $Mad(G) < 24/11$ and G has no 5-cycle, at most 5 if $Mad(G) < 7/3$ and at most 6 if $Mad(G) < 5/2$.

A graph is called *planar* if it can be embedded in the plane. The *girth* of a graph is the smallest length of a cycle in G. Planar graph with presrcibed girth have bounded maximum average degree:

Proposition 2 *Every planar graph with girth at least g has maximum average degree less than $2 + \frac{4}{g-2}$.*

Hence the results of Dvořák, Škrekovski and Tancer imply that the choose number of the square of a planar graph with girth g is at most 6 if $g \geq 13$, at most 5 if $g \geq 14$ and at most 4 if $g \geq 24$. The two later results had been previously proved by Montassier and Raspaud [3].

In this paper, we improve some of these results. We first show (Theorem 4) that the choose number of the square of a subcubic graph with maximum average degree less than 18/7 is at most 6. As a corollary, we get that the choose number of the square of a planar graph with girth at least 9 is at most 6. We then show (Theorem 9) that the choose number of the square of a subcubic planar graph with girth at least 13 is at most 5.

2 The main results

The general frame of the proofs are classical. We consider a k*-minimal graph*, that is a subcubic graph such that its square is not k-choosable, but the square of every proper subgraph is k-choosable. We prove that some *configurations* (i.e. induced subgraphs) are forbidden in such a graph and then deduce a contradiction.

An i*-vertex* is a vertex of degree i. We denote by V_i the set of i-vertices of G and by v_i its cardinality. Let v be a vertex. An i*-neighbour* of v is a neighbour of v with degree i. The i*-neighbourhood* of v is $N_i(v) = N(v) \cap V_i$ and its i*-degree* is $d_i(v) = |N_i(v)|$.

Some properties of 6- and 5-minimal graphs have already been proved in [1]. The easy first one is that $V_0 \cup V_1 = \emptyset$, so G has minimum degree 2. This will allow us to use the following definitions for 6- and 5-minimal graphs.

Definition 3 Let G be a subcubic graph with minimum degree 2.

A *thread* of G is a maximal induced path of G. In other words, it is a a path whose endvertices are 3-vertices and whose internal vertices are 2-vertices.

The *kernel* of G is the weighted graph K such that $V(K) = V_3(G)$ and xy is an edge in K with weight l if and only if x and y are connected by a thread of length l in G. An edge of weight l is also called l*-edge*.

Let x be a 3-vertex of G. The *type* of x is the triple (l_1, l_2, l_3) such that $l_1 \leq l_2 \leq l_3$ and the three edges (A loop is counted twice.) incident to x have weight l_1, l_2 and l_3 in K. We denote by Y_{l_1,l_2,l_3} the set of 3-vertices of type (l_1, l_2, l_3) and y_{l_1,l_2,l_3} its cardinality. Moreover, for every integer i, we define $Z_i := \bigcup_{l_1+l_2+l_3=i} Y_{l_1,l_2,l_3}$ and $z_i = |Z_i|$.

The number of vertices and edges and thus the average degree of a subcubic graph G with minimum degree 2 may be easily expressed in terms of the z_i:

$$\begin{aligned}
|V(G)| &= \sum_{i\geq 3} \frac{i-1}{2} z_i \\
2|E(G)| &= \sum_{i\geq 3} i.z_i \\
Ad(G) &= \frac{\sum_{i\geq 3} i.z_i}{\sum_{i\geq 3} \frac{i-1}{2} z_i} \qquad (1)
\end{aligned}$$

2.1 6-choosability

The aim of this subsection is to prove the following result.

Theorem 4 *Let G be a subcubic graph of maximum average degree $d < 18/7$. Then G^2 is 6-choosable.*

In order to prove this theorem, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [1].

Lemma 5 (Dvořák, Škrekovski and Tancer [1]) *Let G be a 6-minimal graph. Then the following hold:*

1) all the edges of K have weight at most 2;

2) every 3-cycle of G has its vertices in V_3;

3) every 4-cycle of G has at least three vertices in V_3;

4) a vertex of $Y_{2,2,2}$ is not adjacent to a vertex of $Y_{1,2,2} \cup Y_{2,2,2}$.

We will prove in Subsection 3.2 some new properties.

Lemma 6 *Let G be a 6-minimal graph. Then the following hold:*

5) if $(v_1, v_2, v_3, v_4, v_1)$ is a 4-cycle with $v_2 \in V_2$ then v_1 or v_3 is not in Z_5;

6) a vertex of $Y_{1,2,2}$ is adjacent to at most one vertex of $Y_{1,2,2}$ by 2-edges.

Proof of Theorem 4. Let G be a 6-minimal planar graph. G has minimum degree 2, so its kernel K is defined. Moreover by Lemma 5-1), Z_i is empty for $i \geq 7$ and $Z_6 = Y_{2,2,2}$ and $W_5 = Y_{1,2,2}$.

Let us consider a vertex of $Z_4 = Y_{1,1,2}$. Its neighbour via the 2-edge is in $Z_4 \cup Z_5 \cup Z_6$ because a vertex of $Z_3 = Y_{1,1,1}$ is incident to no edge of weight 2.

For $i = 4, 5, 6$, let Z_4^i be the set of vertices of Z_4 which are incident to a vertex of Z_i by their unique 2-edge and z_4^i its cardinality. (Z_4^4, Z_4^5, Z_4^6) is a partition of Z_4 so $z_4 = z_4^4 + z_4^5 + z_4^6$.

Hence Equation (1) becomes

$$Ad(G) = \frac{6z_6 + 5z_5 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{\frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4^6 + \frac{5}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma 5-4), the three neighbours of a vertex of Z_6 is not in $Z_6 \cup Z_5$. So they must be in Z_4^6. It follows that $3z_6 = z_4'$. So

$$Ad(G) = \frac{5z_5 + 6z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{2z_5 + \frac{7}{3}z_4^6 + \frac{5}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma 6-6), a vertex of Z_5 is adjacent to at least one vertex of Z_4^5. Thus $z_5 \leq z_4^5$.

But $Ad(G)$ is decereasing as a function of z^5 since z_4^6, z_4^5, z_4^4 and z_3 are non-negative. It follows that

$$Ad(G) \geq \frac{6z_4^6 + 9z_4^5 + 4z_4^4 + 3z_3}{\frac{7}{3}z_4^6 + \frac{7}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3} \geq \frac{18}{7}.$$

□

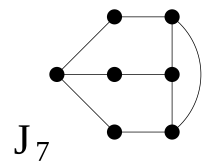

Figure 1: The graph J_7

Remark 7 Theorem 4 is tight. Indeed, the graph J_7 depicted Figure 1 has average degree 18/7 and its square is the complete graph on seven vertices K_7 which is not 6-choosable (nor 6-colourable).

Theorem 4 and Proposition 2 yield that the square of a planar graph with girth 9 is 6-choosable.

Corollary 8 *The square of a planar graph with girth* 9 *is* 6*-choosable.*

2.2 5-choosability

Dvořák, Škrekovski and Tancer [1] proved that the square of a subcubic graph G with maximum average degree less than 7/3 is 5-choosable. This result is tight since the graph J_6 depicted Figure 2 has average degree 7/3 and its square is the complete graph on six vertices K_6 which is not 5-choosable (nor 5-colourable).

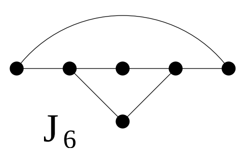

Figure 2: The graph J_6

However, we will prove that the square of a planar graph with girth at least 13 is 5-choosable, which improves the result of Montassier and Raspaud.

Theorem 9 *The square of a planar graph with girth* 13 *is* 5*-choosable.*

In order to prove this theorem, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [1].

Lemma 10 (Dvořák, Škrekovski and Tancer [1]) *Let G be a* 5*-minimal graph of girth* 13*. Then the following hold:*

1) all the edges of K have weight at most 3;

2) if $i \geq 8$, Z_8 is empty.

We will prove in Subsection 3.3 some new properties.

Lemma 11 *Let K be the G be a 5-minimal graph of girth at least 13. Then the following holds:*

3) A vertex of $Y_{2,2,3}$ and a vertex of $Y_{1,2,3} \cup Y_{2,2,3}$ are not linked by a 2-edge.

4) A vertex of $Y_{1,3,3}$ and a vertex of $Y_{1,2,3} \cup Y_{1,3,3}$ are not linked by a 1-edge.

5) A vertex of $Y_{2,2,2}$ is not adjacent to three vertices of $Y_{2,2,3}$ (by 2-edges).

Proof of Theorem 9. Let G be a 5-minimal planar graph with girth at least 13. G has minimum degree 2, so its kernel K is defined. Moreover, by Lemma 10-1), $Z_7 = Y_{2,2,3} \cup Y_{1,3,3}$, so

$$z_7 = y_{2,2,3} + y_{1,3,3}. \tag{2}$$

Let us count the number e_2 of 2-edges incident to vertices of $Y_{2,2,3}$. Since 2-edges may not link two vertices of type $(2,2,3)$ according to Lemma 11-3), we have $e_2 = 2y_{2,2,3}$. Moreover, the end of such edges which is not in $Y_{2,2,3}$ has to be in $Y_{2,2,2} \cup Y_{1,2,2} \cup Z_4$ by Lemmas 10 and 11-3). Furthermore, a vertex of $Y_{2,2,2}$ is incident to at most two edges of e_2 according to Lemma 11-5) and a vertex of $Y_{1,2,2}$ (resp. Z_4) is incident to at most two (resp. one) 2-edges. Therefore $e_2 \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4$. So,

$$2y_{2,2,3} \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4 \tag{3}$$

Let us now count the number e_1 of 1-edges incident to vertices of $Y_{1,3,3}$. Since 1-edges may not link two vertices of type $(1,3,3)$ according to Lemma 11-4), we have $e_1 = y_{1,3,3}$. Moreover, the end of such edges which is not in $Y_{2,2,3}$ has to be in $Y_{1,2,2} \cup Y_{1,1,3} \cup Z_4 \cup Z_3$ by Lemmas 10 and 11-4). Furthermore, vertices of $Y_{1,2,2}$ (resp. $Y_{1,1,3} \cup Z_4$, Z_3) are incident to at most one (resp. two, three) 1-edges. Thus $e_1 \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3$. So,

$$y_{1,3,3} \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3 \tag{4}$$

$2 \times (4) + (3)$ yields $2y_{2,2,3} + 2y_{1,3,3} \leq 2y_{2,2,2} + 4y_{1,2,2} + 4y_{1,1,3} + 5z_4 + 6z_3$. Hence by Equation 2, $2z_7 \leq 2z_6 + 4z_5 + 5z_4 + 6z_3$, so

$$z_7 \leq z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3.$$

Now by Equation 1 the average degree of G is

$$Ad(G) = \frac{7z_7 + 6z_6 + 5z_5 + 4z_4 + 3z_3}{3z_7 + \frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4 + z_3}.$$

As a function of z_7 this is a decreasing function (on $\mathbb{R}^+$), so it is minimum when z_7 is maximum that is equal to $z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3$. So, :

$$Ad(G) \geq \frac{13z_6 + 19z_5 + \frac{43}{2}z_4 + 24z_3}{\frac{11}{2}z_6 + 8z_5 + 9z_4 + 10z_3} \geq \frac{26}{11}.$$

This contradicts the fact that G has girth 13 by Proposition 2. □

It is very likely that using the method below, one can prove that a graph G with maximum average degree less than $\frac{26}{11}$ is 5-choosable unless it contains J_6 as an induced subgraph. However, this will require the tedious study of a large number of configurations.

3 Proofs of Lemmas 6 and 11

In order, to prove Lemmas 6 and 11, we need the following lemma proved in [1]. Let S be a set of vertices of a k-minimal graph G. The function $p_S : S \to N$ is defined by $p_S(v) = k - |N_{G^2}(v) \setminus S|$. Then $p_S(v)$ represents the minimum number of available colours at a vertex $v \in S$ once we have precoloured the square of $G - S$. Hence if $(G-S)^2$ is 6-choosable, $(G-S)^2 = G^2 - S$ (in particular, it happens when two distinct vertices of $G - S$ have no common neighbour in S) and $G[S]^2$ is p_S-choosable, one can extend any k-list-colouring of $G - H$ into a k-list-colouring of G.

Lemma 12 (Dvořák, Škrekovski and Tancer [1]) *Let S be a set of vertices of a k-minimal graph G. If $(G-S)^2 = G^2 - S$, then $G^2[S]$ is not p_S-choosable.*

In order to use Lemma 12, we need some results on the choosability of some graphs.

3.1 Some choosability tools

Definition 13 Let x and y be two vertices of a graph G. An $(x-y)$-*ordering* of G is an ordering such that x is the minimum and y the maximum. An $(x, y-z)$-*ordering* is an ordering such that x is minimum, y is the second minimum and z is maximum.

Let $\sigma = (v_1 < v_2, \ldots < v_n)$ be an ordering of the vertices of G. σ is p-*greedy* if for every i, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| < p(v_i)$. It is p-*nice* if for every i except n, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| < p(v_i)$ and $d(v_n) = p(v_n)$. It is p-*good* if for every $3 \leq i \leq n$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| - \epsilon(v_i) < p(v_i)$ with $\epsilon(v_i) = 1$ if v_i is adjacent to both v_1 and v_2 and $\epsilon(v_i) = 0$ otherwise.

The greedy algorithm according to greedy, nice and good orderings yields the following three lemmas.

Lemma 14 *If G has a p-greedy ordering then G is p-choosable.*

Proof. Applying the greedy algorithm according to the p-greedy ordering gives the desired colouring. □

Lemma 15 *Let xy be an edge of graph G and L be a p-list-assignment of G. If $L(x) \not\subset L(y)$ and G has a p-nice $(x-y)$-ordering, then G is L-colourable.*

Proof. Let a be a colour in $L(x) \setminus L(y)$. Proceed the greedy algorithm starting by assigning a to x. The only vertex which has not more colour in its list than previously coloured neighbours is y for which $|L(y)| = d(y)$. But since $a \notin L(y)$, at most $d(y)-1$ colours of $L(y)$ are assigned to the neighbours of y. Hence one can colour y. □

Lemma 16 *Let x, y and z be three vertices of a graph $G = (V, E)$ such that $xy \notin E$, $xz, yz \in E$. If $L(x) \cap L(y) \neq \emptyset$ and G has a p-good $(x, y-z)$-ordering, then G is L-colourable.*

Proof. Let a be a colour in $L(x) \cap L(y)$ and $\sigma = (v_1 < v_2, \ldots < v_n)$ be a p-good $(x, y-z)$-ordering. (In particular, $v_1 = x$, $v_2 = y$ and $v_n = z$.) Proceed the greedy algorithm according to σ starting by assigning a to x and y. For every $3 \leq i \leq n$, the number of colours assigned to already coloured neighbours of v_i is at most $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| - \epsilon(v_i)$ since v_1 and v_2 are coloured the same. Hence the greedy algorithm gives an L-colouring. □

Remark 17 Note that under the condition $xz, yz \in E$, a p-nice ordering is also a p-good ordering.

Definition 18 The *blocks* of a graph are its maximal 2-connected components. A connected graph is said to be a *Gallai tree* if each of its blocks are either complete graphs or odd cycles.

Theorem 19 (trouver si c'est Lovasz, Borodin, erdos et al) *Let G be a connected graph. Then G is d-choosable if and only if G is not a Gallai tree.*

Lemma 20 *Let $G = (V, E)$ be a graph and $p : V(G) \rightarrow \mathbb{N}$. Let S be a set of vertices such that $p(v) \geq d(v)$ for all $v \in S$. If $G[S]$ is not a Gallai tree and $G - S$ is p-choosable then G is p-choosable.*

Proof. Let L be a p-list-assignment of G. Since $G - S$ is p-choosable, its admits a p-colouring c. Let us now extend it to S. The list $I(v) = L(v) \setminus \{c(w), w \in N(v) \setminus S\}$ of available colours of a vertex $v \in S$ is of size at least $p'(v) = p(v) - |N(v) \setminus S| \geq d_{G[S]}(v)$. Since $G[S]$ is not a Gallai tree, by Theorem 19, $G[S]$ is p'-choosable and thus I-colourable. So, G is L-colourable. □

A 4-regular graph is *cycle+triangles* if it is the edge union of a Hamiltonian cycle and triangles.

Theorem 21 (Fleishner and Stiebitz [2]) *Every cycle+triangles graph is 3-choosable.*

3.2 Proof of Lemma 6

Lemma 22 *Let $q \geq 4$ and $C_{2q} = (v_1, \dots, v_{2q}, v_1)$ be the $2q$-cycle and p defined by $p(v_i) = 4$ if i is odd and $p(v_i) = 3$ otherwise. Then C_{2q}^2 is p-choosable.*

Proof. The set S of vertices v for which $p(v) \geq d_{C_{2q}^2}(v)$ is the set of v_i with odd indices. $C_{2q}^2[S]$ is a q-cycle and thus is not a Gallai tree if $q \geq 4$. Moreover $C_{2q}^2 - S$ is also a q-cycle and is 3-choosable. Hence Lemma 20 gives the result. □

Proposition 23 *Let $P_7 = (v_1, \dots, v_7)$ be a path and p the function defined by $p(v_1) = p(v_2) = p(v_6) = p(v_7) = 2$, $p(v_3) = p(v_5) = 4$ and $p(v_4) = 3$. Then P_7^2 is p-choosable.*

Proof. Since $(v_2 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is p-nice, by Lemma 15, we may assume that $L(v_1) = L(v_2)$, and by symmetry of P_7 and p that $L(v_6) = L(v_7)$.

Since $(v_1 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$ is p-good, by Lemma 16, we may assume that $L(v_1) \cap L(v_4) = \emptyset$, and by symmetry $L(v_7) \cap L(v_4) = \emptyset$.

Now one can find $c(v_1) \in L(v_1)$, $c(v_2)$ in $L(v_2) \setminus \{c(v_1)\}$, $c(v_6)$ in $L(v_6)$, $c(v_7)$ in $L(v_7) \setminus \{c(v_6)\}$, $c(v_3)$ in $L(v_3) \setminus \{c(v_1), c(v_2)\}$, and $c(v_5)$ in $L(v_5) \setminus \{c(v_3), c(v_6), c(v_7)\}$. Now since $L(v_1) \cap L(v_4) = \emptyset$ and $L(v_1) = L(v_2)$, $c(v_2) \notin L(v_4)$. Analogously, $c(v_6) \notin L(v_4)$. Hence, $= L(v_4) \setminus \{c(v_2), c(v_3), c(v_5), c(v_6)\} = L(v_4) \setminus \{c(v_3), c(v_5)\} \neq \emptyset$. So, one can choose $c(v_4)$ in this set to get an L-colouring c of P_7^2. □

Lemma 24 *For $1 \leq i \leq 17$, let F_i be the graphs and p_i be the function depicted Figure 3.*

a) $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$ *is* p_1*-choosable.*

b) $F_2^2 \cup \{v_1v_4\}$ *and* $F_2^2 \cup \{v_4v_7\}$ *are* p_2*-choosable.*

c) $F_3^2 \cup \{v_4v_8\}$ *is* p_3*-choosable.*

d) F_4^2 *is* 6*-choosable.*

e) $F_5^2 \cup \{v_1v_5\}$ *is* p_5*-choosable.*

f) $F_6^2 \cup \{v_1v_4\}$ *and* $F_6^2 \cup \{v_4v_7\}$ *are* p_6*-choosable.*

g) $F_7^2 \cup \{v_4v_8\}$ *is* p_7*-choosable.*

h) F_8^2 *is* 6*-choosable.*

i) F_9^2 *is* p_9*-choosable.*

j) $F_{10}^2 \cup \{v_9v_{10}\}$ *is* p_{10}*-choosable.*

k) F_{11}^2 *is* p_{11}*-choosable.*

l) $F_{12}^2 \cup \{v_2v_9\}$ *and* $F_{12}^2 \cup \{v_6v_9\}$ *are* p_{12}*-choosable.*

m) $F_{13}^2 \cup \{v_4v_8\}$ *is* p_{13}*-choosable.*

n) $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ *is* p_{14}*-choosable.*

o) $F_{15}^2 \cup \{v_4v_8\}$ *is* p_{15}*-choosable.*

p) F_{16}^2 *is* 6*-choosable.*

q) F_{17}^2 *is* p_{17}*-choosable.*

Proof.

a) In $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$, $(v_6 < v_5 < v_4 < v_3 < v_1 < v_2)$ is p_1-greedy. So by Lemma 14, $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$ is p_1-choosable.

b) In $F_2^2 \cup \{v_4v_7\}$, $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_2-nice and $p_2(v_2) > p_2(v_1)$. So by Lemma 15, $F_2^2 \cup \{v_4v_7\}$ is p_2-choosable.

By symmetry, one shows that $F_2^2 \cup \{v_1v_4\}$ is p_2-choosable.

c) In $F_3^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_3-nice and $p_3(v_2) > p_3(v_1)$. So by Lemma 15, $F_3^2 \cup \{v_4v_7\}$ is p_3-choosable.

d) Let L be a 6-list-assignment of F_4^2. $(v_3 < v_2 < v_4 < v_6 < v_8 < v_5 < v_7 < v_1)$ is 6-nice. Thus, by Lemma 15, we may assume that $L(v_3) = L(v_1)$. Now, $(v_5 < v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_3)$ is 6-nice. Hence F_4^2 is L-colourable according to Lemma 16 if $L(v_5) \cap L(v_1) \neq \emptyset$ or Lemma 15 otherwise.

e) In $F_5^2 \cup \{v_1v_5\}$, $(v_1 < v_5 < v_2 < v_4 < v_3)$ is p_5-greedy. So by Lemma 14, $F_5^2 \cup \{v_1v_5\}$ is p_5-choosable.

f) In $F_6^2 \cup \{v_4v_7\}$, $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_6-nice and $p_6(v_2) > p_6(v_1)$. So by Lemma 15, $F_6^2 \cup \{v_4v_7\}$ is p_6-choosable.

By symmetry, one shows that $F_6^2 \cup \{v_1v_4\}$ is p_6-choosable.

g) In $F_7^2 \cup \{v_4v_8\}$. $(v_2 < v_4 < v_8 < v_5 < v_7 < v_6 < v_3 < v_1)$ is p_7-nice and $p_7(v_2) > p_7(v_1)$. So by Lemma 15, $F_7^2 \cup \{v_4v_8\}$ is p_7-choosable.

h) Let L be a p_8-list-assignment of F_8^2, $(v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_5 < v_3)$ is p_8-greedy, so by Lemma 15, we may assume that $L(v_1) = L(v_3)$. Now $(v_3 < v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_5)$, so by Lemma 16, F_8^2 is L-colourable.

i) In F_9^2, $(v_4 < v_2 < v_8 < v_1 < v_3 < v_5)$ is p_9-nice and $p_9(v_4) > p_9(v_5)$. So by Lemma 15, F_9^2 is p_9-choosable.

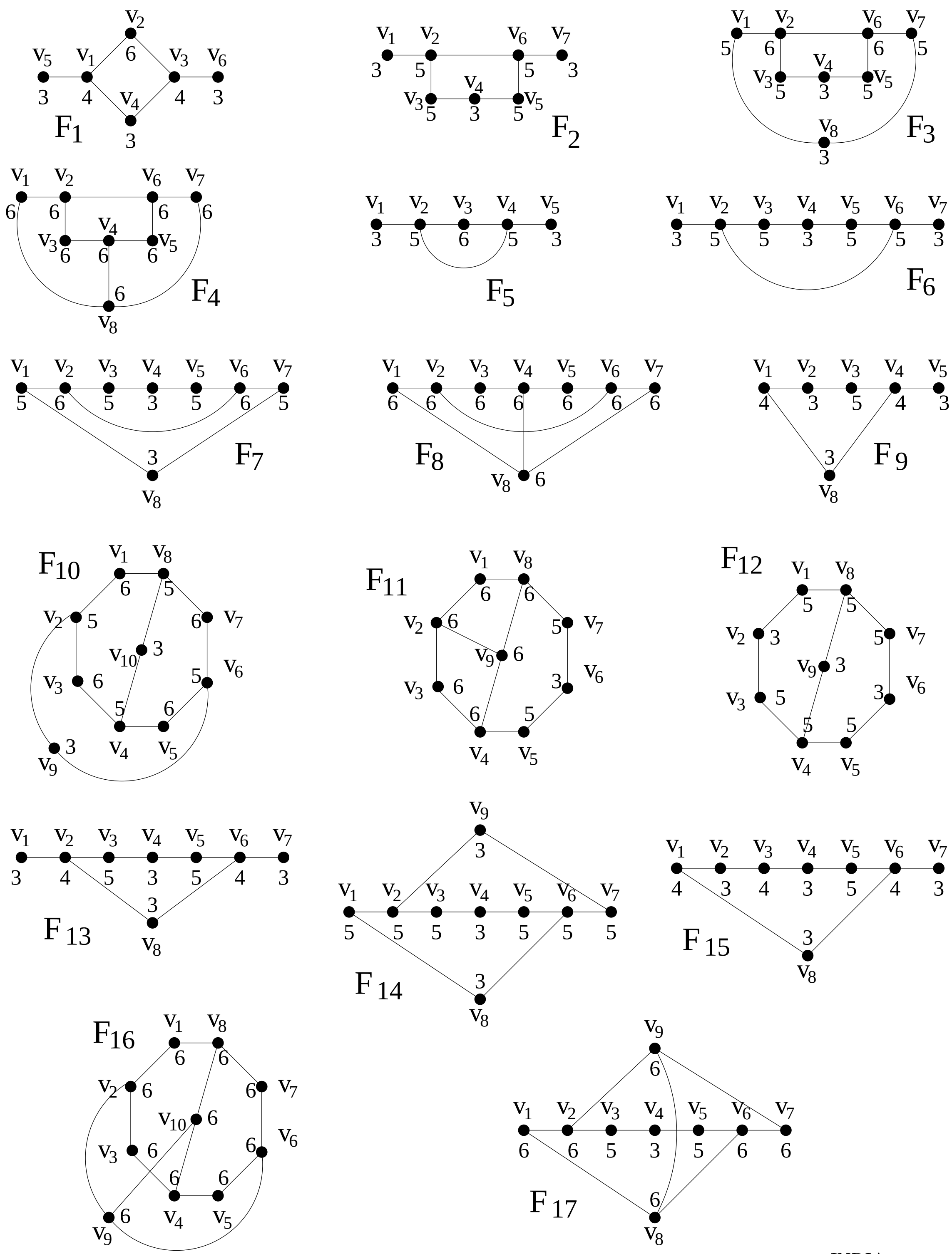

Figure 3: The graphs F_i and functions p_i for $1 \leq i \leq 12$

j) Let L be a p_{10}-list-assignment of $F_{10}^2 \cup \{v_9v_{10}\}$. $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$ is p_{10}-nice. Thus, by Lemma 15, we may assume that $L(v_2) \subset L(v_1)$. Analogously, by symmetry, we may assume that $L(v_2) \subset L(v_3)$ and $L(v_4) \subset L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is p_{10}-good, by Lemma 16, $F_{10}^2 \cup \{v_9v_{10}\}$ is L-colourable.

k) In F_{11}^2, $(v_6 < v_5 < v_7 < v_9 < v_8 < v_4 < v_3 < v_2 < v_1)$ is p_{11}-greedy. So by Lemma 14, F_{11}^2 is p_{11}-choosable.

l) Let L be a p_{12}-list-assignment of $F_{12}^2 \cup \{v_2v_9\}$. Then $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_3 < v_1)$ and $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_1 < v_3)$ are p_{12}-nice so by Lemma 15, we may assume that $L(v_2) \subset L(v_3) \cap L(v_1)$. Moreover, $(v_4 < v_2 < v_9 < v_6 < v_8 < v_7 < v_5 < v_1 < v_3)$ is p_{12}-nice so by Lemma 15, we may assume that $L(v_4) \subset L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Thus, by Lemma 16, since $(v_1 < v_4 < v_2 < v_9 < v_8 < v_6 < v_7 < v_5 < v_3)$ is p_{12}-good, $F_{12}^2 \cup \{v_2v_9\}$ is L-colourable.

By symmetry, one shows that $F_{12}^2 \cup \{v_6v_9\}$ is p_{12}-choosable.

m) In $F_{13}^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is p_{13}-nice and $p_{13}(v_2) > p_{13}(v_1)$. So by Lemma 15, $F_{13}^2 \cup \{v_4v_8\}$ is p_{13}-colourable.

n) Let L be a p_{14}-list-assignment of $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ $(v_1 < v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$ is p_{14}-nice. So by Lemma 15, $L(v_1) = L(v_3)$. Now, $(v_4 < v_1 < v_8 < v_9 < v_2 < v_6 < v_7 < v_5 < v_3)$ is p_{14}-nice. Hence $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ is L-colourable according to Lemma 16 if $L(v_4) \cap L(v_1) \neq \emptyset$ or Lemma 15 otherwise.

o) In $F_{15}^2 \cup \{v_4v_8\}$. $(v_6 < v_8 < v_4 < v_2 < v_1 < v_3 < v_5 < v_7)$ is p_{15}-nice and $p_{15}(v_6) > p_{15}(v_7)$. So by Lemma 15, $F_{15}^2 \cup \{v_4v_8\}$ is p_{15}-choosable.

p) Let L be a 6-list-assignment of F_{16}^2. $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$ is 6-nice. Thus, by Lemma 15, we may assume that $L(v_2) = L(v_1)$. Analogously, by symmetry, we may assume that $L(v_2) = L(v_3) = L(v_4)$. It follows that $L(v_1) = L(v_4)$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is 6-good, by Lemma 16, F_{16}^2 is L-colourable.

q) In F_{17}^2. $(v_9 < v_8 < v_2 < v_6 < v_4 < v_5 < v_3 < v_7 < v_1)$ is p_{17}-greedy, so by Lemma 14, F_{17}^2 is p_{17}-choosable.

□

Proof of Lemma 6.

5)] Suppose for a contradiction that v_1 and v_3 are in Z_5. Let v_5 (resp. v_6) be the neighbour of v_1 (resp. v_3) distinct from v_2 and v_4. By Lemma 5-2), $v_5 \neq v_6$. $(G[S], p_S) = (F_1, p_1)$ and $G^2[S] \subset F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$. So Lemma 24 contradicts Lemma 12.

Suppose for a contradiction that, in K, a vertex v_4 of $Y_{1,2,2}$ is adjacent to two vertices of $Y_{1,2,2}$ v_2 and v_6 by 2-edges. According to 5), $v_2 \neq v_6$. Let v_3 and v_5 be the 2-neighbours of v_4 common with v_2 and v_6 respectively, and v_1 (resp. v_7) be the neighbour of v_2 (resp. v_6) not adjacent to v_4. Set $S = \{v_1, \dots, v_6, v_7\}$.

- Assume that v_2v_4 is an edge. For $i \in \{1, 4, 7\}$, let w_i be the neighbour of v_i not in S. $(G[S], p_S) = (F_2, p_2)$ and $G^2[S] \subset F_2^2 \cup \{v_1v_4, v_4v_7, v_1v_7\}$. Thus, by Lemmas 24 and 12, $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. If $v_8 \neq w_4$, then $(G[T], p_T) = (F_3, p_3)$ and $G^2[T] \subset F_3^2 \cup \{v_4v_8\}$. So Lemma 24 contradicts Lemma 12. If not then $G[T] = G = F_4$, so G is 6-choosable, by Lemma 24. This is a contradiction.

- Assume that $v_1 = v_7$. Clearly $(G-S)^2 = G^2 - S$ because every vertex of C has at most one neighbour in $G-S$. Moreover C has no chord by Lemma 5-2), $C^2 = G^2[S]$. Finally $p_S(v_i) = 4$ if i is even and $p_S(v_i) = 3$ otherwise. C^2 is a cycle+triangle graph, thus, by Theorem 21, it is 3-choosable and so p_S-choosable. This contradicts Lemma 12.

- Assume that $v_1 \neq v_7$. For $i \in \{1, 2, 4, 6, 7\}$, let w_i be the neighbour of v_i not in S. Let $W = \{w_1, w_2, w_4, w_6, w_7\}$.

  Suppose first that $W \cap S \neq \emptyset$. Since G is simple, $w_1 \neq v_2$ and $w_7 \neq v_6$ and by Lemma 5-3), $w_1 \neq v_4$ and $w_7 \neq v_4$. Moreover, by (5), $w_1 \neq v_6$ and $w_7 \neq v_2$. Then, by symmetry, we only need to consider the cases $w_2 = v_4$ and $w_2 = v_6$.

  - Suppose that $w_2 = v_4$. Set $R = \{v_1, v_2, v_3, v_4, v_3\}$, then $(G[R], p_R) = (F_2, p_2)$ and $G^2[R] \subset F_2^2 \cup \{v_1v_5\}$. Thus Lemma 24 contradicts Lemma 12.
  - Suppose that $w_2 = v_6$. Then $(G[S], p_S) = (F_3, p_3)$. If $w_1 \neq w_7$ then $G^2[S] \subset F_3^2 \cup \{v_1v_4\}$ or $G^2[S] \subset F_3^2 \cup \{v_4v_7\}$. So Lemma 24 contradicts Lemma 12.

    Hence we may assume that $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. If $v_8 \neq w_4$, then $(G[T], p_T) = (F_4, p_4)$ and $G^2[T] \subset F_4^2 \cup \{v_4v_8\}$. So Lemma 24 contradicts Lemma 12. If $v_8 = w_4$ then $G = G[T] = F_5$ which, according to Lemma 24, is 6-choosable, a contradiction.

  Hence, we may assume that $W \cap S = \emptyset$.

  Note that by Lemma 5-2), $w_1 \neq w_2$ and $w_6 \neq w_7$.

  Suppose $w_1 = w_4 = v_8$. Then let $R = \{v_1, v_2, v_3, v_4, v_5, v_8\}$ and w_8 the neighbour of v_8. Then $(G[R], p_R) = (F_6, p_6)$. Recall that $w_2 \neq v_6$. So $G^2[R] = F_6^2$. Thus Lemma 24 contradicts Lemma 12.

  Therefore, we may assume that $w_1 \neq w_4$ and, by symmetry, $w_4 \neq w_7$.

  Suppose $w_1 = w_7$. Let $T = S \cup \{v_8\}$. Then $G[T]$ is the cycle C_8 and p_T is the function p defined in Lemma 22. So by Lemmas 22 and 12, $G^2[T] \neq C_8^2$. It follows that either $w_2 = w_6$ or $w_4 = w_8$ with w_8 be the neighbour of v_8 not in S.

- Suppose $w_2 = w_6 = v_9$ and $w_4 = w_8 = v_{10}$. Set $W = \{v_1, \dots v_{10}\}$. If $v_9 v_{10} \notin E(G)$ then $(G[W], p_W) = (F_7, p_7)$ and $G^2[W] \subset F_7^2 \cup \{v_9 v_{10}\}$; so Lemma 24 contradicts Lemma 12. If not, $G = G[W] = F_{13}$, so G^2 is 6-choosable, according to Lemma 24, a contradiction.
- Suppose $w_2 = w_4 = w_6 = v_9$. Setting $U = \{v_1, \dots v_9\}$, we have $G[U] = F_8$ and $G^2[U] = F_8^2$. Hence Lemma 24 contradicts Lemma 12.

  By symmetry, we get a contradiction if $w_2 = w_6 = w_8$, $w_2 = w_4 = w_8$ or $w_4 = w_6 = w_8$.
- Suppose $w_4 = w_8 = v_9$, $w_2 \neq v_9$, $w_6 \neq v_9$ and $w_2 \neq w_6$. Setting $U = \{v_1, \dots v_9\}$, we have $(G[U], p_U) = (F_9, p_9)$ and $G^2[U] \subset F_9^2 \cup \{v_2 v_9\}$ or $G^2[U] \subset F_9^2 \cup \{v_6 v_9\}$. Hence Lemma 24 contradicts Lemma 12.

  By symmetry, we get a contradiction if $w_2 = w_6 = v_9$, $w_4 \neq v_9$, $w_8 \neq v_9$ and $w_4 \neq w_8$.

Therefore, we may assume that $w_1 \neq w_7$.

Suppose that $w_2 = w_6 = v_8$. Let $T = S \cup \{v_8\}$. Then $(G[T], p_T) = (F_{10}, p_{10})$, and $G^2[T] \subset F_{10}^2 \cup \{v_4 v_8\}$, since w_1, w_4 and w_7 are distinct vertices. Hence Lemma 24 contradicts Lemma 12.
Therefore, we may assume that $w_2 \neq w_6$.

Suppose that $w_1 = w_6 = v_8$ and $w_2 = w_7 = v_9$. Let $U = S \cup \{v_8, v_9\}$. If $v_8 v_9 \notin E(G)$ then $(G[U], p_U) = (F_{11}, p_{11})$, and $G^2[U] \subset F_{11}^2 \cup \{v_4 v_8, v_8 v_9, v_9 v_4\}$. If not, then $(G[U], p_U) = (F_{14}, p_{14})$, and $G^2[U] = F_{14}^2$. In both cases, Lemma 24 contradicts Lemma 12.

Therefore, we may assume that $w_1 \neq w_6$ or $w_2 \neq w_7$. By symmetry, we may assume that $w_2 \neq w_7$.

Suppose $w_1 = w_6 = v_8$. Let $T = S \cup \{v_8\}$ and let w_8 be the neighbour of v_8 not in S. Then $(G[T], p_T) = (F_{12}, p_{12})$ and $G^2[T] \subset F_{12}^2 \cup \{v_4 v_8\}$. Hence Lemma 24 contradicts Lemma 12.
Therefore, we may assume that $w_1 \neq w_6$.

Hence we have $w_1 \neq w_4, w_6, w_7$, $w_2 \neq w_6, w_7$ and $w_4 \neq w_7$ then $G[S]^2 = G^2[S]$. Thus Proposition 23 contradicts Lemma 12.

□

3.3 Proof of Lemma 11

Definition 25 For $1 \leq j \leq 4$, let I_j and q_j be the graphs and function depicted Figure 4

Lemma 26 *For $1 \leq j \leq 4$, I_j^2 is q_j-choosable.*

Proof.

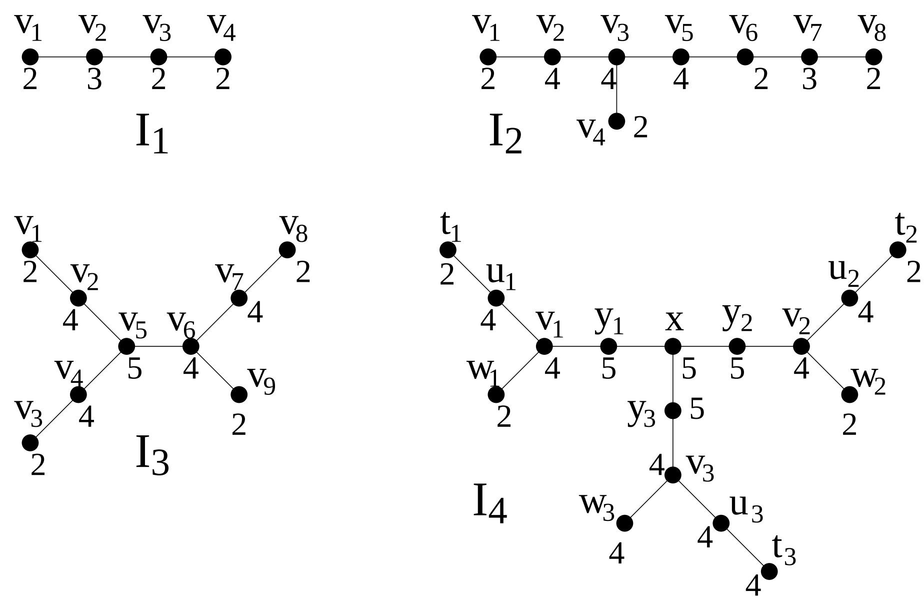

Figure 4: The graphs I_j and functions q_j, $1 \leq j \leq 4$

- Let L be a q_1-list-assignment of I_1^2. $(v_4 < v_3 < v_1 < v_2)$ and $(v_1 < v_3 < v_4 < v_2)$ are q_1-nice, so by Lemma 15, we may assume that $L(v_1) \cup L(v_4) \subset L(v_2)$. Hence $L(v_1) \cap L(v_4) \neq \emptyset$. But $(v_4 < v_1 < v_3 < v_2)$ is q_1-good. Thus by Lemma 16, I_1^2 is L-colourable.
- Let L be a q_2-list-assignment of I_2^2.

  Suppose first that $L(v_3) \not\subset L(v_1) \cup L(v_6)$. Then choose $c(v_3)$ in $L(v_3) \setminus (L(v_1) \cup L(v_6))$ and $c(v_4) \in L(v_4) \setminus \{c(v_3)\}$. Since I_1^2 is q_1-choosable, one can extend c to $\{v_5, v_6, v_7, v_8\}$. Then one can find $c(v_2) \in L(v_2) \setminus \{c(v_3), c(v_4), c(v_5)\}$ and $c(v_1) \in L(v_1) \setminus \{c(v_2), c(v_3)\} = L(v_1) \setminus \{c(v_2)\}$. So, we may assume that $L(v_3) \subset L(v_1) \cup L(v_6)$, so $L(v_3) = L(v_1) \cup L(v_6)$ and $L(v_1) \cap L(v_6) = \emptyset$.

  Now colour v_3 and v_6 the same colour $c_6 \in L(v_6)$. Then proceed greedily according to $(v_4 < v_8 < v_7 < v_5 < v_2 < v_1)$. It is possible since $c_6 \notin L(v_1)$.
- Let L be a q_3-list-assignment of I_3^2. Assign to v_5 a colour c_5 in $L(v_5) \setminus (L(v_1) \cup L(v_9))$ and to v_6 a colour in $L(v_6) \setminus (L(v_8) \cup \{c_5\})$. Then colour the remaining vertices greedily according to $(v_3 < v_4 < v_2 < v_1 < v_1 < v_9 < v_7 < v_8)$ to get an L-colouring of I_3^2.
- Let L be q_4-list-assignment of I_4^2. Pick $c(y_1)$ in $L(y_1) \setminus L(w_1)$, $c(y_2)$ in $L(y_2) \setminus (L(w_2) \cup \{c(v_1)\})$, $c(y_3)$ in $L(y_3) \setminus (L(w_3) \cup \{c(v_1), c(v_2)\})$ and $c(x)$ in $L(x) \setminus \{c(v_1), c(v_2), c(v_3)\}$. Since I_1^2 is q_1-choosable, one can extend c to a colouring of I_4^2.

□

Proof of Lemma 6.

3) Suppose that a vertex v_3 of $Y_{2,2,3}$ and v_6 of $Y_{1,2,3}$ are adjacent via a 2-edge in K. Then the subgraph of G induced by v_3, v_6 and the 2-vertices of their incident threads is I_2. Since G has girth at least 13, then $G^2[I_2] = I_2^2$ and $(G - V(I_2))^2 = G^2 - V(I_2)$, so Lemma 26 contradicts Lemma 12.

4) Suppose that a vertex v_5 of $Y_{1,3,3}$ and v_6 of $Y_{1,2,3}$ are adjacent via a 1-edge in K. Then the subgraph of G induced by v_5, v_6 and the 2-vertices of their incident threads is I_3. Then Lemma 26 contradicts Lemma 12.

5) Suppose that a vertex x of $Y_{2,2,2}$ is adjacent to three vertices v_1, v_2 and v_3 of $Y_{2,2,3}$ in K. Then the subgraph of G induced by x, v_1, v_2, v_3 and the 2-vertices of their incident threads is I_4. Then Lemma 26 contradicts Lemma 12.

□

Acknowledgement

The author would like to thank André Raspaud and Riste Škrekovski for stimulating discussions.

References

[1] Z. Dvořák, R. Škrekovski and M. Tancer, List-colouring squares of sparse subcubic graphs, IMFM Preprint PS-985, 2005.

[2] H. Fleischner and M. Stiebitz, A solution to a Colouring problem of P. Erdös, *Discrete Math.*, **101** (1992), 39–48.

[3] M. Montassier and A. Raspaud, A note on 2-facial coloring of plane graphs, *Technical report* RR-1341-05, LaBRI, 2005.

[4] G. Wegner, Graphs with given diameter and a coloring problem, *Technical Report*, University of Dortmund, Germany, 1977.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399